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by

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Fractional Differential Inclusions on Closed Sets

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Dedecation

All words cannot express gratitude, respect, love, it is all simply that i dedicate this work to:

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Hanane Addouche

Contents

Notations

$(\mathbb{R}^n, \|\cdot\|)$	The Euclidean space.
$B(x,\rho)$	The closed ball of centre x and radius ρ.
$\mathbb{B}$	The unit closed ball, i.e. $\mathbb{B} = B(0,1)$.
$\mathbb{N}$	The set of natural numbers, i.e., 0,1,...
$\mathbb{R}$	The set of real numbers.
$L^1((a,b);\mathbb{R}^n)$	The space of all integrable function defined on (a,b) with values in $\mathbb{R}^n$.
$L^\infty((a,b);\mathbb{R}^n)$	The space of measurable bounded functions defined on (a,b) with values in $\mathbb{R}^n$.
$C([a,b];\mathbb{R})$	The space of all continuous functions f defined on $[a,b]$ with values in $\mathbb{R}$, equipped with the norm $\|f\|_\infty = \sup_{t\in[a,b]} \|f(t)\|$.
$AC([a,b];\mathbb{R}^n)$	The space of all absolutely continuous functions defined on $[a,b]$ with values in $\mathbb{R}^n$.
2^Y	The family of all subset of Y.
$\Gamma(\cdot)$	The Euler Gamma function.
D_c^q	The left-sided Caputo derivative of order q.
$\mathcal{T}_k(y_0)$	The Bouligand-severi tangent set to K at y_0.
usc	Upper semicontinuous.
lsc	Lower semicontinuous.
ODE	Ordinary Differential Equations.
$\mathring{K}$	Interior of K.

Abstract

In this thesis, we investigate the viability problem of a cylindrical domain with respect to fractional differential inclusions having the form

$$D_c^q y(t) \in F(t, y(t)), \quad 0 < q < 1, \quad t \in J = [a, b), \qquad (E)$$

where $F : G \longrightarrow 2^{\mathbb{R}^n}$ is a given set-valued map such that $G = J \times K$, $K \subset \mathbb{R}^n$ is locally closed and $D_c^q y$ stands for the left–sided Caputo derivative of y of order q.

In the case when $q = 1$, the viability problem of G with respect to (E) was studied by many authors by using various frameworks and techniques. It has been shown that viability of G with respect to (E) is equivalent to an appropriate tangency concept expressed in terms of Bouligand-Severi cone.

The case when $q \in (0, 1)$ is poorly considered in literature. Sufficient conditions for viability for G with respect to (E) are provided in [8, 2], by using the Bouligand-Severi cone. Unfortunately, as it is pointed out in [13], their proofs are incorrect.

Here, we propose a new tangency condition more general then the one used in [13] and prove the viability of a cylindrical domain with respect to (E).

The result established in this thesis is submitted in journal MATHEMATICAL REPORTS; ROMANIAN ACADEMY.

Résumé

Dans cette thèse, nous avons étudié le problème de la viabilité d'un domaine cylindrique par rapport à des inclusions différentielles fractionnaires ayant la forme

$$D_c^q y(t) \in F(t, y(t)), \quad 0 < q < 1, \quad t \in J = [a, b), \qquad (E)$$

tel que $F : G \longrightarrow 2^{\mathbb{R}^n}$ est une fonction multivoque avec $G = J \times K$, $K \subset \mathbb{R}^n$ est localement fermé et $D_c^q y$ représente la dérivée de Caputo de y d'order q.

Dans le cas où $q = 1$, le problème de viabilité de G par rapport à (E) a été étudié par plusieurs auteurs en utilisant différentes approchoes et techniques. Il a été prouvé que la viabilité de G est équivalente à une condition tangentielle basé sur le cone tangent de Bouligand and severi.

Le cas où $q \in (0, 1)$ n'est pas suffisamment considéré dans la littérature. Des conditions suffisantes de viabilité de G par apport à (E) ont été établi dans [8, 2], en utilisant le cone tangent de Bouligand-Severi. Malheureusement, comme il a été signalé dans [13], les preuves sont incorrectes.

Ici, on propose une nouvelle condition tangentielle plus générale que celle proposée par [13] et on démontre la viabilité de G par rapport à (E).

Les résultats de la thèse sont soumis dans le journal MATHEMATICAL REPORTS; ROMANIAN ACADEMY.

ملخص

في هذه الأطروحة، درسنا مشكلة حياة المجال الأسطواني فيما يتعلق بالتضمينات التفاضلية الكسرية التي من الشكل

$$D_c^q y(t) \in F(t, y(t)), \quad 0 < q < 1, \quad t \in J = [a, b), \qquad (E)$$

حيث $F : G \longrightarrow 2^{\mathbb{R}^n}$ دالة متعددة مع $G = J \times K$، $K \subset \mathbb{R}^n$ مغلقة محلياً و $D_c^q y$ يمثل مشتق كابوتو لـ y من الترتيب q.

في الحالة التي يكون فيها $q = 1$، تمت دراسة مشكلة حياة G فيما يتعلق بـ (E) من قبل العديد من المؤلفين باستخدام أساليب و تقنيات مختلفة. لقد ثبت أن حياة G هي مكافئ لشرط مماسي يعتمد على مخروط المماس لبوليجاند و سيفيري.

في حالة $q \in (0,1)$ لا يتم فيها النظر بشكل كافي في الأدب. تم وضع شروط ملائمة للحياة لـ G فيما يتعلق بـ (E) في [8, 2]، باستخدام مخروط الظل بواسطة بوليجاند-سيفيري. لسوء الحظ، كما هو موضح في [13] ، فإن الأدلة غير صحيحة.

هنا، نقترح شرطاً مماسياً جديداً أكثر عمومية من ذلك الذي اقترحه [13] ويتم إثبات حياة G فيما يتعلق بـ (E).

تم نشر نتائج الأطروحة في مجلة MATHEMATICAL REPORTS; ROMANIAN ACADEMY.

Introduction

We are concerned with fractional differential inclusions of the form

$$D_c^q y(t) \in F(t, y(t)), \quad 0 < q < 1, \quad t \in J = [a, b), \tag{1}$$

where $F : G \longrightarrow 2^{\mathbb{R}^n}$ is a given set-valued map and $G = J \times K$, $K \subset \mathbb{R}^n$ is locally closed. Here and thereafter $D_c^q y$ stands for the left–sided Caputo derivative of y of order q.
We recall that $K \subset \mathbb{R}^n$ is said to be locally closed if for each $x \in K$ there exists $\rho > 0$ such that $B(x, \rho) \cap K$ is closed.

For various reasons (economical, physical, ect), solutions of (1) need to satisfy constraints, called viability constraints, which are described in the theoretical approach, by a locally closed subset.

Roughly speaking, G is said to viable with respect to (1) if for every $(t_0, y_0) \in G$, there exist $T > t_0$ with $[t_0, T] \subset [a, b)$ and a solution $y : [t_0, T] \longrightarrow \mathbb{R}^n$ of (1) satisfying $y(t_0) = y_0$ and $y(t) \in K$ for every $t \in [t_0, T]$.

Traditionally, criteria of viability for ordinary differential equations or more general inclusions have been expressed in terms of tangency condition. We recall the pioneering contribution of Nagumo [10] in 1942, who considered the particular case of a differential equation

$$y'(t) = f(t, y(t)), \qquad (q = 1), \tag{2}$$

with $f : G \longrightarrow \mathbb{R}^n$ is continuous and K is locally closed, and proved that a necessary and sufficient condition in order that G be viable with respect to (2) is the tangency condition:

$$\forall (t_0, y_0) \in G,\ f(t_0, y_0) \in \mathcal{T}_K((t_0, y_0)),$$

where $\mathcal{T}_K((t_0, y_0))$ stands for the classical Bouligand–Severi tangent cone to K at (t_0, y_0).
We recall that the vector $\eta \in \mathbb{R}^n$ is tangent in the sense of Bouligand–Severi to the set K at the point y_0 if

$$\liminf_{h \to o^+} \frac{1}{h} dist\left(y_0 + h\eta; K\right) = 0.$$

Here and thereafter, $\mathrm{dist}(v, K)$ stands for the distance from $v \in \mathbb{R}^n$ to the set K, i.e.

$$\mathrm{dist}(v, K) = \inf_{l \in K} \|v - l\|.$$

The result of Nagumo was extended to the set-valued setting

$$y'(t) \in F(t, y(t)), \tag{3}$$

by Bebernes–Schuur [16]. Actually, Bebernes–Schuur showed that if F is upper semicontinuous with nonempty, convex and compact values then a necessary and sufficient condition in order that G be viable with respect to (3) is the tangency condition

$$\forall (t_0, y_0) \in G,\ F(t_0, y_0) \cap \mathcal{T}_K((t_0, y_0)) \neq \emptyset.$$

In the fractional setting, the situation is more complicated. Indeed, up to our knowledge, the problem of viability was studied by Vasundgara Devi and Lakshmikantham [8] and Girejko et al. [2] in which some explicit sufficient conditions are obtained. However, it seems that the proof of these results are incomplete as pointed out in [13].

In [13], the authors obtain sufficient abstract conditions ensuring viability. However, it seems not easy to deduce from their conditions an explicit one.

In this thesis, we combine the approach of [13] with the one developed in [11], to get a result referring to viability in the fractional setting.

The thesis is divided into three chapters, in chapter 1, we give some result on set-valued analysis which we need in the sequel.

In chapter 2, we investigate the tangency concept which introduced by Carja et al. [13]. We will also recall the first tangency concepts was introduced by Bouligand [4] and Severi [3].

In chapter 3, we are interested in the study of viability problem for fractional differential inclusions (1). To this end, we first introduce a definition of viability. Then, we will show that the viability is equivalent to an appropriate tangency condition.

1 Introduction to set-valued analysis

In this chapter, we recall some definitions and results of set-valued analysis we need in the sequel. For more details, we refer the reader to [6, 7, 9, 12, 15] and references therein.

1.1 Set-valued maps

We gather in this paragraph some proprieties of set-valued maps needed on the study of fractional differential inclusions.

Definition 1.1
Let X and Y be two nonempty sets. By a set-valued map we mean an application: $F : X \longrightarrow 2^Y$, where 2^Y denotes the family of all subsets of Y.

Notations:

If F is a set-valued map then one may use one of these notations:

$$F : X \rightsquigarrow Y \text{ or } F : X \rightrightarrows Y.$$

For a set-valued map $F : X \longrightarrow 2^Y$, we define:

1) ***Domain*** of F: $Dom(F) = \{x \in X \mid F(x) \neq \emptyset\}$.

2) ***Image*** of F: $Im(F) = \bigcup_{x \in Dom(F)} F(x)$.

3) ***Graph*** of F: $Graph(F) = \{(x, y) \in X \times Y, y \in F(x)\}$.

4) ***The weak inverse image*** of subset $A \subset Y$: $F^{-}(A) = \{x \in Dom(F) : F(x) \cap A \neq \emptyset\}$.

5) ***The strong inverse image*** of subset $A \subset Y$: $F^{+}(A) = \{x \in Dom(F) : F(x) \subset A\}$.

6) ***The inverse set-valued map*** F^{-1} defined by: $F^{-1} : Y \to 2^X$, $x \in F^{-1}(y)$ if and only if $y \in F(x)$.

Two illustrative examples are given below to clarify the above notions.

Example 1 : Let $F:\mathbb{R}\longrightarrow 2^{\mathbb{R}}$, be the set-valued map defined by:

$$F(x)=[x-1,x+1],$$

we have

$$\begin{cases} Dom(F)=\mathbb{R}, \\ Im(F)=\bigcup_{x\in\mathbb{R}}[x-1,x+1]=\mathbb{R}, \\ Graph(F)=\big\{(x,y)\in\mathbb{R}^2;\ y\in[x-1,x+1]\big\}. \end{cases}$$

Let us evaluate $F^{-}(A)$ and $F^{+}(A)$ where $A=[0,+\infty[$. We recall that

$$\begin{cases} F^{-}(A)=\{x\in\mathbb{R}; F(x)\cap A\neq\emptyset\}, \\ F^{+}(A)=\{x\in\mathbb{R}. F(x)\subset A\}. \end{cases}$$

Consequently, one has

$$x\in F^{-}(A)\Leftrightarrow F(x)\cap A\neq\emptyset\Longleftrightarrow[x-1,x+1]\cap[0,+\infty[\neq\emptyset.$$

Hence, $F^{-}(A)=[-1,+\infty[$. Further,

$$x\in F^{+}(A)\Leftrightarrow F(x)\subset A\Longleftrightarrow[x-1,x+1]\subset[0,+\infty[.$$

That means $F^{+}(A)=[1,+\infty[$.

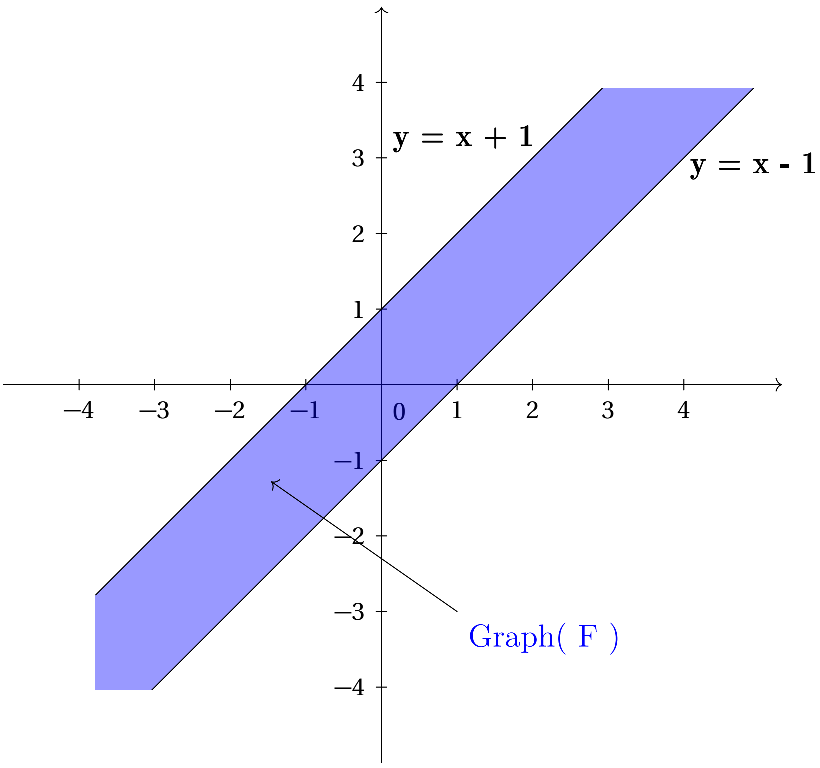

Figure 1.1: *Graph of the set-valued map F.*

Example 2 : Let $F : \mathbb{R} \longrightarrow 2^{\mathbb{R}}$ be a set-valued map defined by:

$$F(x) = \left[-\sqrt{x}, \sqrt{x}\right].$$

One gets

$$\begin{cases} Dom(F) = \mathbb{R}_+, \\ Im(F) = \bigcup_{x \in \mathbb{R}_+} \left[-\sqrt{x}, \sqrt{x}\right] = \mathbb{R}, \\ Graph(F) = \left\{(x, y) \in \mathbb{R}^2;\ y \in \left[-\sqrt{x}, \sqrt{x}\right]\right\}, \\ F^-([0,1]) = \{x \in \mathbb{R} : F(x) \cap [0,1] \neq \emptyset\} = \mathbb{R}_+, \\ F^+([0,1]) = \{x \in \mathbb{R} : F(x) \subset [0,1]\} = \{0\}. \end{cases}$$

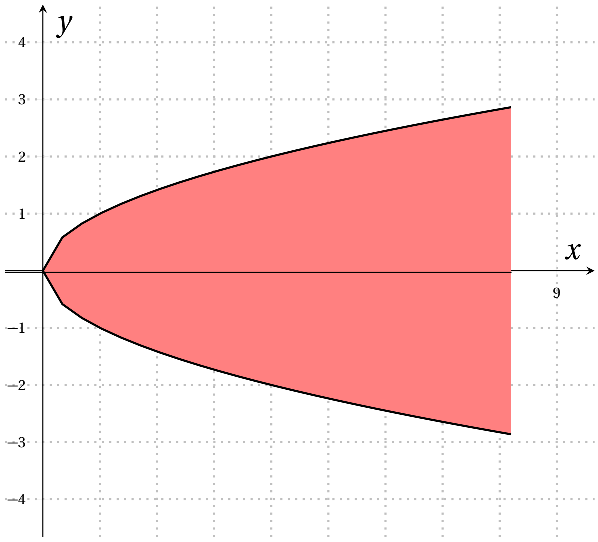

Figure 1.2: *Graph of the set-valued map F*

1.2 Notions of continuity

Defining continuity of set valued maps is one of problems that we may encounter in set-valued analysis. We recall that if $f : X \to Y$ is a given single valued map where X and Y are two topological spaces, then continuity of f at $x_0 \in X$ is characterized by two equivalent properties:

(a) for every neighbourhood N of $f(x_0)$, there exists a neighbourhood M of x_0 such that $f(M) \subset N$;

(b) for any generalised sequence of elements x_μ converging to x_0, the sequence $f(x_\mu)$ converges to $f(x_0)$.

These two properties can be adapted to the case of set valued map $F : X \to 2^Y$ as follows:

(A) for any neighbourhood N of $F(x_0)$ there exists a neighbourhood M of x_0 such that $F(M) \subset N$;

(B) for any generalised sequence of elements x_μ converging to x_0 and for each $y_0 \in F(x_0)$, there exists a sequence of elements $y_\mu \in F(x_\mu)$ that converges to y_0.

These two properties are no longer equivalent. More precisely we have the following definitions. In all that follows X and Y are two topological spaces and $F : X \to 2^Y$ is a given set valued-map.

1.2.1 Upper semicontinuity

Definition 1.2

1) We say that F is upper semicontinuous, (***usc*** for shortness) at $x_0 \in X$, if for any open V in Y containing $F(x_0)$, there exists a neighbourhood $N(x_0)$ of x_0 such that:

$$\forall x \in N(x_0), \quad F(x) \subset V.$$

2) F is upper semicontinuous on X, if it is upper semicontinuous at every $x_0 \in X$.

Example 3 : The set-valued map $F : \mathbb{R} \longrightarrow 2^{\mathbb{R}}$ defined by

$$F(x) = \begin{cases} \{0\} & \text{for} \quad x \neq 0 \\ [-1,1] & \text{for} \quad x = 0 \end{cases}$$

is upper semicontinuous on $\mathbb{R}$. Indeed, if $x_0 \neq 0$, then $F(x_0) = \{0\}$. Let V be an open set containing $F(x_0)$, i.e. containing $\{0\}$. We take $N(x_0) = \mathbb{R}^\star$ a neighbourhood of x_0. Then, it is clear that:

$$\forall x \in N(x_0) : F(x) \subset V.$$

Furher, if $x_0 = 0$, then $F(0) = [-1,1]$. Let V be an open set containing $F(0)$, i.e. containing $[-1,1]$. We take $N(x_0) = \mathbb{R}$ which is a neighbourhood of x_0. Then, it is clear that

$$\forall x \in N(x_0) : F(x) \subset V.$$

The next proposition gives us an other characterizations of the concept of upper semicontinuous set-valued map.

Proposition 1.2.1 *[15]*
If F is upper semicontinuous on X, then $F^{-}(A)$ is closed in X whenever $A \subset Y$ is closed.

1.2.2 Lower semicontinuity

Definition 1.3

1) We say that F is lower semicontinuous, (***lsc*** for shortness) at $x_0 \in X$ if for any $y_0 \in F(x_0)$ and any neighbourhood $N(y_0)$ of y_0, there exists neighbourhood $N(x_0)$ of x_0 such that:

$$\forall x \in N(x_0), \quad F(x) \cap N(y_0) \neq \emptyset.$$

2) F is lower semicontinuous on X, if it is lower semicontinuous at every $x_0 \in X$.

Example 4 : The set-valued map $F : \mathbb{R} \longrightarrow 2^{\mathbb{R}}$ defined by

$$F(x) = \begin{cases} \{0\} & \text{for} \quad x = 0 \\ [-1,1] & \text{for} \quad x \neq 0 \end{cases}$$

is lower semicontinuous on $\mathbb{R}$. Indeed, if $x_0 \neq 0$, for any neighbourhood $N(y_0)$ of $y_0 \in [-1,1]$, we have

$$F(x) \cap N(y_0) \neq \emptyset, \quad \forall x \in \mathbb{R}^{\star}.$$

If $x_0 = 0$, for any neighbourhood $N(y_0)$ of $y_0 = 0$, we have

$$F(x) \cap N(y_0) \neq \emptyset, \quad \forall x \in \mathbb{R}.$$

As the case of upper semicontinuous function, the next proposition gives us an other characterizations of the concept of upper semicontinuous.

Proposition 1.2.2 *[15]*
If F is lower semicontinuous on X, then $F^{+}(A)$ is closed in X whenever $A \subset Y$ is closed.

Definition 1.4
A set-valued map $F : X \longrightarrow 2^{Y}$ is said to be continuous at $x_0 \in X$ if it is both ***usc*** and ***lsc*** at x_0. We say that F is continuous on X if it is continuous at every point $x_0 \in X$.

If X and Y are two normed spaces, then we can give others concepts of continuity.

Definition 1.5

1) We say that F is $(\varepsilon\text{-}\delta)$ upper semicontinuous ($(\varepsilon\text{-}\delta)$-$usc$ for shortness) at $x_0 \in X$, if for every $\varepsilon > 0$, there exist $\delta = \delta(x_0, \varepsilon) > 0$, such that

$$\forall x \in x_0 + \delta\mathbb{B}, \quad F(x) \subset F(x_0) + \varepsilon\mathbb{B},$$

where $\mathbb{B}$ is the closed unit ball of X.

2) F is said to be $(\varepsilon\text{-}\delta)$ upper semicontinuous on X, if it is $(\varepsilon\text{-}\delta)$ upper semicontinuous at every $x_0 \in X$.

Lemma 1.2.1: Let $\mathbb{B}$ be closed unit ball of the normed space X. For every x and y in $\mathbb{R}$ one has

$$x\mathbb{B} \subset y\mathbb{B} \Longleftrightarrow |x| \leq |y|.$$

Proof

• First, assume that $x\mathbb{B} \subset y\mathbb{B}$ and prove that $|x| \leq |y|$.

Since $x\mathbb{B} \subset y\mathbb{B}$, this means that

$$\forall b \in S, \exists b' \in \mathbb{B};\ xb = yb',$$

such that $S = \{b \in \mathbb{R}^n;\ \|b\| = 1\}$, with $S \subset \mathbb{B}$.
We have

$$\begin{aligned}\|xb\| = \|yb'\| &\Leftrightarrow |x|\|b\| = |y|\|b'\|\\ &\Leftrightarrow |x|\|b\| \leq |y| \Rightarrow |x| \leq |y|.\end{aligned}$$

• Now, We assume that $|x| \leq |y|$ and we prove that $x\mathbb{B} \subset y\mathbb{B}$.

Let

$$z \in x\mathbb{B} \Longleftrightarrow z = xb \text{ with } \|b\| \leq 1.$$

We try to prove that $z \in y\mathbb{B}$. Without loss of generality we assume $y \neq 0$.
Let

$$xb = y\frac{x}{y}b = yb',$$

where $b' = \frac{x}{y}b \in \mathbb{B}$.
So

$$\begin{aligned}\|b'\| &= \left|\frac{x}{y}\right|\|b\|\\ &\leq \frac{|x|}{|y|} \leq 1.\end{aligned}$$

Finally $z \in y\mathbb{B}$, $x\mathbb{B} \subset y\mathbb{B}$.
Remark: if $y = 0$ the relation is trivial. ■

Example 5 : The set-valued map $F : \mathbb{R} \longrightarrow 2^{\mathbb{R}^2}$ defined by

$$F(x) = x\mathbb{B},$$

is $(\varepsilon$-$\delta)$ upper semicontinuous on $\mathbb{R}$. Let $x_0 \in \mathbb{R}$ be arbitrary, and $\varepsilon > 0$.

$$\begin{aligned}\forall x \in \mathbb{R} :\ F(x) = x\mathbb{B} &= (x - x_0 + x_0)\mathbb{B}\\ &= (x - x_0)\mathbb{B} + x_0\mathbb{B}\\ &= F(x_0) + (x - x_0)\mathbb{B}.\end{aligned}$$

We have

$$\begin{aligned}F(x) \subset F(x_0) + \varepsilon\mathbb{B} &\Longleftrightarrow F(x_0) + (x - x_0)\mathbb{B} \subset F(x_0) + \varepsilon\mathbb{B}\\ &\Longleftrightarrow (x - x_0)\mathbb{B} \subset \varepsilon\mathbb{B}\\ &\Longleftrightarrow |x - x_0| \leq \varepsilon.\end{aligned}$$

Then we take $\delta = \varepsilon$.

Clearly, a set-valued map that is ***usc*** at $x_0 \in X$ is also $(\varepsilon$-$\delta)$-***usc*** at $x_0 \in X$. In fact, $F(x_0)+\varepsilon\mathbb{B}$ and $x_0+\delta\mathbb{B}$ are special neighborhoods respectively for $F(x_0)$ and x_0. The converse is not true as we can see from the example below (see e.g.,[6, p.45])

Example 6 : The set-valued map $F:\mathbb{R}\longrightarrow 2^{\mathbb{R}^2}$ defined by

$$F(x)=\{(x,y);\ y\in\mathbb{R}\},$$

is $(\varepsilon$-$\delta)$ upper semicontinuous at 0 but not upper semicontinuous at 0. Let check that F is $(\varepsilon$-$\delta)$ upper semicontinuous at 0. One has

$$\begin{aligned} F(x)\subset F(0)+\varepsilon\mathbb{B} &\Longleftrightarrow \{(x,y);\ y\in\mathbb{R}\}\subset\{(0,y);\ y\in\mathbb{R}\}+\varepsilon\mathbb{B}\\ &\Longleftrightarrow \{(x,0)\}+\{(0,y);\ y\in\mathbb{R}\}\subset\{(0,y);\ y\in\mathbb{R}\}+\varepsilon\mathbb{B}\\ &\Longleftrightarrow x\mathbb{B}+\{(0,y);\ y\in\mathbb{R}\}\subset\{(0,y);\ y\in\mathbb{R}\}+\varepsilon\mathbb{B}\\ &\Longleftrightarrow x\mathbb{B}\subset\varepsilon\mathbb{B}. \end{aligned}$$

Its suffices to take $\delta=\varepsilon$. Therefore, F is $\varepsilon-\delta$ usc on 0. However, F is not usc at 0. It suffices to take V defined by

$$V=\left\{(x,y);\ |y|<\frac{1}{|x|}\right\}\cup F(0).$$

V is open set containing $F(0)$, but one remarks that for every $x\neq 0$, $F(x)\not\subset V$.

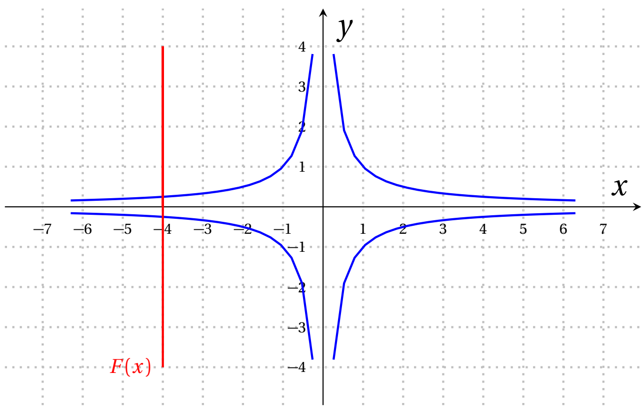

Figure 1.3: *The Graph of V.*

The next theorem shows situations in which we have equivalence between the two notions.

Theorem 1.1: [6]
If $F(x_0)$ is compact, then F is upper semicontinuous at $x_0\in X$ if and only if F is $(\varepsilon$-$\delta)$ upper semicontinuous at $x_0\in X$.

Definition 1.6

1) We say that F is $(\varepsilon\text{-}\delta)$ lower semicontinuous, ($(\varepsilon\text{-}\delta)$-$lsc$ for shortness) at $x_0 \in X$, if for every $\varepsilon > 0$, there exists $\delta = \delta(x_0, \varepsilon) > 0$ such that

$$\forall x \in x_0 + \delta\mathbb{B}, \qquad F(x_0) \subset F(x) + \varepsilon\mathbb{B},$$

where $\mathbb{B}$ is the closed unit ball on X.

2) F is said to be $(\varepsilon\text{-}\delta)$ lower semicontinuous on X, if it is $(\varepsilon\text{-}\delta)$ lower semicontinuous at every $x_0 \in X$.

Example 7 : The set-valued map $F : \mathbb{R} \longrightarrow 2^{\mathbb{R}^2}$ defined by

$$F(x) = \sqrt{x}\mathbb{B},$$

is $(\varepsilon\text{-}\delta)$ lower semicontinuous at 0. Let $\varepsilon > 0$.

$$\forall x \geq 0 : \ F(0) \subset F(x) + \varepsilon\mathbb{B}$$
$$\{0\} \subset (\sqrt{x} + \varepsilon)\mathbb{B} \cdots\cdots (\star)$$

Clearly $(\star)$ is always satisfied, we may take $\varepsilon = \delta$.

Clearly, a set-valued map$F$ which is $(\varepsilon\text{-}\delta)$ lower semicontinuous at $x_0 \in X$ it is also lower semicontinuous at $x_0 \in X$. The converse is not always true The converse is not true as we can see from the example below.

Example 8 : The set-valued map $F : [0,1] \longrightarrow 2^{\mathbb{R}^2_+}$ defined by

$$F(x) = \{[t, xt];\ t \in \mathbb{R}\},$$

is lower semicontinuous but not $(\varepsilon\text{-}\delta)$ lower semicontinuous. See for instance [15, P.61].

Theorem 1.2: [6]
If $F(x_0)$ is compact, F is lower semicontinuous at $x_0 \in X$ if and only if it is $(\varepsilon\text{-}\delta)$ lower semicontinuous at $x_0 \in X$.

Definition 1.7
A set-valued map $F : X \longrightarrow 2^Y$ is said to be $(\varepsilon\text{-}\delta)$-continuous at $x_0 \in X$ if it is both $(\varepsilon\text{-}\delta)$-***usc*** and $(\varepsilon\text{-}\delta)$-***lsc*** at x_0. We say that F is $(\varepsilon\text{-}\delta)$-continuous on X if it is $(\varepsilon\text{-}\delta)$-continuous at every point $x_0 \in X$.

1.2.3 Selections

Definition 1.8
Let $F : X \to 2^Y$ be a set-valued map. We say that $f : X \longrightarrow Y$ is a selection of F if:

$$\forall x \in X, \quad f(x) \in F(x).$$

Example 9 : Let X be the set of all students of the faculty of Science and Technology at Khemis Miliana University.
We define a set-valued map $F : X \longrightarrow 2^X$ as follows:
If $x \in X$, then $F(x)$ contains all students which have the same level as x.
If x is a student of Master 2 Mathematical Analysis and Applications, then

$$F(x) = \{\text{student of Master 2 Mathematical Analysis and Applications}\}.$$

We can define a selection of F as follows: $f : X \longrightarrow Y$
$f(x)$= the valedictorian of students Master 2 Mathematical Analysis and Applications.

Thanks to the choice axiom, each set-valued map admits at least one selection.
However, find selection with some regularity (continuity, measurability,...) is a delicate problem.
The next example illustrates the above results.

Example 10 : The set-valued map $F : \mathbb{R} \longrightarrow 2^{\mathbb{R}}$ defined by

$$F(x) = \begin{cases} \{-1\} & \text{for} \quad x < 0 \\ [-1,1] & \text{for} \quad x = 0 \\ \{1\} & \text{for} \quad x > 0, \end{cases}$$

is upper semicontinuous in $\mathbb{R}$ with compact, convex values, however each selection of F is not continuous.
Let $f : \mathbb{R} \longrightarrow \mathbb{R}$, defined by

$$f(x) = \begin{cases} -1, & x < 0 \\ y_0 \in [-1,1] & \\ 1, & x > 0. \end{cases}$$

For all $x \in \mathbb{R}$, $f(x) \in F(x)$.
for any choise of y_0 the selection f is not continuous.

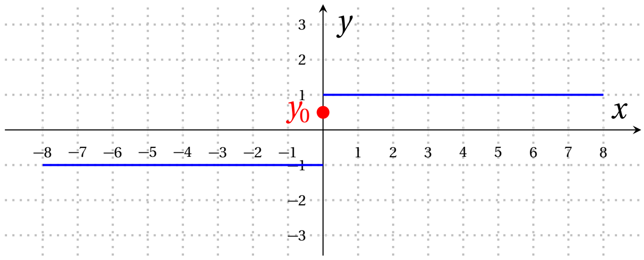

Figure 1.4: *Graph representing selection of the set-valued map F.*

The following theorem gives us conditions for the existence of continuous selections for lower semicontinuous set-valued maps.

Theorem 1.3: (Michael) [9]
Let $F : X \longrightarrow 2^Y$ be lower semicontinuous set-valued map with convex closed values. F admits a continuous selection, i.e. there exist $f : X \longrightarrow Y$ continuous such that

$$\forall x \in X, \qquad f(x) \in F(x).$$

1.3 Measurability

We study in this paragraph measurable set-valued maps and give result concerning the existence of measurable selections.

Let Ω be nonempty set and $\mathscr{A}$ be a family of subsets of Ω. Recall that $\mathscr{A}$ is said to be σ-algebra in Ω if it satisfies the following properties:

(i) $\emptyset \in \mathscr{A}$,

(ii) if $B \in \mathscr{A}$, then $B^c \in \mathscr{A}$, where B^c is the complement of B with respect to Ω,

(iii) if $(B_i)_{i\in\mathbb{N}} \subset \mathscr{A}$, then $\bigcup_{i\in\mathbb{N}} B_i \in \mathscr{A}$.

In this case the pair $(\Omega, \mathscr{A})$ is called measurable space.

Definition 1.9
Let $(\Omega, \mathscr{A})$ be measurable space and let $\mu : \mathscr{A} \longrightarrow \mathbb{R}_+ \cup \{+\infty\}$. The function μ is said to be measure on Ω if it satisfies the following properties:

(i) $\mu(\emptyset) = 0$,

(ii) if $(B_i)_{i\in\mathbb{N}}$ is a countable sequence of pairwise disjoint sets in $\mathscr{A}$, then

$$\mu(\bigcup_{i\geq 1} B_i) = \sum_{i\geq 1} \mu(B_i).$$

Remark 1.3.1

If μ is a measure in Ω, then the triple $(\Omega, \mathscr{A}, \mu)$ is called a measure space.

Definition 1.10
Let $(\Omega, \mathscr{A})$ be a measurable space and Y be a metric space.

(i) The set-valued function $F : \Omega \longrightarrow 2^Y$ is said to be strongly measurable, if $F^-(C) \in \mathscr{A}$ for every closed set $C \subset Y$, where

$$F^-(C) = \{x \in \Omega : F(x) \cap C \neq \emptyset\}.$$

(ii) The set-valued function $F : \Omega \longrightarrow 2^Y$ is said to be measurable, if $F^-(O) \in \mathscr{A}$ for every open set $O \subset Y$, where

$$F^-(O) = \{x \in \Omega : F(x) \cap O \neq \emptyset\}.$$

Proposition 1.3.1 *[15]*
Let $F : \Omega \longrightarrow 2^Y$ be a set-valued map strongly measurable, then F is measurable.

The converse of the proposition is in general false, i.e. measurability does not imply strong measurability, as we can see from the example below.

Example 11 : Let $\Omega = Y = [0,1]$ and let $\mathscr{A}$ be the σ–algebra of Lebesgue measurable subsets of $[0,1]$. We take $A \subset [0,1]$ nonmeasurable and we define the set–valued function $F : \Omega \longrightarrow 2^Y$ by

$$F(x) = \begin{cases} [0,1) & \text{if} \quad x \in A^c \\ [0,1] & \text{if} \quad x \in A. \end{cases}$$

Let $O \subset Y$ be an open set, so we have $F^-(O) = \Omega$, then F is measurable. However, F is not strongly measurable because

$$F^-(\{1\}) = A \notin \mathscr{A}.$$

1.3.1 Measurable selections

One issue that we face in set-valued analysis, is the existence of measurable selection of a given measurable set valued map. The "Kuratowski–Ryll Nardzewski theorem", answers this question.

Theorem 1.4: [15, 7]
Let $(\Omega, \mathscr{A})$ be a measurable space and let Y be complete separable metric space. If $F : \Omega \longrightarrow 2^Y$ is a measurable set–valued function with nonempty and closed values, then F admits at least one measurable selection.

2 Tangency concepts

This chapter is devoted to the study of some concepts of tangency which proves useful to characterize the concept of viability of a set with respect to fractional differential inclusions. We mainly recall the notion of vector tangent to a set at a given point introduced by Bouligand [4] and Severi [3] at the same time in 1932; the notion of set tangent to a set at a given point introduced by Carja et al in 2006 and the more general one, which is appropriate to fractional setting, introduced by Carja et al [13] in 2014.

2.1 Tangent vectors

Definition 2.1
Let $K \subset \mathbb{R}^n$ and $G = [a, b) \times K$. Let $(t_0, y_0) \in G$. The vector $\eta \in \mathbb{R}^n$ is tangent in the sense of Bouligand–Severi to the set K at the point (t_0, y_0) if

$$\liminf_{h \to 0^+} \frac{1}{h} \mathrm{dist}(y_0 + h\eta; K) = 0. \tag{2.1}$$

Here and thereafter, $\mathrm{dist}(v, K)$ stands for the distance from $v \in \mathbb{R}^n$ to the set K, i.e.

$$\mathrm{dist}(v, K) = \inf_{l \in K} \|v - l\|.$$

We denote by $\mathcal{T}_K((t_0, y_0))$ the set of all vectors tangent in the sense of Bouligand–Severi to the set K at the point (t_0, y_0).

Remark 2.1.1

Without loss of generality, one may denotes $\mathcal{T}_K((t_0, y_0))$ by $\mathcal{T}_K(y_0)$. However, if we connect the tangency concept with a differential equation or inclusion, one need to consider $\mathcal{T}_K((t_0, y_0))$ instead of $\mathcal{T}_K(y_0)$.

Example 12 : Let $K = [a, b]$. One shows that

$$\mathcal{T}_{[a,b]}(a) = \mathbb{R}_+ \text{ and } \mathcal{T}_{[a,b]}(b) = \mathbb{R}_-.$$

Example 13 : Let $K = \{(x,y) \in \mathbb{R}^2 : x^2+y^2=1\}$. Let us compute $\mathcal{T}_K((1,0))$. If $\eta \in \mathcal{T}_K((1,0))$, then

$$\liminf_{h\to 0^+} \frac{1}{h}\operatorname{dist}\big((1,0)+h\eta;K\big)=0.$$

Assume that $\eta=(\eta_1,\eta_2)$. We have:

$$\begin{aligned}\operatorname{dist}\big((1+h\eta_1,h\eta_2);K\big) &= |\operatorname{dist}\big((1+h\eta_1,h\eta_2);(0,0)\big)-1\,| \\ &= |\sqrt{(1+h\eta_1)^2+(h\eta_2)^2}-1\,|.\end{aligned}$$

Then

$$\begin{aligned}&\liminf_{h\to 0^+}\frac{1}{h}\operatorname{dist}\big((1+h\eta_1,h\eta_2);K\big)=0\\ \Leftrightarrow\ &\lim_{h\to 0^+}\frac{|\sqrt{(1+h\eta_1)^2+(h\eta_2)^2}-1\,|}{h}=0\\ \Leftrightarrow\ &\liminf_{h\to 0^+}\frac{|(1+h\eta_1)^2+(h\eta_2)^2-1\,|}{h(\sqrt{(1+h\eta_1)^2+(h\eta_2)^2}+1)}=0\\ \Leftrightarrow\ &\liminf_{h\to 0^+}\frac{|h\eta_1^2+2\eta_1+h\eta_2^2\,|}{\sqrt{(1+h\eta_1)^2+(h\eta_2)^2}+1}=0\\ \Leftrightarrow\ &\frac{|2\eta_1|}{2}=0 \Leftrightarrow |\eta_1|=0.\end{aligned}$$

Hence

$$\mathcal{T}_K((1,0))=\{(0,\eta_2);\quad \eta_2\in\mathbb{R}\}$$

.

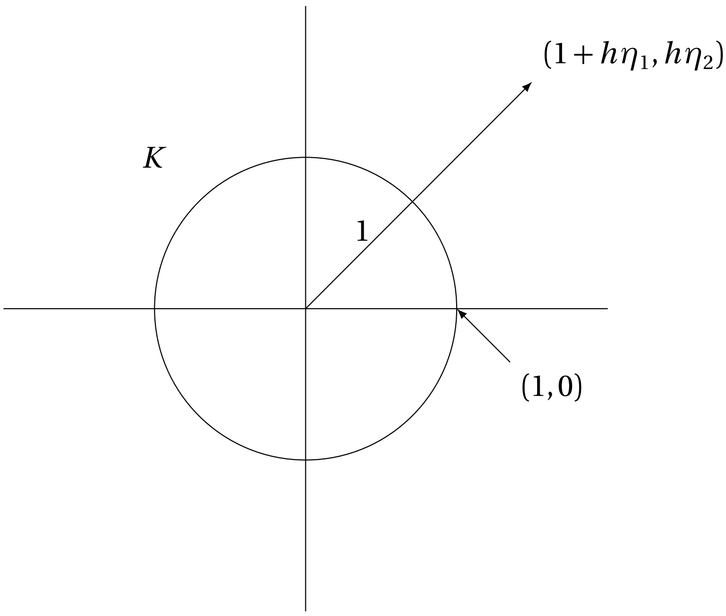

Figure 2.1: *Example of a Bouligand–Severi tangent set to K at the point* $(1,0)$.

Proposition 2.1.1
For each $y_0 \in K$, the set $\mathcal{T}_K(y_0)$ is a closed cone.

Proof

We recall that $\mathcal{T}_K(y_0)$ is a cone if for each $\eta \in \mathcal{T}_K(y_0)$, and for each $\lambda > 0$ then $\lambda\eta \in \mathcal{T}_K(y_0)$.
• First, let us prove that $\mathcal{T}_K(y_0)$ is a cone. Let $\eta \in \mathcal{T}_K(y_0)$ and $\lambda > 0$. One has

$$\begin{aligned}\liminf_{h\to 0^+} \frac{1}{h}\operatorname{dist}(y_0 + h\lambda\eta; K) &= \liminf_{s\to 0^+} \frac{\lambda}{s}\operatorname{dist}(y_0 + s\eta; K) \\ &= \lambda \liminf_{s\to 0^+} \frac{1}{s}\operatorname{dist}(y_0 + s\eta; K) = 0.\end{aligned}$$

Hence $\lambda\eta \in \mathcal{T}_K(y_0)$.
• Second, it remains to prove that $\mathcal{T}_K(y_0)$ is a closed set. Let $(\eta_n)_n$ be a sequence in $\mathcal{T}_K(y_0)$ converging to η. We have to prove that $\eta \in \mathcal{T}_K(y_0)$. Indeed,

$$\begin{aligned}\frac{1}{h}\operatorname{dist}\big(y_0 + h\eta; K\big) &\le \frac{1}{h}\|(y_0 + h\eta) - (y_0 + h\eta_n)\| + \frac{1}{h}\operatorname{dist}\big(y_0 + h\eta_n; K\big) \\ &\le \|\eta - \eta_n\| + \frac{1}{h}\operatorname{dist}\big(y_0 + h\eta_n; K\big),\end{aligned}$$

for each $n \in \mathbb{N}$ and every $h > 0$. Therefore,

$$\liminf_{h\to 0^+} \frac{1}{h}\operatorname{dist}(y_0 + h\eta; K) \le \|\eta - \eta_n\|,$$

for each $n \in \mathbb{N}$. Now we let $n \to +\infty$, we then get

$$\liminf_{h\to 0^+} \frac{1}{h}\operatorname{dist}(y_0 + h\eta; K) = 0.$$

This means that $\eta \in \mathcal{T}_k(y_0)$. ■

In what follows, we shall give some characterizations of the set $\mathcal{T}_K(y_0)$.

Proposition 2.1.2
Let $K \subset \mathbb{R}^n$, $y_0 \in K$ and $B(0,\varepsilon)$ be the closed ball in $\mathbb{R}^n$ of center 0 and of radius $\varepsilon > 0$. The following conditions are equivalent:

(i) the vector $\eta \in \mathbb{R}^n$ is tangent to K at y_0;

(ii) for every $\varepsilon > 0$ there exist $h \in (0,\varepsilon)$ and $p_h \in B(0,\varepsilon)$ such that

$$y_0 + h(\eta + p_h) \in K;$$

(iii) for every $\tilde{\varepsilon} > 0$ there exist $\tilde{h} \in (0,\tilde{\varepsilon})$ and $p_{\tilde{h}} \in B(0,\tilde{\varepsilon})$ with the property

$$y_0 + \frac{\tilde{h}^q}{\Gamma(q+1)} \cdot (\eta + p_{\tilde{h}}) \in K;$$

(iv) there exist two sequences $(h_n)_n$ in $\mathbb{R}_+$ and $(p_n)_n$ in $\mathbb{R}^n$ with $\lim_{n\to\infty} h_n = 0$ and $\lim_{n\to\infty} p_n = 0$, such that

$$y_0 + \frac{h_n^q}{\Gamma(q+1)} \cdot (\eta + p_n) \in K \qquad \text{for each} \quad n \in \mathbb{N}.$$

We recall that Γ stands for the Euler gamma function defined by

$$\forall z \in \mathbb{R}, \quad \Gamma(z) = \int_0^{\infty} t^{z-1} e^{-t} \, \mathrm{d}t.$$

Proof

(i) $\Longrightarrow$ (ii). Let $\eta \in \mathbb{R}^n$ be tangent to K at y_0. This means that,

$$\liminf_{h\to 0^+} \frac{1}{h} \operatorname{dist}(y_0 + h\eta; K) = 0,$$

or equivalently,

$$\sup_{\varepsilon>0} \inf_{h\in(0,\varepsilon)} \frac{1}{h} \operatorname{dist}(y_0 + h\eta; K) = 0.$$

Hence, for each $\varepsilon > 0$

$$\inf_{h\in(0,\varepsilon)} \frac{1}{h} \operatorname{dist}(y_0 + h\eta; K) = 0$$

Therefore, for each $\varepsilon > 0$, there exists $h \in (0,\varepsilon)$ such that

$$\frac{1}{h} \operatorname{dist}(y_0 + h\eta; K) \leq \varepsilon.$$

This is true if and only if there exists $z \in K$, such that

$$\frac{1}{h} \| y_0 + h\eta - z \| \leq \varepsilon.$$

Now, let us define

$$p_h = \frac{z - y_0 - h\eta}{h},$$

then we get $\|p_h\| \leq \varepsilon$ and

$$z = y_0 + h(\eta + p_h) \in K.$$

(ii) $\Longrightarrow$ (iii). Now, assume that (ii) holds true. Let $\tilde{\varepsilon} > 0$. Take $\tilde{\varepsilon} = \varepsilon$. Then, there exists $h \in (0, \varepsilon)$ such that $y_0 + h(\eta + p_h) \in K$ holds. We construct $\tilde{h} = \left[h \cdot \Gamma(q+1)\right]^{\frac{1}{q}}$, as we have $h < \varepsilon$, thus $h \cdot \Gamma(q+1) < \varepsilon \cdot \Gamma(q+1)$ and consequently $\left[h \cdot \Gamma(q+1)\right]^{\frac{1}{q}} < \left[\varepsilon \cdot \Gamma(q+1)\right]^{\frac{1}{q}}$. Therefore $\tilde{h} < \left[\varepsilon \cdot \Gamma(q+1)\right]^{\frac{1}{q}} < \varepsilon = \tilde{\varepsilon}$. Hence $\tilde{h} < \tilde{\varepsilon}$. Take $p_{\tilde{h}} = p_h$. Then $p_{\tilde{h}} \in B(0, \tilde{\varepsilon})$. So we get

$$y_0 + \frac{\tilde{h}^q}{\Gamma(q+1)} \cdot (\eta + p_{\tilde{h}}) \in K.$$

Conversely, let us assume that (iii) holds. Let $\varepsilon > 0$. Take $\tilde{\varepsilon} = \left[\varepsilon \cdot \Gamma(q+1)\right]^{\frac{1}{q}}$. Then, there exists $\tilde{h} < \tilde{\varepsilon}$ and $p_{\tilde{h}} \in B(0, \tilde{\varepsilon})$ such that $y_0 + \frac{\tilde{h}^q}{\Gamma(q+1)} \cdot (\eta + p_{\tilde{h}}) \in K$ holds. Pose $h = \frac{\tilde{h}^q}{\Gamma(q+1)}$. Observe that $\tilde{h} = \left[h \cdot \Gamma(q+1)\right]^{\frac{1}{q}} < \tilde{\varepsilon}$. Then $h \cdot \Gamma(q+1) < \tilde{\varepsilon}^q$ and consequently $h < \frac{\tilde{\varepsilon}^q}{\Gamma(q+1)} = \varepsilon$. Moreover, take $p_h = p_{\tilde{h}}$. Then $p_h \in B(0, \varepsilon)$ and such that

$$y_0 + h(\eta + p_h) \in K.$$

Hence, (ii) is satisfied.
(iii) $\Longrightarrow$ (iv)
Now, to prove that (iii) implies (iv), it suffices to take in (iii): $\tilde{\varepsilon} = \frac{1}{n+1}$ with $n \in \mathbb{N}$. Then, there exists $h_n \in \left(0, \frac{1}{n+1}\right)$ and $p_n \in B\left(0, \frac{1}{n+1}\right)$, with $\lim_{n\to\infty} p_n = 0$ and $\lim_{n\to\infty} h_n = 0$, such that

$$y_0 + \frac{h_n^q}{\Gamma(q+1)} \cdot (\eta + p_n) \in K,$$

for every $n \in \mathbb{N}$, the proof is complete.

■

Remark 2.1.2

Let $K \subset \mathbb{R}^n$. We notice that, if y_0 is an interior point of the set K, then $\mathcal{T}_K(y_0) = \mathbb{R}^n$. Indeed, in this case there exists $r > 0$ with $B(y_0, r) \subset K$. Let $\eta \in \mathbb{R}^n$ and $\eta \neq 0$. Therefore, for h satisfying $0 < h \leq \frac{r}{\|\eta\|}$ one has,

$$y_0 + h\eta \in B(y_0, r) \subset K.$$

This means that

$$\operatorname{dist}(y_0 + h\eta; K) = 0.$$

Thus

$$\liminf_{h\to 0^+} \frac{1}{h} \operatorname{dist}(y_0 + h\eta; K) = 0.$$

Consequently,

$$\mathbb{R}^n \subset \mathcal{T}_K(y_0).$$

Notice that If $\eta = 0$, then for each $h > 0$ one has

$$\operatorname{dist}(y_0 + h\eta; K) = 0$$

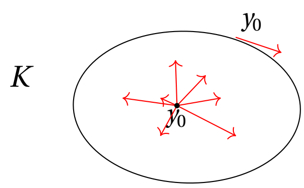

Figure 2.2: *Diagram representing the tangent vector to the set K at the point y_0.*

Example 14 : Let $K = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$, and $(\frac{1}{2}, \frac{1}{2}) \in \mathring{K}$.
Then, clearly $\mathscr{T}_K(\frac{1}{2}, \frac{1}{2}) = \mathbb{R}^2$.

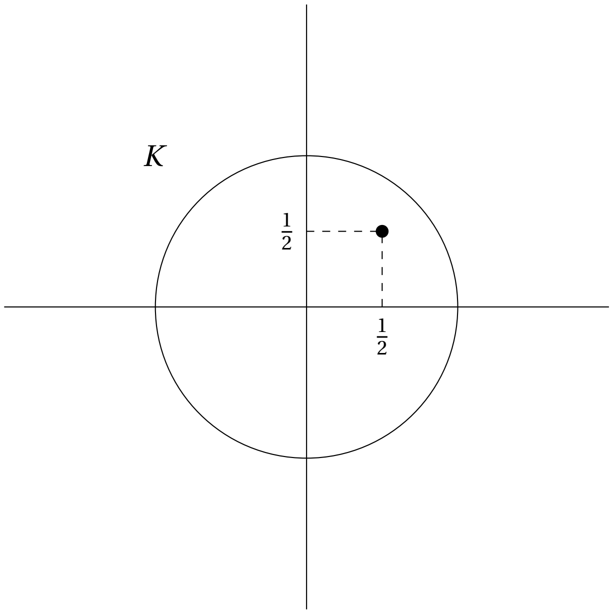

Figure 2.3: *Example of a Bouligand–Severi tangent set to K at the point $(\frac{1}{2}, \frac{1}{2})$.*

2.2 Tangent sets

Definition 2.2
Let $K \subset \mathbb{R}^n$ and $y_0 \in K$. The bounded set $E \subset \mathbb{R}^n$ is said to be tangent to the set K at the point y_0 if

$$\liminf_{h \to 0^+} \frac{1}{h} \operatorname{dist}(y_0 + hE; K) = 0. \tag{2.2}$$

Here and thereafter, the distance between two subsets A and D in $\mathbb{R}^n$ is define by

$$\operatorname{dist}(A, D) = \inf_{a \in A, d \in D} \|a - d\|.$$

We denote by $\mathscr{T}_K(y_0)$ the class of all sets which are tangent to the set K at the point y_0.

The above definition is equivalent to

Proposition 2.2.1
Let $K \subset \mathbb{R}^n$, $y_0 \in K$ and the bounded set $E \subset \mathbb{R}^n$. Then, the following conditions are equivalent:

(i) the bounded set $E \subset \mathbb{R}^n$ is tangent to the set K at the point y_0;

(ii) there exist two sequences $(h_n)_n$ in $\mathbb{R}_+$ with $h_n \downarrow 0$, and $(\eta_n)_n \in E$, such that

$$\liminf_n \frac{1}{h_n} dist(y_0 + h_n \eta_n; K) = 0;$$

(iii) for each $\varepsilon > 0$ there exist $\eta \in E$, $h \in (0, \varepsilon)$ and $p \in \mathbb{R}^n$ with $\| p \| \leq \varepsilon$ such that

$$y_0 + h(\eta + p) \in K;$$

(iv) there exist three sequences $(h_n)_n$ in $\mathbb{R}_+$ with $h_n \downarrow 0$, $(\eta_n)_n \in E$ and $(p_n)_n$ in $\mathbb{R}^n$ with $\lim_n p_n = 0$, such that

$$y_0 + h_n(\eta_n + p_n) \in K \qquad \text{for each} \quad n \in \mathbb{N}.$$

Proof

Let us observe that, the bounded set $E \subset \mathbb{R}^n$ is tangent to K at y_0 if and only if

$$\liminf_{h \to 0^+} \frac{1}{h} \text{dist}(y_0 + hE; K) = 0,$$

or equivalently,

$$\sup_{\varepsilon > 0} \inf_{h \in (0,\varepsilon)} \frac{1}{h} \text{dist}(y_0 + hE; K) = 0,$$

which is equivalent to: for each $\varepsilon > 0$, there exists $h \in (0, \varepsilon)$ such that

$$\text{dist}(y_0 + hE; K) \leq \varepsilon.$$

This is true if and only if there exist $\eta \in E$ and $z \in K$, such that

$$\| y_0 + h\eta - z \| \leq h\varepsilon.$$

Therefore, if we let

$$p = \frac{z - y_0 - h\eta}{h},$$

then we get $\| p \| \leq \varepsilon$ and

$$z = y_0 + h(\eta + p) \in K.$$

which means that (i) is equivalent to (iii).
Now, to prove that (iii) implies (iv), let us take in (iii): $\varepsilon = \frac{1}{n}$ with $n \in \mathbb{N}$. Then, there exists $(h_n)_n$ in $\mathbb{R}_+$ with $h_n \downarrow 0$, $(\eta_n)_n \in E$ and $(p_n)_n$ in $\mathbb{R}^n$ with $\lim_n p_n = 0$, such that

$$y_0 + h_n(\eta_n + p_n) \in K,$$

for each $n \in \mathbb{N}$. Conversely, let us assume that (iv) holds. Let $\varepsilon > 0$. Then, there exists $n_0 \in \mathbb{N}$ such that $\eta_{n_0} \in E$, $h_{n_0} \in (0, \varepsilon)$, $p_{n_0} \in \mathbb{R}^n$ with $\| p_{n_0} \| \leq \varepsilon$, and such that

$$y_0 + h_{n_0}(\eta_{n_0} + p_{n_0}) \in K,$$

hence, (iii) is satisfied. Since the equivalence between (i) and (ii) are straightforward, the proof is complete. ■

Example 15 : Let the sets $E=\{(y,\alpha)\in\mathbb{R}^2;\ f(y)\le\alpha\}$ and $K=\{(y,\beta)\in\mathbb{R}^2;\ g(y)\ge\beta\}$, where

$$f(y)=\begin{cases} 0 & \text{if} \quad |y|\ge 2 \\ 2-|y| & \text{if} \quad |y|<2, \end{cases}$$

and

$$g(y)=\begin{cases} 0 & \text{if} \quad |y|\ge 1 \\ -|y| & \text{if} \quad |y|<1. \end{cases}$$

Then, the set E is tangent to K at $y_0=(0,0)\in K$.

2.3 A more general tangency concept

Now we investigate the tangency concept introduced by Carja et al [13]. Later on, we show that that tangency concept is more adequate to study viability of fractional differential inclusions. Let $[t_0,T]\subset[a,b)$, $E\subset\mathbb{R}^n$ be bounded and $K\subset\mathbb{R}^n$. We consider a function $y:[t_0,T]\longrightarrow\mathbb{R}^n$ defined by

$$y(t)=y_0+\int_{t_0}^{t}(t-s)^{q-1}g(s)\,ds,\ \forall\, t\in[t_0,T].$$

where $q\in[0,1]$, $y_0\in K$ and $g\in L^\infty([a,b);\mathbb{R}^n)$.

Definition 2.3
Let $t_0\le\bar{t}<T$ and assume that $y(\bar{t})=\bar{y}\in K$. We say that the pair (g,E) is tangent to $J\times K$ at $(\bar{t},\bar{y})$ if

$$\liminf_{h\to 0^+}\frac{1}{h^q}\operatorname{dist}\left(\bar{y}+\phi(\bar{t},g)(h)+\frac{h^q}{\Gamma(q+1)}E;K\right)=0, \tag{2.3}$$

where

$$\phi(\bar{t},g)(h)=\frac{1}{\Gamma(q)}\int_{t_0}^{\bar{t}}\left((\bar{t}+h-s)^{q-1}-(\bar{t}-s)^{q-1}\right)g(s)\,ds.$$

Remark 2.3.1

It is clear that if $q=1$ then equation (2.3) becomes equation (2.2).

The next proposition is a handy characterization of the above tangency concept.

Proposition 2.3.1
The pair (g,E) is tangent to $J\times K$ at $(\bar{t},\bar{y})\in J\times K$ if and only if for each $\delta>0$ and each $\varepsilon>0$, there exist $h\in(0,\delta)$ and $\eta\in E$, $p\in\mathbb{R}^n$ satisfying $|p|<\varepsilon$ such that

$$\bar{y}+\phi(\bar{t},g)(h)+\frac{h^q}{\Gamma(q+1)}(\eta+p)\in K.$$

Proof

The relation (2.3) is equivalent to

$$\sup_{\delta>0}\inf_{h\in(0,\delta)}\frac{1}{h^q}\text{dist}(\bar{y}+\phi(\bar{t},g)(h)+\frac{h^q}{\Gamma(q+1)}E;K)=0,$$

The latter relation is also equivalent to: for each $\delta > 0$

$$\inf_{h\in(0,\delta)}\frac{1}{h^q}\text{dist}\left(\bar{y}+\phi(\bar{t},g)(h)+\frac{h^q}{\Gamma(q+1)}E;K\right)=0.$$

Which in turn is equivalent to: for each $\delta > 0$ and each $\varepsilon > 0$, there exist $h\in(0,\delta)$ such that

$$\text{dist}\left(\bar{y}+\phi(\bar{t},g)(h)+\frac{h^q}{\Gamma(q+1)}E;K\right) < \frac{\varepsilon h^q}{\Gamma(q+1)}.$$

This is true if and only if there exists $\eta\in E$ and $k\in K$, such that:

$$\left\|\bar{y}+\phi(\bar{t},g)(h)+\frac{h^q\eta}{\Gamma(q+1)}-k\right\| < \frac{\varepsilon h^q}{\Gamma(q+1)}.$$

Therefore, if we let

$$p=\frac{k-\bar{y}-\phi(\bar{t},g)(h)-\frac{h^q\eta}{\Gamma(q+1)}}{h^q}\Gamma(q+1)$$

Then, we gets $|p| < \varepsilon$, and

$$\bar{y}+\phi(\bar{t},g)(h)+\frac{h^q}{\Gamma(q+1)}(\eta+p)\in K.$$

■

3

Viability of fractional differential inclusions

3.1 Introduction

We consider a fractional differential inclusion of the form

$$D_c^q y(t) \in F(t, y(t)), \quad 0 < q < 1, \quad t \in J = [a, b), \tag{3.1}$$

where $F : G \longrightarrow 2^{\mathbb{R}^n}$ is a given set-valued map and $G = J \times K$ with $K \subset \mathbb{R}^n$ is locally closed set and $D_c^q y$ stands for the left–sided Caputo derivative of y of order q and the lower limit t_0 defined by

$$D_c^q y(t) = \frac{1}{\Gamma(1-q)} \int_{t_0}^{t} y'(s)(t-s)^{-q} \, \mathrm{d}s,$$

where $t_0 \in [a, b)$ and Γ stands for the Euler gamma function defined by

$$\forall z \in \mathbb{R}, \quad \Gamma(z) = \int_0^{\infty} t^{z-1} e^{-t} \, \mathrm{d}t.$$

We recall that $K \subset \mathbb{R}^n$ is said to be locally closed if for each $x \in K$ there exists $\rho > 0$ such that $B(x, \rho) \cap K$ is closed. We use the following notion of solution to (3.1).

Definition 3.1
The absolutely continuous function $y : [t_0, T] \subset J \longrightarrow \mathbb{R}^n$ is said to be a solution of (3.1) if there exists a measurable selection $f_y(t) \in F(t, y(t))$ such that for all $t \in [t_0, T]$ we have

$$y(t) = y_0 + \frac{1}{\Gamma(q)} \int_{t_0}^{t} (t-s)^{q-1} f_y(s) \, \mathrm{d}s.$$

We recall that a function $y : [t_0, T] \longrightarrow \mathbb{R}^n$ is said to be absolutely continuous if:

$$\forall t, s \in [t_0, T], y(t) = y(s) + \int_s^t y'(\tau) \, \mathrm{d}\tau.$$

In this chapter, we are concerned with the concept of viability of a cylindrical domain G with respect to (3.1). Let us first recall the notion of viability of G with respect to (3.1).

Definition 3.2

1) We say that G is viable with respect to (3.1) if for every $(t_0, y_0) \in G$, there exist $T > t_0$ with $[t_0, T] \subset [a, b)$ and a solution $y : [t_0, T] \longrightarrow \mathbb{R}^n$ of (3.1) satisfying $y(t_0) = y_0$ and
$$\forall t \in [t_0, T] : y(t) \in K.$$

2) The set G is said to be globally viable with respect to (3.1) if for each $(t_0, y_0) \in G$ and $T > t_0$ with $[t_0, T] \subset [a, b)$, there exists a solution $y : [t_0, T] \longrightarrow \mathbb{R}^n$ of (3.1) satisfying $y(t_0) = y_0$ and
$$\forall t \in [t_0, T] : y(t) \in K.$$

As far as (3.1) is concerned, the viability problem was studied by many authors by using various frameworks and techniques. We begin by noticing the pioneering contribution of Nagumo [10] who considered the particular case of an ordinary differential equation

$$y'(t) = f(t, y(t)), \tag{3.2}$$

with $f : G \longrightarrow \mathbb{R}^n$ and proved that if f is continuous, then a necessary and sufficient condition for G to be viable with respect to (3.2) is the tangency condition

$$\forall (t_0, y_0) \in G, \; f(t_0, y_0) \in \mathcal{T}_K(t_0, y_0),$$

where $\mathcal{T}_K(t_0, y_0)$ stands for the Bouligand–Severi tangent cone to K at (t_0, y_0) (see Definition 2.1) in Chapter 2.[1] The result of Nagumo was extended by Bebernes–Schuur [16] to ordinary differential inclusion

$$y'(t) \in F(t, y(t)). \tag{3.3}$$

The authors showed that if F is upper semicontinuous with nonempty, convex and compact values, then a necessary and sufficient condition for G to be viable with respect to (3.3) is the tangency condition
$$\forall (t_0, y_0) \in G, \; F(t_0, y_0) \cap \mathcal{T}_K(t_0, y_0) \neq \emptyset.$$

For $q \in (0, 1)$, the problem seems more complicated. For ordinary fractional differential equations, the problem of viability was studied first by Vasundgara Devi and Lakshmikantham and Girejko et al. [8, 2]. Actually, the authors have showed that the classical tangency condition of Nagumo is sufficient (but a priori also necessary) for the cylindrical domain G to be viable with respect to (3.1). Unfortunately, their proofs are incorrect as it is pointed out in [13].
For fractional differential inclusion, up to our knowledge, [13] was the only paper in which sufficient conditions for G to be viable with respect to (3.1) are expressed in terms of a new tangency concept.

Here, we use the tangency concept introduced by Carja et al. in[13] and the technique developed in [11] for viability with respect to delay fractional differential inclusion to provide a necessary and sufficient condition for the cylindrical domain G to be viable with respect to (3.1). Let us recall the notion of ε-solution to (3.1), we need in the sequel.

[1] For convenience of the reader we recall that the vector $\eta \in \mathbb{R}^n$ is tangent in the sense of Bouligand–Severi to the set K at the point y_0, i.e. $\eta \in \mathcal{T}_K(t_0, y_0)$ if $\liminf\limits_{h \to o^+} \frac{1}{h} dist\big(y_0 + h\eta; K\big) = 0$.

Definition 3.3
Let $\varepsilon > 0$. By an ε-solution of (3.1), we mean any absolutely continuous function $y : [t_0, T] \to \mathbb{R}^n$ solution of the fractional differential inclusion

$$D_c^q y(t) \in F(t, y(t) + \varepsilon \mathbb{B}),$$

where $\mathbb{B}$ denotes the closed unit ball of $\mathbb{R}^n$.

3.2 Sufficient conditions for viability

The goal of this section is to state sufficient conditions for viability for the cylindrical domain set G with respect to (3.1).

We begin by formulating the standing assumptions on the set-valued map F we need in the sequel.

(H) The set-valued map $F : [a, b) \times \mathbb{R}^n \longrightarrow 2^{\mathbb{R}^n}$ is upper semicontinuous with nonempty convex and compact values. Further, there exists a constant α such that

$$\|F(t, y)\| \leq \alpha(1 + \|y\|_\infty) \text{ for all } (t, y) \in [a, b) \times \mathbb{R}^n. \tag{3.4}$$

3.2.1 Construction of approximate solutions

To prove viability of G with respect to (3.1), one needs to construct approximate solutions to the problem (3.1). Next definition is crucial in all what follows.

Definition 3.4
Let $\varepsilon > 0$, $T > t_0$ be such that $[t_0, T] \subset [a, b)$ and $y_0 \in K$. Let $\sigma : [t_0, T] \to [t_0, T]$ be nondecreasing; $f : [t_0, T] \to \mathbb{R}^n$ be measurable; $g : [t_0, T] \to \mathbb{R}^n$ be integrable and $y : [t_0, T] \to \mathbb{R}^n$ is absolutely continuous. The quarter (σ, f, g, y) is said to be an ε-solution of (3.1) on the interval $[t_0, T]$ if the following conditions are satisfied

(i) $t - \varepsilon \leq \sigma(t) \leq t$ for every $t \in [t_0, T]$;

(ii) $\|g(t)\| \leq \varepsilon$ for every $t \in [t_0, T]$;

(iii) $y(\sigma(t)) \in K$ for every $t \in [t_0, T]$ and $y(T) \in K$;

(iv) $f(t) \in F(\sigma(t), y(\sigma(t)))$ such that

$$y(t) = y_0 + \frac{1}{\Gamma(q)} \int_{t_0}^{t} (t - s)^{q-1} [f(s) + g(s)] \mathrm{d}s,$$

for every $t \in [t_0, T]$.

The next lemma plays a crucial role in the proof of the existence of approximate solutions of the fractional differential inclusion (3.1).

Lemma 3.2.1: There exists $M>0$ such that for every $\varepsilon>0$, $[t_0,T]\subset[a,b)$ and every ε-solution (σ,f,g,y) to (3.1) on the interval $[t_0,T]$, we have

$$\|y\|_\infty \le M.$$

The proof of the above lemma is based on an auxiliary result, which is interesting by itself; see for instance [1].

Lemma 3.2.2: [1] Suppose that $w(t)$ is a nonnegative, locally integrable function on $[t_0,a]$ and $u:[0,a]\longrightarrow[0,\infty)$ be a real function. Assume that there are constants $b>0$ and $0<\alpha<1$ such that

$$u(t)\le w(t)+b\int_{t_0}^{t}(t-s)^{\alpha-1}u(s)\mathrm{d}s.$$

Then, there exists a constant $\theta=\theta(\alpha)$ such that

$$u(t)\le w(t)+\theta b\int_{t_0}^{t}(t-s)^{\alpha-1}w(s)\mathrm{d}s.$$

Proof

Let $\varepsilon>0$, (σ,f,g,y) be an ε-solution of (3.1) on $[t_0,T]\subset[a,b)$ and $y_0\in K$. From (iv), one gets

$$\forall t\in[t_0,T];\ y(t)=y_0+\frac{1}{\Gamma(q)}\int_{t_0}^{t}(t-s)^{q-1}[f(s)+g(s)]\mathrm{d}s,$$

such that for a.e. $s\in[t_0,T]$,

$$f(s)\in F(\sigma(s),y(\sigma(s))).$$

It follows from (3.4) that

$$\|y(t)\|\le\|y_0\|+\frac{1}{\Gamma(q)}\int_{t_0}^{t}(t-s)^{q-1}\left(\alpha+1+\alpha\|y(\sigma(s))\|\right)\mathrm{d}s,\ t_0\le t\le T.$$

Put $v(t)=\sup\{\|y(s)\|:t_0\le s\le T\}$. Using the above inequality and the definition of v, we have that

$$v(t)\le\|y_0\|+\frac{\alpha+1}{\Gamma(q+1)}+\frac{\alpha}{\Gamma(q)}\int_{t_0}^{t}(t-s)^{q-1}v(s)\mathrm{d}s.$$

The Lemma 3.2.1 implies

$$\begin{aligned}v(t)&\le\|y_0\|+\frac{\alpha+1}{\Gamma(q+1)}+\frac{\theta\alpha}{\Gamma(q)}\left(\|y_0\|+\frac{\alpha+1}{\Gamma(q+1)}\right)\int_{t_0}^{t}(t-s)^{q-1}\mathrm{d}s\\&\le\|y_0\|+\frac{\alpha+1}{\Gamma(q+1)}+\frac{\theta\alpha(T-t_0)^q}{\Gamma(q+1)}\left(\|y_0\|+\frac{\alpha+1}{\Gamma(q+1)}\right):=M,\ \text{for all } t\in[t_0,T].\end{aligned}$$

Thus

$$\|y\|_\infty\le M.$$

■

In the next proposition, we show how the tangency condition introduced by Carja et al. allows us to extend an ε-solution.

Proposition 3.2.1
Let $\varepsilon \in (0,1)$ and (σ, f, g, y) be ε-solution of (3.1) on $[t_0, \bar{t}] \subset [a, b)$. If the pair $\left(D_c^q y(t), F(\bar{t}, y(\bar{t}))\right)$ is tangent to G at $(\bar{t}, \bar{y}) \in G$, then there exist $h > 0$ and an extension $(\sigma_1, f_1, g_1, y_1)$ of (σ, f, g, y) which is ε-solution of (3.1) on $[t_0, \bar{t} + h]$.

Proof

Let $\varepsilon \in (0,1)$. From Proposition 2.3.1, one deduces that there exist $h \in (0, \delta)$, $\eta \in F(\bar{t}, y(\bar{t}))$ and $p \in \mathbb{R}^n$ such that $|p| < \varepsilon$ and

$$y(\bar{t}) + \phi(\bar{t}, D_c^q y)(h) + \frac{h^q}{\Gamma(q+1)}(\eta + p) \in K. \tag{3.5}$$

We define $z : [\bar{t}, \bar{t} + h] \longrightarrow \mathbb{R}^n$ by

$$z(t) = y(\bar{t}) + \phi(\bar{t}, D_c^q y)(t - \bar{t}) + \frac{(t - \bar{t})^q}{\Gamma(q+1)}(\eta + p), \; t \in [\bar{t}, \bar{t} + h], \tag{3.6}$$

such that

$$\phi(\bar{t}, D_c^q y)(h) = \frac{1}{\Gamma(q)} \int_{t_0}^{\bar{t}} \left((\bar{t} + h - s)^{q-1} - (\bar{t} - s)^{q-1}\right) D_c^q y(s) \mathrm{d}s. \text{(See Definition 2.4).}$$

We define $(\sigma_1, f_1, g_1, y_1) : [t_0, \bar{t} + h] \longrightarrow [t_0, \bar{t} + h] \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n$ as follows:

$$\sigma_1(t) = \begin{cases} \sigma(t), & \text{for} \quad t \in [t_0, \bar{t}], \\ \bar{t}, & \text{for} \quad t \in [\bar{t}, \bar{t} + h], \end{cases}$$

$$f_1(t) = \begin{cases} f(t), & \text{for} \quad t \in [t_0, \bar{t}], \\ \eta, & \text{for} \quad t \in [\bar{t}, \bar{t} + h], \end{cases}$$

$$g_1(t) = \begin{cases} g(t), & \text{for} \quad t \in [t_0, \bar{t}], \\ p, & \text{for} \quad t \in [\bar{t}, \bar{t} + h], \end{cases}$$

$$y_1(t) = \begin{cases} y(t), & \text{for} \quad t \in [t_0, \bar{t}], \\ z(t), & \text{for} \quad t \in [\bar{t}, \bar{t} + h]. \end{cases}$$

Let us check that $(\sigma_1, f_1, g_1, y_1)$ satisfy conditions (i)~(iv) in Definition 3.4 on the interval $[t_0, \bar{t} + h]$.
Clearly, condition (i) is satisfied on $[t_0, \bar{t}]$. Further, if $t \in [\bar{t}, \bar{t} + h]$, one has $\sigma_1(t) = \bar{t}$ and $\bar{t} \leq t$. On the other hand, we have $t \leq \bar{t} + h$,
(we assume from Proposition 2.3.1 that $\delta < \varepsilon$), so

$$t \leq \bar{t} + \varepsilon \implies t - \varepsilon \leq \sigma_1(t).$$

Finally, we have proved that

$$\forall t \in [t_0, \bar{t} + h], \; t - \varepsilon \leq \sigma_1(t) \leq t.$$

Condition (ii), holds true on $[t_0, \bar{t}+h]$.
For the condition (iii), we have for each $t \in [t_0, \bar{t}]$,

$$y_1(\sigma_1(t)) = y_1(\sigma(t)) = y(\sigma(t)) \in K.$$

On the other hand, if $t \in [\bar{t}, \bar{t}+h]$,

$$y_1(\sigma_1(t)) = y(\bar{t}) \in K.$$

Thanks to (3.5) one gets

$$\begin{aligned} y_1(\bar{t}+h) = z(\bar{t}+h) &= y(\bar{t}) + \phi(\bar{t}; D_c^q y)(\bar{t}+h-\bar{t}) + \frac{(\bar{t}+h-\bar{t})^q}{\Gamma(q+1)}(\eta+p) \\ &= y(\bar{t}) + \phi(\bar{t}; D_c^q y)(h) + \frac{h^q}{\Gamma(q+1)}(\eta+p) \in K \end{aligned}$$

Hence condition (iii) is satisfied. Now, we check that condition (iv) is satisfied, that means

$$\forall t \in [t_0, \bar{t}+h],\ y_1(t) = y_0 + \frac{1}{\Gamma(q)} \int_{t_0}^{t} (t-s)^{q-1}[f_1(s)+g_1(s)]\,\mathrm{d}s.$$

So $D_c^q y(t) = f_1(t) + g_1(t)$ for every $t \in [t_0, \bar{t}]$. If $t \in [t_0, \bar{t}]$, then

$$y_1(t) = y(t) = y_0 + \frac{1}{\Gamma(q)} \int_{t_0}^{t} (t-s)^{q-1}[f(s)+g(s)]\,\mathrm{d}s.$$

Since $f = f_1$ and $g = g_1$ on the interval $[t_0, \bar{t}]$, then

$$y_1(t) = y(t) = y_0 + \frac{1}{\Gamma(q)} \int_{t_0}^{t} (t-s)^{q-1}[f_1(s)+g_1(s)]\,\mathrm{d}s.$$

If $t \in [\bar{t}, \bar{t}+h]$, one gets

$$\begin{aligned} y_1(t) = z(t) &= y(\bar{t}) + \phi(\bar{t}, D_c^q y)(t-\bar{t}) + \frac{(t-\bar{t})^q}{\Gamma(q+1)}(\eta+p) \\ &= y(\bar{t}) + \frac{1}{\Gamma(q)} \int_{t_0}^{\bar{t}} \left[(t-s)^{q-1} - (\bar{t}-s)^{q-1}\right] D_c^q y(s)\,\mathrm{d}s + \frac{(t-\bar{t})^q}{\Gamma(q+1)}(\eta+p). \end{aligned}$$

Morever,

$$\begin{aligned} y(\bar{t}) &= y_0 + \frac{1}{\Gamma(q)} \int_{t_0}^{\bar{t}} (\bar{t}-s)^{q-1}[f_1(s)+g_1(s)]\,\mathrm{d}s \\ &= y_0 + \frac{1}{\Gamma(q)} \int_{t_0}^{\bar{t}} D_c^q y(s)\,\mathrm{d}s. \end{aligned}$$

Then

$$\begin{aligned} y_1(t) = y_0 &+ \frac{1}{\Gamma(q)} \int_{t_0}^{\bar{t}} (\bar{t}-s)^{q-1} D_c^q y(s)\,\mathrm{d}s + \frac{1}{\Gamma(q)} \int_{t_0}^{\bar{t}} \left[(t-s)^{q-1} - (\bar{t}-s)^{q-1}\right] D_c^q y(s)\,\mathrm{d}s \\ &+ \frac{(t-\bar{t})^q}{\Gamma(q+1)}(\eta+p) \end{aligned}$$

$$= y_0 + \frac{1}{\Gamma(q)}\int_{t_0}^{\bar{t}} (\bar{t}-s)^{q-1} D_c^q y(s)\mathrm{d}s + \frac{1}{\Gamma(q)}\int_{t_0}^{\bar{t}} \big[(t-s)^{q-1}\big] D_c^q y(s)\mathrm{d}s$$

$$- \frac{1}{\Gamma(q)}\int_{t_0}^{\bar{t}} \big[(\bar{t}-s)^{q-1}\big] D_c^q y(s)\mathrm{d}s + \frac{(t-\bar{t})^q}{\Gamma(q+1)}(\eta+p)$$

$$= y_0 + \frac{1}{\Gamma(q)}\int_{t_0}^{\bar{t}} \big[(t-s)^{q-1}\big] D_c^q y(s)\mathrm{d}s + \frac{(t-\bar{t})^q}{\Gamma(q+1)}(\eta+p)$$

$$= y_0 + \frac{1}{\Gamma(q)}\int_{t_0}^{\bar{t}} \big[(t-s)^{q-1}\big][f_1(s)+g_1(s)]\mathrm{d}s + \frac{1}{\Gamma(q)}\int_{\bar{t}}^{t} (t-s)^{q-1}[f_1(s)+g_1(s)]\mathrm{d}s$$

$$= y_0 + \frac{1}{\Gamma(q)}\int_{t_0}^{t} (t-s)^{q-1}[f_1(s)+g_1(s)]\mathrm{d}s.$$

■

To prove the existence of ε-solutions of (3.1), one needs to define a tangency condition.

Definition 3.5
The system (3.1) is said to satisfy the tangency condition at $(\bar{t},\bar{y}) \in G$ if for every $\varepsilon > 0$, there exists an ε-solution y of (3.1) defined on $[t_0,\bar{t}]$ such that the pair $(D_c^q y, F(\bar{t},\bar{y}))$ is tangentto G at $(\bar{t},\bar{y})$.

Lemma 3.2.3: If the problem (3.1) satisfies the tangency condition at every $(\bar{t},\bar{y}) \in G$, then for each $\varepsilon \in (0,1)$, there exists ε-solution (σ, f, g, y) of (3.1) on the whole interval $[t_0,T] \subset [a,b)$.

To begin with, let us recall some definitions and notations. Let $\mathtt{S}$ be a nonempty set. A binary relation $\preceq$ defined on $\mathtt{S}\times\mathtt{S}$ is said to be a preorder on $\mathtt{S}$ if it is reflexive, i.e., $x \preceq x$ for each $x \in \mathtt{S}$, and transitive, i.e., $x \preceq y$ and $y \preceq z$ imply $x \preceq z$.

Definition 3.6
Let $\mathtt{S}$ be a nonempty set, $\preceq$ a preorder on $\mathtt{S}$, and let $\mathtt{M}:\mathtt{S}\longrightarrow \mathbb{R}\cup\{+\infty\}$ be an increasing function. An $\mathtt{M}$–maximal element is an element $\bar{x}\in\mathtt{S}$ satisfying $\mathtt{M}(x)=\mathtt{M}(\bar{x})$ for every $x\in\mathtt{S}$ with $\bar{x}\preceq x$.

We may now proceed to the statement of Brezis–Browder Principal.

Theorem 3.1: (Brezis–Browder Theorem [5])
Let $\mathtt{S}$ be a nonempty set, $\preceq$ a preorder on $\mathtt{S}$ and let $\mathtt{M}:\mathtt{S}\longrightarrow \mathbb{R}\cup\{+\infty\}$ be a given function. suppose that:

(i) for any increasing sequence $(x_k)_k \subset \mathtt{S}$, there exists some $\xi \in \mathtt{S}$ such that $x_k \preceq \xi$, for all $k\in\mathbb{N}$;

(ii) the function $\mathtt{M}$ is increasing.

Then, for each $x_0\in\mathtt{S}$, there exists a $\mathtt{M}$–maximal element $\bar{x}\in\mathtt{S}$ satisfying $x_0\preceq\bar{x}$.

We now pass to the proof of Lemma 3.2.1.

Proof

Let $\varepsilon \in (0,1)$. We first prove that there exists at least one ε-solutions $\big(\sigma, f, g, y\big)$ defined on some $[t_0, t_0+h]$ where $t_0+h < T$. Let $(t_0, y_0) \in G$. Since the tangency condition is satisfies at (t_0, y_0), then there exists an ε-solutions $\big(\sigma, f, g, y\big)$ defined on some $[t_0, \bar{t}]$ such that $(D_c^q y, F(t_0, y_0))$ is tangent to G at (t_0, y_0). Therefore, due to proposition 2.3.1, there exists $h \in (0,\varepsilon)$, $\eta \in F(t_0, y_0)$ and $p \in \mathbb{R}^n$ with $\|p\| < \varepsilon$ and

$$y_0 + \phi(\bar{t}, D_c^q y)(h) + \frac{h^q}{\Gamma(q+1)}(\eta + p) \in K. \tag{3.7}$$

Now let us define the functions $\sigma : [t_0, t_0+h] \longrightarrow [t_0, t_0+h]$, $g : [t_0, t_0+h] \longrightarrow \mathbb{R}^n$, f : $[t_0, t_0+h] \longrightarrow \mathbb{R}^n$ and $y : [t_0, t_0+h] \longrightarrow \mathbb{R}^n$ as follows:

$$\begin{cases} \sigma(t) = t_0, & \text{for} \quad t \in [t_0, t_0+h], \\ f(t) = \eta, & \text{for} \quad t \in [t_0, t_0+h], \\ g(t) = p, & \text{for} \quad t \in [t_0, t_0+h], \\ y(t) = y_0 + \phi(\bar{t}, D_c^q y)(t-t_0) + \frac{(t-t_0)^q}{\Gamma(q+1)}(\eta+p), & \text{for} \quad t \in [t_0, t_0+h]. \end{cases}$$

Let us check that $\big(\sigma, f, g, y\big)$ satisfy conditions (i) $\sim$ (iv) in Definition 3.4 on the interval $[t_0, t_0+h]$. First, if $t \in [t_0, t_0+h]$, one has $\sigma(t) = t_0 \leq t$. On the other hand, we have $t \leq t_0 + h$. So

$$t \leq t_0 + \varepsilon \implies t - \varepsilon \leq \sigma(t).$$

Hence, we have proved that

$$\forall t \in t \in [t_0, t_0+h],\ t - \varepsilon \leq \sigma(t) \leq t.$$

Condition (ii), holds true on $t \in [t_0, t_0+h]$.
Let us check that condition (iii) holds true. It is clear that

$$\forall t \in t \in [t_0, t_0+h],\ y(\sigma(t)) = y(t_0) = y_0 \in K.$$

Thanks to (3.7), one gets

$$\begin{aligned} y(t_0+h) &= y_0 + \phi(\bar{t}, D_c^q y)(t_0+h-t_0) + \frac{(t_0+h-t_0)^q}{\Gamma(q+1)}(\eta+p) \\ &= y_0 + \phi(\bar{t}, D_c^q y)(h) + \frac{h^q}{\Gamma(q+1)}(\eta+p) \in K. \end{aligned}$$

Hence, condition (iii) is satisfied.
Notice that for $\varepsilon \in (0,1)$ the ε-solution $\big(\sigma, f, g, y\big)$ defined on $[t_0, \bar{t}]$ where $\bar{t}$ depend on ε.
Our aim now is to prove that we can construct $\big(\sigma, f, g, y\big)$ defined on $[t_0, T]$ which does not depends on ε. To this aim we shall make use of Brezis-Browder Theorem 3.2.1, as follows.
Let S be the set of all $\varepsilon-$solutions to the problem (3.1) defined on $[t_0, c]$ where $c \in [t_0, T]$.
On S we define the preorder $\preceq$. Let $\big(\sigma_1, f_1, g_1, y_1\big)$ and $\big(\sigma_2, f_2, g_2, y_2\big)$ defined respectively on $[t_0, c_1]$ and $[t_0, c_2]$, then

$$\big(\sigma_1, f_1, g_1, y_1\big) \preceq \big(\sigma_2, f_2, g_2, y_2\big),$$

if and only if $c_1 \leq c_2$ and the two ε-solutions coincide on $[t_0, c_1]$. Obviously $\preceq$ is a preorder on S.

Let $\big((\sigma_k, f_k, g_k, y_k)\big)_k$ be an increasing sequence defined on $[t_0, c_k]$. We show that $\big((\sigma_k, f_k, g_k, y_k)\big)_k$ is bounded from above that means there exists at least one element, $\big(\sigma^*, f^*, g^*, y^*\big) \in \mathbb{S}$, defined on $[t_0, c^*]$ and satisfying

$$\big(\sigma_k, f_k, g_k, y_k\big) \preceq \big(\sigma^*, f^*, g^*, y^*\big)$$

for each $k \in \mathbb{N}$. We set $c^* = \lim_{k\to\infty} c_k$, clearly $c^* \in [t_0, T]$. $\lim_{k\to\infty} c_k$ exist, notice that $\big((\sigma_k, f_k, g_k, y_k)\big)_k$ is increasing sequence, we deduce that the sequence $(c_k)_k$ is increasing and is also bounded from above by T. Therefore $(c_k)_k$ is convergent, so $\lim_{k\to\infty} c_k$ exist.
One needs to prove that there exists $\lim_{k\to\infty} y_k(c_k)$. Let $k, m \in \mathbb{N}, k \le m$, we define:

$$\begin{cases} \sigma_k(s) = \sigma_m(s), & \text{if} \quad s \in [t_0, c_k], \\ g_k(t) = g_m(s), & \text{if} \quad s \in [t_0, c_k], \\ y_k(t) = y_m(s), & \text{if} \quad s \in [t_0, c_k], \\ f_k(t) = f_m(s), & \text{if} \quad s \in [t_0, c_k]. \end{cases}$$

Morever, $c_k - \varepsilon \le \sigma_k(c_k) \le c_k$.
Noticing, for every $k \in \mathbb{N}$,

$$y_k(t) = y_0 + \frac{1}{\Gamma(q)} \int_{t_0}^{t} (t-s)^{q-1} [f_k(s) + g(s)]\, \mathrm{d}s,$$

where $f_k(s) \in F\big(\sigma_k(s), y_k(\sigma_k(s))\big)$ and $y_k(t_0) = y_0$.
From Lemma 3.2.1, we have

$$\|y_k\|_\infty \le M.$$

This implies

$$\|y_k(\sigma_k(s))\|_\infty \le M, \forall s \in [t_0, c_k].$$

Morever $\|F(t,y)\| \le \alpha\big(1 + \|y\|_\infty\big), \forall y \in \mathbb{R}^n$ so

$$\|F(\sigma_k(s), y_k(\sigma_k(s)))\| \le \alpha(1+M) := N, \forall s \in [t_0, c_k].$$

We have

$$\begin{aligned}
\big\|y_m(c_m) - y_k(c_k)\big\| &= \frac{1}{\Gamma(q)} \left\| \int_{t_0}^{c_m} (c_m - s)^{q-1} \cdot \big[f_m(s) - g(s)\big]\, \mathrm{d}s - \int_{t_0}^{c_k} (c_k - s)^{q-1} \cdot \big[f_k(s) - g(s)\big]\, \mathrm{d}s \right\| \\
&= \frac{1}{\Gamma(q)} \left\| \int_{t_0}^{c_k} \big[(c_m - s)^{q-1} - (c_k - s)^{q-1}\big]\big[f_k(s) - g(s)\big]\, \mathrm{d}s + \int_{c_k}^{c_m} (c_m - s)^{q-1} \cdot \big[f_m(s) - g(s)\big]\, \mathrm{d}s \right\| \\
&\le \frac{N+\varepsilon}{\Gamma(q)} \cdot \left| \int_{t_0}^{c_k} \big[(c_m - s)^{q-1} - (c_k - s)^{q-1}\big]\, \mathrm{d}s + \int_{c_k}^{c_m} (c_m - s)^{q-1}\, \mathrm{d}s \right| \\
&= \frac{N+\varepsilon}{\Gamma(q+1)} \cdot \left| \int_{t_0}^{c_m} (c_m - s)^{q-1}\, \mathrm{d}s - \int_{t_0}^{c_k} (c_k - s)^{q-1}\, \mathrm{d}s \right| \\
&= \frac{N+\varepsilon}{\Gamma(q+1)} \cdot \left| (c_m - t_0)^q - (c_k - t_0)^q \right|.
\end{aligned}$$

for every $m,k\in\mathbb{N}$. We know that $\lim\limits_{k\to\infty} c_k = c^*$, so $\lim\limits_{k\to\infty}(c_k - t_0)^q = (c^* - t_0)^q$. Then for $|(c_m - t_0)^q - (c_k - t_0)^q| \le \frac{\Gamma(q+1)}{N+\varepsilon}\cdot\varepsilon_1$ we get

$$\|y_m(c_m) - y_k(c_k)\| \le \varepsilon_1,$$

which proves that there exists $\lim\limits_{k\to\infty} y_k(c_k)$. Since for every $k\in\mathbb{N}$, $y_k(c_k)\in K$, and the latter is closed, it readily follows that $\lim\limits_{k\to\infty} y_k(c_k)\in K$.

Now, all the functions in the set $\{\sigma_k : k\in\mathbb{N}\}$ are nondecreasing, with values in $[t_0, c^\star]$. This means that $\sigma_k(c_k)\le\sigma_m(c_m)$ for every $k,m\in\mathbb{N}$, $k\le m$, and then $\lim\limits_{k\to\infty}\sigma_k(c_k)$ exists and belongs to $[t_0,c^*]$. This shows that we can define the quartet of function $\big(\sigma^*, f^*, g^*, y^*\big)$: $[t_0,c^*]\longrightarrow[t_0,c^*]\times\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}^n$ by

$$\sigma^*(t) = \begin{cases} \sigma_k(t), & \text{for} \quad t\in[t_0,c_k],\ k\in\mathbb{N},\\ \lim_k \sigma_k(c_k), & \text{for} \quad t = c^*,\end{cases}$$

$$f^*(t) = \begin{cases} f_k(t), & \text{for} \quad t\in[t_0,c_k],\ k\in\mathbb{N},\\ \eta^*, & \text{for} \quad t = c^*,\end{cases}$$

$$g^*(t) = \begin{cases} g_k(t), & \text{for} \quad t\in[t_0,c_k],\ k\in\mathbb{N},\\ 0, & \text{for} \quad t = c^*,\end{cases}$$

$$y^*(t) = \begin{cases} y_k(t), & \text{for} \quad t\in[t_0,c_k],\ k\in\mathbb{N},\\ \lim_k y_k(c_k), & \text{for} \quad t = c^*.\end{cases}$$

Where η^* is an arbitrary but fixed element in $F\big(c^*, y^*(\sigma^*(c^*))\big)$.

One can see that $\big(\sigma^*, f^*, g^*, y^*\big)$ is an ε-approximate solution which is an upper bound for the sequence $\big((\sigma_k, f_k, g_k, y_k)\big)_{k\in\mathbb{N}}$.

Now, let us define the function $\mathtt{M} : \mathtt{S}\longrightarrow\mathbb{R}\cup\{+\infty\}$ by $\mathtt{M}\big((\sigma,f,g,y)\big) = c$, where $[t_0,c]$ is the domain of definition of (σ,f,g,y). The function $\mathtt{M}$ is increasing. According to Brezis–Browder Theorem 3.2.1, there exists at least one $\mathtt{M}$–maximal element of $\mathtt{S}$ that we will denote $\big(\bar\sigma,\bar f,\bar g,\bar y\big)$ defined on $[t_0,\bar c]$, which means that if $\big(\tilde\sigma,\tilde f,\tilde g,\tilde y\big)\in\mathtt{S}$, defined on $[t_0,\tilde c]$, satisfies $\big(\bar\sigma,\bar f,\bar g,\bar y\big)\preceq\big(\tilde\sigma,\tilde f,\tilde g,\tilde y\big)$, then necessarily have $\bar c = \tilde c$.

Next, We will show that $\bar c = T$. To this aim, let us assume by contradiction that $\bar c < T$. Then taking into account the fact that $\bar y(\bar c)\in K$ and $f(\bar c)\in\mathscr{T}_k(\bar c)$. Thanks to Proposition 2.3.1, there exist $\delta\in(t_0, T-\bar c)$, $\delta\le\varepsilon$ and $p\in\mathbb{R}^n$ such that $\|p\|<\varepsilon$ and

$$\bar y(\bar c) + \phi(\bar c, D_c^q\bar y)(\delta) + \frac{\delta^q}{\Gamma(q+1)}(f(\bar c)+p)\in K.$$

Let us define the functions $\hat\sigma : [t_0,\bar c+\delta]\longrightarrow[t_0,\bar c+\delta]$, $\hat g : [t_0,\bar c+\delta]\longrightarrow\mathbb{R}^n$ and $\hat f : [t_0,\bar c+\delta]\longrightarrow\mathbb{R}^n$ by

$$\hat\sigma(t) = \begin{cases} \bar\sigma(t), & \text{for} \quad t\in[t_0,\bar c],\\ \bar c, & \text{for} \quad t\in(\bar c,\bar c+\delta],\end{cases}$$

$$\hat g(t) = \begin{cases} \bar g(t), & \text{for} \quad t\in[t_0,\bar c],\\ p, & \text{for} \quad t\in(\bar c,\bar c+\delta],\end{cases}$$

$$\hat{f}(t) = \begin{cases} \bar{f}(t), & \text{for} \quad t \in [t_0, \bar{c}], \\ \eta, & \text{for} \quad t \in (\bar{c}, \bar{c}+\delta]. \end{cases}$$

Clearly, $\hat{g}$ is integrable on $[t_0, \bar{c}+\delta]$ and $\|\hat{g}\| \leq \varepsilon$ for every $t \in [t_0, \bar{c}+\delta]$.
Therefore $\hat{y}(\hat{\sigma}(t))$ is well defined and $\bar{y}(\hat{\sigma}(t)) \in K$. We can define $\hat{y} : [t_0, \bar{c}+\delta] \longrightarrow \mathbb{R}^n$ by

$$\hat{y}(t) = y_0 + \frac{1}{\Gamma(q)} \int_{t_0}^{t} (t-s)^{q-1} [f(s) + \hat{g}(s)] \mathrm{d}s.$$

Note, that $\hat{y}(t) = \bar{y}(t)$ for $t \in [t_0, \bar{c}]$ and $\hat{y}$, $\hat{\sigma}$ and $\hat{g}$ satisfy conditions (i) and (ii) in Definition 3.4 Observe that

$$\hat{y}(t) = \begin{cases} \bar{y}(t), & \text{for} \quad t \in [t_0, \bar{c}], \\ \bar{y}(\bar{c}) + \phi(\bar{c}, D_c^q \bar{y})(t-\bar{c}) + \frac{(t-\bar{c})^q}{\Gamma(q+1)}(f(\bar{c})+p), & \text{for} \quad t \in (\bar{c}, \bar{c}+\delta]. \end{cases}$$

Let us observe that

$$\hat{y}(\hat{\sigma}(t)) = \begin{cases} \bar{y}(\hat{\sigma}(t)), & \text{for} \quad t \in [t_0, \bar{c}], \\ \bar{y}(\bar{c}), & \text{for} \quad t \in (\bar{c}, \bar{c}+\delta], \end{cases}$$

it follows that $\hat{y}(\hat{\sigma}(t)) \in K$. Furthermore, from the choice of δ and p, we have

$$\hat{y}(\bar{c}+\delta) = \bar{y}(\bar{c}) + \phi(\bar{c}, D_c^q \bar{y})(\delta) + \frac{\delta^q}{\Gamma(q+1)}(f(\bar{c})+p) \in K$$

consequently $\hat{y}$ satisfies (iii) in Definition 3.4. Thus $\left(\hat{\sigma}, \hat{f}, \hat{g}, \hat{y}\right) \in \mathtt{S}$. Since $\left(\bar{\sigma}, \bar{f}, \bar{g}, \bar{y}\right) \preceq \left(\hat{\sigma}, \hat{f}, \hat{g}, \hat{y}\right)$ with $\bar{c} < \bar{c}+\delta$, it follows that $\left(\bar{\sigma}, \bar{f}, \bar{g}, \bar{y}\right)$ is not M-maximal element. But this is absurd. This contradiction can be eliminated only if each maximal element in the set S is defined on $[t_0, T]$. Therefore the existence of an ε-approximate solution defined on the whole interval $[t_0, T]$ was proved. ■

3.3 Convergence of approximate solutions

In this section, we state sufficient conditions for G to be viable with respect to (3.1). The idea is to construct a suitably sequence of ε-approximate solutions and show its convergence to a solution which remain inside G for at least a short time. More precisely, we have the following result.

Theorem 3.2
Let (3.4) holds true. If the inclusion (3.1) satisfies the tangency condition at every $(\bar{t}, \bar{y}) \in G$, then G is viable with respect to (3.1).

Proof

Let $(t_0, y_0) \in G$. Let us consider a sequence $(\varepsilon_n)_{n\in\mathbb{N}}$ in $(0,1)$, decreasing to 0, and let $\left((\sigma_n, f_n, g_n, y_n)\right)_{n\in\mathbb{N}}$ be a sequence of ε_n-approximate solutions defined on the interval $[t_0, T]$. We will first show that $(y_n)_{n\in\mathbb{N}}$ has at least one convergent subsequence. Indeed, from (i) and

(iii) of Definition 3.4, we get the following uniform convergence on $[t_0, T]$:

$$\lim_{n\to\infty} \sigma_n(t) = t, \tag{3.8}$$

$$\lim_{n\to\infty} g_n(t) = 0, \tag{3.9}$$

Thus, the sequence $(y_n)_{n\in\mathbb{N}}$ uniformly bounded on $[t_0, T]$. We now check that $(y_n)_{n\in\mathbb{N}}$ is equicontinuous on $[t_0, T]$. For any $n \in \mathbb{N}$, $t_0 \le \tau \le T$, with $f_n(s) \in F\big(\sigma_n(s), y_n(\sigma_n(s))\big)$, one has

$$\begin{aligned}
\|y_n(t) - y_n(\tau)\| &= \frac{1}{\Gamma(q)} \left\| \int_{t_0}^{t} (t-s)^{q-1} \cdot \big[f_n(s) - g_n(s)\big]\, \mathrm{d}s - \int_{t_0}^{\tau} (\tau - s)^{q-1} \cdot \big[f_n(s) - g_n(s)\big]\, \mathrm{d}s \right\| \\
&= \frac{1}{\Gamma(q)} \left\| \int_{t_0}^{\tau} \big[(t-s)^{q-1} - (\tau-s)^{q-1}\big]\big[f_n(s) - g_n(s)\big]\, \mathrm{d}s + \int_{\tau}^{t} (t-s)^{q-1} \cdot \big[f_n(s) - g_n(s)\big] ds \right\| \\
&\le \frac{N+\varepsilon_n}{\Gamma(q)} \cdot \left| \int_{t_0}^{\tau} \big[(t-s)^{q-1} - (\tau-s)^{q-1}\big]\, \mathrm{d}s + \int_{\tau}^{t} (t-s)^{q-1}\, \mathrm{d}s \right| \\
&\le \frac{N+1}{\Gamma(q+1)} \cdot (2(t-\tau)^q - (t-t_0)^q + (\tau - t_0)^q) \\
&\le \frac{N+1}{\Gamma(q+1)} \cdot 3(t-\tau)^q.
\end{aligned}$$

Thus, if $|t-\tau|^q \le \delta = \left[\frac{\varepsilon\Gamma(q+1)}{3(N+1)}\right]^{\frac{1}{q}}$, then

$$\|y_n(t) - y_n(\tau)\| \le \frac{N+1}{\Gamma(q+1)} \cdot 3\delta^q < \varepsilon.$$

Hence the sequence $(y_n)_{n\in\mathbb{N}}$ is equicontinuous on $[t_0, T]$. Conclusion: by Arzela– Ascoli Theorem, $(y_n)_{n\in\mathbb{N}}$ has at least one uniformly convergent subsequence, denoted again by $(y_n)_{n\in\mathbb{N}}$, which converges uniformly to $y : [t_0, T] \longrightarrow \mathbb{R}^n$. Now from (iii) in Definition 3.4. one has for every $n \in \mathbb{N}$ and every $t \in [t_0, T]$, $y_n(\sigma_n(t)) \in K$, $y_n(T) \in K$ and condition (3.8). Taking into account that K is locally closed, passing to limit in the previous relation, one deduces that $y(t) \in K$ for every $t \in [t_0, T]$. In other hand, one has for every $n \in \mathbb{N}$, for every $t \in [t_0, T]$,

$$y_n(t) = y_0 + \frac{1}{\Gamma(q)} \int_{t_0}^{t} (t-s)^{q-1} \cdot \big[f_n(s)) + g_n(s)\big]\, \mathrm{d}s.$$

Passing to limit when $n \to \infty$ in the above equation, one gets for every $t \in [t_0, T]$,

$$y(t) = y_0 + \frac{1}{\Gamma(q)} \int_{t_0}^{t} (t-s)^{q-1} f(s))\mathrm{d}s.$$

This means that y is a solution of (3.1) satisfying $y(t_0) = y_0$ and for all $t \in [t_0, T]$, $y(t) \in K$. This is true for every $(t_0, y_0) \in G$. Therefore, G is viable with respect to (3.1). ■

conclusion

In this thesis, we have stated sufficient conditions for a cylindrical domain to be viable with respect to a fractional differential inclusion of the form

$$D_c^q y(t) \in F(t, y(t)),$$

where $D_c^q y$ stands for the Caputo derivative and $F : [a, b) \times \mathbb{R}^n \longrightarrow 2^{\mathbb{R}^n}$ is a given set-valued map. The approach that we used for the proof of our result is different from that in [13]. Notice that the tangency concept which is the base of the construction of approximate solution to the above problem was criticized in the literature. In future works, we develop a new tangency concept and studied other notions of viability as approximate and near viability for the above fractional differential inclusion.

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