

Université Djilali Bounaama, Khemis Miliana
Faculté des Sciences de la Matière et d'informatique
Conseil Scientifique de la Faculté



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كلية علوم المادة والإعلام الآلي
المجلس العلمي للكلية

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**EXTRAIT DU PV
DE LA REUNION ORDINAIRE DU CONSEIL SCIENTIFIQUE
Du 02/05/2026**

Objet : : Expertise de polycopié pédagogique

En l'an deux mille vingt-six (2026), le deux (02) mai 09 h 30, une réunion ordinaire du Conseil Scientifique de la Faculté des Sciences de la Matière et de l'Informatique s'est tenue dans la salle de réunion de la faculté (Bloc B).

Suite aux rapports favorables reçus de la part des experts cités ci-après concernant l'expertise du polycopié pédagogique, le CSF a prononcé favorablement pour la conformité du polycopié pédagogique en vue de préparer son professorat.

- **Auteur du polycopié** : Dr. BEZZIOU Mohamed (MCA)
- **Intitulé du polycopié** : Course and Exercises on the Theory of Linear Operators
- **Destiné aux étudiants de** : L3 Mathématiques.
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People's Democratic Republic of Algeria

MINISTRY OF HIGHER EDUCATION AND SCIENTIFIC RESEARCH

Djilali Bounaama University of Khemis Miliana (Algeria)

Faculty of Matter Sciences and Computer Science

Department of Mathematics

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Course and exercises on the theory of linear operators

For 3rd Year bachelor's degree in mathematics - Semester 6

Teaching Unit: Fundamental

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Introduction

This manuscript educational, entitled "Course and exercises on the theory of linear operators" is the result of several years of teaching Topology of metric spaces, normed vector spaces and Hilbert analysis. This booklet is a collection courses to familiarize the student with the basic notions of linear operators theory to provide a foundation for potential future studies in PDEs, spectral theory, and abstract differential equations. It is intended for third-year bachelor's degree in mathematics students, and was taught at Djilali Bounaama University in Khemis Miliana. I have followed the official program published in the ministerial decision in the semester course organization sheet for the academic year: 2025-2026.

This manuscript is divided into three chapters. The goal of the first Chapter is to present the basic results in topology such as infinite-dimensional Banach and finite-dimensional normed vector spaces, the $L(E,F)$ space of linear operators, dense domain operators and continuity extension with Point convergence, Uniform bound principle and invertibility of linear operators.

In the second chapter, we turn to the Hahn–Banach theorem in its analytic form and its consequences, as well as the first and second geometric forms of this theorem. Adjoint operator, Hilbert spaces with the properties of the adjoint of a linear operator and we study the Spectrum of an operator.

The last chapter introduces a very important concept in analysis: compact operators, including the Fredholm alternative, the spectrum of a compact operator and the finite-rank operators.

At the end of each chapter, we provide a series of diverse and rich exercises to ensure a better understanding of the material. Some of these exercises have been addressed in

tutorial sessions.

Finally, I hope that the student and mathematics teacher find their needs in this manuscript and I believe that this work is always in need of revision, so I welcome all constructive and useful criticism.

Chapter 1

Linear Operators

1.1 Reminders on Banach spaces

1.1.1 Definitions and preliminary results

Definition 1.1. Let E be a vector space over the field $\mathbb{K}$ (where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$). A **norm** on E is an application $N : E \rightarrow \mathbb{R}_+$ satisfying the following axioms:

$$N(x) = 0 \Leftrightarrow x = 0 \quad \forall x \in E \quad (\text{N1})$$

$$N(\lambda x) = |\lambda|N(x) \quad \forall \lambda \in \mathbb{K}, \forall x \in E \quad (\text{N2})$$

$$N(x + y) \leq N(x) + N(y) \quad \forall x, y \in E \quad (\text{N3})$$

The pair (E, N) is called a **normed vector space**.

Example 1.1. The standard norms on $\mathbb{R}^n$:

$$N_1(x) = \sum_{i=1}^n |x_i| \quad (1)$$

$$N_2(x) = \left(\sum_{i=1}^n |x_i|^2 \right)^{1/2} \quad (2)$$

$$N_\infty(x) = \max_{1 \leq i \leq n} |x_i| \quad (3)$$

Example 1.2. The norm defined on $\ell^p(\mathbb{C})$, the space of sequences:

$$\ell^p(\mathbb{C}) = \left\{ (x_n) \subset \mathbb{C} : \sum_{n=0}^{\infty} |x_n|^p < \infty \right\}, \quad \text{where } p \in [1, \infty)$$

The norm is given by:

$$\|x\|_p = \left(\sum_{n=0}^{\infty} |x_n|^p \right)^{1/p}$$

Definition 1.2. Let E be a normed vector space. A sequence (x_n) in E is called a **Cauchy sequence** if and only if:

$$\lim_{m,n \rightarrow \infty} \|x_n - x_m\| = 0$$

i.e., $\forall \varepsilon > 0, \exists n_0 \in \mathbb{N}, \forall m, n \geq n_0, \|x_n - x_m\| < \varepsilon$.

Definition 1.3. A sequence (x_n) in E **converges** to $x \in E$ if and only if:

$$\lim_{n \rightarrow \infty} \|x_n - x\| = 0$$

Definition 1.4. A function $f : E \rightarrow F$ (where F is another normed space) is said to be **continuous** at $x_0 \in E$ if and only if:

$$\forall \varepsilon > 0, \exists \alpha > 0, \|x - x_0\| < \alpha \Rightarrow \|f(x) - f(x_0)\| < \varepsilon$$

1.1.2 Banach Space

Definition 1.5. Let E be a metric space. E is called **Banach Space** if and only if every Cauchy sequence in E converges (in E).

Definition 1.6. A normed space E is called a **Banach space** if it is complete as a metric space with the distance induced by the norm, i.e., if every Cauchy sequence converges.

1.1.3 Examples of infinite-dimensional Banach spaces

Example 1.3. Recall that $\mathbf{R}^N$ or $\mathbf{C}^N$ for $N \in \mathbf{N}^*$, equipped with a norm, is a Banach space.

Example 1.4. Moreover, the space of bounded sequences l^∞ , endowed with the norm $\|x\|_{l^\infty}$ given for $x \in l^\infty$ by

$$\|x\|_{l^\infty} = \sup_{n \in \mathbf{N}} |x_n| < \infty,$$

is a Banach space.

Example 1.5. More generally, the spaces l^p for every $1 \leq p < \infty$, i.e., $x \in l^p$ when $x = (x_n)_{n \in \mathbb{N}}$ with

$$\|x\|_{l^p} \left(\sum_{n \in \mathbb{N}} |x_n|^p \right)^{1/p} < \infty,$$

are Banach spaces.

Example 1.6. $\mathbb{R}^n$ with any of the following norms ($p \in \mathbb{N}^*$) is a Banach space:

$$\|x\|_\infty = \max_{1 \leq i \leq n} |x_i|, \quad \|x\|_p = \left(\sum_{i=1}^n |x_i|^p \right)^{1/p}$$

Example 1.7. For $p \in [1, \infty)$, the space

$$\ell^p(\mathbb{C}) = \left\{ (x_n) \subset \mathbb{C} : \sum_{n=0}^{\infty} |x_n|^p < \infty \right\}$$

with the norm

$$\|x\|_p = \left(\sum_{n=0}^{\infty} |x_n|^p \right)^{1/p}$$

is a Banach space.

Example 1.8. The space of bounded sequences on $\mathbb{C}$:

$$\ell^\infty(\mathbb{C}) = \left\{ (x_n) \subset \mathbb{C} : \sup_{n \geq 0} |x_n| < \infty \right\}$$

with the norm

$$\|x\|_\infty = \sup_{n \geq 0} |x_n|$$

is a Banach space.

Proposition 1.1. Let $(E_1, \|\cdot\|_1)$ and $(E_2, \|\cdot\|_2)$ be Banach spaces over $\mathbb{K}$. Then the product space $E_1 \times E_2$ with the norm

$$\|(x, y)\|_{E_1 \times E_2} = \max(\|x\|_1, \|y\|_2)$$

is a Banach space.

Proof. Let $((x_n, y_n))$ be a Cauchy sequence in $E_1 \times E_2$. Then for $n, m \in \mathbb{N}$ with $n, m > n_0$, we have:

$$\|(x_n, y_n) - (x_m, y_m)\|_{E_1 \times E_2} = \max(\|x_n - x_m\|_{E_1}, \|y_n - y_m\|_{E_2}) < \varepsilon$$

This implies that (x_n) is Cauchy in E_1 and (y_n) is Cauchy in E_2 . Since E_1 and E_2 are complete, $x_n \rightarrow x$ in E_1 and $y_n \rightarrow y$ in E_2 . Then:

$$\|(x_n, y_n) - (x, y)\|_{E_1 \times E_2} = \max(\|x_n - x\|_{E_1}, \|y_n - y\|_{E_2}) \rightarrow 0$$

Hence, $(x_n, y_n) \rightarrow (x, y)$ in $E_1 \times E_2$. □

1.1.4 Finite-dimensional normed vector spaces

For $n \geq 1$, let E be a $\mathbf{K}$ -vector space of dimension n and $\{e_1, \dots, e_n\}$ a basis of E . Consider on E the norm $\|\cdot\|_\infty$ defined by

$$\left\| \sum_{i=1}^n x_i e_i \right\|_\infty = \max\{|x_1|, \dots, |x_n|\}.$$

Proposition 1.2. (i) Let F be a normed vector space and $T : \mathbf{K}^n \rightarrow F$ a linear map. Then T is continuous.

(ii) All norms on E are equivalent.

(iii) E is complete for any norm $\|\cdot\|$.

Proof. (i) For every $x = \sum_{i=1}^n x_i e_i \in \mathbf{K}^n$, we have $Tx = \sum_{i=1}^n x_i T e_i$ and therefore

$$\|Tx\| \leq \sum_{i=1}^n |x_i| \|T e_i\| \leq \sum_{i=1}^n \|T e_i\| \|x\|_\infty.$$

This shows that T is continuous and that $\|T\| \leq \sum_{i=1}^n \|T e_i\|$.

(ii) It suffices to show that every norm $\|\cdot\|$ is equivalent to the norm $\|\cdot\|_\infty$.

For every $x = \sum_{i=1}^n x_i e_i \in \mathbf{K}^n$, we have

$$\|x\| \leq \sum_{i=1}^n |x_i| \|e_i\| \leq C \|x\|_\infty,$$

with $C = \sum_{i=1}^n \|e_i\|$. Let

$$S = \{x \in E : \|x\|_\infty = 1\}.$$

Every sequence $(x_n)_n$ of elements of S contains a convergent subsequence; thus S is compact.

Consider the map

$$f : S \rightarrow [0, +\infty[, \quad x \mapsto \|x\|.$$

Then f is continuous on the compact set S and $f(x) > 0$ for every $x \in S$. Therefore there exists $\epsilon > 0$ such that $f(x) \geq \epsilon$, i.e. $\|x\| \geq \epsilon$, for every $x \in S$.

Let $x \in E$ with $x \neq 0$. Then, since $\frac{1}{\|x\|_\infty}x \in S$, we have

$$\frac{1}{\|x\|_\infty} \|x\| \geq \epsilon,$$

$$\text{i.e. } \|x\|_\infty \leq \frac{1}{\epsilon} \|x\|.$$

(iii) It is clear that E , equipped with the norm $\|\cdot\|_\infty$, is complete. By (ii), E equipped with any norm is complete.

□

1.2 The space $L(E, F)$ of linear operators

Definition 1.7. 1. An operator is a mapping between two vector spaces.

2. Let E and F be two $\mathbb{K}$ -vector spaces; $T : E \rightarrow F$ is a linear operator if it is linear, i.e.,

$$\forall x, y \in E, \quad \forall \lambda \in \mathbb{K}$$

$$T(x + y) = T(x) + T(y) \quad \text{and} \quad T(\lambda x) = \lambda T(x)$$

3. Let E and F be two normed $\mathbb{K}$ -vector spaces. $T : E \rightarrow F$ is a continuous linear operator on E if T is linear and continuous on E , i.e.,

$$\forall x_0 \in E : \quad \forall (x_n)_{n \in \mathbb{N}} \in E : x_n \rightarrow x_0 \Rightarrow \|Tx_n\| \rightarrow \|Tx_0\|.$$

4. Let E and F be two normed $\mathbb{K}$ -vector spaces. $T : E \rightarrow F$ is a bounded linear operator if:

$$\exists C > 0, \forall x \in E : \|Tx\|_F \leq C\|x\|_E.$$

Definition 1.8. • $L(E, F)$ is the set of linear operators.

- $\mathcal{L}(E, F)$ is the set of continuous linear operators.
- If $E = F$, we write $L(E, F) = L(E)$ and $\mathcal{L}(E, F) = \mathcal{L}(E)$.

Theorem 1.1. Let $L(E, F)$. Then, the following statements are equivalent:

- A) T is continuous on E .
- B) T is continuous at 0_E .
- C) T is bounded on $\overline{B}(0, 1)$.
- D) T is bounded on $S(0, 1)$.
- E) There exists $M \in \mathbb{R}^+$ such that $\forall x \in E$, $\|Tx\|_F \leq M\|x\|_E$.
- (F) T is uniformly continuous on E .

Proof. A) $\Rightarrow$ B), is obvious. B) $\Rightarrow$ C), if A is continuous at 0_E , then there exists $\delta > 0$, such that

$$\text{if } x \in \overline{B}(0, \delta) : T(x) \in \overline{B}(0, 1).$$

Let $x \in \overline{B}(0, 1)$, then $\delta x \in \overline{B}(0, \delta)$, so $T(\delta x) \in \overline{B}(0, 1)$, and since $A \in \mathcal{L}(E, F)$, then

$$\forall x \in \overline{B}(0, 1), \quad T(x) \in \overline{B}\left(0, \frac{1}{\delta}\right),$$

therefore T is bounded on $\overline{B}(0, 1)$.

C) $\Rightarrow$ D), is obvious. D) $\Rightarrow$ E), if T is bounded on $S(0, 1)$, then there exists $M > 0$, such that

$$\forall x \in S(0, 1), \quad \|T(x)\| \leq M.$$

Let $x \in E$, with $x \neq 0$, we have $\frac{x}{\|x\|} \in S(0, 1)$, so

$$\left\| T\left(\frac{x}{\|x\|}\right) \right\| \leq M.$$

hence

$$\forall x \in E, \quad \|T(x)\| \leq M\|x\|.$$

E) $\Rightarrow$ F) Let $x, y \in E$, we have

$$\|T(x) - T(y)\| = \|T(x - y)\| \leq M\|x - y\|.$$

Then T is M -Lipschitz, hence T is uniformly continuous on E .

F) $\Rightarrow$ A), is obvious. □

Theorem 1.2. *Let $T \in \mathcal{L}(E, F)$ and suppose that:*

$$\begin{aligned} N_1 &= \sup_{x \in E \setminus \{0\}} \frac{\|Tx\|_F}{\|x\|_E}. \\ N_2 &= \sup_{x \in S(0,1)} \|Tx\|_F. \\ N_3 &= \sup_{x \in \overline{B}(0,1)} \|Tx\|_F. \\ N_4 &= \inf\{M \in \mathbb{R}^+ : \|Tx\|_F \leq M\|x\|_E\}. \end{aligned}$$

Then, we have:

$$N_1 = N_2 = N_3 = N_4.$$

Proof. Let $x \in E$, then

$$\|Tx\|_F \leq N_2, \quad \text{for } x \in S(0,1).$$

If $x \neq 0$, then $\frac{x}{\|x\|} \in S(0,1)$, so

$$\frac{\|Tx\|_F}{\|x\|_E} \leq N_2, \quad \text{for } x \in E \setminus \{0\},$$

therefore

$$N_1 = \sup_{x \in E \setminus \{0\}} \frac{\|Tx\|_F}{\|x\|_E} \leq N_2.$$

Let $x \in \overline{B}(0,1)$, then $\|Tx\|_F \leq N_3$, so

$$\|Tx\|_F \leq N_3, \quad \text{for } x \in S(0,1),$$

hence

$$N_2 \leq \sup_{x \in S(0,1)} \|Tx\|_F \leq N_3.$$

By the definition of N_4 , for all $\varepsilon > 0$, there exists $M_\varepsilon \in \mathbb{R}^+$, such that

$$0 \leq M_\varepsilon \leq \varepsilon + N_4 \quad \text{and} \quad \|Tx\|_F \leq M_\varepsilon \|x\|_E, \quad \text{for all } x \in E.$$

Then,

$$\|Tx\|_F \leq M_\varepsilon \|x\|_E \leq (\varepsilon + N_4) \|x\|_E, \quad \text{for all } x \in E.$$

Consequently

$$N_3 = \sup_{x \in \overline{B}(0,1)} \|Tx\|_F \leq \varepsilon + N_4.$$

When $\varepsilon \rightarrow 0$, we obtain $N_3 \leq N_4$.

By the definition of N_1 , we have

$$\|Tx\|_F \leq N_1\|x\|_E, \quad \text{for all } x \in E.$$

Then

$$N_1 \in \{M \in \mathbb{R}^+ : \|Tx\|_F \leq M\|x\|_E\}.$$

So

$$N_4 = \inf\{M \in \mathbb{R}^+ : \|Tx\|_F \leq M\|x\|_E\} \leq N_1.$$

Finally, we have found the following relation

$$N_1 \leq N_2 \leq N_3 \leq N_4 \leq N_1.$$

This means

$$N_1 = N_2 = N_3 = N_4.$$

□

Definition 1.9. If T is a continuous linear map from E to F , we set:

$$\|T\|_{\mathcal{L}(E,F)} = N_1 = N_2 = N_3 = N_4.$$

$\|T\|_{\mathcal{L}(E,F)}$ is called the operator norm or the norm subordinated to the norms of E and F . We have in particular:

$$\|Tx\|_F \leq \|T\|_{\mathcal{L}(E,F)}\|x\|_E, \quad \text{for all } x \in E.$$

Proposition 1.3. i) $\mathcal{L}(E, F)$ is a vector subspace of $L(E, F)$.

ii) The map $\mathcal{L}(E, F) \rightarrow \mathbb{R}^+, T \rightarrow \|T\|_{\mathcal{L}(E,F)}$ is a norm on $\mathcal{L}(E, F)$.

Proof. Let $\alpha, \beta \in \mathbb{K}$, $T, S \in \mathcal{L}(E, F)$, there exist $M_1 > 0$ and $M_2 > 0$ such that

$$\|Tx\|_F \leq M_1\|x\|_E \quad \text{and} \quad \|Sx\|_F \leq M_2\|x\|_E, \quad \text{for all } x \in E.$$

Then:

$$\|(\alpha T + \beta S)x\|_F = \|\alpha Tx + \beta Sx\|_F \leq |\alpha|\|Tx\|_F + |\beta|\|Sx\|_F \leq (|\alpha|M_1 + |\beta|M_2)\|x\|_E, \quad \text{for all } x \in E.$$

Therefore, the map $\alpha T + \beta B$ is continuous, meaning $\alpha T + \beta B \in \mathcal{L}(E, F)$, hence $\mathcal{L}(E, F)$ is a vector subspace of $L(E, F)$. □

Proposition 1.4. *If F is a complete normed $\mathbb{K}$ -vector space, then $\mathcal{L}(E, F)$ is also a complete normed vector space.*

Proof. Let $(T_n)_{n \in \mathbb{N}}$ be a Cauchy sequence in $\mathcal{L}(E, F)$, for all $\varepsilon > 0$,

$$\exists n_0 \in \mathbb{N}, \quad n, m \geq n_0 : \|T_n - T_m\|_{\mathcal{L}(E, F)} \leq \varepsilon$$

For all $x \in E$, we have

$$\begin{aligned} \|T_n x - T_m x\|_F &= \|(T_n - T_m)x\|_F \\ &\leq \|T_n - T_m\|_{\mathcal{L}(E, F)} \|x\|_E \\ &\leq \varepsilon \|x\|_E. \end{aligned}$$

This means, for all $x \in E$, $(T_n x)_{n \in \mathbb{N}}$ is a Cauchy sequence in F . The completeness of F allows us to assert that $T_n x$ converges in F to a certain limit which we denote Ax .

We will show that the map $T : E \rightarrow F$ is linear. Let $\alpha, \beta \in \mathbb{K}$ and $x, y \in E$, we have

$$\begin{aligned} T(\alpha x + \beta y) &= \lim_{n \rightarrow +\infty} T_n(\alpha x + \beta y) \\ &= \alpha \lim_{n \rightarrow +\infty} T_n x + \beta \lim_{n \rightarrow +\infty} T_n y \\ &= \alpha T x + \beta T y. \end{aligned}$$

Let us show that T is continuous. Let $x \in \overline{B}_E(0, 1)$, then

$$\begin{aligned} \|Tx\|_F &= \lim_{n \rightarrow \infty} \|T_n x\|_F \leq \lim_{n \rightarrow \infty} \|T_n\|_{\mathcal{L}(E, F)} \|x\|_E \\ &\leq M \|x\|_E, \end{aligned}$$

with $M = \sup\{\|T_n\|_{\mathcal{L}(E, F)} : n \in \mathbb{N}\} < \infty$, because $(T_n)_{n \in \mathbb{N}}$ is a bounded sequence (since it is Cauchy), hence the continuity of T .

Let us show that $(T_n)_{n \in \mathbb{N}}$ converges to T in $\mathcal{L}(E, F)$. Let $x \in \overline{B}_E(0, 1)$, we have

$$\exists n_0 \in \mathbb{N}, \quad n, m \geq n_0 : \|T_n x - T_m x\|_F \leq \varepsilon.$$

When $m \rightarrow +\infty$, we find

$$\exists n_0 \in \mathbb{N}, \quad n \geq n_0 : \|T_n x - Tx\|_F \leq \epsilon.$$

Then

$$\exists n_0 \in \mathbb{N}, \quad n \geq n_0 : \|T_n - T\|_{\mathcal{L}(E,F)} = \sup_{\|x\|_E \leq 1} \|T_n x - Tx\|_F \leq \epsilon.$$

This establishes the convergence of the sequence $(T_n)_n$ to T in the space $\mathcal{L}(E, F)$.

□

□

Proposition 1.5. *Let E, F and G , be three normed vector spaces, $T \in \mathcal{L}(E, F)$ and $S \in \mathcal{L}(F, G)$. Then, $S \circ T \in \mathcal{L}(E, G)$ and*

$$\|S \circ T\|_{\mathcal{L}(E,G)} \leq \|T\|_{\mathcal{L}(E,F)} \|S\|_{\mathcal{L}(F,G)}.$$

Proof. Let $x \in \overline{B}_E(0, 1)$, we have

$$\begin{aligned} \|(S \circ T)(x)\|_G &= \|S(T(x))\|_G \\ &\leq \|S\|_{\mathcal{L}(F,G)} \|T(x)\|_F \\ &\leq \|S\|_{\mathcal{L}(F,G)} \|T\|_{\mathcal{L}(E,F)} \|x\|_E \\ &\leq \|S\|_{\mathcal{L}(F,G)} \|T\|_{\mathcal{L}(E,F)}. \end{aligned}$$

hence

$$\begin{aligned} \|S \circ T\|_{\mathcal{L}(E,G)} &= \sup \{ \|(S \circ T)(x)\|_G : x \in \overline{B}_E(0, 1) \} \\ &\leq \|S\|_{\mathcal{L}(F,G)} \|T\|_{\mathcal{L}(E,F)}. \end{aligned}$$

□

Proposition 1.6. *Let E and F , be two normed vector spaces, with E finite-dimensional. Then $L(E, F) = \mathcal{L}(E, F)$, in other words, every linear map on a finite-dimensional space E is continuous.*

Proof. Let $\{e_1, e_2, \dots, e_n\}$ be a basis of E . For all $x \in \overline{B}_E(0, 1)$, we have $x = \sum_{i=1}^n x_i e_i$, and

$$\begin{aligned} \|T(x)\|_F &= \left\| T \left(\sum_{i=1}^n x_i e_i \right) \right\|_F \\ &\leq \sum_{i=1}^n |x_i| \|T(e_i)\|_F \\ &\leq \left(\sum_{i=1}^n \|T(e_i)\|_F \right) \max_{i \in \{1, \dots, n\}} |x_i| \\ &= \left(\sum_{i=1}^n \|T(e_i)\|_F \right) \|x\|_\infty \quad (\text{where } \|x\|_\infty = \max_i |x_i|). \end{aligned}$$

Since $\dim E = n < \infty$, all norms are equivalent, so there exists $\alpha > 0$ such that

$$\forall x \in E, \|x\|_\infty \leq \alpha \|x\|_E.$$

Therefore

$$\|T(x)\|_F \leq \left(\sum_{i=1}^n \|T(e_i)\|_F \right) \alpha \|x\|_E.$$

Hence the continuity of A . $\square$

$\square$

Lemma 1.1. *If the operator T is bounded, with bound M , then $\|T\| \leq M$.*

Indeed:

$$T \text{ bounded on } E \Rightarrow \exists M > 0, \|T(x)\|_F \leq M \|x\|_E,$$

and moreover,

$$\|T(x)\|_F \leq \|T\| \|x\|_E.$$

Since $\|T\|$ is the smallest constant M such that this holds.

Proposition 1.7. *If the space E is finite-dimensional ($\dim E = n$), then every linear operator $T : E \rightarrow F$ is continuous, i.e.,*

$$\mathcal{L}(E, F) = L(E, F).$$

Proof. Let $\dim E = n$. Let $\{v_1, \dots, v_n\}$ be a basis of E , such that:

$$\forall x \in E : x = \sum_{i=1}^n \lambda_i v_i, \quad \lambda_i \in \mathbb{K}.$$

Since E is finite-dimensional, all norms are equivalent. Consider the norm $\|x\|_\infty = \max_{i=1, \dots, n} |\lambda_i|$. Then:

$$\|Tx\|_F = \left\| T \left(\sum_{i=1}^n \lambda_i v_i \right) \right\|_F \leq \sum_{i=1}^n |\lambda_i| \cdot \|T(v_i)\| \leq \left(\max_i |\lambda_i| \right) \left(\sum_{i=1}^n \|T(v_i)\| \right) \leq C \|x\|_\infty.$$

This is sufficient to show that T is continuous on E .

□

Remark 1.1. If $\dim E = \infty$: $T \in L(E, F) \not\Rightarrow T \in \mathcal{L}(E, F)$.

Example 1.9. Let $E = \mathbb{R}[x]$ be the space of real polynomials.

$$E = \{P : \mathbb{R} \rightarrow \mathbb{R} \mid P(x) = a_0 + a_1x + \dots + a_nx^n, \dots\}, \quad \dim E = \infty.$$

Take $\|P_n\| = \max_i |a_i|$. Define the operator $T_n : \mathbb{R}[x] \rightarrow \mathbb{R}[x]$ by:

$$P \mapsto T_n P = \sum_{i=0}^n a_i x^i.$$

Let $\tilde{P} = \sum_{i=0}^n x^i$, then $a_i = 1$, so $\tilde{P} \in E$,

$$\|T_n \tilde{P}\| = \left| \sum_{i=0}^n a_i \right| = n+1 \quad \Rightarrow \quad \|T_n \tilde{P}\| \xrightarrow{n \rightarrow \infty} +\infty.$$

Since $\|T_n \tilde{P}\|$ depends on n and tends to infinity, there does not exist a constant M such that $\|T_n \tilde{P}\| \leq M \|\tilde{P}\|$. Therefore, T_n is not continuous.

Theorem 1.3. Let E be a normed vector space and let F be a Banach space, then $\mathcal{L}(E, F)$ is a Banach space.

Proposition 1.8. If $T \in \mathcal{L}(E, F)$, $S \in \mathcal{L}(F, G)$, then $S \circ T = ST \in \mathcal{L}(E, G)$ and

$$\|ST\| \leq \|S\| \cdot \|T\|$$

.

Lemma 1.2. Let E , F and G be three normed vector spaces.

- If $T \in \mathcal{L}(E, F)$, $S \in \mathcal{L}(F, G)$ then:

$$S \circ T = ST \in \mathcal{L}(E, G) \text{ and we have}$$

$$\|ST\| \leq \|S\| \cdot \|T\|.$$

Proof. Indeed: $\forall x \in E$, $\|ST(x)\| \leq \|S\| \cdot \|T(x)\| \leq \|S\| \cdot \|T\| \cdot \|x\|$.

Moreover we have: $\|ST(x)\| \leq \|ST\| \cdot \|x\|$

and since: $\|ST\|$ is the smallest of the constants M such that: $\|ST(x)\| \leq M\|x\|$ $\forall x \in E$, then:

$$\|ST\| \leq \|S\| \cdot \|T\|$$

□

- If $T \in \mathcal{L}(E)$ we have $\forall n \in \mathbb{N}$: $\|T^n\| \leq \|T\|^n$. (Note that in our case,)

Definition 1.10. Let T be a linear operator on E . We say T is **bounded** if and only if there exists a constant $c > 0$ such that:

$$\|T(x)\|_F \leq c\|x\|_E \quad \forall x \in E$$

Theorem 1.4. A linear operator T is continuous if and only if it is bounded.

Proof. ($\Rightarrow$) Suppose T is continuous but not bounded. Then for every $M > 0$, there exists x_M such that $\|Tx_M\| > M\|x_M\|$. In particular, for $n \in \mathbb{N}$, there exists a sequence (x_n) in E such that:

$$\|Tx_n\| > n\|x_n\|$$

Define $y_n = \frac{x_n}{n\|x_n\|}$. Clearly, $\lim_{n \rightarrow \infty} y_n = 0$. Since T is continuous, we get $Ty_n \rightarrow 0$. But:

$$\|Ty_n\| = \frac{1}{n\|x_n\|} \|Tx_n\| > \frac{1}{n\|x_n\|} \cdot n\|x_n\| = 1$$

which is a contradiction.

($\Leftarrow$) If T is bounded, then there exists $M > 0$ such that $\|Tx\| \leq M\|x\|$. Let (x_n) be a sequence in E converging to x . Then:

$$\|T(x_n - x)\| \leq M\|x_n - x\| \rightarrow 0$$

which shows that $Tx_n \rightarrow Tx$, so T is continuous.

□

Theorem 1.5. *For a linear operator $T \in \mathcal{L}(E, F)$, the following three properties are equivalent:*

1. T is bounded.
2. T is continuous on E .
3. T is continuous at the point $0 \in E$.

Proof. (1) $\Rightarrow$ (2): Already shown. (2) $\Rightarrow$ (3): Trivial. (3) $\Rightarrow$ (1): Suppose T is continuous at 0 but not bounded. Then for every $n \in \mathbb{N}$, there exists a non-zero vector $x_n \in E$ such that $\|Tx_n\| > n\|x_n\|$. Let $y_n = \frac{x_n}{n\|x_n\|}$. Then $\|y_n\| = \frac{1}{n} \rightarrow 0$, so $y_n \rightarrow 0$. But:

$$\|Ty_n\| = \frac{1}{n\|x_n\|} \|Tx_n\| > \frac{1}{n\|x_n\|} \cdot n\|x_n\| = 1$$

so Ty_n does not converge to $T0 = 0$, contradicting continuity at 0. Thus, T must be bounded. $\square$

Theorem 1.6. *If T is an additive and continuous operator on a normed vector space, then it is homogeneous.*

Proof. 1. For $n \in \mathbb{N}$, we have $T(nx) = nT(x)$ (by induction).

2. For $n = 0$, $T(0 \cdot x) = T(0) = 0 = 0 \cdot T(x)$.

3. For $n \in \mathbb{Q}$, let $n = m/k$. Then:

$$T\left(\frac{m}{k}x\right) = mT\left(\frac{x}{k}\right) = m \cdot \frac{1}{k}T(x) = \frac{m}{k}T(x)$$

(since $T(x) = T(k \cdot (x/k)) = kT(x/k)$ implies $T(x/k) = \frac{1}{k}T(x)$).

4. For $\lambda \in \mathbb{R}$ (or $\mathbb{C}$), let $\lambda_n \rightarrow \lambda$ with $\lambda_n \in \mathbb{Q}$. By continuity, $T(\lambda_n x) \rightarrow T(\lambda x)$. But $T(\lambda_n x) = \lambda_n T(x) \rightarrow \lambda T(x)$. Hence, $T(\lambda x) = \lambda T(x)$.

$\square$

1.2.1 Calculating the Norm of an Operator:

We consider a type of operator where the norm can be easily computed.

Definition 1.11. Let E, F be Banach spaces. For $T \in \mathcal{L}(E, F)$, the **norm** of T is defined as the smallest positive constant c such that $\|Tx\|_F \leq c\|x\|_E$, i.e.:

$$\|T\|_{\mathcal{L}} = \inf\{M > 0 : \|Tx\|_F \leq M\|x\|_E\}$$

Proposition 1.9. Let $T \in \mathcal{L}(E, F)$. Then:

1. $\forall x \in E, \|Tx\|_F \leq \|T\|_{\mathcal{L}}\|x\|_E$
2. $\forall \varepsilon > 0, \exists x_\varepsilon \in E, \|Tx_\varepsilon\|_F \geq (\|T\|_{\mathcal{L}} - \varepsilon)\|x_\varepsilon\|_E$
3. $\|T\|_{\mathcal{L}} = \sup_{x \neq 0} \frac{\|Tx\|_F}{\|x\|_E} = \sup_{\|x\|_E \leq 1} \|Tx\|_F$

Proof. Let $M_0 = \|T\|_{\mathcal{L}}$. By definition, $\|Tx\|_F \leq M_0\|x\|_E$ for all $x \in E$, proving (1).

For (2), by the definition of infimum, for every $\varepsilon > 0$, $M_0 - \varepsilon$ is not a lower bound. Thus, there exists $x_\varepsilon \in E$ such that $\|Tx_\varepsilon\|_F > (M_0 - \varepsilon)\|x_\varepsilon\|_E$.

For (3), from (1), for $\|x\|_E \leq 1$, we have $\|Tx\|_F \leq \|T\|_{\mathcal{L}}$, so:

$$\sup_{\|x\|_E \leq 1} \|Tx\|_F \leq \|T\|_{\mathcal{L}}$$

Now, let $y_\varepsilon = \frac{x_\varepsilon}{\|x_\varepsilon\|_E}$. Then $\|y_\varepsilon\|_E = 1$, and:

$$\|Ty_\varepsilon\|_F = \frac{1}{\|x_\varepsilon\|_E} \|Tx_\varepsilon\|_F \geq \frac{1}{\|x_\varepsilon\|_E} (M_0 - \varepsilon)\|x_\varepsilon\|_E = M_0 - \varepsilon$$

Hence, $\sup_{\|x\|_E \leq 1} \|Tx\|_F \geq M_0 - \varepsilon$ for every $\varepsilon > 0$, so:

$$\sup_{\|x\|_E \leq 1} \|Tx\|_F \geq M_0 = \|T\|_{\mathcal{L}}$$

Combining both inequalities gives the result. □

Example 1.10. Let $T : C([a, b], \mathbb{R}) \rightarrow \mathbb{R}$ be defined by $Tx = \int_a^b x(t)dt$. Then:

$$\|Tx\| \leq \int_a^b |x(t)|dt \leq (b-a)\|x\|_\infty$$

So $\|T\| \leq (b-a)$. Consider the constant function $x_0(t) = 1$. Then $\|x_0\|_\infty = 1$ and $Tx_0 = b-a$, so $\|Tx_0\| = b-a$. Thus, $\frac{\|Tx_0\|}{\|x_0\|_\infty} = b-a$, and hence $\|T\| = b-a$.

Lemma 1.3. If $T \in \mathcal{L}(E, F)$ and $\|Tx\|_F = \|x\|_E$ (T is isometric), then T is bounded and $\|T\| = 1$.

In practice, calculating the norm of an operator is not always straightforward; for this, we use the following result:

Proposition 1.10. *Let $T \in \mathcal{L}(E, F)$ and $M > 0$. Suppose $\|T\| \leq M$. To show that $\|T\| = M$, it suffices that one of the following two conditions holds:*

i. $\exists x \in E : \|x\|_E = 1$ and $\|Tx\|_F = M$.

ii. *There exists a sequence (x_n) in E with $\|x_n\| = 1$, and $\lim_{n \rightarrow \infty} \|Tx_n\|_F = M$.*

Example 1.11. *Let $E = C([0, 1], \mathbb{R})$, $T : E \rightarrow E$, equipped with the norm $\|\cdot\|_\infty$.*

For all $f \in C([0, 1], \mathbb{R})$, define:

$$(Tf)(x) = \int_0^x f(t) dt, \quad \forall x \in [0, 1].$$

Then:

$$|Tf(x)| \leq \int_0^x |f(t)| dt \leq \sup_{t \in [0, 1]} |f(t)| \cdot \int_0^x dt \leq \|f\|_\infty \cdot x \leq \|f\|_\infty.$$

So $\|Tf\|_\infty \leq \|f\|_\infty$, hence T is bounded and $\|T\| \leq 1$.

Now show that $\|T\| = 1$. Take $f(t) = 1$ (constant function), $\|f\|_\infty = 1$, and:

$$Tf(x) = \int_0^x 1 dt = x \Rightarrow \|Tf\|_\infty = \sup_{x \in [0, 1]} x = 1.$$

By Proposition 1.8, $\|T\| = 1$.

Example 1.12. *Show that the following operator is well-defined and find its norm:*

$$T : (C([0, 1], \mathbb{R}), \|\cdot\|_\infty) \rightarrow (L^2([0, 1]), \|\cdot\|_2), \quad Tf = xf(x).$$

(Homework).

1.3 Operators with Dense Domain and Extension by Continuity

Definition 1.12. *An operator with dense domain is an operator*

$$T : D(T) \subset E \longrightarrow F \quad (\text{possibly bounded}),$$

with $\overline{D(T)} = E$ (i.e., $D(T)$ is dense in E).

Here, $D(T) = \{x \in E \mid Tx \in \text{Im}(T)\}$.

(If $D(T) \neq E$, then T may be bounded on $D(T)$ but not on E).

Example 1.13. (Unbounded Operator)

Consider the operator:

$$T : \mathcal{P} \subset C([0, 1], \mathbb{R}) \longrightarrow C([0, 1], \mathbb{R}), \quad P(t) \mapsto P'(t),$$

where $\mathcal{P}$ is the subspace of $C([0, 1], \mathbb{R})$ consisting of polynomial functions, and $C([0, 1], \mathbb{R})$ is equipped with the uniform norm $\|\cdot\|_2$.

This operator is not bounded. Indeed, let $P_n(t) = t^n$, $n \in \mathbb{N}$, $\|P_n\| = \sup_{t \in [0, 1]} |P_n(t)| = 1$.

$$\|TP_n\| = \|P'\| = \sup_{t \in [0, 1]} |nt^{n-1}| = n \Rightarrow \|TP_n\| \xrightarrow{n \rightarrow \infty} +\infty.$$

So there is no constant $C > 0$ such that $\|TP_n\| \leq C\|P_n\|$. Hence, T is not bounded on $\mathcal{P}$.

Theorem 1.7. (Extension by Continuity)

Let $T : D(T) \subset E \rightarrow F$, where F is a Banach space. Assume T has a dense domain ($\overline{D(T)} = E$) and is bounded:

$$\exists M > 0, \forall x \in D(T) : \|Tx\|_F \leq M\|x\|_E.$$

Then, the operator T can be extended to the whole space E by continuity, with preservation of the norm, i.e., there exists $\tilde{T} \in \mathcal{L}(E, F)$ such that:

$$\tilde{T}(x) = T(x), \quad \forall x \in D(T), \quad \text{and} \quad \|\tilde{T}\| = \|T\|.$$

The operator $\tilde{T}$ is called the extension by continuity of T .

Definition 1.13. Simple convergence, uniform convergence.

Let E, F be normed vector spaces. We say that a sequence (T_n) in $\mathcal{L}(E, F)$ converges to $T \in \mathcal{L}(E, F)$:

1. *Simply (pointwise) if:*

$$\forall x \in E : \lim_{n \rightarrow \infty} \|T_n x - T x\|_F = 0, \quad \text{denoted } T_n \xrightarrow{cs} T.$$

2. *Uniformly if:*

$$\lim_{n \rightarrow \infty} \|T_n - T\| = 0, \quad \text{denoted } T_n \xrightarrow{cu} T.$$

Proposition 1.11. *If $T_n \xrightarrow{cu} T$, then $T_n \xrightarrow{cs} T$ (the converse is false).*

Assume $T_n \rightarrow T$ in $\mathcal{L}(E, F)$, i.e., $T_n - T \in \mathcal{L}(E, F)$, $T \in \mathcal{L}(E, F)$. Then:

$$\|(T_n - T)(x)\| \leq \|T_n - T\| \cdot \|x\| \Rightarrow \|T_n x - T x\| \leq \|T_n - T\| \|x\|, \quad \forall x \in E.$$

Taking the limit:

$$\lim_{n \rightarrow \infty} \|T_n x - T x\| \leq \lim_{n \rightarrow \infty} \|T_n - T\| \|x\| = 0.$$

Hence, $\lim_{n \rightarrow \infty} \|T_n x - T x\| = 0$, so T_n converges simply to T .

1.4 Pointwise convergence and uniform convergence

Let (T_n) be a sequence of bounded linear operators from X to Y and let T be a bounded linear operator from X to Y .

Definition 1.14. *We say that the sequence (T_n) converges pointwise (or simply, or strongly) to T if*

$$\forall x \in X, T_n x \xrightarrow{n \rightarrow +\infty} T x.$$

Example 1.14. *Consider the sequence of bounded linear operators (T_n) defined on $l_2(\mathbb{R})$ by*

$$T_n(x_1, x_2, \dots) = \frac{n}{n^2 + 1}(x_1, x_2, \dots).$$

Let $T = 0$ be the zero operator on $l_2(\mathbb{R})$ given by:

$$T : l_2(\mathbb{R}) \longrightarrow l_2(\mathbb{R}), \quad (x_1, x_2, \dots) \longmapsto 0.$$

(One can show that T is a bounded linear operator on $l_2(\mathbb{R})$.)

For every $n \in \mathbb{N}$ and $(x_1, x_2, \dots) \in \ell_2(\mathbb{R})$, we have

$$\begin{aligned} \|T_n(x_1, x_2, \dots) - T(x_1, x_2, \dots)\| &= \|T_n(x_1, x_2, \dots) - 0\| \\ &= \|T_n(x_1, x_2, \dots)\| \\ &= \left\| \frac{n}{n^2 + 1}(x_1, x_2, \dots) \right\| \\ &= \frac{n}{n^2 + 1} \|(x_1, x_2, \dots)\| \xrightarrow{n \rightarrow \infty} 0. \end{aligned}$$

That is,

$$\text{For all } (x_1, x_2, \dots) \in \ell_2(\mathbb{R}) : \|T_n(x_1, x_2, \dots) - T(x_1, x_2, \dots)\| \xrightarrow{n \rightarrow \infty} 0.$$

This means that (T_n) converges pointwise to the zero operator $A = 0$.

Definition 1.15. We say that the sequence (T_n) converges uniformly (or in norm) to T if

$$\|T_n - T\| \xrightarrow{n \rightarrow +\infty} 0.$$

Example 1.15. Consider the sequence of bounded linear operators (T_n) and the zero operator

0 on $\ell_2(\mathbb{R})$ given in the previous example. For all $n \in \mathbb{N}$ we have

$$\begin{aligned} \|T_n - T\| &= \|T_n - 0\| = \|T_n\| = \sup_{0 \neq (x_1, x_2, \dots) \in \ell_2(\mathbb{R})} \frac{\|A_n(x_1, x_2, \dots)\|}{\|(x_1, x_2, \dots)\|} \\ &= \sup_{0 \neq (x_1, x_2, \dots) \in \ell_2(\mathbb{R})} \frac{\|x_{n+1}^n(x_1, x_2, \dots)\|}{\|(x_1, x_2, \dots)\|} = \sup_{0 \neq (x_1, x_2, \dots) \in \ell_2(\mathbb{R})} \frac{n}{n+1} \|(x_1, x_2, \dots)\| \\ &= \sup_{0 \neq (x_1, x_2, \dots) \in \ell_2(\mathbb{R})} \frac{n}{n^2+1} = \frac{n}{n^2+1} \xrightarrow{n \rightarrow \infty} 0. \end{aligned}$$

That is,

$$\|T_n - 0\| \xrightarrow{n \rightarrow \infty} 0.$$

This means that (T_n) converges uniformly to the null operator $T = 0$.

Lemma 1.4. If (T_n) converges uniformly to T , then it converges pointwise to T .

Proof. Let $x \in X$. We have

$$0 \leq \|T_n x - T x\| = \|(T_n - T)x\| \leq \|T_n - T\| \|x\|.$$

Since (T_n) converges uniformly to A , we have

$$\|T_n - T\| \xrightarrow{n \rightarrow +\infty} 0.$$

Hence,

$$\|T_n - T\| \|x\| \xrightarrow{n \rightarrow +\infty} 0.$$

Thus,

$$\text{For all } x \in X : \quad \|T_n x - T x\| \xrightarrow{n \rightarrow +\infty} 0.$$

This means that (T_n) converges strongly to T .

□

Lemma 1.5. *If (T_n) is a sequence that converges pointwise, then one cannot conclude anything about its uniform convergence.*

Proof. Consider the sequence of bounded linear operators (T_n) defined on $\ell_2(\mathbb{R})$ by

$$T_n(x_1, x_2, \dots) = (x_1, x_2, \dots, x_n, 0, 0, \dots) \quad \text{for all } n.$$

Let I be the identity operator on $\ell_2(\mathbb{R})$ given as follows:

$$I : \ell_2(\mathbb{R}) \longrightarrow \ell_2(\mathbb{R}), \quad (x_1, x_2, \dots) \longmapsto I(x_1, x_2, \dots) = (x_1, x_2, \dots).$$

- One can show that I is a bounded linear operator on $\ell_2(\mathbb{R})$.
- Let us show that (T_n) converges pointwise to the identity operator I . For all $n \in \mathbb{N}$ and $(x_1, x_2, \dots) \in \ell_2(\mathbb{R})$ we have

$$\begin{aligned} 0 \leq \|T_n(x_1, x_2, \dots) - I(x_1, x_2, \dots)\|^2 &= \|(x_1, x_2, \dots, x_n, 0, 0, \dots) - (x_1, x_2, \dots)\|^2 \\ &= \|- (x_{n+1}, x_{n+2}, \dots)\|^2 = \sum_{k=n+1}^{+\infty} |x_k|^2. \end{aligned}$$

Since $(x_1, x_2, \dots) \in \ell_2(\mathbb{R})$, the series $\sum_{k=1}^{+\infty} |x_k|^2$ is convergent. Thus, its remainder satisfies

$$\sum_{k=n+1}^{+\infty} |x_k|^2 \xrightarrow{n \rightarrow +\infty} 0.$$

This implies that

$$\|T_n(x_1, x_2, \dots) - I(x_1, x_2, \dots)\| \xrightarrow{n \rightarrow +\infty} 0.$$

Thus (T_n) converges pointwise to the identity operator I .

- Let us show that (T_n) does not converge uniformly to the identity I : Initially, we

□

set

$$0 \neq x^* = \left(0, 0, \dots, 0, \underbrace{1}_{n+1}, 0, 0, \dots\right) \in \ell_2(\mathbb{R}).$$

For all n , we have

$$\begin{aligned} \|T_n - I\| &= \sup_{0 \neq (x_1, x_2, \dots) \in \ell_2(\mathbb{R})} \frac{\|(T_n - I)(x_1, x_2, \dots)\|}{\|(x_1, x_2, \dots)\|} \geq \frac{\|(T_n - I)x^*\|}{\|x^*\|} \\ &= \left\| (T_n - I) \left(0, 0, \dots, 0, \underbrace{1}_{n+1}, 0, 0, \dots\right) \right\| = \left\| T_n \left(0, 0, \dots, 0, \underbrace{1}_{n+1}, 0, 0, \dots\right) - I \left(0, 0, \dots, 0, \underbrace{1}_{n+1}, 0, 0, \dots\right) \right\| \\ &= \left\| (0, 0, \dots, 0, 0, 0, 0, \dots) - \left(0, 0, \dots, 0, \underbrace{1}_{n+1}, 0, 0, \dots\right) \right\| = \left\| - \left(0, 0, \dots, 0, \underbrace{1}_{n+1}, 0, 0, \dots\right) \right\| = 1. \end{aligned}$$

That is,

$$\text{For all } n : \quad \|T_n - I\| \geq 1.$$

This implies that

$$\|T_n - I\| \not\rightarrow_{n \rightarrow +\infty} 0.$$

Thus, (T_n) does not converge uniformly to the identity.

1.5 Uniform Boundedness Principle

Definition 1.16. A family $(T_i)_{i \in I}$ in $\mathcal{L}(E, F)$ is said to be:

1. *Uniformly bounded* if:

$$\exists M > 0, \forall i \in I : \|T_i\| \leq M.$$

2. *Simply bounded (pointwise bounded)* if:

$$\forall x \in E, \exists M_x > 0, \forall i \in I : \|T_i(x)\| \leq M_x.$$

Remark 1.2. It is clear that if (T_i) is uniformly bounded, then it is simply bounded. The converse is not true. However, the uniform boundedness principle (which we will state) provides a situation where these two notions coincide.

Theorem 1.8. (*Banach-Steinhaus*)

Let $(T_i)_{i \in I} \subset \mathcal{L}(E, F)$. Assume that E is a Banach space. Then:

$$(T_i)_{i \in I} \text{ is uniformly bounded } \iff (T_i)_{i \in I} \text{ is simply bounded.}$$

That is:

$$\exists M > 0, \forall i \in I : \|T_i\| \leq M \iff \forall x \in E, \exists M_x > 0, \forall i \in I : \|T_i(x)\| \leq M_x.$$

1.6 inverse linear operators

Definition 1.17. Let $(E, \|\cdot\|_E)$, $(F, \|\cdot\|_F)$ be two normed K -vector spaces. A linear operator $T : D(T) \rightarrow \text{Im}(T)$ is said to be invertible if there exists an operator $S : \text{Im}(T) \rightarrow D(T)$ such that:

$$S \circ T = I_d, \quad T \circ S = I_d.$$

S is called the inverse of T and is denoted $S = T^{-1}$.

Proposition 1.12. Let E and F be two normed K -vector spaces, and $T : D(T) \subset E \rightarrow \text{Im}(T) \subset F$ a linear operator.

- T^{-1} exists on $\text{Im}(T)$ if and only if $Tx = 0 \Rightarrow x = 0$.
- If T^{-1} exists, then T^{-1} is linear.
- If $\dim D(T) = n < \infty$ and T^{-1} exists, then $\dim(\text{Im } T) = \dim D(T) = n$

Lemma 1.6. If $T \in \mathcal{L}(E, F)$, $S \in \mathcal{L}(F, G)$ are invertible, then $S \circ T$ is invertible and:

$$(S \circ T)^{-1} = T^{-1} \circ S^{-1}.$$

Remark 1.3.

$$T \in \mathcal{L}(E, F) \not\Rightarrow T^{-1} \in \mathcal{L}(F, E).$$

(i.e., T bounded does not imply T^{-1} bounded).

Example 1.16. Define $T : \ell^2(\mathbb{R}) \rightarrow \ell^2(\mathbb{R})$ by:

$$(x_n) = (x_1, x_2, \dots) \mapsto T(x) = \left(x_1, \frac{x_2}{2}, \dots, \frac{x_n}{n}, \dots\right).$$

For all $x = (x_1, x_2, \dots) \in \ell^2(\mathbb{R})$:

$$\|Tx\|_2 = \left(\sum_{k=1}^{\infty} \left| \frac{x_k}{k} \right|^2 \right)^{1/2} \leq \left(\sum_{k=1}^{\infty} |x_k|^2 \right)^{1/2} = \|x\|_2.$$

So T is bounded. Moreover:

$$Tx = 0 \iff \left(x_1, \frac{x_2}{2}, \dots, \frac{x_n}{n}, \dots \right) = 0 \Rightarrow x = (0, 0, \dots),$$

so T is injective, hence invertible.

The inverse of T is:

$$T^{-1} : \ell^2(\mathbb{R}) \rightarrow \ell^2(\mathbb{R}), \quad (x_n) \mapsto (x_1, 2x_2, \dots, nx_n, \dots).$$

Now, T^{-1} is not bounded (controversial). Let $x = (0, 0, \dots, 1, 0, \dots)$ with 1 in the n th position. Then:

$$\|x\|_2 = 1 \quad \Rightarrow \quad \|T^{-1}x\|_2 = n.$$

There is no $M > 0$ such that $\|T^{-1}x\|_2 \leq M\|x\|_2$ for all such x . Hence, T^{-1} is not bounded.

Remark 1.4. If T is a bounded linear operator from a Banach space E to a Banach space F , and T is bijective, then the inverse operator T^{-1} is bounded.

Proposition 1.13. Let E and F be two normed vector spaces. Let $T \in \mathcal{L}(E, F)$ be surjective. Then:

1. T has a bounded inverse if and only if there exists $C > 0$ such that:

$$\|Tx\|_F \geq C\|x\|_E, \quad \forall x \in E.$$

2. In this case, $\|T^{-1}\| \leq \frac{1}{C}$.

Proof. • (1) Suppose $\|Tx\|_F \geq C\|x\|_E$. Then:

$$Tx = 0 \Rightarrow 0 \geq C\|x\|_E \Rightarrow \|x\|_E = 0 \Rightarrow x = 0,$$

so T is injective. Since T is surjective, it is bijective, hence invertible.

- (2) For all $y \in F$, there exists $x \in E$ such that $y = Tx$. Then:

$$\|y\|_F = \|Tx\|_F \geq C\|x\|_E = C\|T^{-1}y\|_E \Rightarrow \|T^{-1}y\|_E \leq \frac{1}{C}\|y\|_F.$$

This shows that T^{-1} is bounded and $\|T^{-1}\| \leq \frac{1}{C}$.

□

Corollary 1.1. *If T is invertible, then:*

$$\|Tu\|_F \geq \|T^{-1}\|^{-1}\|u\|_E, \quad \forall u \in E.$$

Indeed:

$$\|u\|_E = \|T^{-1}(Tu)\|_E \leq \|T^{-1}\|\|Tu\|_F \Rightarrow \frac{\|u\|_E}{\|T^{-1}\|} \leq \|Tu\|_F.$$

Remark 1.5. *Theorem: Let E and F be Banach spaces and $T \in \mathcal{L}(E, F)$ be bijective. Then $T^{-1} \in \mathcal{L}(F, E)$.*

Example 1.17. *Let $(E, \|\cdot\|_E)$ be a Banach space and $T \in \mathcal{L}(E)$ with $\|T\| < 1$.*

Then $I_E - T$ is invertible and:

$$(I_E - T)^{-1} = \sum_{k=0}^{\infty} T^k.$$

Proposition 1.14. *Let $S \in \mathcal{L}(F, H)$, $T \in \mathcal{L}(E, F)$. Then $ST \in \mathcal{L}(E, H)$ and $\|ST\| \leq \|S\|\|T\|$.*

Proof. Linearity: $ST(\alpha x + y) = S(\alpha Tx + Ty) = \alpha STx + STy$. Boundedness:

$$\|STx\| \leq \|S\|\|Tx\| \leq \|S\|\|T\|\|x\|$$

so $\|ST\| \leq \|S\|\|T\|$. □

Theorem 1.9. *Let E be a Banach space and $T \in \mathcal{L}(E)$. If $\|T\| < 1$, then $I - T$ is invertible and:*

$$(I - T)^{-1} = \sum_{n=0}^{\infty} T^n = I + T + T^2 + \dots$$

Proof. Since $\|T\| < 1$, the series $\sum_{n=0}^{\infty} \|T^n\|$ converges because $\|T^n\| \leq \|T\|^n$ and $\sum \|T\|^n = \frac{1}{1-\|T\|} < \infty$. Since $\mathcal{L}(E)$ is a Banach space, the series $\sum_{n=0}^{\infty} T^n$ converges in $\mathcal{L}(E)$. Now:

$$\begin{aligned} (I - T) \sum_{n=0}^N T^n &= \sum_{n=0}^N T^n - \sum_{n=1}^{N+1} T^n = I - T^{N+1} \\ \sum_{n=0}^N T^n (I - T) &= I - T^{N+1} \end{aligned}$$

As $N \rightarrow \infty$, $T^{N+1} \rightarrow 0$ because $\|T^{N+1}\| \leq \|T\|^{N+1} \rightarrow 0$. Hence:

$$(I - T) \sum_{n=0}^{\infty} T^n = I = \left(\sum_{n=0}^{\infty} T^n \right) (I - T)$$

so $(I - T)^{-1} = \sum_{n=0}^{\infty} T^n$. □

1.6.1 Extension of an Operator by Continuity

Let E be a normed vector space, $D \subseteq E$ a linear subspace, and $T : D \rightarrow F$ a linear operator. We say T is **bounded** on D if there exists $M > 0$ such that:

$$\|Tx\|_F \leq M\|x\|_E \quad \forall x \in D$$

The smallest such M is called the norm of T on D , denoted $\|T\|_D$.

Theorem 1.10. *Let E be a Banach space, D a dense linear subspace of E , and $T : D \rightarrow F$ a bounded linear operator into a Banach space F . Then T admits a unique extension $\hat{T} \in \mathcal{L}(E, F)$ such that $\hat{T}|_D = T$ and $\|\hat{T}\|_E = \|T\|_D$.*

Proof. Let $x \in E$. Since D is dense, there exists a sequence $(x_n) \subset D$ such that $x_n \rightarrow x$. Since (x_n) is convergent, it is Cauchy. For $\varepsilon > 0$, $\exists n_0$ such that $\forall n, m > n_0$, $\|x_n - x_m\| < \varepsilon$. Then:

$$\|Tx_n - Tx_m\| = \|T(x_n - x_m)\| \leq \|T\|_D \|x_n - x_m\| \rightarrow 0$$

So (Tx_n) is Cauchy in F . Since F is complete, $Tx_n \rightarrow y \in F$. Define $\hat{T}x = y$. This definition is independent of the approximating sequence: if $(y_n) \subset D$ also converges to x , then:

$$\|x_n - y_n\| \leq \|x_n - x\| + \|y_n - x\| \rightarrow 0$$

and

$$\|Tx_n - Ty_n\| = \|T(x_n - y_n)\| \leq M\|x_n - y_n\| \rightarrow 0$$

so $Ty_n \rightarrow y$ as well.

Linearity of $\hat{T}$ follows from the linearity of T and the limit. For boundedness:

$$\|\hat{T}x\| = \lim_{n \rightarrow \infty} \|Tx_n\| \leq \lim_{n \rightarrow \infty} \|T\|_D \|x_n\| = \|T\|_D \|x\|$$

so $\|\hat{T}\|_E \leq \|T\|_D$. Conversely, for $x \in D$, $\hat{T}x = Tx$, so:

$$\|\hat{T}\|_E = \sup_{x \in E} \frac{\|\hat{T}x\|}{\|x\|} \geq \sup_{x \in D} \frac{\|Tx\|}{\|x\|} = \|T\|_D$$

Hence, $\|\hat{T}\|_E = \|T\|_D$. □

1.6.2 Convergence in the Space of Bounded Linear Operators

Proposition 1.15. *Let E, F be normed spaces. Then $\mathcal{L}(E, F)$ is a vector space.*

Proof. Clearly, $\mathcal{L}(E, F)$ is a vector space with pointwise operations. The zero operator is bounded. If $T, S \in \mathcal{L}(E, F)$, then αT and $T + S$ are linear, and:

$$\|(\alpha T)x\| = |\alpha| \|Tx\| \leq |\alpha| \|T\| \|x\|$$

$$\|(T + S)x\| \leq \|Tx\| + \|Sx\| \leq (\|T\| + \|S\|) \|x\|$$

so they are bounded. The norm $\|\cdot\|_{\mathcal{L}}$ satisfies the norm axioms:

- $\|T\|_{\mathcal{L}} = 0 \Leftrightarrow \sup_{x \neq 0} \frac{\|Tx\|}{\|x\|} = 0 \Leftrightarrow T = 0$
- $\|\alpha T\|_{\mathcal{L}} = |\alpha| \|T\|_{\mathcal{L}}$
- $\|T + S\|_{\mathcal{L}} \leq \|T\|_{\mathcal{L}} + \|S\|_{\mathcal{L}}$

Thus, $(\mathcal{L}(E, F), \|\cdot\|_{\mathcal{L}})$ is a normed space. □

Pointwise Convergence

Let $T_n, T \in \mathcal{L}(E, F)$. We say the sequence (T_n) **converges pointwise** (or strongly) to T if and only if:

$$\forall x \in E, \quad T_n x \rightarrow Tx$$

i.e., $\forall x \in E, \forall \varepsilon > 0, \exists n_0, \forall n \geq n_0, \|T_n x - Tx\|_F < \varepsilon$. We denote this by $T_n \xrightarrow{sc} T$.

Example 1.18. Let $T_n : \ell^2(\mathbb{C}) \rightarrow \ell^2(\mathbb{C})$ be defined by:

$$T_n x = (x_1, x_2, \dots, x_n, 0, 0, \dots)$$

Show that $T_n \xrightarrow{sc} I_{\ell^2}$. First, $T_n \in \mathcal{L}(\ell^2)$ because:

$$\|T_n x\| = \left(\sum_{k=1}^n |x_k|^2 \right)^{1/2} \leq \|x\|_{\ell^2}$$

and for $z = (1, 0, 0, \dots)$, $\|T_n z\| = 1 = \|z\|$, so $\|T_n\| = 1$. For any $x \in \ell^2$:

$$\|T_n x - Ix\| = \left(\sum_{k=n+1}^{\infty} |x_k|^2 \right)^{1/2} \rightarrow 0 \quad \text{as } n \rightarrow \infty$$

Hence, $T_n \xrightarrow{sc} I$.

Uniform Convergence

Let $T_n, T \in \mathcal{L}(E, F)$. We say the sequence (T_n) **converges uniformly** to T if and only if:

$$\lim_{n \rightarrow \infty} \sup_{\|x\| \leq 1} \|T_n x - T x\|_F = 0$$

i.e., $\lim_{n \rightarrow \infty} \|T_n - T\|_{\mathcal{L}} = 0$. We denote this by $T_n \xrightarrow{uc} T$.

Weak Convergence

Definition 1.18. Let E be a normed space over $\mathbb{K}$. The **topological dual space** of E , denoted E^* , is the space of all continuous linear functionals from E to $\mathbb{K}$.

We say a sequence (T_n) in $\mathcal{L}(E, F)$ **converges weakly** to T if and only if:

$$\forall f \in F^*, \forall x \in E, \quad f(T_n x) \rightarrow f(T x)$$

We denote this by $T_n \xrightarrow{w} T$.

Theorem 1.11. Let (T_n) be a sequence in $\mathcal{L}(E, F)$. Then:

$$T_n \xrightarrow{uc} T \quad \Rightarrow \quad T_n \xrightarrow{sc} T \quad \Rightarrow \quad T_n \xrightarrow{w} T$$

Proof. If $T_n \xrightarrow{uc} T$, then $\|T_n - T\| \rightarrow 0$. For any $x \in E$:

$$\|T_n x - T x\| \leq \|T_n - T\| \|x\| \rightarrow 0$$

so $T_n \xrightarrow{sc} T$. If $T_n \xrightarrow{sc} T$, then for any $f \in F^*$ and $x \in E$:

$$|f(T_n x) - f(T x)| = |f(T_n x - T x)| \leq \|f\| \|T_n x - T x\| \rightarrow 0$$

so $T_n \xrightarrow{w} T$. □

1.7 Exercises

Exercise 1

Show that the following operators are linear and bounded:

1. $Af(x) = x^2 f(x)$, $A : C[0, 1] \rightarrow C[0, 1]$.
2. $Af(x) = \int_0^1 f(s) ds$, $A : C[-1, 1] \rightarrow C[0, 1]$.
3. $Af(x) = f(x^2)$, $A : C[0, 1] \rightarrow C[0, 1]$.
4. $Af(x) = f(x)$, $A : C^1[0, 1] \rightarrow C[0, 1]$.
5. $Af(x) = x \int_0^1 f(s) ds$, $A : L^2[-1, 1] \rightarrow L^2[0, 1]$.

Exercise 2

Consider the operator T defined as follows:

$$Tf(x) = xf(x), \quad T : L^2[0, 1] \rightarrow L^2[0, 1].$$

Show that T is a continuous linear operator and compute its norm.

Exercise 3

Consider the following sequence of operators:

$$A_n : C[0, 1] \rightarrow C[0, 1], \quad A_n x(t) = x\left(t^{1+\frac{1}{n}}\right), \quad n \in \mathbb{N}^*.$$

1. Show that the sequence of operators $(A_n)_{n \geq 1}$ is linear and bounded.
2. Show that $(A_n)_{n \geq 1}$ converges strongly to the identity operator.
3. The convergence of $(A_n)_{n \geq 1}$ to I_d is it uniform?

Exercise 4

Find the adjoint of the operator A if:

1. $Af(x) = xf(x)$, $A : L^2[0, 1] \rightarrow L^2[0, 1]$.
2. $Af(x) = \int_0^x f(s)ds$, $A : L^2[0, 1] \rightarrow L^2[0, 1]$.
3. $Af(x) = \int_0^1 sf(s)ds$, $A : L^2[0, 1] \rightarrow L^2[0, 1]$.
4. $Af(x) = x \int_0^1 f(s)ds$, $A : L^2[0, 1] \rightarrow L^2[0, 1]$.
5. $A(x_1, x_2, \dots) = (0, 4x_1, x_2, 4x_3, x_4, \dots)$, $A : \ell_2 \rightarrow \ell_2$.

Exercise 5

Let $(\lambda_n)_{n \geq 1}$ be a bounded sequence in $\mathbb{C}$ and $M = \sup_n |\lambda_n|$. Let $T : \ell_2 \rightarrow \ell_2$ be defined by:

$$T(x_1, x_2, \dots) = (\lambda_1 x_1, \lambda_2 x_2, \dots).$$

1. Show that T is linear, continuous, and compute its norm.
2. Show that if the set $\{|\lambda_n|, n \geq 1\}$ is bounded below by a strictly positive number, then T is bijective. Specify in this case T^{-1} and determine its norm.
3. Suppose that one of the λ_n is zero. Show that T is neither injective nor surjective and that $\overline{\text{Im}(T)} \neq \ell_2$.

Exercise 6

Let $E = C^1[0, 1]$ and $F = C[0, 1]$, both equipped with the norm $\|\cdot\|_\infty$. Define the derivative operator $T : E \rightarrow F$ such that $\forall f \in E, Tf = f'$.

Let $G(T) = \{(f, Tf), f \in E\}$ be the graph of T .

1. Show that $G(T)$ is closed in $E \times F$.
2. Show that T is not continuous. (Consider the sequence $f_n \in E, f_n(x) = x^n$.)
3. Explain the result.

Exercise 7

Let $E = C[0, 1]$ equipped with the norm $\|\cdot\|_\infty$. and for $f \in E$, define

$$Tf(x) = \int_0^x K(x, t)f(t)dt,$$

where $K(\cdot, \cdot) \in C([0, 1] \times [0, 1])$. Let $M = \sup_{0 \leq x, t \leq 1} |K(x, t)|$.

1. Show that $T \in \mathcal{L}(E)$.
2. Show that for all $n \geq 1$, we have $|T^n f(x)| \leq \frac{M^n}{n!} x^n \|f\|$.
3. Deduce that for all $n \geq 1$, $\|T^n\| \leq \frac{M^n}{n!}$.
1. Let E, F be normed spaces and $T : E \rightarrow F$ a linear operator. Show that if T is continuous at one point, then it is continuous everywhere.
2. Let E, F be Banach spaces and $T \in \mathcal{L}(E, F)$. Show that if T is bijective, then T^{-1} is bounded.
3. Let E be a Banach space and $T \in \mathcal{L}(E)$. Suppose $\|T\| < 1$. Show that $I - T$ is invertible and $\|(I - T)^{-1}\| \leq \frac{1}{1 - \|T\|}$.
4. Let E, F be Banach spaces and $(T_n) \subset \mathcal{L}(E, F)$ such that $T_n \rightarrow T$ pointwise. Show that $\sup_n \|T_n\| < \infty$.
5. Let E, F be Banach spaces and $T : E \rightarrow F$ a linear operator. Show that if the graph of T is closed, then T is continuous.

Exercise 8

Let $(X, \|\cdot\|)$ be a normed $\mathbb{C}$ -vector space, $x, y \in X$ and $\lambda \in \mathbb{C}$. Let (x_n) and (y_n) be two sequences in X and (λ_n) a sequence in $\mathbb{C}$.

1. Show that if $x_n \xrightarrow{n \rightarrow +\infty} x$ then (x_n) is bounded in X and $\|x_n\| \xrightarrow{n \rightarrow +\infty} \|x\|$. State the first result literally.
2. Show that if $x_n \xrightarrow{n \rightarrow +\infty} x$ and $\lambda_n \xrightarrow{n \rightarrow +\infty} \lambda$ then $\lambda_n x_n \xrightarrow{n \rightarrow +\infty} \lambda x$.
3. Show that if $x_n \xrightarrow{n \rightarrow +\infty} x$ and $y_n \xrightarrow{n \rightarrow +\infty} y$ then $\|x_n - y_n\| \xrightarrow{n \rightarrow +\infty} \|x - y\|$.
4. Show that if $x_n \xrightarrow{n \rightarrow +\infty} x$ and $\|x_n - y_n\| \xrightarrow{n \rightarrow +\infty} 0$ then $y_n \xrightarrow{n \rightarrow +\infty} x$.
5. Show that if $x_n \xrightarrow{n \rightarrow +\infty} x$ then $\|x_n - y\| \xrightarrow{n \rightarrow +\infty} \|x - y\|$.

Exercise 9

Let X and Y be two normed vector spaces over K and $A : X \rightarrow Y$ a linear operator.

1. Show that A is bounded if and only if

$$\exists M > 0, \forall x \in X \text{ such that } \|x\| \leq 1, \|Ax\| \leq M.$$

What can we deduce? .

Now show that if

$$\exists M > 0, \forall x \in X \text{ with } \|x\| \leq 1, \|Ax\| \leq M,$$

then A is bounded.

2. Show that if $\dim X < +\infty$ then A is bounded. (Hint: For all $x \in X$, define $\|x\|_* := \|x\| + \|Ax\|$, then show that $\|\cdot\|$ and $\|\cdot\|_*$ are two equivalent norms on X .)
3. Show that if A is bounded then the kernel of A , is a closed subset of X . (The kernel of A may also be denoted $\ker A$.)

Exercise 10

Define $A : C[0, 1] \rightarrow C[0, 1]$ by: for every $f \in C[0, 1]$, Af is defined by $(Af)(x) = f(x) - f(0)$ for every $x \in [0, 1]$.

1. Show that A is well-defined and linear.
2. Show that for $C[0, 1]$ equipped with the norm $\|\cdot\|_\infty$, the operator A is bounded.
3. Show that for $C[0, 1]$ equipped with the norm $\|\cdot\|_1$, the operator A is unbounded.

(Consider the sequence of functions $f_n(x) = (n+1)(1-x)^n$ for every $n \in \mathbb{N}^*$ and $x \in [0, 1]$.)

1.8 Solution

Solution of Exercise 1

All operators are clearly linear. We check boundedness:

$$(1) \|Af\|_\infty = \max_{x \in [0,1]} |x^2 f(x)| \leq \|f\|_\infty, \text{ so } \|A\| \leq 1.$$

$$(2) |Af(x)| \leq \int_{-1}^1 |f(s)| ds \leq 2\|f\|_\infty, \text{ so } \|A\| \leq 2.$$

$$(3) \|Af\|_\infty = \max_{x \in [0,1]} |f(x^2)| \leq \|f\|_\infty, \text{ so } \|A\| \leq 1.$$

$$(4) \|Af\|_\infty = \|f\|_\infty, \text{ so } \|A\| = 1.$$

(5) By Cauchy–Schwarz,

$$|Af(x)| \leq x \int_{-1}^1 |f(s)| ds \leq x\sqrt{2}\|f\|_{L^2} \implies \|Af\|_{L^2[0,1]}^2 \leq \int_0^1 2x^2 \|f\|_{L^2}^2 dx = \frac{2}{3} \|f\|_{L^2}^2.$$

$$\text{So } \|A\| \leq \sqrt{2/3}.$$

Solution of Exercise 2

Linearity is clear. For $f \in L^2[0, 1]$,

$$\|Tf\|_{L^2}^2 = \int_0^1 x^2 |f(x)|^2 dx \leq \int_0^1 |f(x)|^2 dx = \|f\|_{L^2}^2,$$

so $\|T\| \leq 1$. Taking $f \equiv 1$, $\|Tf\|_{L^2}^2 = \int_0^1 x^2 dx = \frac{1}{3} \neq 1$, but for f_n approximating the indicator of $[1 - \frac{1}{n}, 1]$, $\frac{\|Tf_n\|}{\|f_n\|} \rightarrow 1$. Hence $\|T\| = 1$.

Solution of Exercise 3

- (a) Linearity clear. $\|A_n x\|_\infty = \max_{t \in [0,1]} |x(t^{1+1/n})| \leq \|x\|_\infty$, so $\|A_n\| \leq 1$.
- (b) For each $x \in C[0,1]$, $t^{1+1/n} \rightarrow t$ uniformly, so $x(t^{1+1/n}) \rightarrow x(t)$ uniformly. Thus $A_n x \rightarrow x$ in $C[0,1]$.
- (c) Not uniform: take $x_n(t) = t^n$. Then $A_n x_n(t) = t^{n(1+1/n)} = t^{n+1}$. $\|A_n x_n - x_n\|_\infty = \max_t |t^{n+1} - t^n| = \max_t t^n |t - 1|$. For $t = 1 - \frac{1}{n}$, $(1 - \frac{1}{n})^n \frac{1}{n} \sim \frac{1}{ne}$, not tending to 0 uniformly in n .

Solution of Exercise 4

For $A : L^2 \rightarrow L^2$, $\langle Af, g \rangle = \langle f, A^*g \rangle$.

- (1) $\int_0^1 xf(x)\overline{g(x)}dx = \int_0^1 f(x)\overline{xg(x)}dx$, so $A^*g(x) = xg(x)$.
- (2) $\int_0^1 (\int_0^x f(s)ds) \overline{g(x)}dx = \int_0^1 f(s) \left(\int_s^1 \overline{g(x)}dx \right) ds$, so $A^*g(s) = \int_s^1 g(x)dx$.
- (3) $\int_0^1 \left(\int_0^1 sf(s)ds \right) \overline{g(x)}dx = \int_0^1 f(s) \overline{\int_0^1 g(x)dx} ds$, so $A^*g(s) = s \int_0^1 g(x)dx$.
- (4) $\int_0^1 x \left(\int_0^1 f(s)ds \right) \overline{g(x)}dx = \int_0^1 f(s) \overline{\int_0^1 xg(x)dx} ds$, so $A^*g(s) = \int_0^1 xg(x)dx$ (constant function).
- (5) For $x, y \in \ell_2$, $\langle Ax, y \rangle = 4x_1\overline{y_2} + x_2\overline{y_3} + 4x_3\overline{y_4} + \dots = \langle x, (0, 4y_2, y_3, 4y_4, y_5, \dots) \rangle$, so $A^*(y_1, y_2, \dots) = (0, 4y_2, y_3, 4y_4, y_5, \dots)$.

Solution of Exercise 4

- (a) $\|Tx\|^2 = \sum |\lambda_n x_n|^2 \leq M^2 \|x\|^2$, so $\|T\| \leq M$. Taking $x = e_n$ (unit vector), $\|Te_n\| = |\lambda_n|$, so $\|T\| \geq |\lambda_n|$ for all n , hence $\|T\| = M$.
- (b) If $\inf |\lambda_n| = \delta > 0$, define $T^{-1}(y) = (\lambda_1^{-1}y_1, \lambda_2^{-1}y_2, \dots)$. Then $\|T^{-1}y\|^2 \leq \delta^{-2} \|y\|^2$, so T^{-1} bounded. Clearly T bijective. $\|T^{-1}\| = 1/\inf |\lambda_n|$.

- (c) If $\lambda_k = 0$, then $Te_k = 0$, so not injective. Not surjective because any y with $y_k \neq 0$ not in image. Closure of image consists of sequences with $y_k = 0$ whenever $\lambda_k = 0$, so not all of ℓ_2 .

Solution of Exercise 5

- (a) Let $(f_n, f'_n) \rightarrow (f, g)$ in $E \times F$. Then $f_n \rightarrow f$ uniformly, $f'_n \rightarrow g$ uniformly. By uniform convergence, $g = f'$. So $(f, g) \in G(T)$, hence $G(T)$ closed.
- (b) $f_n(x) = x^n$, $\|f_n\|_\infty = 1$, but $\|Tf_n\|_\infty = \|nx^{n-1}\|_\infty = n \rightarrow \infty$, so T unbounded.
- (c) Closed graph theorem requires E, F Banach. Here E is not complete under $\|\cdot\|_\infty$, so theorem does not apply.

Solution of Exercise 6

- (a) $|Tf(x)| \leq M \int_0^x |f(t)| dt \leq Mx\|f\|_\infty \leq M\|f\|_\infty$, so $\|Tf\|_\infty \leq M\|f\|_\infty$.
- (b) By induction: for $n = 1$, done. Assume true for n , then
- $$|T^{n+1}f(x)| \leq \int_0^x M|T^n f(t)| dt \leq \int_0^x M \frac{M^n}{n!} t^n \|f\| dt = \frac{M^{n+1}}{(n+1)!} x^{n+1} \|f\|.$$
- (c) $\|T^n f\|_\infty \leq \frac{M^n}{n!} \|f\|_\infty$, so $\|T^n\| \leq \frac{M^n}{n!}$.

Solution of Exercise 7

- (1) Suppose T continuous at x_0 . For any $x \in E$, let $x_n \rightarrow x$. Then $x_n - x + x_0 \rightarrow x_0$, so $T(x_n - x + x_0) \rightarrow T(x_0)$. By linearity, $T(x_n) - T(x) + T(x_0) \rightarrow T(x_0)$, so $T(x_n) \rightarrow T(x)$.
- (2) By open mapping theorem, T open, so T^{-1} continuous, hence bounded.
- (3) Neumann series: $(I - T)^{-1} = \sum_{k=0}^{\infty} T^k$ converges because $\|T^k\| \leq \|T\|^k$. Then
- $$\|(I - T)^{-1}\| \leq \sum_{k=0}^{\infty} \|T\|^k = \frac{1}{1 - \|T\|}.$$

- (4) By uniform boundedness principle: for each x , $(T_n x)$ convergent hence bounded, so $\sup_n \|T_n x\| < \infty$. Thus $\sup_n \|T_n\| < \infty$.
- (5) Closed graph theorem.

Solution of Exercise 8

1. Show that (x_n) is bounded in X : Since $x_n \xrightarrow{n \rightarrow +\infty} x$, we have

$$\|x_n - x\| \xrightarrow{n \rightarrow +\infty} 0.$$

This means that the sequence $(\|x_n - x\|)$ is convergent in $\mathbb{C}$, hence it is bounded in $\mathbb{C}$ (Why?). That is,

$$\exists M_1 \geq 0, \forall n, \|x_n - x\| \leq M_1.$$

But for all n we have

$$\|x_n\| = \|x_n - x + x\| \leq \|x_n - x\| + \|x\| \leq M_1 + \|x\| = M_2.$$

That is,

$$\exists M_2 \geq 0, \forall n, \|x_n\| \leq M_2.$$

This means that (x_n) is bounded in X .

2. Show that $\|x_n\| \xrightarrow{n \rightarrow +\infty} \|x\|$: We have

$$v_n 0 \leq u_n \|x_n\| - \|x\| \leq w_n \|x_n - x\| \quad \text{for all } n.$$

Since

$$v_n = 0 \xrightarrow{n \rightarrow +\infty} 0 \quad \text{and} \quad w_n = \|x_n - x\| \xrightarrow{n \rightarrow +\infty} 0,$$

then by the squeeze theorem we obtain

$$u_n = \|x_n\| - \|x\| \xrightarrow{n \rightarrow +\infty} 0.$$

This implies that

$$\|x_n\| - \|x\| \xrightarrow{n \rightarrow +\infty} 0,$$

hence

$$\|x_n\| \xrightarrow{n \rightarrow +\infty} \|x\|.$$

Literally, the first result means: Every convergent sequence in a normed space is bounded (in that space).

1. For all n , we have

$$\begin{aligned} 0 \leq \|\lambda_n x_n - \lambda x\| &= \|\lambda_n(x_n - x) + (\lambda_n - \lambda)x\| \leq \|\lambda_n(x_n - x)\| + \|(\lambda_n - \lambda)x\| \\ &= |\lambda_n| \|x_n - x\| + |\lambda_n - \lambda| \|x\|. \end{aligned}$$

Since $\lambda_n \xrightarrow{n \rightarrow +\infty} \lambda$, the sequence (λ_n) is bounded, so

$$\exists M \geq 0, \forall n, |\lambda_n| \leq M.$$

Thus,

$$0 \leq \|\lambda_n x_n - \lambda x\| \leq M \|x_n - x\| + |\lambda_n - \lambda| \|x\| \quad \text{for all } n.$$

Since

$$[M \|x_n - x\| + |\lambda_n - \lambda| \|x\|] \xrightarrow{n \rightarrow +\infty} 0$$

then

$$\|\lambda_n x_n - \lambda x\| \xrightarrow{n \rightarrow +\infty} 0.$$

This means that

$$\lambda_n x_n \xrightarrow{n \rightarrow +\infty} \lambda x.$$

2. For all n , we have

$$\begin{aligned} 0 \leq \|x_n - y_n\| - \|x - y\| &\leq \|(x_n - y_n) - (x - y)\| = \|(x_n - x) - (y_n - y)\| \\ &\leq \|x_n - x\| + \|y_n - y\|. \end{aligned}$$

Since $x_n \xrightarrow{n \rightarrow +\infty} x$ and $y_n \xrightarrow{n \rightarrow +\infty} y$, we have

$$\|x_n - x\| + \|y_n - y\| \xrightarrow{n \rightarrow +\infty} 0,$$

hence

$$\|x_n - y_n\| \xrightarrow{n \rightarrow +\infty} \|x - y\|.$$

3. For all n , we have

$$\begin{aligned} 0 \leq \|y_n - x\| &= \|(y_n - x_n) + (x_n - x)\| \leq \|y_n - x_n\| + \|x_n - x\| \\ &= \|x_n - y_n\| + \|x_n - x\| \quad . \end{aligned}$$

Since

$$x_n \xrightarrow{n \rightarrow +\infty} x \quad \text{and} \quad \|x_n - y_n\| \xrightarrow{n \rightarrow +\infty} 0,$$

then

$$\|x_n - y_n\| + \|x_n - x\| \xrightarrow{n \rightarrow +\infty} 0.$$

Thus, by the squeeze theorem, we obtain

$$\|y_n - x\| \xrightarrow{n \rightarrow +\infty} 0,$$

hence

$$y_n \xrightarrow{n \rightarrow +\infty} x.$$

4. For all n , we have

$$0 \leq \|x_n - y\| - \|x - y\| \leq \|(x_n - y) - (x - y)\| = \|x_n - x\|.$$

Since

$$\|x_n - x\| \xrightarrow{n \rightarrow +\infty} 0 \quad ,$$

then

$$\|x_n - y\| - \|x - y\| \xrightarrow{n \rightarrow +\infty} 0,$$

hence

$$\|x_n - y\| \xrightarrow{n \rightarrow +\infty} \|x - y\| \quad .$$

Solution of Exercise 9

1. We show that if A is bounded then

$$\exists M > 0, \forall x \in X \text{ with } \|x\| \leq 1, \|Ax\| \leq M.$$

Since A is bounded,

$$\exists M > 0, \forall x \in X, \|Ax\| \leq M\|x\|.$$

This implies that

$$\exists M > 0, \forall x \in X \text{ with } \|x\| \leq 1, \|Ax\| \leq M\|x\| \quad .$$

Thus

$$\exists M > 0, \forall x \in X \text{ with } \|x\| \leq 1, \|Ax\| \leq M\|x\| \leq M \cdot 1 = M.$$

Therefore,

$$\exists M > 0, \forall x \in X \text{ with } \|x\| \leq 1, \|Ax\| \leq M.$$

Note that

$$\forall x \in X \setminus \{0\}, \left\| \frac{x}{\|x\|} \right\| = \left\| \frac{1}{\|x\|} x \right\| = \frac{1}{\|x\|} \|x\| = 1 \leq 1.$$

Since

$$\exists M > 0, \forall x \in X \text{ with } \|x\| \leq 1, \|Ax\| \leq M,$$

then

$$\exists M > 0, \forall x \in X \setminus \{0\}, \left\| A \left(\frac{x}{\|x\|} \right) \right\| \leq M.$$

But

$$\left\| A \left(\frac{x}{\|x\|} \right) \right\| = \left\| A \left(\frac{1}{\|x\|} x \right) \right\| = \left\| \frac{1}{\|x\|} Ax \right\| = \frac{1}{\|x\|} \|Ax\|.$$

Hence

$$\exists M > 0, \forall x \in X \setminus \{0\}, \frac{1}{\|x\|} \|Ax\| \leq M.$$

So

$$\exists M > 0, \forall x \in X \setminus \{0\}, \|Ax\| \leq M\|x\|. \quad (*)$$

For $x = 0$, we have

$$\|Ax\| = \|A0\| = \|0\| = 0 \quad \text{and} \quad M\|x\| = M\|0\| = M \cdot 0 = 0.$$

Thus, for $x = 0$,

$$\|Ax\| \leq M\|x\|. \quad (**)$$

From (*) and (**), we find

$$\exists M > 0, \forall x \in X, \|Ax\| \leq M\|x\|.$$

This means that A is bounded.

Conclusion: An operator is bounded if and only if it is a bounded map on the closed unit ball.

2. We can show that $\|\cdot\|_*$ is also a norm on X (to be done). Since $\dim X < +\infty$, the norms $\|\cdot\|$ and $\|\cdot\|_*$ are equivalent. Therefore,

$$\exists M > 0, \forall x \in X, \|x\|_* \leq M\|x\|.$$

But

$$\forall x \in X, \|Ax\| \leq \|x\| + \|Ax\| = \|x\|_*.$$

Thus

$$\exists M > 0, \forall x \in X, \|Ax\| \leq \|x\|_* \leq M\|x\|.$$

Hence

$$\exists M > 0, \forall x \in X, \|Ax\| \leq M\|x\|.$$

This means that A is bounded.

3. To show that $N(A)$ is closed in X , it suffices to show that

$$\forall (x_n) \subset N(A), (x_n) \text{ converges to } x \implies x \in N(A).$$

Let $(x_n) \subset N(A)$. Then

$$\forall n, Ax_n = 0 \quad .$$

Thus

$$\lim_{n \rightarrow +\infty} Ax_n = \lim_{n \rightarrow +\infty} 0 = 0.$$

But A is linear and bounded, hence continuous, so

$$\lim_{n \rightarrow +\infty} Ax_n = A \left(\lim_{n \rightarrow +\infty} x_n \right) = Ax.$$

Therefore,

$$Ax = 0,$$

which implies $x \in N(A)$.

Solution of Exercise 10

1. We show that A is well-defined, we must show that

$$\forall f \in C[0, 1], Af \in C[0, 1].$$

Let $f \in C[0, 1]$. Then

$$Af : [0, 1] \longrightarrow \mathbb{R}, \quad x \longmapsto (Af)(x) = f(x) - f(0) = f(x) + (-f(0)).$$

Thus Af is the sum of two continuous functions on $[0, 1]$: the function f and the constant function $-f(0)$. Therefore, Af is continuous on $[0, 1]$, meaning $Af \in C[0, 1]$.

- Show that A is linear: We must show that

$$\forall \alpha \in \mathbb{R}, \forall f, g \in C[0, 1], A(\alpha f + g) = \alpha Af + Ag.$$

Let $\alpha \in \mathbb{R}$ and $f, g \in C[0, 1]$. We show that

$$A(\alpha f + g) = \alpha Af + Ag \quad (\text{equality of two functions}).$$

To do so, we show that

$$[A(\alpha f + g)](x) = [\alpha Af + Ag](x) \quad \text{for every } x \in [0, 1].$$

Let $x \in [0, 1]$. Then

$$\begin{aligned} [A(\alpha f + g)](x) &= (\alpha f + g)(x) - (\alpha f + g)(0) \\ &= [\alpha f(x) + g(x)] - [\alpha f(0) + g(0)] \\ &= \alpha[f(x) - f(0)] + [g(x) - g(0)] \\ &= \alpha(Af)(x) + (Ag)(x) = [\alpha Af + Ag](x). \end{aligned}$$

2. We must show that

$$\exists M \geq 0, \forall f \in C[0, 1], \|Af\|_\infty \leq M\|f\|_\infty.$$

Let $f \in C[0, 1]$. Then

$$\begin{aligned}
 \|Af\|_\infty &= \sup_{x \in [0, 1]} |(Af)(x)| \\
 &= \sup_{x \in [0, 1]} |f(x) - f(0)| \\
 &\leq \sup_{x \in [0, 1]} |f(x)| + |f(0)| \\
 &\leq \sup_{x \in [0, 1]} |f(x)| + \sup_{x \in [0, 1]} |f(0)| \\
 &= \|f\|_\infty + |f(0)| \\
 &\leq \|f\|_\infty + \|f\|_\infty \\
 &= 2\|f\|_\infty.
 \end{aligned}$$

Thus we have shown that

$$\exists M = 2 \geq 0, \forall f \in C[0, 1], \|Af\|_\infty \leq M\|f\|_\infty.$$

3. Before showing that A is unbounded, we note:

(*) For every $n \in \mathbb{N}^*$, we have

$$\begin{aligned}
 \|f_n\|_1 &= \int_0^1 |f_n(x)| dx \\
 &= \int_0^1 |(n+1)(1-x)^n| dx \\
 &= (n+1) \int_0^1 (1-x)^n dx \\
 &= 1,
 \end{aligned}$$

that is,

$$\forall n \in \mathbb{N}^*, \|f_n\|_1 = 1.$$

(**) For every $n \in \mathbb{N}^*$, we have

$$\begin{aligned}
 \|Af_n\|_1 &= \int_0^1 |(Af_n)(x)| dx = \int_0^1 |f_n(x) - f_n(0)| dx \\
 &= \int_0^1 |(n+1)(1-x)^n - (n+1)| dx \\
 &= (n+1) \int_0^1 [1 - (1-x)^n] dx \\
 &= n,
 \end{aligned}$$

that is,

$$\forall n \in \mathbb{N}^*, \|Af_n\|_1 = n.$$

Now we show that A is unbounded: Assume, for contradiction, that A is bounded, so

$$\exists M \geq 0, \forall f \in C[0, 1], \|Af\|_1 \leq M\|f\|_1.$$

Then

$$\exists M \geq 0, \forall n \in \mathbb{N}^*, \|Af_n\|_1 \leq M\|f_n\|_1 \quad (\text{why?}).$$

From (*) and (**) this implies that

$$\exists M \geq 0, \forall n \in \mathbb{N}^*, n \leq M \cdot 1 = M.$$

This means $\mathbb{N}^*$ is bounded above by M , a contradiction.

Chapter 2

Linear operators and applications

2.1 Dual of a normed vector space

Definition 2.1. Let E be a normed vector space over $\mathbb{K}$ ($\mathbb{R}$ or $\mathbb{C}$). A **linear form** is a linear map $f : E \rightarrow \mathbb{K}$.

Example 2.1. Let $E = C([0, 1])$, the space of continuous functions on $[0, 1]$.

- $f : E \rightarrow \mathbb{R}$ defined by $f(p) = p(1)$ for $p \in E$ (evaluation at 1) is a linear form.
- $f' : E \rightarrow \mathbb{R}$ defined by $f'(g) = \int_0^1 g(t)dt$ is a linear form.

Definition 2.2. The set of all linear forms on E forms a vector space over $\mathbb{K}$ under the operations:

- **Addition:** $(f + g)(x) = f(x) + g(x)$
- **Scalar Multiplication:** $(\lambda f)(x) = \lambda f(x)$

This vector space is called the **algebraic dual** of E , denoted E^* or $L(E, \mathbb{K})$.

Proposition 2.1. If E is finite-dimensional with $\dim E = n$, then $\dim L(E, \mathbb{K}) = n$.

Let E be a normed space. The space of **continuous** (or **bounded**) linear forms on E is called the **topological dual** of E , denoted:

$$E' = \{T : E \rightarrow \mathbb{K} \mid T \text{ is linear and continuous}\} = L(E, \mathbb{K})$$

Remark 2.1. *If E is finite-dimensional, then the algebraic dual E^* and the topological dual E' coincide, i.e., $L(E, \mathbb{K}) = L(E, \mathbb{K})$ (every linear form is continuous).*

We are primarily interested in the topological dual, often simply called the **dual**. The dual E' is a Banach space when equipped with the operator norm:

$$\|f\| = \sup_{\|x\| \leq 1} |f(x)| \quad \text{for } f \in E'$$

2.2 Hahn-Banach Theorems

2.2.1 Analytic form of the Hahn-Banach theorem

Definition 2.3. *We call a **semi-norm** on a vector space E every mapping $p : E \rightarrow \mathbb{R}_+$ that satisfies the following properties:*

1. $p(\lambda x) = \lambda p(x), \quad \forall \lambda \geq 0, \quad \forall x \in E.$
2. $p(x + y) \leq p(x) + p(y), \quad \forall x, y \in E.$

Every norm on E is a semi-norm.

[Zorn's Lemma] Every non-empty partially ordered set in which every chain has an upper bound contains a maximal element.

Theorem 2.1 (Hahn-Banach Theorem (Real Case)). *Let E be a real vector space, G a subspace of E , and $p : E \rightarrow \mathbb{R}$ a semi-norm on E . Let $f \in G^*$ be a linear form on G such that*

$$\forall x \in G, \quad f(x) \leq p(x).$$

Then there exists an extension $\bar{f} \in E^*$ (a linear form on E) such that

$$\forall x \in G, \quad \bar{f}(x) = f(x);$$

$$\forall x \in E, \quad \bar{f}(x) \leq p(x).$$

Proof. Assume $G \neq E$. Then there exists $x_0 \in E \setminus G$. Define the subspace G_1 of E as follows:

$$G_1 = \{tx + x_0 : x \in G, t \in \mathbb{R}\}.$$

We now show the existence of an extension f_1 on G_1 .

For $y = tx + x_0 \in G_1$, define

$$f_1(y) = f_1(tx + x_0) = tf(x) + f_1(x_0) = tc + f(x),$$

where $c = f_1(x_0)$ is a constant to be chosen later so that

$$f_1(tx + x_0) \leq p(tx + x_0).$$

By definition, f_1 is linear and satisfies $\forall x \in G, f_1(x) = f(x)$.

For $t > 0$, inequality (1.2) is equivalent to

$$f_1\left(x + \frac{x_0}{t}\right) \leq p\left(x + \frac{x_0}{t}\right),$$

since f_1 is linear and $x_0/t \notin G$. This previous inequality is equivalent to

$$c + f\left(\frac{x_0}{t}\right) \leq p\left(x + \frac{x_0}{t}\right),$$

i.e.,

$$p\left(x + \frac{x_0}{t}\right) - f\left(\frac{x_0}{t}\right) \geq c. \quad (1.2)$$

Similarly, for $t < 0$, we obtain the inequality:

$$-p\left(-x - \frac{x_0}{t}\right) - f\left(\frac{x_0}{t}\right) \leq c. \quad (1.3)$$

For $x', x'' \in G$, from the property of f on G , we have

$$f(x'') - f(x') \leq p(x'' - x') = p((x'' + x_0) - (x' + x_0)) \leq p(x'' + x_0) + p(-x' - x_0),$$

which implies

$$-f(x'') + p(x'' + x_0) \geq -f(x') - p(-x' - x_0).$$

Define

$$c' = \sup_{x' \in G} (-f(x') - p(-x' - x_0)),$$

$$c'' = \inf_{x'' \in G} (-f(x'') + p(x'' + x_0)).$$

c' and c'' exist. The constant $c = f_1(x_0)$ can be chosen such that $c' \leq c \leq c''$. Then f_1 , defined by $f_1(tx + x_0) = tc + f(x)$, is a linear form on G_1 which is an extension of f on G and satisfies

$$f_1(y) \leq p(y), \quad \forall y \in G_1.$$

To complete the proof of the theorem, we consider the following two cases:

1. If E is separable, then there exists a countable sequence of subspaces G_n such that $G_1 = G$, G_{n+1} is constructed from G_n by adding one vector (as with G_1 above), and $E = \bigcup_{n=1}^{\infty} G_n$. By induction, we obtain extensions f_n on G_n satisfying $f_n(y) \leq p(y)$ for all $y \in G_n$. Since $E = \bigcup_{n=1}^{\infty} G_n$, there exists an extension f^* on E satisfying $f^*(x) \leq p(x)$ for all $x \in E$.
2. In the general case, let $\mathcal{A}$ be the set of all possible extensions g of f , defined on subspaces $D(g) \supset G$, which satisfy

$$\forall x \in D(g), \quad g(x) \leq p(x).$$

Define a partial order on $\mathcal{A}$ by:

$$g_1 \prec g_2 \quad \Leftrightarrow \quad \begin{cases} D(g_1) \subset D(g_2), \\ \forall x \in D(g_1), \quad g_1(x) = g_2(x). \end{cases}$$

Let $\{g_i\}_{i \in I}$ be a totally ordered subset of $\mathcal{A}$. Define g' on $D(g') = \bigcup_{i \in I} D(g_i)$ by $g'(x) = g_i(x)$ if $x \in D(g_i)$. Then $g' \in \mathcal{A}$ and is an upper bound for the chain $\{g_i\}$. By Zorn's Lemma, $\mathcal{A}$ has a maximal element F .

We show that the domain of this maximal extension F must be all of E . Suppose, for contradiction, that $D(F) \neq E$. Then there exists $x_0 \in E \setminus D(F)$. As shown in the first part of the proof, F can be extended to a larger subspace $D(F) \oplus \mathbb{R}x_0$, contradicting the maximality of F . Therefore, $D(F) = E$, and F is the desired linear form on E satisfying:

$$\begin{aligned} \forall x \in G, \quad F(x) &= f(x); \\ \forall x \in E, \quad F(x) &\leq p(x). \end{aligned}$$

□

Theorem 2.2. *Let E be a normed vector space and G a subspace of E . Let f be a continuous linear form on G . Then f has an extension $\bar{f}$ on E such that $\|\bar{f}\|_{E^*} = \|f\|_{G^*}$.*

Proof. Simply take $p(x) = \|f\|_{G^*} \cdot \|x\|$, which is a semi-norm on E , and apply the Hahn-Banach theorem to obtain the desired result. □

Theorem 2.3 (Hahn–Banach). *The maximal element f^* is defined on the whole space E , i.e., $E^* = E$, and satisfies*

$$f^*(x) \leq p(x), \quad \forall x \in E.$$

Moreover, if f is defined on a subspace $G \subset E$, then f^* extends f and

$$\|f^*\|_{E^*} = \|f\|_{G^*}.$$

Proof. Suppose for contradiction that $\text{dom}(f^*) \neq E$. Let $x_0 \in E \setminus \text{dom}(f^*)$. Define $\tilde{G} = \text{dom}(f^*) \oplus \mathbb{R}x_0$ and extend f^* to $\tilde{f}$ on $\tilde{G}$ by setting

$$\tilde{f}(x + \lambda x_0) = f^*(x) + \lambda \alpha,$$

where $\alpha \in \mathbb{R}$ is chosen so that $\tilde{f} \leq p$ on $\tilde{G}$. This contradicts the maximality of f^* . Thus $\text{dom}(f^*) = E$.

For the norm equality, take $p(x) = \|f\|_{G^*}\|x\|$. Then $\|f^*\|_{E^*} \leq \|f\|_{G^*}$, and the reverse inequality is obvious. $\square$

Corollary 2.1. *Let E be a normed space and $x_0 \in E$, $x_0 \neq 0$. Then there exists $f \in E^*$ such that*

$$\|f\| = 1 \quad \text{and} \quad f(x_0) = \|x_0\|.$$

Proof. Define $G = \mathbb{R}x_0$ and $f(\lambda x_0) = \lambda\|x_0\|$. Then $\|f\|_{G^*} = 1$. Extend f to E by the Hahn–Banach theorem with $p(x) = \|x\|$. $\square$

Theorem 2.4 (Extension with Norm Preservation). *Let E be a normed space over $\mathbb{R}$ (or $\mathbb{C}$), and let $F \subseteq E$ be a linear subspace. Then:*

$$\forall f \in F', \exists g \in E' \text{ such that } g|_F = f \text{ and } \|g\|_{E'} = \|f\|_{F'}$$

Proof. Let $f \in F'$ and let $M = \|f\|_{F'}$. Apply the Hahn–Banach theorem with the seminorm $p(x) = M\|x\|$. We have $f(x) \leq |f(x)| \leq M\|x\| = p(x)$ for all $x \in F$. The theorem guarantees an extension $g : E \rightarrow \mathbb{K}$ such that $g|_F = f$ and $g(x) \leq p(x) = M\|x\|$ for all $x \in E$. Furthermore, $-g(x) = g(-x) \leq M\| -x \| = M\|x\|$, so $|g(x)| \leq M\|x\|$ for all $x \in E$. Thus, g is continuous and $\|g\|_{E'} \leq M = \|f\|_{F'}$. Since g extends f , we also have $\|g\|_{E'} \geq \|f\|_{F'}$. Therefore, $\|g\|_{E'} = \|f\|_{F'}$. $\square$

2.2.2 Hahn-Banach complex theorem

Theorem 2.5. *Let E be a complex vector space, G a subspace of E , and p a function defined on E satisfying the following conditions:*

$$\forall x \in E, p(x) \geq 0. \quad (1)$$

$$\forall x, y \in E, p(x + y) \leq p(x) + p(y). \quad (2)$$

$$\forall x \in E, \forall \lambda \in \mathbb{C}, p(\lambda x) = |\lambda|p(x). \quad (3)$$

If f is a linear form on G such that for every $x \in G$, $|f(x)| \leq p(x)$, then there exists a complex linear form $\tilde{f}$ on E satisfying:

$$\forall x \in E, |\tilde{f}(x)| \leq p(x), \quad \text{and} \quad \forall x \in G, \tilde{f}(x) = f(x).$$

Proof. According to the hypothesis, we have:

$$f(x) = g(x) + ih(x),$$

where g and h are real linear functionals. On the other hand, for every $x \in G$ we have

$$\begin{aligned} f(x) &= g(ix) + ih(x) = ig(x) - h(x) \\ &= if(x) \end{aligned}$$

and hence, for $x \in G$, we have $x = g(ix)$ without correction.

According to the real Hahn-Banach theorem, there exists an extension $\tilde{g}$ of g on E satisfying

$$\begin{aligned} \forall x \in G, \tilde{g}(x) &\leq p(x) \\ \Rightarrow -\tilde{g}(x) &= \tilde{g}(-x) \leq p(-x) = p(x) \\ \Rightarrow |\tilde{g}(x)| &\leq p(x). \end{aligned}$$

Define $\tilde{f}$ as follows:

$$\tilde{f}(x) = \tilde{g}(x) - i\tilde{g}(ix).$$

But

$$\begin{aligned}\tilde{f}(ix) &= \tilde{g}(ix) - i\tilde{g}(i(ix)) = \tilde{g}(ix) - i\tilde{g}(-x) \\ &= \tilde{g}(ix) + i\tilde{g}(x) = i\tilde{f}(x).\end{aligned}$$

Thus, $\tilde{f}$ is a complex linear functional. For every $x \in G$, we have

$$\begin{aligned}\tilde{f}(x) &= \tilde{g}(x) - i\tilde{g}(ix) = g(x) - ig(ix) \\ &= g(x) + ih(x) = f(x).\end{aligned}$$

Hence, $\tilde{f}$ is an extension of f . Now, we prove that

$$\forall x \in E, |\tilde{f}(x)| \leq p(x).$$

Indeed, using the exponential form of a complex number, we find

$$\begin{aligned}\tilde{f}(x) &= \tilde{g}(x) - i\tilde{g}(ix) = re^{-i\theta} \\ \Rightarrow |\tilde{f}(x)| &= r = e^{i\theta}\tilde{f}(x) = \tilde{f}(xe^{i\theta}).\end{aligned}$$

The last value is positive real, and we also have

$$\begin{aligned}\tilde{f}(xe^{i\theta}) &= \tilde{g}(xe^{i\theta}) - i\tilde{g}(ixe^{i\theta}) \\ \Rightarrow |\tilde{f}(x)| &= \tilde{f}(xe^{i\theta}) = |\tilde{g}(xe^{i\theta})| \\ \Rightarrow |\tilde{f}(x)| &= |\tilde{g}(xe^{i\theta})| \leq p(xe^{i\theta}) = |e^{i\theta}|p(x) = p(x).\end{aligned}$$

Therefore,

$$\forall x \in E, |\tilde{f}(x)| \leq p(x).$$

□

2.2.3 Geometric Form of the Hahn-Banach Theorem

Theorem 2.6. *Let E be a normed vector space, and let G and C be non-empty disjoint subsets of E such that C is convex, compact, and G is closed and convex. Then, there exists a continuous linear functional $\varphi \in E^*$ such that:*

$$\sup_{x \in G} \operatorname{Re} \varphi(x) < \inf_{y \in C} \operatorname{Re} \varphi(y).$$

2.3 Adjoint Operators

Definition 2.4. *Let E, F be normed spaces, and let $T \in L(E, F)$ be a bounded linear operator. The **adjoint** (or **dual**) of T , denoted T^* , is the bounded linear operator from F' to E' defined by:*

$$T^* : F' \rightarrow E'$$

$$g \mapsto T^*(g) = g \circ T$$

That is, for $g \in F'$ and $x \in E$:

$$(T^*(g))(x) = g(T(x))$$

Example: Let $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ be a linear operator represented by the matrix $A \in M_{m \times n}(\mathbb{R})$. Its adjoint $T' : (\mathbb{R}^m)' \rightarrow (\mathbb{R}^n)'$ is represented by the transpose matrix A^T .

Proposition 2.2. *Let $T \in L(E, F)$. Then $T' \in L(F', E')$ and $\|T'\| = \|T\|$.*

Proof (Sketch):

1. **Linearity:** For $g_1, g_2 \in F'$, $\alpha, \beta \in \mathbb{K}$:

$$\begin{aligned} T^*(\alpha g_1 + \beta g_2)(x) &= (\alpha g_1 + \beta g_2)(T(x)) \\ &= \alpha g_1(T(x)) + \beta g_2(T(x)) \\ &= \alpha T^*(g_1)(x) + \beta T^*(g_2)(x) \end{aligned}$$

So $T^*(\alpha g_1 + \beta g_2) = \alpha T^*(g_1) + \beta T^*(g_2)$.

2. **Boundedness:** For $g \in F'$:

$$\begin{aligned}
 \|T^*(g)\|_{E'} &= \|g \circ T\|_{E'} = \sup_{\|x\| \leq 1} |g(T(x))| \\
 &\leq \sup_{\|x\| \leq 1} \|g\|_{F'} \|T(x)\|_F \\
 &\leq \|g\|_{F'} \|T\| \sup_{\|x\| \leq 1} \|x\|_E = \|g\|_{F'} \|T\|
 \end{aligned}$$

Thus, $\|T^*\| \leq \|T\|$.

3. **Equality of Norms** ($\|T'\| = \|T\|$): Using a corollary of the Hahn-Banach theorem, for any $x \in E$ with $T(x) \neq 0$, there exists $g \in F'$ with $\|g\| = 1$ such that $g(T(x)) = \|T(x)\|_F$. Then:

$$\begin{aligned}
 \|T(x)\|_F &= |g(T(x))| = |T'(g)(x)| \\
 &\leq \|T'(g)\|_{E'} \|x\|_E \\
 &\leq \|T'\| \|g\|_{F'} \|x\|_E = \|T'\| \|x\|_E
 \end{aligned}$$

So $\|T(x)\|_F \leq \|T'\| \|x\|_E$ for all $x \in E$, implying $\|T\| \leq \|T'\|$. Combined with the previous inequality, $\|T'\| = \|T\|$.

2.3.1 Properties of the Adjoint

Let $T, T_1, T_2 \in L(E, F)$, $S \in L(F, G)$, and $\lambda \in \mathbb{K}$.

1. $(T_1 + T_2)' = T_1' + T_2'$
2. $(\lambda T)' = \lambda T'$
3. $(S \circ T)' = T' \circ S'$
4. If T is an isomorphism, then $(T^{-1})' = (T')^{-1}$

2.4 operator spectrum

In the rest of this section, assume that X is a complex Banach space, and $T \in B(X)$.

Let $\lambda \in \mathbb{K}$. λ is called a *spectral value* of T , if $T - \lambda I$ is not invertible (or equivalently, $T - \lambda I$ is bijective, since X is a Banach space).

Definition 2.5. Let $\lambda \in \mathbb{K}$. We say that λ is an eigenvalue of T , if $T - \lambda I$ is not injective, or equivalently, there exists a nonzero vector $u \in X$ such that $(T - \lambda I)(u) = 0$.

Definition 2.6. We define the following sets:

1. The spectrum of T , denoted by $\sigma(T)$, is defined to be:

$$\sigma(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not invertible}\}.$$

2. The resolvent of T is given by:

$$\rho(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is invertible}\}.$$

3. The point spectrum, denoted by $\sigma_p(T)$, the set of all eigenvalues, is defined as:

$$\sigma_p(T) = \{\lambda \in \mathbb{C} : \ker(T - \lambda I) \neq \{0\}\}.$$

Remark 2.2. The spectral radius, denoted by $r(T)$, is the nonnegative real number given by:

$$r(T) = \sup_{\lambda \in \sigma(T)} |\lambda|.$$

Remark 2.3. We have:

1. $\sigma(T)$ is a nonempty and compact subset of $\mathbb{C}$.
2. It is clear that $\sigma_p(T) \subset \sigma(T)$.
3. If $\dim X < \infty$, then $\sigma_p(T) = \sigma(T)$.

Proof. Proof of (ii): We want to show that $\sigma_p(T) \subset \sigma(T)$.

For any $\lambda \in \mathbb{C}$, if $\lambda \in \sigma_p(T)$, then $T - \lambda I$ is not injective. Since invertibility implies injectivity (and surjectivity), if $T - \lambda I$ is not injective, it cannot be invertible. Hence, $\lambda \in \sigma(T)$. Therefore, $\sigma_p(T) \subset \sigma(T)$.

Proof of (iii): Let $T \in L(X)$. If $\dim X < \infty$, we have:

$$T \text{ invertible} \Leftrightarrow T \text{ bijective} \Leftrightarrow T \text{ injective} \Leftrightarrow T \text{ surjective}.$$

Since

$$\sigma(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ not invertible}\} = \{\lambda \in \mathbb{C} : T - \lambda I \text{ not injective}\} = \sigma_p(T),$$

it follows that $\sigma_p(T) = \sigma(T)$. □

If $A \subset \mathbb{C}^*$, we define $A^{-1} = \{\frac{1}{a} : a \in A\}$.

Theorem 2.7. *If T is an invertible operator, then we have:*

$$\sigma(T^{-1}) = (\sigma(T))^{-1}, \quad \text{where} \quad (\sigma(T))^{-1} = \{\lambda^{-1} : \lambda \in \sigma(T)\}.$$

Proof. First, we prove that $\sigma(T^{-1}) \subset (\sigma(T))^{-1}$.

Let $\lambda \in \sigma(T^{-1})$. Then $T^{-1} - \lambda I$ is not invertible. Since T is invertible, the operator $T(T^{-1} - \lambda I)$ is not invertible. Hence, $I - \lambda T$ is not invertible. Then $(-\lambda)(T - \frac{1}{\lambda}I)$ is not invertible.

Since $-\lambda \neq 0$, it follows that $T - \frac{1}{\lambda}I$ is not invertible, hence $\frac{1}{\lambda} \in \sigma(T)$. Therefore, $\lambda \in (\sigma(T))^{-1}$. Thus,

$$\sigma(T^{-1}) \subset (\sigma(T))^{-1}.$$

Now, we show that $(\sigma(T))^{-1} \subset \sigma(T^{-1})$.

Let $\lambda \in (\sigma(T))^{-1}$. Then $\frac{1}{\lambda} \in \sigma(T)$, so $T - \frac{1}{\lambda}I$ is not invertible (and $\lambda \neq 0$). Then $\lambda(T - \frac{1}{\lambda}I) = \lambda T - I$ is not invertible. Hence,

$$T^{-1} - \lambda^{-1}I = (-\lambda)^{-1}T^{-1}(\lambda T - I)$$

is not invertible (since $\lambda T - I$ is not invertible and $(-\lambda)^{-1}T^{-1}$ is invertible). Thus, $\lambda^{-1} \in \sigma(T^{-1})$. Therefore,

$$(\sigma(T))^{-1} \subset \sigma(T^{-1}).$$

Hence, we conclude:

$$\sigma(T^{-1}) = (\sigma(T))^{-1}.$$

□

2.5 Special case: Hilbert space

2.5.1 generalities on Hilbert spaces

2.6 Definition

Let E be a vector space over the field $\mathbb{K}$ ($\mathbb{R}$ or $\mathbb{C}$). An inner product on E is a mapping $\langle \cdot, \cdot \rangle : E \times E \rightarrow \mathbb{K}$ satisfying the following properties for all $x, y, z \in E$ and $\alpha \in \mathbb{K}$:

1. $\langle x, x \rangle = 0 \Leftrightarrow x = 0$
2. $\langle \alpha x, y \rangle = \alpha \langle x, y \rangle$
3. $\langle x, y \rangle = \overline{\langle y, x \rangle}$ (Conjugate symmetry)
4. $\langle x + z, y \rangle = \langle x, y \rangle + \langle z, y \rangle$

The pair $(E, \langle \cdot, \cdot \rangle)$ is called a pre-Hilbert space.

Example 2.2. Let $\ell^2(\mathbb{C})$ be the space of complex sequences (x_n) such that $\sum_{n=1}^{\infty} |x_n|^2 < \infty$. The inner product on $\ell^2(\mathbb{C})$ is defined by:

$$\langle x, y \rangle = \sum_{n=1}^{\infty} x_n \overline{y_n}$$

Example 2.3. On $E = \mathbb{R}^n$, the standard inner product is:

$$\langle x, y \rangle = \sum_{i=1}^n x_i y_i$$

Example 2.4. On $E = \mathbb{C}^n$, the standard inner product is:

$$\langle x, y \rangle = \sum_{i=1}^n x_i \overline{y_i}$$

Example 2.5. Let $E = L^2([a, b], \mathbb{C})$ be the space of square-integrable complex-valued functions on $[a, b]$. The inner product is defined by:

$$\langle f, g \rangle = \int_a^b f(t) \overline{g(t)} dt$$

2.6.1 Cauchy-Schwarz Inequality

Theorem 2.8. *If E is a pre-Hilbert space, then for all $x, y \in E$:*

$$|\langle x, y \rangle| \leq \|x\| \|y\|$$

where $\|x\| = \sqrt{\langle x, x \rangle}$.

Proof. For any $\alpha \in \mathbb{K}$, and $x, y \in E$, we have:

$$\langle x + \alpha y, x + \alpha y \rangle = \langle x, x \rangle + \langle x, \alpha y \rangle + \langle \alpha y, x \rangle + \langle \alpha y, \alpha y \rangle \geq 0$$

For $y \neq 0$, choose $\alpha = -\frac{\langle x, y \rangle}{\langle y, y \rangle}$. Then:

$$\langle x, x \rangle - \frac{|\langle x, y \rangle|^2}{\langle y, y \rangle} \geq 0 \implies |\langle x, y \rangle|^2 \leq \langle x, x \rangle \langle y, y \rangle$$

The inequality holds trivially if $y = 0$. □

Corollary 2.2. *Let H be a Hilbert space. For each $y \in H$, the linear functional $f_y : H \rightarrow \mathbb{K}$ defined by $f_y(x) = \langle x, y \rangle$ is continuous, and $\|f_y\| = \|y\|$.*

Proof. By the Cauchy-Schwarz inequality:

$$|f_y(x)| = |\langle x, y \rangle| \leq \|x\| \|y\| \implies \|f_y\| \leq \|y\|$$

Also, $f_y(y) = \langle y, y \rangle = \|y\|^2$, so $\|f_y\| \geq \|y\|$. Hence, $\|f_y\| = \|y\|$. □

Definition 2.7. *Two elements x, y in a Hilbert space H are said to be orthogonal if $\langle x, y \rangle = 0$. For a subset $A \subseteq H$, the orthogonal complement is defined as:*

$$A^\perp = \{x \in H : \langle x, y \rangle = 0 \quad \forall y \in A\}$$

Definition 2.8. *A pre-Hilbert space H is called a Hilbert space if it is complete with respect to the norm induced by the inner product.*

Example 2.6. • $\mathbb{C}^n$ with the inner product $\langle x, y \rangle = \sum_{i=1}^n x_i \overline{y_i}$ is a Hilbert space.

• $L^2([a, b], \mathbb{C})$ with $\langle f, g \rangle = \int_a^b f(t) \overline{g(t)} dt$ is a Hilbert space.

2.6.2 Riesz Representation Theorem

Theorem 2.9. *Let H be a Hilbert space. For every continuous linear functional $f \in H^*$, there exists a unique $y \in H$ such that:*

$$f(x) = \langle x, y \rangle \quad \forall x \in H$$

and $\|f\| = \|y\|$.

Proof. If $f = 0$, take $y = 0$. Otherwise, let $F = \ker f$, which is a closed subspace. Then $H = F \oplus F^\perp$, and $F^\perp$ is one-dimensional. Take $x_0 \in F^\perp$ with $\|x_0\| = 1$. For any $x \in H$, we have:

$$x - \frac{f(x)}{f(x_0)}x_0 \in F \implies \left\langle x - \frac{f(x)}{f(x_0)}x_0, x_0 \right\rangle = 0 \implies \langle x, x_0 \rangle = \frac{f(x)}{f(x_0)}$$

So, $f(x) = f(x_0)\langle x, x_0 \rangle = \langle x, \overline{f(x_0)}x_0 \rangle$. Let $y = \overline{f(x_0)}x_0$. Uniqueness follows from the properties of the inner product. $\square$

2.6.3 Properties of the adjoint of a linear operator

Let $(H, \langle \cdot, \cdot \rangle)$ be a Hilbert space (a complete inner product space, where the norm is given by $\|x\| = \sqrt{\langle x, x \rangle}$).

Definition 2.9. *Let H_1, H_2 be Hilbert spaces, and let $T \in L(H_1, H_2)$ be a bounded linear operator. The **(Hilbert) adjoint** of T , denoted T^* , is the unique operator in $L(H_2, H_1)$ defined by the relation:*

$$\langle T(x), y \rangle_{H_2} = \langle x, T^*(y) \rangle_{H_1} \quad \forall x \in H_1, \forall y \in H_2$$

Example 2.7. *Let $H_1 = H_2 = \mathbb{C}^n$ with the standard inner product. If T is represented by a matrix $A \in M_n(\mathbb{C})$, then T^* is represented by the conjugate transpose matrix A^* .*

Example 2.8. *(Shift Operator): Consider the **right shift operator** S on $\ell^2(\mathbb{N})$:*

$$S : \ell^2(\mathbb{N}) \rightarrow \ell^2(\mathbb{N})$$

$$(x_1, x_2, x_3, \dots) \mapsto (0, x_1, x_2, x_3, \dots)$$

Its adjoint is the **left shift operator** S^* :

$$S^* : \ell^2(\mathbb{N}) \rightarrow \ell^2(\mathbb{N})$$

$$(x_1, x_2, x_3, \dots) \mapsto (x_2, x_3, x_4, \dots)$$

One can verify that $\langle Sx, y \rangle = \langle x, S^*y \rangle$ for all $x, y \in \ell^2$.

Example 2.9. (Integral Operator): Consider the integral operator T on $L^2([a, b])$ with kernel K :

$$T : L^2([a, b]) \rightarrow L^2([a, b])$$

$$f \mapsto Tf, \quad \text{where } (Tf)(s) = \int_a^b K(s, t)f(t)dt$$

If the kernel satisfies $K(s, t) = \overline{K(t, s)}$, then T is self-adjoint ($T^* = T$). The adjoint T^* is given by:

$$(T^*f)(s) = \int_a^b \overline{K(t, s)}f(t)dt$$

Corollary 2.3. Let $T \in L(H)$.

1. $\|T^*T\| = \|TT^*\| = \|T\|^2$
2. $T^*T = 0$ if and only if $T = 0$

Proposition 2.3. Let $T \in L(H_1, H_2)$. Then $(T^*)^* = T$ and $\|T^*\| = \|T\|$.

Proof. Let $y \in H$, then $f : x \mapsto \langle Tx, y \rangle$ is an element of H^* , therefore by the Riesz representation theorem, there exists a unique $T^*y \in H$ satisfying

$$\forall x \in H, \quad \langle Tx, y \rangle = f(x) = \langle x, T^*y \rangle.$$

We also have

$$\|T^*y\| \leq \|T^*\| \|y\| \quad \Rightarrow \quad \|T^*\| \leq \|T\|.$$

On the other hand,

$$\|Tx\|^2 = \langle Tx, Tx \rangle = \langle x, T^*(Tx) \rangle,$$

$$\Rightarrow \|Tx\|^2 \leq \|x\| \|T^*\| \|Tx\|,$$

$$\Rightarrow \|Tx\| \leq \|x\| \|T^*\|,$$

$$\Rightarrow \|T\| \leq \|T^*\|,$$

□

Theorem 2.10. *Let S and T be two operators on a Hilbert space.*

1. $(T + S)^* = T^* + S^*$
2. $(\alpha T)^* = \bar{\alpha}T^*, \quad \alpha \in \mathbb{C}$
3. $(T^*)^* = T$
4. $(ST)^* = T^*S^*$

Proof. 1. $\langle (S + T)x, y \rangle = \langle Sx, y \rangle + \langle Tx, y \rangle$

$$= \langle x, S^*y \rangle + \langle x, T^*y \rangle = \langle x, (T^* + S^*)y \rangle.$$

$$2. \langle (\alpha T)x, y \rangle = \langle T(\alpha x), y \rangle$$

$$= \langle \alpha x, T^*y \rangle = \langle x, \bar{\alpha}T^*y \rangle.$$

3. It is evident

$$4. \langle (ST)x, y \rangle = \langle Tx, S^*y \rangle$$

$$= \langle x, T^*(S^*y) \rangle = \langle x, (T^*S^*)y \rangle.$$

$$\Rightarrow (ST)^* = T^*S^*.$$

□

Proposition 2.4. *Let $\mathcal{H}$ be a Hilbert space over $\mathbf{K}$ and $T \in \mathcal{L}(\mathcal{H})$. Then*

- (a) $\text{Ker}T = (\text{Im}T^*)^\perp, \quad \text{Ker}T^* = (\text{Im}T)^\perp.$
- (b) $\overline{\text{Im}T} = (\text{Ker}T^*)^\perp, \quad \overline{\text{Im}T^*} = (\text{Ker}T)^\perp.$
- (c) $\text{Ker}T^* = \{0\}$ if and only if $\text{Im}T$ is dense in $\mathcal{H}$.

Here, $\text{Ker}T = \{x \in \mathcal{H} : Tx = 0\}$, $\text{Im}T = \{y \in \mathcal{H} : y = Tx, x \in \mathcal{H}\}$ and $\overline{\text{Im}T}$ is the closure of $\text{Im}T$ in $\mathcal{H}$.

Proof. (a) Let $x \in \text{Ker}T$, then $Tx = 0$, so for all $y \in \mathcal{H}$

$$\langle x, T^*y \rangle = \langle Tx, y \rangle = 0,$$

which means $x \in (\text{Im}T^*)^\perp$ and thus $\text{Ker}T \subset (\text{Im}T^*)^\perp$. If $x \in (\text{Im}T^*)^\perp$ then for all $y \in \mathcal{H}$

$$\langle Tx, y \rangle = \langle x, T^*y \rangle = 0,$$

which implies $x \in \text{Ker}T$ and therefore $(\text{Im}T^*)^\perp \subset \text{Ker}T$. Hence $\text{Ker}T = (\text{Im}T^*)^\perp$.

From the above, we have

$$\text{Ker}T^* = ((\text{Im}T^*)^*)^\perp = (\text{Im}T)^\perp.$$

(b) By part (a), we have

$$\overline{\text{Im}T} = ((\text{Im}T)^\perp)^\perp = (\text{Ker}T^*)^\perp$$

$$\overline{\text{Im}T^*} = ((\text{Im}T^*)^{\text{perp}})^\perp = (\text{Ker}T)^\perp.$$

(c) If $\text{Ker}T^* = \{0\}$, then by part (b)

$$\overline{\text{Im}T^*} = (\text{Ker}T^*)^\perp = \{0\}^\perp = \mathcal{H}.$$

Therefore $\text{Im}T$ is dense in $\mathcal{H}$. Conversely, if $\text{Im}T$ is dense in $\mathcal{H}$, then by (a),

$$\text{Ker}T^* = (\text{Im}T)^\perp = (((\text{Im}T)^\perp)^\perp)^\perp = (\mathcal{H})^\perp = \{0\}.$$

□

Corollary 2.4. *Let $\mathcal{H}$ be a Hilbert space, and $T \in \mathcal{L}(\mathcal{H})$. Then the orthogonal decomposition becomes:*

$$\mathcal{H} = \overline{\text{Im}T} \oplus \text{Ker}T^* = \overline{\text{Im}T^*} \oplus \text{Ker}T.$$

2.7 Spectrum of an operator

In the rest of this section, assume that X is a complex Banach space, and $T \in B(X)$.

Definition 2.10. *Let $\lambda \in \mathbb{K}$. λ is called a spectral value of T , if $T - \lambda I$ is not invertible (or equivalently, $T - \lambda I$ is bijective, since X is a Banach space).*

Definition 2.11. Let $\lambda \in \mathbb{K}$. We say that λ is an eigenvalue of T , if $T - \lambda I$ is not injective, or equivalently, there exists a nonzero vector $u \in X$ such that $(T - \lambda I)(u) = 0$.

Definition 2.12. We define the following sets:

1. The spectrum of T , denoted by $\sigma(T)$, is defined to be:

$$\sigma(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not invertible}\}.$$

2. The resolvent of T is given by:

$$\rho(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is invertible}\}.$$

3. The point spectrum, denoted by $\sigma_p(T)$, the set of all eigenvalues, is defined as:

$$\sigma_p(T) = \{\lambda \in \mathbb{C} : \ker(T - \lambda I) \neq \{0\}\}.$$

Remark 2.4. The spectral radius, denoted by $r(T)$, is the nonnegative real number given by:

$$r(T) = \sup_{\lambda \in \sigma(T)} |\lambda|.$$

Remark 2.5. We have:

1. $\sigma(T)$ is a nonempty and compact subset of $\mathbb{C}$.
2. It is clear that $\sigma_p(T) \subset \sigma(T)$.
3. If $\dim X < \infty$, then $\sigma_p(T) = \sigma(T)$.

Proof. *Proof of (2):* We want to show that $\sigma_p(T) \subset \sigma(T)$.

For any $\lambda \in \mathbb{C}$, if $\lambda \in \sigma_p(T)$, then $T - \lambda I$ is not injective. Since invertibility implies injectivity (and surjectivity), if $T - \lambda I$ is not injective, it cannot be invertible. Hence, $\lambda \in \sigma(T)$. Therefore, $\sigma_p(T) \subset \sigma(T)$.

Proof of (iii): Let $T \in L(X)$. If $\dim X < \infty$, we have:

$$T \text{ invertible} \Leftrightarrow T \text{ bijective} \Leftrightarrow T \text{ injective} \Leftrightarrow T \text{ surjective}.$$

Since

$$\sigma(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ not invertible}\} = \{\lambda \in \mathbb{C} : T - \lambda I \text{ not injective}\} = \sigma_p(T),$$

it follows that $\sigma_p(T) = \sigma(T)$. □

If $A \subset \mathbb{C}^*$, we define $A^{-1} = \{\frac{1}{a} : a \in A\}$.

Theorem 2.11. *If T is an invertible operator, then we have:*

$$\sigma(T^{-1}) = (\sigma(T))^{-1}, \quad \text{where} \quad (\sigma(T))^{-1} = \{\lambda^{-1} : \lambda \in \sigma(T)\}.$$

Proof. First, we prove that $\sigma(T^{-1}) \subset (\sigma(T))^{-1}$.

Let $\lambda \in \sigma(T^{-1})$. Then $T^{-1} - \lambda I$ is not invertible. Since T is invertible, the operator $T(T^{-1} - \lambda I)$ is not invertible. Hence, $I - \lambda T$ is not invertible. Then $(-\lambda)(T - \frac{1}{\lambda}I)$ is not invertible.

Since $-\lambda \neq 0$, it follows that $T - \frac{1}{\lambda}I$ is not invertible, hence $\frac{1}{\lambda} \in \sigma(T)$. Therefore, $\lambda \in (\sigma(T))^{-1}$. Thus,

$$\sigma(T^{-1}) \subset (\sigma(T))^{-1}.$$

Now, we show that $(\sigma(T))^{-1} \subset \sigma(T^{-1})$.

Let $\lambda \in (\sigma(T))^{-1}$. Then $\frac{1}{\lambda} \in \sigma(T)$, so $T - \frac{1}{\lambda}I$ is not invertible (and $\lambda \neq 0$). Then $\lambda(T - \frac{1}{\lambda}I) = \lambda T - I$ is not invertible. Hence,

$$T^{-1} - \lambda^{-1}I = (-\lambda)^{-1}T^{-1}(\lambda T - I)$$

is not invertible (since $\lambda T - I$ is not invertible and $(-\lambda)^{-1}T^{-1}$ is invertible). Thus, $\lambda^{-1} \in \sigma(T^{-1})$. Therefore,

$$(\sigma(T))^{-1} \subset \sigma(T^{-1}).$$

Hence, we conclude:

$$\sigma(T^{-1}) = (\sigma(T))^{-1}.$$

□

2.8 Exercises

Exercise 1

Find the adjoint T^* in the following cases:

1. $T : L^2([0, 1]) \rightarrow L^2([0, 1])$, $(Tx)(t) = x\left(\frac{t+1}{2}\right)$
2. $T : \ell^2(\mathbb{C}) \rightarrow \ell^2(\mathbb{C})$, $Tx = (\alpha_1 x_1, \alpha_2 x_2, \dots)$, where $(\alpha_n) \in \ell^\infty$
3. $T : L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R})$, $(Tx)(t) = a(t)x(t+h)$, $h \in \mathbb{R}$

Exercise 2

Find the eigenvalues and spectrum for:

1. $Tx(t) = x(-t)$
2. $Tx(t) = \int_{-\pi}^{\pi} \sin(t+s)x(s) ds$
3. $Tx(t) = \int_0^1 \phi(t)x(t) dt$, where $\phi > 0$ on $[0, 1]$

Exercise 3

Let $T \in L(L^2([0, 1]))$ such that $T^2 = \lambda T$. Show that there exists λ , and compute the spectrum.

Exercise 4

Let $T : \ell^2(\mathbb{C}) \rightarrow \ell^2(\mathbb{C})$ be the left shift: $T(x_1, x_2, \dots) = (x_2, x_3, \dots)$. Show that the point spectrum is empty, and the spectrum is $\{\lambda \in \mathbb{C} : |\lambda| \leq 1\}$. Also, consider the right shift $S(x_1, x_2, \dots) = (0, x_1, x_2, \dots)$, and show its spectrum is the same.

Exercise 5

Let $T : \ell^2(\mathbb{C}) \rightarrow \ell^2(\mathbb{C})$ be defined by $Tx = (\alpha_1 x_1, \alpha_2 x_2, \dots)$, with (α_n) bounded. Show that the eigenvalues are exactly the α_n , and $\sigma(T) = \overline{\{\alpha_n\}}$.

Exercise 6

Let $E = L^2([0, 1], \mathbb{C})$, and $Tf(t) = tf(t)$. Show that T has no eigenvalues, and $\sigma(T) = [0, 1]$.

Exercise 7

1. Show that the operator $A : (C[0, 1], \|\cdot\|_\infty) \rightarrow \mathbb{R}$ defined by $Af = \int_0^1 f(t) \sin t \, dt$ is linear, bounded, continuous, and Lipschitz, and determine its norm.

2. Same questions for $A : (L^2[0, 1], \|\cdot\|_2) \rightarrow \mathbb{R}$ with $Af = \int_0^1 f(t) \sin t \, dt$ (Lebesgue integral).

3. Same questions for $A : C[0, 1] \rightarrow C[0, 1]$ defined by $Af(t) = \int_0^t f(s) \, ds$.

4. Same questions for the left-shift operator $A : (\ell_2, \|\cdot\|_2) \rightarrow (\ell_2, \|\cdot\|_2)$ defined by

$$A(x_1, x_2, \dots) = (x_2, x_3, \dots).$$

5. Same questions for the integral operator $A : L^2[0, 1] \rightarrow L^2[0, 1]$ defined by

$$Ay(t) = \int_0^1 K(t, s)y(s) \, ds,$$

where the kernel K is continuous on $[0, 1] \times [0, 1]$.

2.9 Solutions

Solution of Exercise 1

We find the adjoint T^*

1. $T : L^2([0, 1]) \rightarrow L^2([0, 1])$, $(Tx)(t) = x\left(\frac{t+1}{2}\right)$.

Let $x, y \in L^2([0, 1])$. We have:

$$\langle Tx, y \rangle = \int_0^1 x\left(\frac{t+1}{2}\right) \overline{y(t)} \, dt.$$

Substitute $u = \frac{t+1}{2} \Rightarrow t = 2u - 1$, $dt = 2 \, du$.

The limits become $t = 0 \Rightarrow u = \frac{1}{2}$, $t = 1 \Rightarrow u = 1$.

Then:

$$\langle Tx, y \rangle = \int_{1/2}^1 x(u) \overline{y(2u-1)} \cdot 2 du.$$

Hence the adjoint is:

$$(T^*y)(u) = \begin{cases} 2y(2u-1), & u \in [\frac{1}{2}, 1], \\ 0, & \text{otherwise.} \end{cases}$$

2. $T : \ell^2(\mathbb{C}) \rightarrow \ell^2(\mathbb{C})$, $Tx = (\alpha_1 x_1, \alpha_2 x_2, \dots)$, with $(\alpha_n) \in \ell^\infty$.

Let $x, y \in \ell^2$. Then:

$$\langle Tx, y \rangle = \sum_{n=1}^{\infty} \alpha_n x_n \overline{y_n}.$$

So clearly:

$$\langle x, T^*y \rangle = \sum_{n=1}^{\infty} x_n \overline{\alpha_n y_n},$$

hence $T^*y = (\overline{\alpha_1} y_1, \overline{\alpha_2} y_2, \dots)$.

3. $T : L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R})$, $(Tx)(t) = a(t)x(t+h)$, $h \in \mathbb{R}$.

Let $x, y \in L^2(\mathbb{R})$. Then:

$$\langle Tx, y \rangle = \int_{\mathbb{R}} a(t)x(t+h) \overline{y(t)} dt.$$

Let $u = t+h \Rightarrow t = u-h$, $dt = du$. Then:

$$\langle Tx, y \rangle = \int_{\mathbb{R}} a(u-h)x(u) \overline{y(u-h)} du.$$

Thus:

$$(T^*y)(u) = \overline{a(u-h)} y(u-h).$$

Solution of Exercise 2

We find eigenvalues and spectrum

1. $Tx(t) = x(-t)$ on $L^2([-a, a])$ (assume symmetric interval).

Let λ be an eigenvalue: $x(-t) = \lambda x(t)$ for almost all t .

Applying twice: $x(t) = \lambda x(-t) = \lambda^2 x(t) \Rightarrow \lambda^2 = 1 \Rightarrow \lambda = \pm 1$.

Check eigenfunctions: $\lambda = 1$: even functions; $\lambda = -1$: odd functions.

The spectrum $\sigma(T) = \{\pm 1\}$ (since T is unitary and bounded).

$$2. Tx(t) = \int_{-\pi}^{\pi} \sin(t+s)x(s) ds.$$

Note $\sin(t+s) = \sin t \cos s + \cos t \sin s$. Let:

$$u_1(t) = \sin t, \quad u_2(t) = \cos t, \quad v_1(s) = \cos s, \quad v_2(s) = \sin s.$$

Then $Tx = \langle x, v_1 \rangle u_1 + \langle x, v_2 \rangle u_2$. So T is a rank-2 operator.

Eigenvalues come from the matrix representation in basis $\{u_1, u_2\}$. Compute:

$$\int_{-\pi}^{\pi} \sin(t+s) \cos(s) ds = \pi \sin t, \quad \int_{-\pi}^{\pi} \sin(t+s) \sin(s) ds = \pi \cos t.$$

Thus the matrix of T in basis $\{\sin t, \cos t\}$ is $\begin{pmatrix} \pi & 0 \\ 0 & \pi \end{pmatrix}$.

Eigenvalues: $\lambda = \pi$ (multiplicity 2). Spectrum $\sigma(T) = \{\pi, 0\}$ (0 is in spectrum since T is compact and non-surjective).

$$3. Tx(t) = \int_0^1 \phi(t)x(t) dt \text{ with } \phi > 0 \text{ on } [0, 1].$$

Here $Tx = \langle x, 1 \rangle \phi(t)$, a rank-1 operator.

Eigenvalue equation: $\langle x, 1 \rangle \phi(t) = \lambda x(t)$.

If $\lambda \neq 0$, then $x(t) = c\phi(t)$ with $c = \langle x, 1 \rangle / \lambda$.

Compute $\langle x, 1 \rangle = c \int_0^1 \phi(t) dt$.

The equation becomes $c \int_0^1 \phi(t) dt \cdot \phi(t) = \lambda c \phi(t)$.

Thus $\lambda = \int_0^1 \phi(t) dt$ (non-zero eigenvalue).

If $\lambda = 0$, then $\langle x, 1 \rangle = 0$, so any $x \perp 1$ is an eigenfunction for $\lambda = 0$.

Thus point spectrum: $\lambda = 0$ and $\lambda = \int_0^1 \phi(t) dt$.

Spectrum $\sigma(T) = \{0, \int_0^1 \phi(t) dt\}$.

Solution of Exercise 3

Let $T \in \mathcal{L}(L^2([0, 1]))$ such that $T^2 = \lambda T$.

If $\lambda = 0$, then $T^2 = 0$. Spectrum may contain 0 only.

If $\lambda \neq 0$, let $\mu \in \mathbb{C} \setminus \{0, \lambda\}$. Then:

$$(T - \mu I)(T - (\lambda - \mu)I) = T^2 - \lambda T + \mu(\lambda - \mu)I = \mu(\lambda - \mu)I.$$

Thus $T - \mu I$ is invertible with inverse $\frac{1}{\mu(\lambda - \mu)}(T - (\lambda - \mu)I)$.

So $\sigma(T) \subseteq \{0, \lambda\}$.

If $T = 0$, spectrum is $\{0\}$.

If $T \neq 0$ and satisfies $T^2 = \lambda T$, then any non-zero x with $Tx \neq 0$ satisfies $T(Tx) = \lambda(Tx)$, so λ is an eigenvalue.

Also, if $Tx = 0$ for some non-zero x , then 0 is eigenvalue.

Thus $\sigma(T) = \{0, \lambda\}$ or subset thereof.

Solution of Exercise 4

Left shift $T(x_1, x_2, \dots) = (x_2, x_3, \dots)$ on $\ell^2(\mathbb{C})$.

No eigenvalues: Suppose $Tx = \lambda x$. Then $x_{n+1} = \lambda x_n$ for all n , so $x_n = \lambda^{n-1}x_1$.

If $|\lambda| < 1$, $x \in \ell^2$. But $x_1 \neq 0$ yields $\|x\|^2 = |x_1|^2 \sum_{n=0}^{\infty} |\lambda|^{2n} < \infty$, so indeed eigenvector exists? Wait, check: For $|\lambda| < 1$, $x = (1, \lambda, \lambda^2, \dots) \in \ell^2$. Then $Tx = (\lambda, \lambda^2, \dots) = \lambda x$. So λ is eigenvalue for $|\lambda| < 1$. Similarly for $|\lambda| = 1$, $x \notin \ell^2$. So point spectrum $\sigma_p(T) = \{\lambda : |\lambda| < 1\}$.

But spectrum $\sigma(T) = \{\lambda : |\lambda| \leq 1\}$ for the left shift.

Right shift $S(x_1, x_2, \dots) = (0, x_1, x_2, \dots)$.

Eigenvalues: $Sx = \lambda x \Rightarrow (0, x_1, x_2, \dots) = (\lambda x_1, \lambda x_2, \dots)$.

If $\lambda \neq 0$, then $x_1 = 0$, $\lambda x_2 = x_1 = 0 \Rightarrow x_2 = 0$, etc., so $x = 0$. If $\lambda = 0$, then $Sx = 0$ means $x_1 = x_2 = \dots = 0$, so no non-zero eigenvectors. So $\sigma_p(S) = \emptyset$.

Spectrum $\sigma(S) = \{\lambda : |\lambda| \leq 1\}$, same as left shift.

Solution of Exercise 5

$Tx = (\alpha_1 x_1, \alpha_2 x_2, \dots)$ with (α_n) bounded.

Eigenvalue equation: $\alpha_n x_n = \lambda x_n$ for all n .

If for some n , $\alpha_n = \lambda$, choose $x = e_n$ (standard basis vector), then $Tx = \lambda e_n$, so λ is eigenvalue.

Conversely, if λ is eigenvalue with eigenvector x , then for each n either $x_n = 0$ or $\alpha_n = \lambda$. Since $x \neq 0$, some $x_n \neq 0$, so $\lambda = \alpha_n$ for that n . Thus eigenvalues are exactly $\{\alpha_n\}$.

Spectrum $\sigma(T) = \overline{\{\alpha_n\}}$ (closure in $\mathbb{C}$).

Solution of Exercise 6

$E = L^2([0, 1], \mathbb{C})$, $Tf(t) = tf(t)$.

Eigenvalue equation: $tf(t) = \lambda f(t)$ a.e.

Then $(t - \lambda)f(t) = 0$ a.e., so $f(t) = 0$ for $t \neq \lambda$. Since λ is fixed and $\{t : t \neq \lambda\}$ has full measure, $f = 0$ a.e. So no eigenvalues.

Spectrum: T is multiplication by t , so $\sigma(T) = \text{ess range of } t = [0, 1]$.

Solution of Exercise 7

Linearity: Let $\alpha \in \mathbb{R}$ and $f, g \in C[0, 1]$. Then

$$A(\alpha f + g) = \int_0^1 (\alpha f + g)(t) \sin t \, dt = \alpha \int_0^1 f(t) \sin t \, dt + \int_0^1 g(t) \sin t \, dt = \alpha Af + Ag.$$

Boundedness: Let $f \in C[0, 1]$. Then

$$|Af| = \left| \int_0^1 f(t) \sin t \, dt \right| \leq \int_0^1 |f(t) \sin t| \, dt = \int_0^1 |f(t)| |\sin t| \, dt \leq \|f\|_\infty \int_0^1 |\sin t| \, dt.$$

Thus there exists $M = \int_0^1 |\sin t| \, dt \geq 0$ such that

$$|Af| \leq M\|f\|_\infty \quad \text{for all } f \in C[0, 1]. \quad (*)$$

Continuity and Lipschitz property: Since A is linear and bounded, it is continuous and Lipschitz.

Norm of A : From $(*)$ we obtain

$$\|A\| \leq M = \int_0^1 |\sin t| \, dt. \quad (1)$$

Now consider the sequence of continuous functions on $[0, 1]$ defined by

$$f_n(t) = \frac{n \sin t}{n|\sin t| + 1}, \quad t \in [0, 1], \quad n \geq 1.$$

On one hand,

$$Af_n = \int_0^1 f_n(t) \sin t \, dt = \int_0^1 \frac{n \sin^2 t}{n|\sin t| + 1} \, dt.$$

Hence,

$$0 \leq \left| Af_n - \int_0^1 |\sin t| \, dt \right| = \left| \int_0^1 \left[\frac{n \sin^2 t}{n|\sin t| + 1} - |\sin t| \right] \, dt \right| = \int_0^1 \frac{|\sin t|}{n|\sin t| + 1} \, dt \leq \frac{1}{n}.$$

This implies

$$Af_n \xrightarrow{n \rightarrow \infty} \int_0^1 |\sin t| \, dt.$$

On the other hand,

$$\|f_n\|_\infty = \sup_{t \in [0, 1]} |f_n(t)| = \sup_{t \in [0, 1]} \frac{n|\sin t|}{n|\sin t| + 1} \leq 1.$$

Therefore,

$$Af_n \leq \frac{\|Af_n\|}{\|f_n\|_\infty} \leq \|A\|,$$

which gives

$$\int_0^1 |\sin t| \, dt \leq \|A\|. \quad (2)$$

From (1) and (2) we conclude

$$\|A\| = \int_0^1 |\sin t| \, dt.$$

2. Recall that the Lebesgue space

$$L^2[0, 1] = \left\{ f : [0, 1] \rightarrow \mathbb{R} \mid f \text{ measurable, } \int_0^1 |f(t)|^2 \, dt < \infty \right\}$$

is a Hilbert space with inner product

$$\langle f, g \rangle = \int_0^1 f(t)g(t) \, dt, \quad f, g \in L^2[0, 1],$$

and norm

$$\|f\|_2 = \sqrt{\langle f, f \rangle} = \left(\int_0^1 |f(t)|^2 \, dt \right)^{\frac{1}{2}}.$$

Linearity: As before, $A(\alpha f + g) = \alpha Af + Ag$.

Boundedness: For $f \in L^2[0, 1]$,

$$|Af| = \left| \int_0^1 f(t) \sin t \, dt \right| = |\langle f, \sin \rangle| \leq \|f\|_2 \|\sin\|_2 \quad (\text{Cauchy-Schwarz}) = \|f\|_2 \left(\int_0^1 |\sin t|^2 dt \right)^{\frac{1}{2}}.$$

Thus

$$|Af| \leq M \|f\|_2 \quad \text{with} \quad M = \left(\int_0^1 |\sin t|^2 dt \right)^{\frac{1}{2}}. \quad (*)$$

Continuity and Lipschitz property: Follows from linearity and boundedness.

Norm of A : From $(*)$ we have

$$\|A\| \leq M = \left(\int_0^1 |\sin t|^2 dt \right)^{\frac{1}{2}}. \quad (1)$$

Now take $f_* = \sin \in L^2[0, 1]$. Then

$$\frac{|Af_*|}{\|f_*\|_2} = \frac{\left| \int_0^1 \sin^2 t \, dt \right|}{\left(\int_0^1 \sin^2 t \, dt \right)^{\frac{1}{2}}} = \left(\int_0^1 \sin^2 t \, dt \right)^{\frac{1}{2}}.$$

Since $\frac{|Af_*|}{\|f_*\|_2} \leq \|A\|$, we obtain

$$\left(\int_0^1 \sin^2 t \, dt \right)^{\frac{1}{2}} \leq \|A\|. \quad (2)$$

From (1) and (2),

$$\|A\| = \left(\int_0^1 \sin^2 t \, dt \right)^{\frac{1}{2}}.$$

3.

Linearity: For $\alpha \in \mathbb{R}$, $f, g \in C[0, 1]$, and any $t \in [0, 1]$,

$$A(\alpha f + g)(t) = \int_0^t (\alpha f + g)(s) \, ds = \alpha \int_0^t f(s) \, ds + \int_0^t g(s) \, ds = \alpha Af(t) + Ag(t).$$

Boundedness: For $f \in C[0, 1]$,

$$\|Af\|_\infty = \sup_{t \in [0, 1]} \left| \int_0^t f(s) \, ds \right| \leq \sup_{t \in [0, 1]} \int_0^t |f(s)| \, ds \leq \sup_{t \in [0, 1]} \int_0^t \|f\|_\infty \, ds = \sup_{t \in [0, 1]} (\|f\|_\infty t) = \|f\|_\infty.$$

Thus $\|Af\|_\infty \leq 1 \cdot \|f\|_\infty$.

Continuity and Lipschitz property: Follows from linearity and boundedness.

Norm of A : From (*), $\|A\| \leq 1$.

Take $f_*(t) = 1$. Then

$$\frac{\|Af_*\|_\infty}{\|f_*\|_\infty} = \frac{\sup_{t \in [0,1]} \left| \int_0^t 1 \, ds \right|}{\sup_{t \in [0,1]} 1} = \frac{\sup_{t \in [0,1]} t}{1} = 1.$$

Since $\frac{\|Af_*\|_\infty}{\|f_*\|_\infty} \leq \|A\|$, we get $1 \leq \|A\|$.

From (1) and (2), $\|A\| = 1$.

4.

Linearity: For $\alpha \in \mathbb{R}$ and $x = (x_1, x_2, \dots), y = (y_1, y_2, \dots) \in \ell_2$,

$$A(\alpha x + y) = (\alpha x_2 + y_2, \alpha x_3 + y_3, \dots) = \alpha(x_2, x_3, \dots) + (y_2, y_3, \dots) = \alpha Ax + Ay.$$

Boundedness: For $x \in \ell_2$,

$$\|Ax\|_2^2 = \sum_{i=2}^{\infty} |x_i|^2 \leq \sum_{i=1}^{\infty} |x_i|^2 = \|x\|_2^2.$$

Thus $\|Ax\|_2 \leq 1 \cdot \|x\|_2$.

Continuity and Lipschitz property: Follows from linearity and boundedness.

Norm of A : From (*), $\|A\| \leq 1$.

Take $x_* = (0, 1, 0, 0, \dots) \in \ell_2$. Then

$$\frac{\|Ax_*\|_2}{\|x_*\|_2} = \frac{\|(1, 0, 0, \dots)\|_2}{\|(0, 1, 0, \dots)\|_2} = \frac{1}{1} = 1.$$

Since $\frac{\|Ax_*\|_2}{\|x_*\|_2} \leq \|A\|$, we obtain $1 \leq \|A\|$.

From (1) and (2), $\|A\| = 1$.

5.

Linearity: For $\alpha \in \mathbb{R}$, $f, g \in L^2[0, 1]$, and any $t \in [0, 1]$,

$$[A(\alpha f + g)](t) = \int_0^1 K(t, s)(\alpha f + g)(s) \, ds = \alpha \int_0^1 K(t, s)f(s) \, ds + \int_0^1 K(t, s)g(s) \, ds = \alpha(Af)(t) + (Ag)(t)$$

Boundedness: For $f \in L^2[0, 1]$ and any $t \in [0, 1]$,

$$|(Af)(t)| = \left| \int_0^1 K(t, s)f(s) ds \right| = |\langle K(t, \cdot), f \rangle| \leq \|K(t, \cdot)\|_2 \|f\|_2 = \left(\int_0^1 |K(t, s)|^2 ds \right)^{\frac{1}{2}} \|f\|_2.$$

Hence,

$$\|Af\|_2^2 = \int_0^1 |(Af)(t)|^2 dt \leq \int_0^1 \left(\int_0^1 |K(t, s)|^2 ds \right) \|f\|_2^2 dt = \left(\iint_{[0,1]^2} |K(t, s)|^2 ds dt \right) \|f\|_2^2.$$

Therefore,

$$\|Af\|_2 \leq M \|f\|_2 \quad \text{with} \quad M = \left(\iint_{[0,1]^2} |K(t, s)|^2 ds dt \right)^{\frac{1}{2}}. \quad (*)$$

Continuity and Lipschitz property: Follows from linearity and boundedness.

Chapter 3

Introduction to the spectral theory of compact operators

3.1 Definitions and results

3.1.1 compact operators

We recall that $L(E, F)$ denotes the Banach space over $\mathbb{K}$ of linear operators from E to F , while $\mathcal{L}(E, F)$ denotes the Banach space over $\mathbb{K}$ of bounded linear operators from E to F .

Definition 3.1. We say that T is a **compact operator** from E to F if $T \in L(E, F)$ and $\overline{T(B_E)}$ is a compact subset of F . The set of compact operators from E to F is denoted $K_c(E, F)$. When $E = F$, we simply write $K_c(E, E) = K_c(E)$.

We recall that if (X, d) is a metric space, a subset Y of X is said to be **relatively compact** if its closure in X is compact. Then, Definition 4.1.1 is equivalent to: $T \in K_c(E, F)$ if and only if $T(B_E)$ is relatively compact.

In particular, it is immediate that $T \in K_c(E, F)$ if and only if $T(B)$ is relatively compact for every bounded subset B of E . We deduce the following result:

Proposition 3.1 (Characterization of compact operators). *If $T \in L(E, F)$, then the following statements are equivalent:*

1. $T \in K_c(E, F)$;

2. For every bounded sequence $(x_n)_{n \in \mathbb{N}}$ in E , the sequence $(Tx_n)_{n \in \mathbb{N}}$ admits a subsequence that converges in F .

Below we recall Ascoli's theorem, which is a classic and powerful tool for showing that an operator is compact.

Theorem 3.1 (Ascoli). *Let (K, d) be a compact metric space and (X, d') a complete metric space. A subset $\mathcal{H} \subset C(K; X)$ is relatively compact for the uniform convergence norm if and only if:*

1. $\mathcal{H}$ is equicontinuous, i.e., for every $x \in K$,

$$\forall \epsilon > 0, \exists V \in \mathcal{V}(x), \forall g \in \mathcal{H}, \forall y \in V, \quad d'(g(x), g(y)) < \epsilon,$$

where $\mathcal{V}(x)$ denotes the set of neighborhoods of x for the distance $d(\cdot, \cdot)$.

2. For every $x \in K$, the set $\mathcal{H}(x) = \{g(x) \mid g \in \mathcal{H}\}$ is relatively compact.

Example 3.1. *The Volterra operator T (primitive vanishing at 0) defined on $E = C([0, 1]; \mathbb{K})$, equipped with the uniform convergence norm $\|\cdot\|_\infty$, is compact.*

Indeed, we use Ascoli's theorem with $K = [0, 1]$, $X = \mathbb{K}$, and $\mathcal{H} = T(B_E)$. Let $f \in B_E$. For $x, y \in [0, 1]$, we have

$$|Tf(x) - Tf(y)| = \left| \int_x^y f(t) dt \right| \leq |x - y| \|f\|_\infty \leq |x - y|.$$

Let $x \in K$ be fixed. Then, for every $\epsilon > 0$, there exists $V = \{y \in K \mid |x - y| < \epsilon\} \in \mathcal{V}(x)$ such that for every $g \in \mathcal{H}$ and every $y \in V$, we have $|g(x) - g(y)| < \epsilon$. In other words, $\mathcal{H}$ is indeed equicontinuous. Moreover, for every $x \in [0, 1]$, $\mathcal{H}(x)$ is a subset of $\mathbb{K}$, so if $\mathcal{H}(x)$ is bounded then $\mathcal{H}(x)$ is relatively compact. Now, for every $x \in [0, 1]$, we have

$$\forall f \in B_E, \quad |Tf(x)| \leq \int_0^x |f(t)| dt \leq x \leq 1,$$

which gives the result.

Example 3.2. *Let $E = C([a, b], K)$ equipped with the uniform convergence norm $\|\cdot\|_\infty$ and $K \in C([a, b]^2, K)$. Then, the integral operator T_K defined by*

$$\forall f \in E, \quad T_K f(x) = \int_a^b K(x, y) f(y) dy, \quad \forall x \in [a, b],$$

is compact.

We will repeatedly use Riesz's theorem, which we recall below in the form of a lemma.

Lemma 3.1 (Riesz's theorem). *Let X be a normed vector space. Then, the closed unit ball of X is compact if and only if X is finite-dimensional.*

Remark 3.1. 1. *A direct application of Riesz's theorem is that the identity map on X is compact if and only if X is finite-dimensional.*

2. *If E and F are infinite-dimensional, Riesz's theorem implies (see tutorial sheet no. 3) that if $T \in L(E, F)$ is invertible then T is not compact.*

Theorem 3.2. *$Kc(E, F)$ is a closed vector subspace of $L(E, F)$; in particular, $Kc(E, F)$ is a Banach space over K .*

Remark 3.2. *If F is not complete, the result may be false.*

Proof. For every $T \in Kc(E, F)$, we have

$$\sup_{x \in B_E} \|Tx\|_F = \sup_{y \in T(B_E)} \|y\|_F \leq \sup_{y \in \overline{T(B_E)}} \|y\|_F < +\infty,$$

hence $Kc(E, F) \subset L(E, F)$. Moreover, $0 \in Kc(E, F)$. Let us show that $Kc(E, F)$ is stable under linear combinations. Let $S, T \in Kc(E, F)$ and $\lambda \in \mathbb{K}$, for every bounded sequence $(x_n)_{n \in \mathbb{N}}$ in E , there exists a subsequence $(Sx_{\phi(n)})_{n \in \mathbb{N}}$ of $(Sx_n)_{n \in \mathbb{N}}$ that converges to a point y in F . Similarly, there exists a subsequence $(Tx_{\phi(n)})_{n \in \mathbb{N}}$ of $(Tx_n)_{n \in \mathbb{N}}$ that converges to a point z in F . Finally, we obtain

$$(S + \lambda T)x_{\psi(\phi(n))} \xrightarrow{n \rightarrow +\infty} y + \lambda z.$$

In conclusion, for every bounded sequence $(x_n)_{n \in \mathbb{N}}$ in E , the sequence $((S + \lambda T)x_n)_{n \in \mathbb{N}}$ admits a subsequence $((S + \lambda T)x_{\psi(\phi(n))})_{n \in \mathbb{N}}$ that converges in F . We deduce, by Proposition 4.1.2, that $S + \lambda T \in Kc(E, F)$; hence $Kc(E, F)$ is a vector subspace of $L(E, F)$.

It remains to show that the vector space $Kc(E, F)$, equipped with the norm induced by that of $L(E, F)$, is closed. Let $T \in \overline{Kc(E, F)}$. Then, there exists a sequence $(T_n)_{n \in \mathbb{N}}$ in $Kc(E, F)$ that converges in $L(E, F)$ to T . We want to show that $T \in Kc(E, F)$, i.e., that $\overline{T(B_E)}$ is a compact subset of F .

Lemma 3.2. *If K is a closed subset of a complete metric space (X, d) , then K is compact if and only if for every $\epsilon > 0$, there exist $n \in \mathbb{N}^*$ and $y_1, \dots, y_n \in X$ such that*

$$K \subset \bigcup_{i=1}^n B(y_i, \epsilon),$$

where $B(y_i, \epsilon) = \{x \in X \mid d(x, y_i) < \epsilon\}$.

Let $\epsilon > 0$. There exists $n_0 \in \mathbb{N}$ such that $\|T_{n_0} - T\| < \epsilon/4$. Now $T_{n_0} \in Kc(E, F)$, so $\overline{T_{n_0}(B_E)}$ is a compact subset of F . In particular, by Lemma 3.2, there exist $n \in \mathbb{N}^*$ and $y_1, \dots, y_n \in F$ such that

$$T_{n_0}(B_E) \subset \bigcup_{i=1}^n B\left(y_i, \frac{\epsilon}{4}\right).$$

If $x \in B_E$, there exists $i_0 \in \{1, \dots, n\}$ such that

$$T_{n_0}x \in B\left(y_{i_0}, \frac{\epsilon}{4}\right).$$

Thus, we obtain

$$\|Tx - y_{i_0}\| \leq \|Tx - T_{n_0}x\| + \|T_{n_0}x - y_{i_0}\| \leq \|T - T_{n_0}\| + \frac{\epsilon}{4} \leq \frac{\epsilon}{4} + \frac{\epsilon}{4} = \frac{\epsilon}{2}.$$

Hence,

$$Tx \in B\left(y_{i_0}, \frac{\epsilon}{2}\right) \subset \bigcup_{i=1}^n B\left(y_i, \frac{\epsilon}{2}\right).$$

We deduce

$$T(B_E) \subset \bigcup_{i=1}^n B\left(y_i, \frac{\epsilon}{2}\right).$$

Therefore,

$$\overline{T(B_E)} \subset \bigcup_{i=1}^n B\left(y_i, \frac{\epsilon}{2}\right) \subset \bigcup_{i=1}^n \overline{B\left(y_i, \frac{\epsilon}{2}\right)} \subset \bigcup_{i=1}^n B(y_i, \epsilon),$$

which gives the result by Lemma 3.2. □

We conclude this section with a property on the composition of compact operators.

Proposition 3.2. *Let $T \in L(E, F)$ and $S \in L(F, G)$. If S or T is compact then ST is compact.*

Proof. If $T \in Kc(E, F)$. Then, for every bounded sequence $(x_n)_{n \in \mathbb{N}}$ in E , the sequence $(Tx_n)_{n \in \mathbb{N}}$ admits a subsequence $(Tx_{\phi(n)})_{n \in \mathbb{N}}$ that converges to a point $y \in F$. Since S is continuous, $(S(Tx_{\phi(n)}))_{n \in \mathbb{N}}$ converges to $Sy \in G$. Hence $ST \in Kc(E, G)$.

If $S \in Kc(F, G)$. Then, for every sequence $(x_n)_{n \in \mathbb{N}}$ of elements of E bounded by a constant M , the sequence $(Tx_n)_{n \in \mathbb{N}}$ is a bounded sequence in F . Indeed, for every $n \in \mathbb{N}$, we have

$$\|Tx_n\|_F \leq \|T\| \|x_n\|_E \leq M \|T\|.$$

Since $S \in Kc(F, G)$, the sequence $(S(Tx_n))_{n \in \mathbb{N}}$ admits a subsequence $(S(Tx_{\phi(n)}))_{k \in \mathbb{N}}$ that converges to a point $z \in G$. Hence $ST \in Kc(E, G)$. $\square$

3.2 Finite-Rank Operators

In this section, we focus on particular compact operators.

Definition 3.2. Let $T \in L(E, F)$. We say that T is a **finite-rank operator** if $Im(T)$ is finite-dimensional.

Example 3.3. Let $E = L^p(]0, 1[, \mathbb{R})$ and T the operator defined on E by

$$\forall f \in E, \quad Tf(x) = \int_0^1 xy(1 - xy)f(y) dy, \quad \forall x \in]0, 1[.$$

Then, T takes values in $F = \mathbb{R}_2[X]$ (the space of polynomials with real coefficients of degree at most 2). Since F is finite-dimensional, the operator T is of finite rank.

Proposition 3.3. Every bounded finite-rank operator is compact.

Proof. Let $T \in L(E, F)$ be a finite-rank operator. The operator T is continuous, so for every $x \in B_E$, $\|Tx\| \leq \|T\|$. Then, $T(B_E)$ is bounded in F and, consequently, $\overline{T(B_E)}$ is also bounded. Moreover, $Im(T)$ is closed because it is a finite-dimensional vector space, whence $\overline{T(B_E)} \subset \overline{Im(T)} = Im(T)$. Finally, $\overline{T(B_E)}$ is a closed and bounded subset of the finite-dimensional vector space $Im(T)$; hence it is a compact subset of $Im(T)$, and therefore $T \in Kc(E, F)$. $\square$

The space $Kc(E, F)$ is closed and every bounded finite-rank operator is compact. We deduce the following result:

Corollary 3.1. Every limit in $L(E, F)$ of finite-rank operators is a compact operator.

The converse is false in general, but if F is a Hilbert space, we have the following result:

Theorem 3.3. *Let $(H, (\cdot, \cdot))$ be a Hilbert space over K and $T \in Kc(E, H)$. Then, there exists a sequence $(T_n)_{n \in \mathbb{N}}$ in $L(E, H)$ of finite-rank operators that converges to T in $L(E, H)$.*

Proof. Let $n \in \mathbb{N}$ be fixed. By Lemma 3.2, since $\overline{T(B_E)}$ is a compact subset of H , there exist $k_n \in \mathbb{N}^*$ and $y_1, \dots, y_{k_n} \in H$ such that

$$\overline{T(B_E)} \subset \bigcup_{i=1}^{k_n} B\left(y_i, \frac{1}{n+1}\right). \quad (3.1)$$

Now $L_n = \text{vect}\{y_1, \dots, y_{k_n}\}$ is a finite-dimensional vector space, hence a closed subspace of the Hilbert space H , so $H = L_n \oplus L_n^\perp$. Let $P_n \in \mathcal{L}(H)$ be the orthogonal projection of H onto L_n . Then, $T_n = P_n T \in L(E, H)$ is a finite-rank operator since

$$\dim(\text{Im}(T_n)) = \dim(\text{Im}(P_n T)) \leq \dim(L_n) \leq k_n.$$

Finally, for $x \in B_E$, by (3.1), there exists $i_0 \in \{1, \dots, k_n\}$ such that

$$\|Tx - y_{i_0}\| < \frac{1}{n+1}.$$

But $y_{i_0} \in L_n$ and $P_n Tx$ is the orthogonal projection of Tx onto L_n , so

$$\|Tx - P_n Tx\| = d(Tx, L_n) \leq \|Tx - y_{i_0}\| < \frac{1}{n+1}.$$

Finally, we obtain

$$\forall x \in B_E, \quad \|Tx - T_n x\| < \frac{1}{n+1},$$

hence $\|T - T_n\| \leq \frac{1}{n+1}$. □

We conclude this section with a last property on finite-rank operators.

Proposition 3.4. *Let $T \in Kc(E, F)$. Then, (T) is closed if and only if T is of finite rank.*

Proof. If $\text{Im}(T)$ is closed in the Banach space F , it is also a Banach space. Then, T is a continuous and surjective operator from the Banach space E onto the Banach space

$Im(T)$. By the open mapping theorem, T is open, i.e., T maps every open set of E to an open set of F . The closed ball B_E is a neighborhood of 0 in E and T is open, so $T(B_E)$ is a neighborhood of $T(0) = 0$ in $Im(T)$. Then, there exists $r > 0$ such that

$$B_{Im(T)}(0, r) = \{y \in Im(T) \mid \|y\| < r\} \subset T(B_E).$$

Then, the closure $\overline{B_{Im(T)}(0, r)}^F$ of $B_{Im(T)}(0, r)$ in F is a closed subset of the compact set $\overline{T(B_E)}$, hence it is a compact subset of F . Moreover, we have

$$\overline{B_{Im(T)}(0, r)}^F \subset \overline{T(B_E)}^F \subset \overline{Im(T)}^F = Im(T),$$

so $\overline{B_{Im(T)}(0, r)}^F$ is a compact subset of $Im(T)$. We deduce that the unit ball of $Im(T)$ is a compact subset of $Im(T)$, which proves that $Im(T)$ is finite-dimensional by Lemma 4.1.6. The converse is evident: if T is a finite-rank operator, then $Im(T)$ is a finite-dimensional vector space and therefore is closed. $\square$

3.3 Spectrum of a compact operator

Theorem 3.4. *Let $T \in K(E)$.*

1. *If E is infinite-dimensional, then $0 \in \sigma(T)$.*
2. *$Vp(T) \setminus \{0\} = \sigma(T) \setminus \{0\}$, and for every $\lambda \in \sigma(T) \setminus \{0\}$, the corresponding eigenspace $Ker(\lambda I - T)$ is finite-dimensional.*
3. *The spectrum $\sigma(T)$ is countable. Moreover, if it is infinite, the elements of $\sigma(T) \setminus \{0\}$ form a sequence $(\lambda_n)_{n \in \mathbb{N}}$ in K such that*

$$|\lambda_{n+1}| \leq |\lambda_n| \quad \text{and} \quad \lim_{n \rightarrow +\infty} \lambda_n = 0.$$

Proof. 1. If $0 \notin \sigma(T)$, then T is invertible in $L(E)$ and $I = T^{-1}T \in K(E)$, then the identity operator on E is compact if and only if E is finite-dimensional.

2. We have $\lambda \in \sigma(T) \setminus \{0\}$ if and only if $\lambda \neq 0$ and $\lambda I - T$ is not invertible in $L(E)$, which is equivalent to $\lambda \neq 0$ and $I - \lambda^{-1}T$ not being invertible in $L(E)$, the operator $I - \lambda^{-1}T$ is not invertible if and only if it is not injective. Hence $\lambda \in \sigma(T) \setminus \{0\}$ iff

$\lambda \neq 0$ and $\lambda \in Vp(T)$. Then, $Ker(I - \lambda^{-1}T)$ is finite-dimensional, so $Ker(\lambda I - T)$ is finite-dimensional.

3. For every $\epsilon > 0$ the set $\{\lambda \in \sigma(T) \mid |\lambda| \geq \epsilon\}$ is finite. Fix $n \in \mathbb{N}^*$. Then the set $\{\lambda \in \sigma(T) \mid |\lambda| \geq 1/n\}$ is finite; denote its elements (ordered by decreasing modulus) by

$$|\lambda_0| \geq \dots \geq |\lambda_{n_0}| \geq \frac{1}{n}.$$

Similarly, the set

$$\Lambda_n = \left\{ \lambda \in \sigma(T) \mid \frac{1}{n+1} \leq |\lambda| < \frac{1}{n} \right\}$$

is finite. Write $\Lambda_n = \{\lambda_{n_0+1}, \dots, \lambda_{n_1}\}$ with

$$\frac{1}{n+1} \leq |\lambda_{n_1}| \leq \dots \leq |\lambda_{n_0+1}| < \frac{1}{n} \leq |\lambda_{n_0}| \leq \dots \leq |\lambda_0|.$$

Proceeding inductively, we can arrange all elements of $\sigma(T) \setminus \{0\}$ into a sequence $(\lambda_n)_{n \in \mathbb{N}}$ whose moduli decrease to 0.

□

Lemma 3.3. *Let $T \in K(E)$ and $\epsilon > 0$. Then the set*

$$\{\lambda \in Vp(T) \mid |\lambda| \geq \epsilon\}$$

is finite.

Proof. Assume the contrary. Then there exists a sequence $(\lambda_n)_{n \in \mathbb{N}}$ of distinct eigenvalues of T such that $|\lambda_n| \geq \epsilon$ for all $n \in \mathbb{N}$. By definition of $Vp(T)$, there exists a sequence $(e_n)_{n \in \mathbb{N}}$ in $E \setminus \{0\}$ such that

$$Te_n = \lambda_n e_n.$$

Since the eigenvalues are distinct, the family $(e_n)_{n \in \mathbb{N}}$ is linearly independent. For each $n \in \mathbb{N}$, set

$$E_n = \text{vect}\{e_0, \dots, e_n\}.$$

By Lemma 4.3.9, there exists a linearly independent family $(f_n)_{n \in \mathbb{N}}$ in E such that for all

$n \in \mathbb{N}$,

$$E_n = \text{vect}\{f_0, \dots, f_n\}, \quad (3.2)$$

$$\|f_n\| = 1, \quad d(f_{n+1}, E_n) \geq \frac{1}{2}, \quad (3.3)$$

$$(\lambda_{n+1}I - T)f_{n+1} \in E_n. \quad (3.4)$$

Let $y_n = \lambda_n^{-1}f_n \in E_n$. Since $|\lambda_n| \geq \epsilon$ and $\|f_n\| = 1$, we have $\|y_n\| \leq \epsilon^{-1}$.

Now take $p, q \in \mathbb{N}^*$ with $q > p$. From (3.3),

$$\lambda_q^{-1}(\lambda_q I - T)f_q \in E_{q-1}.$$

From (3.2) we have the inclusions

$$Ty_p \in T(E_p) \subset E_p \subset E_{q-1}.$$

Hence $Ty_p + \lambda_q^{-1}(\lambda_q I - T)f_q \in E_{q-1}$. Using (3.3) we obtain

$$\begin{aligned} \|Ty_q - Ty_p\| &= \|f_q - (Ty_p + f_q - Ty_q)\| \\ &= \|f_q - (Ty_p + f_q - \lambda_q^{-1}Ty_q)\| \\ &= \|f_q - (Ty_p + \lambda_q^{-1}(\lambda_q I - T)f_q)\| \\ &\geq d(f_q, E_{q-1}) \geq \frac{1}{2}. \end{aligned} \quad (3.5)$$

But $(y_n)_{n \in \mathbb{N}}$ is bounded (since $\|y_n\| \leq \epsilon^{-1}$), and because T is compact, the sequence $(Ty_n)_{n \in \mathbb{N}}$ admits a convergent subsequence, contradicting. This completes the proof. $\square$

3.4 Fredholm's theorems

Definition 3.3. Let $F \subset E$ be a vector subspace (not necessarily closed). The codimension of F is the dimension of the quotient space E/F . We denote $\text{codim } F = \dim(E/F)$, where E/F is the quotient vector space, an algebraic notion. When F is closed, we have the direct sum $E = F \oplus F^\perp$, hence E/F is isomorphic to $F^\perp$ and $\text{codim } F = \dim(F^\perp)$.

Definition 3.4. Let $T \in \mathcal{L}(E)$. We say that T is a Fredholm operator if the following three conditions are equivalent:

- (a) The image of T is closed,

- (b) the kernel of T is finite-dimensional,
- (c) the image of T has finite codimension.

We then denote $\text{Ind}(T) = \dim(\text{Ker}(T)) - \text{codim}(\text{Im}(T))$, and the integer $\text{I}(T)$ is called the index of T .

Example 3.4. The operator $f \mapsto f''$ is a Fredholm operator from $\mathcal{C}^2([0, 1], \mathbb{R})$ into $\mathcal{C}([0, 1], \mathbb{R})$. Clearly it is surjective and its kernel is of dimension 2 (affine functions).

Theorem 3.5 (Fredholm Alternative). Let $\mathcal{H}$ be a Hilbert space over $\mathbb{K}$ and $T \in K(\mathcal{H})$ a compact operator. Then

- i) $\text{Ker}(I_H - T)$ is finite-dimensional.
- ii) $\text{Im}(I_H - T)$ is closed.
- iii) $\text{Ker}(I_H - T) = 0$ if and only if $\text{Im}(I_H - T) = \mathcal{H}$.

Proof. i) Suppose that $\mathcal{M} = \text{Ker}(I_H - T)$ is infinite-dimensional. Since the kernel of a bounded linear operator is closed, the space $\mathcal{M}$ is then an infinite-dimensional Hilbert space, and there exists an orthonormal sequence $\{e_n\}$ in $\mathcal{M}$. It follows that $Te_n = e_n$ for all $n \in \mathbb{N}$. As $\{e_n\}$ is orthonormal, the sequence $\{e_n\}$ has no convergent subsequence. This contradicts the compactness of T . Hence $\dim \text{Ker}(I_H - T) < +\infty$.

ii) Let $\{y_n\}$ be a sequence in $\text{Im}(I_H - T)$ with $y_n \rightarrow y$ as $n \rightarrow \infty$. We must show that $y \in \text{Im}(I_H - T)$.

For each $n \in \mathbb{N}$, there exists $x_n \in \mathcal{H}$ such that $y_n = (I_H - T)x_n$. From the above, $\text{Ker}(I_H - T)$ is closed, so that $x_n = u_n + v_n$ with $u_n \in \text{Ker}(I_H - T)$ and $v_n \in \text{Ker}(I_H - T)^\perp$. We claim that the sequence $\{v_n\}$ is bounded. Otherwise, by passing to a subsequence, we may assume that $\|v_n\| \neq 0$ for all n and that $\|v_n\| \rightarrow \infty$. Let $w_n = v_n/\|v_n\|$, then $w_n \in \text{Ker}(I_H - T)^\perp$ with $\|w_n\| = 1$ for all $n \in \mathbb{N}$ and

$$(I_H - T)w_n = \frac{y_n}{\|v_n\|} \rightarrow 0 \quad \text{as } n \rightarrow \infty$$

since $\{y_n\}$ is bounded. By the compactness of T , $\{Tw_n\}$ converges (after passing to a subsequence if necessary). Thus we obtain that $\{w_n\}$ converges to an element $w \in \mathcal{H}$. Clearly $\|w\| = 1$ and

$$(I_H - T)w = \lim_{n \rightarrow \infty} (I_H - T)w_n = 0,$$

which gives $w \in \text{Ker}(\lambda I_H - T)$. However $w_n \in (\lambda I_H - T)^\perp$ implies

$$\|w - w_n\|^2 = \langle w - w_n, w - w_n \rangle = 1 + 1 = 2,$$

which contradicts $w_n \rightarrow w$ as $n \rightarrow \infty$. Hence the sequence $\{v_n\}$ is bounded.

Now, by the compactness of T we may assume that $\{Tv_n\}$ is convergent (after passing to a subsequence if necessary). From $v_n = ((I_H - T)v_n + Tv_n) = (y_n + Tv_n)$ for all $n \in \mathbb{N}$, it follows that the sequence $\{v_n\}$ converges to $v \in \mathcal{H}_r$. Then

$$y = \lim_{n \rightarrow \infty} y_n = \lim_{n \rightarrow \infty} (I_H - T)v_n = (I_H - T)v,$$

This proves that $\text{Im}(I_H - T)$ is closed.

iii) Let $V = I_H - T$ and suppose that $V^{-1}(\{0\}) = \{0\}$. Set $E_n = V^n(\mathcal{H})$ and note that

$$E_n = V^n(\mathcal{H}) = V^{n-1}(V(\mathcal{H})) \subset V^{n-1}(\mathcal{H}) = E_{n-1} \quad \forall n \geq 1.$$

From i), we know that E_n is closed for all $n \geq 1$. Suppose that $E_{n+1} \neq E_n$ for all $n \geq 1$. Then there exists a sequence $\{x_n\} \subset E_n$ with $\|x_n\| = 1$ such that

$$\|Tx_n - Tx_m\|_{\mathcal{H}} > \frac{1}{2} \quad \text{for all } n \neq m,$$

contradicting the compactness of T . Hence, $E_{n+1} = E_n$ for some $n \geq 1$. We will show that $\mathcal{H} = E_0 = E_1$. Suppose this is not true, i.e., $E_0 \neq E_1$. Let $m \geq 1$ be the smallest positive integer such that

$$E_{m-1} \neq E_m = E_{m+1}.$$

We choose $y \in E_{m-1} \setminus E_m$, then $V(y) \in E_m = E_{m+1}$. Consequently, we can find $z \in E_m$ such that

$$V(y) = V(z) \quad \text{and} \quad y \neq z$$

since $y \notin E_m = E_{m+1}$. Thus $V(y - z) = 0$ and so $y - z \in \text{Ker} V$, contradicting the hypothesis that $V^{-1}(\{0\}) = \{0\}$. Hence V is surjective.

Conversely, if $\text{Im}(I_H - T) = \mathcal{H}$, then $\{0\} = \text{Im}(I - T)^\perp = \text{Ker}(I_H - T^*)$ whence $\text{Im}(I_H - T^*) = \mathcal{H}$ and then $\text{Im}(I_H - T^*)^\perp = \{0\}$ whence $\text{Ker}(I_H - T) = \{0\}$. $\square$

Theorem 3.6 (Adjoint of Fredholm operator on Hilbert spaces). *Let H_1, H_2 be Hilbert spaces and $T \in L(H_1, H_2)$ a Fredholm operator. Then $T^* \in L(H_2, H_1)$ is Fredholm and*

$$\text{ind}(T^*) = -\text{ind}(T).$$

Proof. For Hilbert spaces, by the closed range theorem:

$$\overline{\text{ran } T} = (\text{ker } T^*)^\perp, \quad \overline{\text{ran } T^*} = (\text{ker } T)^\perp.$$

Since T Fredholm implies $\text{ran } T$ closed, we have $\text{ran } T = (\text{ker } T^*)^\perp$. Thus $\text{ker } T^* \cong H_2 / \text{ran } T = \text{coker } T$, so

$$\dim \text{ker } T^* = \dim \text{coker } T.$$

Similarly, $\text{ran } T^*$ is closed (since T^* has finite-dimensional kernel and cokernel as we will see) and $\text{coker } T^* \cong \text{ker } T$, so

$$\dim \text{coker } T^* = \dim \text{ker } T.$$

Thus

$$\begin{aligned} \text{ind}(T^*) &= \dim \text{ker } T^* - \dim \text{coker } T^* \\ &= \dim \text{coker } T - \dim \text{ker } T \\ &= -(\dim \text{ker } T - \dim \text{coker } T) \\ &= -\text{ind}(T). \end{aligned}$$

This shows T^* is Fredholm because both $\dim \text{ker } T^*$ and $\dim \text{coker } T^*$ are finite, and the range is closed. $\square$

Remark 3.3. *These properties reflect the homological nature of the index: it is additive under composition and changes sign under dualization (adjoint).*

3.5 Exercises

Exercise 1

Let $E = \mathcal{C}([a, b], \mathbb{K})$ equipped with the uniform convergence norm $\|\cdot\|_\infty$. Consider the integral operator with kernel $k \in \mathcal{C}([a, b]^2, \mathbb{K})$ defined by

$$\forall f \in E : Tf(x) = \int_a^b k(x, t)f(t) dt, \quad \forall x \in [a, b].$$

Show that $T \in \mathcal{K}(E)$ (i.e., T is compact).

Exercise 2

Let $H = l^2(\mathbb{N}, \mathbb{R})$ be the space of sequences $x = (x_n)_n$ such that $\sum_{n \geq 1} |x_n|^2 < +\infty$, equipped with the norm

$$\|x\| = \left(\sum_{n \geq 1} |x_n|^2 \right)^{\frac{1}{2}}.$$

Define the Hilbert–Schmidt matrix operator T on H by

$$T(x_n)_n = (y_n)_n, \quad \text{with } y_n = \sum_{j \geq 1} a_{nj}x_j, \quad n = 1, 2, \dots$$

where the coefficients (a_{nj}) satisfy $\sum_{n \geq 1} \sum_{j \geq 1} |a_{nj}|^2 < +\infty$.

Show that A is a compact operator on H .

Exercise 3

Let E be a Banach space and $T \in \mathcal{L}(E)$. The goal of this exercise is to show the equivalence of the following two statements:

- (i) There exists an operator $S \in \mathcal{L}(E)$ such that $I_E - TS$ and $I_E - ST$ are compact.
- (ii) $\text{Ker}(T)$ is finite-dimensional and $\text{Im}(T)$ is a closed subspace of finite codimension.

1. Assuming (i), show that $\text{Ker}(ST)$ is finite-dimensional, $\text{Im}(TS)$ has finite codimension, and deduce (ii).

2. Let $F \subset E$ be a closed subspace. Show that F admits a complement if and only if there exists $P \in \mathcal{L}(E)$ satisfying $P^2 = P$ and $\text{Im}(P) = F$.
3. Prove the implication $(ii) \implies (i)$.

Exercise 4

Let $H = L^2((0, 1), \mathbb{R})$. Define the operator T on H by

$$\forall f \in H, \forall x \in (0, 1), \quad Tf(x) = \int_0^1 e^{x+t} f(t) dt.$$

1. Show that T is a bounded operator of finite rank.
2. Determine the eigenvalues and associated eigenvectors. Deduce $\sigma(T)$ (the spectrum of T).

Exercise 5

Let H be a separable Hilbert space and $(x_k)_k$ a Hilbert basis of H . Denote the set of Hilbert–Schmidt operators from H to H by $\mathcal{L}_{HS}(H)$. Recall that a linear operator $T : H \rightarrow H$ is Hilbert–Schmidt (or simply HS) if

$$\sum_{k=1}^{\infty} \|Tx_k\|^2 < +\infty.$$

Define

$$\|T\|_{HS} = \left(\sum_{k=1}^{\infty} \|Tx_k\|^2 \right)^{\frac{1}{2}}.$$

Show that:

- (1) If $T, S \in \mathcal{L}_{HS}(H)$, then $T + S \in \mathcal{L}_{HS}(H)$.
- (2) If $T \in \mathcal{L}_{HS}(H)$, then $T^* \in \mathcal{L}_{HS}(H)$ and $\|T\|_{HS} = \|T^*\|_{HS}$.
- (3) If $T \in \mathcal{L}(H)$ and $S \in \mathcal{L}_{HS}(H)$, then $ST \in \mathcal{L}_{HS}(H)$ and $TS \in \mathcal{L}_{HS}(H)$. Moreover,

$$\max(\|TS\|_{HS}, \|ST\|_{HS}) \leq \|T\| \cdot \|S\|_{HS}.$$

(4) For all $T, S \in \mathcal{L}_{HS}(H)$, the form

$$\langle T, S \rangle = \sum_k \langle T x_k, S x_k \rangle$$

is well defined (independent of the choice of (x_k)) and defines an inner product on $\mathcal{L}_{HS}(H)$.

Exercise 6

Let H be a Hilbert space and $T: H \rightarrow H$ a continuous linear operator. Recall that $\|T\| = \sup\{\|T(x)\|; \|x\| = 1\}$.

1. Prove that $\|T\| = \sup\{\|\langle T(x), y \rangle\|; \|x\| \leq 1, \|y\| \leq 1\}$.
2. Prove that there exists a unique linear operator $T^*: H \rightarrow H$ such that the following three properties are satisfied:

- (a) $\langle T(x), y \rangle = \langle x, T^*(y) \rangle$ for all $x, y \in H$,
- (b) $\|T\| = \|T^*\|$,
- (c) $(T^*)^* = T$.

The continuous linear operator T^* is called the adjoint of T .

3. If $T = T^*$ (we say that T is self-adjoint or symmetric), prove that

$$\|T\| = \sup\{\|\langle T(x), x \rangle\|; \|x\| = 1\}.$$

Exercise 7

Let $K: \mathbb{R}^2 \rightarrow \mathbb{R}$ be a measurable function. Whenever for every $f \in L^2(\mathbb{R})$ the function

$$T(f) = \int_{\mathbb{R}} K(\cdot, y) f(y) dy$$

is well defined in $L^2(\mathbb{R})$, we say that $T: L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R})$ is an integral operator and that K is its kernel.

If $K \in L^2(\mathbb{R}^2)$, we say that T is a Hilbert–Schmidt operator. In the following, we consider a Hilbert–Schmidt operator T with kernel K .

1. If $f \in L^2(\mathbb{R})$, show that for almost every $x \in \mathbb{R}$ the map $y \mapsto K(x, y)f(y)$ is integrable.
2. Show that the linear operator T is continuous and that $\|T\| \leq \|K\|_{L^2(\mathbb{R}^2)}$.
3. Prove that the adjoint T^* of T is an integral operator with kernel $(x, y) \mapsto \overline{K(y, x)}$.

Exercise 8

Let X be a Banach space, $A \in \mathcal{L}(X)$, and $\lambda \in \mathbb{C}$ an element of the spectrum of A that is **not** an eigenvalue. Show that for every compact operator $K \in \mathcal{K}(X)$, λ belongs to the spectrum of $A + K$.

Exercise 9

1. Consider the operator

$$T : \ell^1(\mathbb{N}) \rightarrow \ell^1(\mathbb{N}), \quad (u_0, u_1, \dots) \mapsto (u_1, u_2, \dots).$$

Determine the spectrum of T . Do T and its adjoint T^* have the same spectrum? The same eigenvalues? 2. Let E be the Banach space of bounded continuous functions $f : \mathbb{R} \rightarrow \mathbb{C}$ with the sup-norm. Define $T : E \rightarrow E$ by $T(f)(x) = f(x + 1)$. Determine the spectrum of T .

Exercise 10

Show that the unbounded operator T on $L^2(0, 1)$, taking values in $\mathbb{C}$, whose domain is $C([0, 1])$ and which is defined by $Tf = f(1/2)$, is not closable.

More generally, show that a linear form that defines a closable operator on a Banach space is bounded.

Exercise 11

Consider an arbitrary sequence $(\alpha_n)_{n \in \mathbb{N}}$ of scalars, and the vector space

$$V = \left\{ (x_n)_{n \in \mathbb{N}} \in \mathbb{C}^{\mathbb{N}} : \sum_n (1 + |\alpha_n|^2) |x_n|^2 < +\infty \right\}.$$

For $x = (x_n)_{n \in \mathbb{N}} \in V$, define $Tx = (\alpha_n x_n)_{n \in \mathbb{N}}$; show that T is a closed, densely defined operator on $\ell^2(\mathbb{N})$; determine its adjoint and its spectrum.

Exercise 12

If $T_1 \in L(X, Y)$ and $T_2 \in L(Y, Z)$ are two Fredholm operators, show that $T_2 \circ T_1$ is Fredholm from X to Z , and that

$$\text{ind}(T_2 \circ T_1) = \text{ind}(T_2) + \text{ind}(T_1).$$

If T is Fredholm between two Hilbert spaces, show that its adjoint T^* is Fredholm, with $\text{ind}(T^*) = -\text{ind}(T)$.

3.6 Solutions

Solution of Exercise 1

Let $E = \mathcal{C}([a, b], \mathbb{K})$ with the uniform norm $\|\cdot\|_{\infty}$. Define

$$Tf(x) = \int_a^b k(x, t) f(t) dt,$$

with $k \in \mathcal{C}([a, b]^2, \mathbb{K})$.

Since $[a, b]$ is compact, k is uniformly continuous. Hence, for any bounded sequence (f_n) in E , (Tf_n) is equicontinuous and uniformly bounded. By the Arzelà–Ascoli theorem, (Tf_n) has a uniformly convergent subsequence, so T maps bounded sets to relatively compact sets. Thus T is compact.

Solution of Exercise 2

Let $H = \ell^2(\mathbb{N}, \mathbb{R})$ and define

$$y_n = \sum_{j \geq 1} a_{nj} x_j,$$

with $\sum_{n,j} |a_{nj}|^2 < \infty$.

Define finite-rank operators T_N by

$$(T_N x)_n = \sum_{j=1}^N a_{nj} x_j, \quad n = 1, \dots, N,$$

and zero otherwise. Each T_N is compact (finite rank). One checks that

$$\|T - T_N\|^2 \leq \sum_{n > N} \sum_{j \geq 1} |a_{nj}|^2 \rightarrow 0$$

as $N \rightarrow \infty$. Since the set of compact operators is closed in the operator norm, T is compact.

Solution of Exercise 3

Let E be a Banach space, $T \in \mathcal{L}(E)$.

1. Assume (i): $I_E - TS$ and $I_E - ST$ compact.

Then $ST = I - (I - ST)$ is a Fredholm operator, so $\text{Ker}(ST)$ finite-dimensional. Similarly, TS is Fredholm, so $\text{Im}(TS)$ has finite codimension. Since $\text{Ker}(T) \subset \text{Ker}(ST)$ and $\text{Im}(T) \supset \text{Im}(TS)$, we deduce (ii).

2. Closed subspace $F \subset E$ has a complement iff there exists a bounded projection P with $\text{Im}(P) = F$.

3. Assume (ii): $\dim \text{Ker}(T) < \infty$, $\text{Im}(T)$ closed with finite codimension.

Then T is Fredholm. There exist $S \in \mathcal{L}(E)$ such that $I - TS$ and $I - ST$ are finite-rank, hence compact. Thus (i) holds.

Solution of Exercise 4

Let $H = L^2((0, 1), \mathbb{R})$,

$$Tf(x) = \int_0^1 e^{x+t} f(t) dt.$$

1. Write $Tf(x) = e^x \int_0^1 e^t f(t) dt$. Hence $Tf = \langle f, e \rangle e^x$, so T is rank one, hence bounded and compact.

2. For eigenvalue $\lambda \neq 0$, solve $Tf = \lambda f$:

$$e^x \int_0^1 e^t f(t) dt = \lambda f(x).$$

Thus $f(x) = Ce^x$ with $C = \frac{1}{\lambda} \int_0^1 e^{2t} C dt$. That gives $\lambda = \int_0^1 e^{2t} dt = \frac{e^2-1}{2}$. For $\lambda = 0$, kernel is all f with $\int_0^1 e^t f(t) dt = 0$. So $\sigma(T) = \{0, \frac{e^2-1}{2}\}$.

Solution of Exercise 5

Let H separable Hilbert space, (x_k) orthonormal basis. T is Hilbert–Schmidt if

$$\|T\|_{HS}^2 = \sum_k \|Tx_k\|^2 < \infty.$$

(1) $T, S \in \mathcal{L}_{HS}(H)$:

$$\|T + S\|_{HS} \leq \|T\|_{HS} + \|S\|_{HS} < \infty.$$

(2) $\|T^*\|_{HS}^2 = \sum_k \|T^*x_k\|^2 = \sum_k \|Tx_k\|^2 = \|T\|_{HS}^2$.

(3) For $T \in \mathcal{L}(H)$, $S \in \mathcal{L}_{HS}(H)$:

$$\|ST\|_{HS}^2 = \sum_k \|STx_k\|^2 \leq \|S\|_{HS}^2 \|T\|^2.$$

Similarly for TS .

(4) $\langle T, S \rangle = \sum_k \langle Tx_k, Sx_k \rangle$ is independent of basis choice and defines an inner product.

Solution of Exercise 6

$T : H \rightarrow H$ continuous linear.

1.

$$\|T\| = \sup_{\|x\|=1} \|Tx\| = \sup_{\|x\|=1, \|y\|=1} |\langle Tx, y \rangle|.$$

2. For each y , map $x \mapsto \langle Tx, y \rangle$ is continuous linear functional. By Riesz, exists unique T^*y such that $\langle Tx, y \rangle = \langle x, T^*y \rangle$. T^* is linear, bounded, $\|T^*\| = \|T\|$, $(T^*)^* = T$.

3. If $T = T^*$, then

$$\|T\| = \sup_{\|x\|=1} |\langle Tx, x \rangle|.$$

Solution of Exercise 7

$K \in L^2(\mathbb{R}^2)$, $Tf(x) = \int_{\mathbb{R}} K(x, y)f(y) dy$.

1. For a.e. x , $y \mapsto K(x, y) \in L^2(\mathbb{R})$. By Cauchy–Schwarz,

$$\int |K(x, y)f(y)| dy \leq \|K(x, \cdot)\|_{L^2} \|f\|_{L^2} < \infty.$$

2. By Minkowski integral inequality,

$$\|Tf\|_{L^2} \leq \|K\|_{L^2(\mathbb{R}^2)} \|f\|_{L^2}.$$

Thus $\|T\| \leq \|K\|_{L^2}$.

3. For $f, g \in L^2$,

$$\langle Tf, g \rangle = \iint K(x, y)f(y)\overline{g(x)} dydx.$$

Swap x and y to see

$$\langle f, T^*g \rangle = \iint \overline{K(y, x)}f(y)\overline{g(x)} dydx.$$

Thus T^* has kernel $\overline{K(y, x)}$.

Solution of Exercise 8

Suppose, for contradiction, that $\lambda \notin \sigma(A + K)$. Then $(A + K - \lambda I)$ is invertible. Write

$$A - \lambda I = (A + K - \lambda I) - K.$$

Since $(A + K - \lambda I)$ is invertible and K is compact, the operator

$$I - (A + K - \lambda I)^{-1}K$$

is of the form $I - \text{compact}$. If it is invertible, then $A - \lambda I$ is invertible (by the Fredholm alternative), contradicting $\lambda \in \sigma(A)$. If it is not invertible, then 1 is an eigenvalue of the compact operator $(A + K - \lambda I)^{-1}K$, which implies that λ is an eigenvalue of A (by a standard argument), contradicting the hypothesis. Hence $\lambda \in \sigma(A + K)$.

Solution of Exercise 9

1. The operator T is the (unweighted) backward shift on ℓ^1 . Its norm is $\|T\| = 1$, so $\sigma(T) \subset \overline{\mathbb{D}}$ (the closed unit disk). In fact, one can show that $\sigma(T) = \overline{\mathbb{D}}$: for $|\lambda| < 1$, the

vector $(1, \lambda, \lambda^2, \dots)$ is an eigenvector in ℓ^1 with eigenvalue λ . For $|\lambda| = 1$, $T - \lambda I$ is not surjective (by a standard argument). Hence $\sigma(T) = \overline{\mathbb{D}}$.

The adjoint T^* acts on ℓ^∞ by $T^*(v_0, v_1, \dots) = (0, v_0, v_1, \dots)$. Its spectrum is also $\overline{\mathbb{D}}$, but its eigenvalues are exactly the open disk $\{|\lambda| < 1\}$ (the same as T). However, T^* also has points in $\overline{\mathbb{D}}$ that are not eigenvalues but belong to the residual or continuous spectrum. 2. First, T is invertible with inverse $T^{-1}(f)(x) = f(x - 1)$, and both T and T^{-1} have norm 1. Hence $\sigma(T) \subset \mathbb{S}^1$ (the unit circle). In fact, $\sigma(T) = \mathbb{S}^1$. Indeed, for $\lambda = e^{i\theta}$, the function $f(x) = e^{i\theta x}$ satisfies $Tf = e^{i\theta}f$, but it is not in E because it is not bounded unless θ is real (which it is). Wait: $e^{i\theta x}$ is bounded for $\theta \in \mathbb{R}$. Actually, it is bounded only if θ is real, and then its modulus is 1, so it belongs to E . Thus every $\lambda = e^{i\theta}$ is an eigenvalue. Conversely, if $|\lambda| \neq 1$, then $T - \lambda I$ is invertible (since $\|T\| = 1$, the Neumann series works for $|\lambda| > 1$, and for $|\lambda| < 1$ one uses the inverse operator explicitly). Hence $\sigma(T) = \mathbb{S}^1$.

Solution of Exercise 10

Part 1: The operator T . Domain: $D(T) = C([0, 1]) \subset L^2(0, 1)$, $T : D(T) \rightarrow \mathbb{C}$ is $Tf = f(1/2)$.

Suppose T were closable. Let $(f_n) \subset D(T)$ with $f_n \rightarrow 0$ in L^2 and $Tf_n = f_n(1/2) \rightarrow \ell$ in $\mathbb{C}$ for some ℓ . For closability we would need $\ell = 0$.

We construct two sequences with the same L^2 -limit 0 but different limits of Tf_n :

First sequence: $f_n(x) = 0$ for all n , then $f_n \rightarrow 0$, $Tf_n = 0$.

Second sequence: Let g_n be a continuous "spike" at $x = 1/2$:

$$g_n(x) = \begin{cases} n(1 - n|x - 1/2|), & |x - 1/2| \leq 1/n, \\ 0, & \text{otherwise.} \end{cases}$$

Then $g_n(1/2) = n$ and $\|g_n\|_{L^2}^2 \approx \int_{-1/n}^{1/n} [n(1 - n|t|)]^2 dt$ (substitute $t = x - 1/2$). This is

$$\|g_n\|_2^2 = 2n^2 \int_0^{1/n} (1 - nt)^2 dt = 2n^2 \cdot \frac{1}{3n} = \frac{2}{3n} \rightarrow 0.$$

Thus $g_n \rightarrow 0$ in L^2 , but $Tg_n = g_n(1/2) = n \rightarrow +\infty$. We can adjust to get finite limit: e.g., take $h_n = \frac{1}{n}g_n$: then $h_n \rightarrow 0$ in L^2 (since $\|h_n\|_2 = \frac{1}{n}\|g_n\|_2 \sim \frac{1}{n\sqrt{n}} \rightarrow 0$), but $Th_n = 1$ for all n .

So $h_n \rightarrow 0$ in L^2 but $Th_n \rightarrow 1 \neq 0$. This shows T is not closable.

Part 2: General statement. Let X be a Banach space, $D \subset X$ a dense subspace, $F : D \rightarrow \mathbb{C}$ a linear functional, viewed as an operator $F : D \subset X \rightarrow \mathbb{C}$.

Suppose F is closable. Let $x_n \rightarrow 0$ in X , $x_n \in D$, and $Fx_n \rightarrow \alpha$ in $\mathbb{C}$. Since $0 \in D$ and $F(0) = 0$, closability implies $\alpha = 0$.

Thus: For any sequence $x_n \in D$ with $x_n \rightarrow 0$, if (Fx_n) converges, then its limit is 0.

We now show F is bounded. Suppose not: then $\exists y_n \in D$ with $\|y_n\| = 1$ and $|Fy_n| \rightarrow \infty$. Let $t_n = 1/\sqrt{|Fy_n|} \rightarrow 0$ (if $Fy_n = 0$, modify slightly; generically $Fy_n \neq 0$ for large n). Set $x_n = t_n y_n$, so $\|x_n\| = t_n \rightarrow 0$, so $x_n \rightarrow 0$ in X . But $Fx_n = t_n Fy_n = \sqrt{|Fy_n|} \rightarrow \infty$, so (Fx_n) diverges, not convergent. That alone does not contradict yet, but we can force convergence:

Pick subsequence such that Fy_{n_k} all have same sign and $|Fy_{n_k}| \rightarrow \infty$. Choose $s_{n_k} = 1/|Fy_{n_k}|$. Then $s_{n_k} y_{n_k} \rightarrow 0$, $F(s_{n_k} y_{n_k}) = \pm 1$, so $F(s_{n_k} y_{n_k}) \rightarrow \pm 1 \neq 0$, contradiction to closability.

Thus F must be bounded on D . Since D is dense, F extends uniquely to a bounded linear functional on X . Hence any closable everywhere-defined linear form on a dense subspace of a Banach space is bounded.

Solution of Exercise 11

Consider an arbitrary sequence $(\alpha_n)_{n \in \mathbb{N}}$ of scalars, and the vector space

$$V = \left\{ (x_n)_{n \in \mathbb{N}} \in \mathbb{C}^{\mathbb{N}} : \sum_n (1 + |\alpha_n|^2) |x_n|^2 < +\infty \right\}.$$

For $x = (x_n)_{n \in \mathbb{N}} \in V$, define $Tx = (\alpha_n x_n)_{n \in \mathbb{N}}$; show that T is a closed, densely defined operator on $\ell^2(\mathbb{N})$; determine its adjoint and its spectrum.

We work on $H = \ell^2(\mathbb{N})$ with standard inner product $\langle x, y \rangle = \sum_n x_n \overline{y_n}$.

Domain $D(T) = V = \{x \in \ell^2 : \sum_n |\alpha_n|^2 |x_n|^2 < \infty\}$, i.e., $(\alpha_n x_n) \in \ell^2$.

Density: Let $e_k = (\delta_{nk})_{n \geq 0}$; each $e_k \in D(T)$ since $|\alpha_k|^2 |1|^2 < \infty$. Finite linear combinations of e_k (i.e., sequences with finitely many nonzero terms) are in $D(T)$ and are dense in ℓ^2 . So $D(T)$ is dense.

Closedness: Suppose $x^{(m)} \in D(T)$, $x^{(m)} \rightarrow x$ in ℓ^2 , and $Tx^{(m)} \rightarrow y$ in ℓ^2 . We must show $x \in D(T)$ and $Tx = y$.

Convergence in ℓ^2 implies componentwise convergence: for each n , $x_n^{(m)} \rightarrow x_n$ as $m \rightarrow \infty$, and $(Tx^{(m)})_n = \alpha_n x_n^{(m)} \rightarrow y_n$. So $\alpha_n x_n = y_n$ for each n . Hence $y = (\alpha_n x_n)_n$, so $x \in D(T)$ and $Tx = y$. So T is closed.

Adjoint: For $x \in D(T)$, $y \in D(T^*)$, we need $\langle Tx, y \rangle = \langle x, T^*y \rangle$ for all $x \in D(T)$.

$\langle Tx, y \rangle = \sum_n \alpha_n x_n \overline{y_n} = \sum_n x_n \overline{\alpha_n y_n}$ when the series converges absolutely for all $x \in D(T)$.

This equals $\langle x, z \rangle$ with $z_n = \overline{\alpha_n y_n}$ provided $z \in \ell^2$ and y is such that $\sum_n |x_n| |\alpha_n| |y_n| < \infty$ for all $x \in D(T)$. The largest y for which this holds for all $x \in D(T)$ is those with $(\overline{\alpha_n y_n}) \in \ell^2$, i.e., $\sum_n |\alpha_n|^2 |y_n|^2 < \infty$, so $y \in D(T)$ and then $T^*y = (\overline{\alpha_n y_n})$.

Thus $D(T^*) = D(T)$ and $T^*y = (\overline{\alpha_n y_n})$.

Spectrum: Since T is a multiplication operator on its domain, we analyze $\sigma(T)$.

Point spectrum $\sigma_p(T)$: Solve $(T - \lambda I)x = 0$, $x \in D(T)$, $x \neq 0$: $(\alpha_n - \lambda)x_n = 0$ for all n . If $\lambda = \alpha_k$ for some k , choose $x = e_k$; $e_k \in D(T)$, so $\alpha_k \in \sigma_p(T)$. Thus $\{\alpha_n : n \in \mathbb{N}\} \subset \sigma_p(T)$.

Now, is every $\lambda \notin \{\alpha_n\}$ in $\rho(T)$? We check if $T - \lambda I$ is bijective from $D(T)$ to ℓ^2 with bounded inverse.

For $\lambda \neq \alpha_n$, $(T - \lambda I)^{-1}$ would be multiplication by $1/(\alpha_n - \lambda)$ on its range. But we need this inverse defined on all ℓ^2 and bounded, with range exactly $D(T)$.

Given $y \in \ell^2$, define $x_n = y_n/(\alpha_n - \lambda)$. We need $x \in D(T)$: (1) $x \in \ell^2$ — requires $\sup_n 1/|\alpha_n - \lambda| < \infty$, i.e., $|\alpha_n - \lambda| \geq \delta > 0$ for all n . (2) $(\alpha_n x_n) \in \ell^2$ automatically if $x \in \ell^2$ and α_n bounded? Not necessarily: if α_n unbounded, need also $\alpha_n/(\alpha_n - \lambda)$ bounded, i.e., $|\alpha_n|/|\alpha_n - \lambda| \leq M$. This fails if λ is a limit point of α_n from "outside" such that ratios blow up.

Thus $\lambda \in \sigma(T)$ if either:

- $\lambda \in \{\alpha_n\}$ (point spectrum).
- $\lambda \notin \{\alpha_n\}$ but $\inf_n |\alpha_n - \lambda| = 0$ (then inverse exists but not bounded, so $\lambda \in \sigma_c(T)$), or inverse exists, bounded, but range not dense? Actually, if $\inf_n |\alpha_n - \lambda| > 0$, then $1/(\alpha_n - \lambda)$ bounded, so $R(T - \lambda I) = \ell^2$ and bounded inverse, so $\lambda \in \rho(T)$. Wait:

If $\inf_n |\alpha_n - \lambda| > 0$, then $1/(\alpha_n - \lambda)$ bounded; then $x_n = y_n/(\alpha_n - \lambda)$ is in ℓ^2 for $y \in \ell^2$, but is $(\alpha_n x_n) \in \ell^2$? $(\alpha_n x_n) = (\alpha_n y_n/(\alpha_n - \lambda)) = y_n + \lambda x_n$, so clearly in ℓ^2 since $y \in \ell^2$ and $x \in \ell^2$. So indeed $x \in D(T)$. So $\lambda \in \rho(T)$.

Thus $\sigma(T) = \overline{\{\alpha_n\}}$, the closure in $\mathbb{C}$. $\sigma_p(T) = \{\alpha_n\}$, $\sigma_c(T) = \overline{\{\alpha_n\}} \setminus \{\alpha_n\}$ (limit points not in the sequence), no residual spectrum because T is normal on its domain (and T^* has same domain).

Solution of Exercise 12

A bounded linear operator T is Fredholm if $\dim \ker T < \infty$, $\text{coker } T = Z/\text{ran } T$ satisfies $\dim \text{coker } T < \infty$, and $\text{ran } T$ is closed. Its index is

$$\text{ind}(T) = \dim \ker T - \dim \text{coker } T.$$

Step 1: Kernel of $T_2 \circ T_1$.

We have $\ker(T_2 \circ T_1) = T_1^{-1}(\ker T_2)$. Restrict T_1 to $\tilde{T}_1 : \ker(T_2 \circ T_1) \rightarrow \ker T_2$, which is surjective by definition. Its kernel is $\ker T_1$. The exact sequence of vector spaces

$$0 \rightarrow \ker T_1 \hookrightarrow \ker(T_2 \circ T_1) \xrightarrow{\tilde{T}_1} \ker T_2 \rightarrow 0$$

implies

$$\dim \ker(T_2 \circ T_1) = \dim \ker T_1 + \dim \ker T_2.$$

Step 2: Cokernel of $T_2 \circ T_1$.

Consider the restriction $\hat{T}_2 : Y/\text{ran } T_1 \rightarrow Z/\text{ran}(T_2 \circ T_1)$ induced by T_2 . One checks this is well-defined and surjective, and its kernel is $(\ker T_2 + \text{ran } T_1)/\text{ran } T_1 \cong \ker T_2/(\ker T_2 \cap \text{ran } T_1)$.

Also consider $\hat{T}_1 : \text{coker } T_1 \rightarrow (\ker T_2 + \text{ran } T_1)/\text{ran } T_1$ by $y + \text{ran } T_1 \mapsto y + \text{ran } T_1$ (for $y \in \ker T_2$), which is surjective with kernel $\text{ran } T_1 \cap \ker T_2/\text{ran } T_1 = 0$, so actually

$$\text{coker } T_1 \cong (\ker T_2 + \text{ran } T_1)/\text{ran } T_1.$$

Then using $\hat{T}_2$ above, we get exact sequence

$$0 \rightarrow \text{coker } T_1 \rightarrow \text{coker}(T_2 \circ T_1) \rightarrow \text{coker } T_2 \rightarrow 0,$$

where the last map comes from $Z/\text{ran}(T_2 \circ T_1) \rightarrow Z/\text{ran } T_2$ being surjective with kernel $\text{ran } T_2/\text{ran}(T_2 \circ T_1) \cong \text{coker } T_1$ (by the isomorphisms above).

Hence

$$\dim \text{coker}(T_2 \circ T_1) = \dim \text{coker } T_1 + \dim \text{coker } T_2.$$

Step 3: Closed range.

Since T_1 and T_2 have closed range, one can show $T_2 \circ T_1$ has closed range using the finite dimensionality of kernels and cokernels in previous steps and the closed range theorem applied suitably.

Step 4: Index formula.

From Steps 1 and 2,

$$\begin{aligned} \text{ind}(T_2 \circ T_1) &= \dim \ker(T_2 \circ T_1) - \dim \text{coker}(T_2 \circ T_1) \\ &= [\dim \ker T_1 + \dim \ker T_2] - [\dim \text{coker } T_1 + \dim \text{coker } T_2] \\ &= (\dim \ker T_1 - \dim \text{coker } T_1) + (\dim \ker T_2 - \dim \text{coker } T_2) \\ &= \text{ind}(T_1) + \text{ind}(T_2). \end{aligned}$$

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