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بتاريخ يوم الثلاثاء 03 فيفري 2026 على الساعة 09:00 صباحا انعقد
المجلس العلمي للكلية بقاعة الاجتماعات للكلية بحضور السادة والسيدات أعضاء المجلس لدراسة
جدول الأعمال الذي تم فيه:

- تحديد لجنة لتحكيم المطبوعة العلمية المعدة باللغة الانجليزية من طرف الأستاذة:

د. بن فريحة نجة

بمعنوان: Detailed lessons in mathematics 1

الموجهة لطلبة: السنة الأولى جذع مشترك

وبعد الردود الإيجابية للجنة التحكيم على المطبوعة قرر المجلس اعتماد المطبوعة العلمية.

رفعت الجلسة على الساعة 14:30 مساء من نفس اليوم ونفس التاريخ.

رئيس المجلس العلمي



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People's Democratic Republic of Algeria
Ministry of Higher Education and Scientific Research
Djilali Bou Naama University – Khemis Miliana



FACULTY OF ECONOMIC, COMMERCIAL, AND MANAGEMENT SCIENCES



Course handout:

**DETAILED LESSONS IN MATHEMATICS 1
FOR THE FIRST YEAR OF THE COMMON CORE**

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Academic year: 2025–2026

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INTRODUCTION

Mathematics 1 is one of the methodological education units offered in the LMD system. It is taught in the first semester of the first year of the common core for the Faculty of Economic Sciences, Business, and Management. Mathematics is a set of abstract knowledge resulting from logical deductions applied to sets, numbers, shapes, data, and transformations. It aims to study quantitative topics, helping us understand, compare, and comprehend the world around us, and develop the ability to think soundly and logically. Mathematics participates in many fields such as technology and other sciences, thereby assisting researchers and students in finding solutions to problems and making sound decisions.

Mathematics helps provide logical solutions and address daily issues such as project planning, project management, budgeting, and even managing discussions.

The overall objective of the course is to enable the student to understand mathematics and apply it in various administrative, financial, economic fields, and other related measures to the specialization. Under this objective, there are a set of skill goals to be achieved, which can be clarified as follows:

- Introducing the concept of factorial and how to calculate it so that the student can use it in Statistics 1, 2, and 3.
- Enabling the student to use the laws of sequences and apply them in the economic aspect in examples of interest-bearing savings.
- Employing the fundamentals of mathematical analysis in fields related to the specialization of economic sciences, such as statistics, operations research, financial mathematics, etc. etc.
- Distinguishing between the characteristics of functions and their economic applications.

The module contains five main chapters, each of which is part of a pedagogical sequence that allows students to understand and assimilate the basic concepts covered in the lecture. Through guided work sessions, learning lessons are reinforced with a series of exercises that help students better understand and absorb concepts, in addition to applying theory to practice and drawing conclusions. These chapters are as follows:

- **First chapter: Combinatorial analysis**
- **Second chapter: Generalities of numerical sequences**
- **Third chapter: Logarithmic and exponential functions.**
- **Fourth chapter: Derivatives.**
- **Fifth chapter: Original functions and integration**

CHAPTER ONE: COMBINATORIAL ANALYSIS

CHAPTER ONE: COMBINATORIAL ANALYSIS

Combinatorial analysis is only a method or technique that helps to count and limit the possible results of a random variable and an accident in case they cannot be found by direct counting, so Combinatorial analysis is used in order to facilitate the calculation of the number of elements of a set if they cannot be found directly.

And the analysis of the combinatorial is concerned with giving the number of possible ways for the groups under certain conditions through some mathematical rules that facilitate this formation on the one hand and enables the study of the finite groups by simplifying the counting in them and deducing more effective ways, more effective ways, the number of suitable cases and the number of possible cases associated with that incident on the other hand, and thus the calculation of probabilities becomes one of the most important practical applications of the combinatorial analysis, so the necessity of presenting those mathematical rules to benefit from them when addressing the topic of probability and its laws becomes clear.

In this chapter we will try to cover the following topics:

permutation

Arrangements

Combinations

CHAPTER ONE: COMBINATORIAL ANALYSIS

1. FACTORIALS :

The sequences of consecutive numbers in products are: the factorials.

EXAMPLE

$$3! = 3 \times 2 \times 1 = 6$$

$$5! = 5 \times 4 \times 3 \times 2 \times 1 = 120$$

Definition:

Let n be a positive integer. We define " n factorial" as:

$$n! = 1 \times 2 \times 3 \times \dots \times (n-1)n \quad \text{if } n > 0$$
$$0! = 1$$

EXAMPLE

$$\frac{15!}{12!} = \frac{15 \times 14 \times 13 \times 12!}{12!} = 15 \times 14 \times 13 = 2730$$

Practical exercise1:

Calculate the following:

$$(4 \times 3)!, \quad \frac{20!}{3! \times 5! \times 2!}, \quad \frac{300!}{3! \times 297!}, \quad 4! \times 3!, \quad (4+3)!, \quad 4! + 3!$$



$$\frac{20!}{3! \times 5! \times 2!} = 1.69 \times 10^{15}$$
$$\frac{300!}{3! \times 297!} = 4455100$$
$$(4 \times 3)! = 12! = 479\,001\,600$$
$$4! \times 3! = 144$$
$$(4+3)! = 7! = 5040$$
$$4! + 3! = 24 + 6 = 30$$

Practical exercise2 :

Simplify the expressions below (n is a positive integer):

$$\frac{n!}{(n-1)!}, \quad \frac{n!}{(n-2)!}, \quad \frac{(n+1)!}{n!}, \quad \frac{(n+1)!}{(n-1)!}$$

CHAPTER ONE: COMBINATORIAL ANALYSIS

Solution :



$$\begin{aligned}\frac{n!}{(n-1)!} &= \frac{n(n-1)!}{(n-1)!} = n \\ \frac{n!}{(n-2)!} &= \frac{n(n-1)(n-2)!}{(n-2)!} = n(n-1) = n^2 - n \\ \frac{(n+1)!}{n!} &= \frac{(n+1)n!}{n!} = n+1 \\ \frac{(n+1)!}{(n-1)!} &= \frac{(n+1) \times n \times (n-1)!}{(n-1)!} = (n+1)n = n^2 + n\end{aligned}$$

2.THE COUNTING METHODS:



The basic law of counting allows us to calculate the number of possible outcomes only and does not allow us to determine them. Therefore, we resort to a tree diagram to determine the elements of possible outcomes. This diagram is a graphical representation of the. It shows each stage and its possible outcomes through branches, with each branch of the trees having only one outcome of each stage. All the outcomes in the branch are the final outcome of the random experiment

3.PERMUTATION AND ARRANGEMENTS:

There is another counting rule that is sometimes useful, which is the method of calculating arrangements. It allows us to calculate the number of possible outcomes when selecting n elements from N elements ($n \leq N$), taking into account the order of the elements drawn. If the number of elements drawn n is equal to N , i.e., we draw all the elements, then we have a system consisting of one element. If we arrange it in a certain order, we get a permutation. and each time we swap the elements, we get another permutation different from the previous one. The first system is called the permutation of n elements taken from N elements, in which we have selected some but not all of the elements ($n < N$). If we draw all the elements, the system in this case is called the permutation of n elements ($n = N$).



If we have the following letters: A, B, C, D, how many different words can be formed from these letters without repeating them (these words do not have to have meaning)? **Name three of them.**

CHAPTER ONE: COMBINATORIAL ANALYSIS

Solution:

Each word consists of four letters, and since letters cannot be repeated, there are four possible cases for choosing the first letter, three possible cases for choosing the second letter, two cases for the third letter, and one case for the last letter. Therefore, the number of possible words (the number of permutations of four elements) is equal to:

$$P_4 = 4! = 4 \times 3 \times 2 \times 1 = 24$$

There are 24 permutations of four letters, three of which are mentioned here as examples:

A B C D, B C D A, C D A B.



Each permutation consists of the same elements, i.e., the letters A, B, C, and D. The difference between one permutation and another lies in the order of these elements.



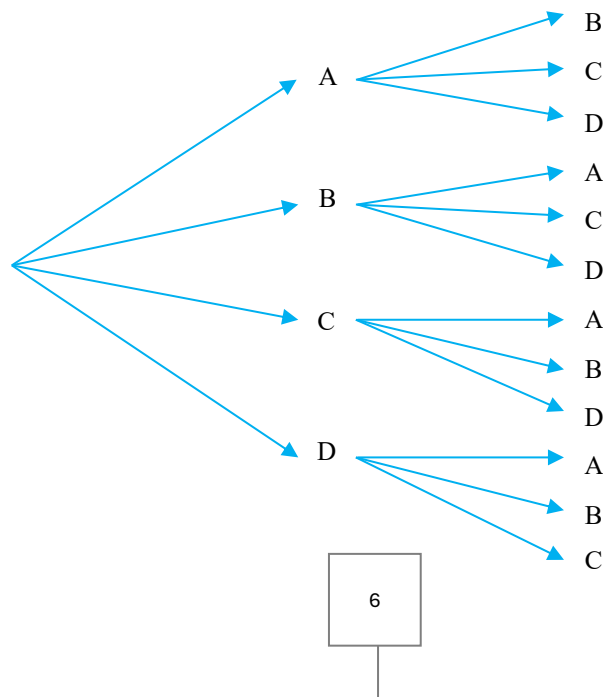
If we have the same letters as before: A, B, C, D, how many different two-letter words can be formed without repeating letters (these words do not have to make sense)? **List all these words.**

Solution:

Each word consists of two letters taken from four letters in a arrangement. There are four possible cases for choosing the first letter and three cases for the second letter, resulting in the number of possible words (number of arrangement):

$$A_4^2 = 4 \times 3 = 12$$

All possible cases can be found using a tree diagram:



CHAPTER ONE: COMBINATORIAL ANALYSIS

The tree diagram shows the arrangement of two of the four elements: The set of these words is:

(A B, A C, A D, B A, B C, B D, C A, C B, C D, D A, D B, D C)



Each arrangement differs from the other either in terms of elements or order. The result (A B) differs from (B A) in order, while the result (A B) differs from (B D) in terms of elements.

Theory:

The number of permutations (P_n) of n elements is equal to:

$$P_n = n \times (n - 1) \times (n - 2) \times \dots \times 1 = n!$$



Three students, Ahmed, Muhammad, and Mahmoud, are nominated for a competition for a certain position. Calculate the number of possible permutations for these students if we arrange them according to their exam scores, and list all possible results.

Solution:

Based on the above data, there are three ranks: The first, second, and third ranks. The first rank has three possible cases: Ahmed, Muhammad, or Mahmoud. If we place one of them in the first rank, only two cases remain for the second rank. As for the third and last rank, it has only one possible case after placing the first and second ranks. Therefore, the number of possible permutations for three elements is:

$$P_n = n \times (n - 1) \times (n - 2) \times \dots \times 1 = n! \Rightarrow P_3 = 3 \times 2 \times 1 = 6$$

3.1. permutations with repetition:

If we have set A containing n such that element i is repeated n_i times and we have: $\sum_{i=1}^r n_i = n$, then the number of possible permutations is equal to:

$$P_n^{n_1, n_2, \dots, n_r} = \frac{n!}{n_1! \times n_2! \times \dots \times n_r!}$$



How many words can be formed from the word « Tomat »?

The letter “t” is repeated twice, and the letter “o”, “m”, “a” is repeated once so the number of substitutions is equal to:

$$P_5^{2,1,1,1} = \frac{5!}{2! \times 1! \times 1! \times 1!} = 60$$

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3.2. Arrangement with repetition:

The number of samples that can be formed by repeated drawing is equal to (n^r) . The first individual has n chances of being drawn, and the same applies to the second individual because we have returned the first individual to the population. Thus, individuals are drawn one after another until we reach the r^{th} element of the sample, which has the same possibilities as the previous elements. Therefore, the number of samples formed in this way is equal to:

$$n^r = \underbrace{n \times n \times \dots \times n}_r$$

3.3. Arrangement without repetition:

If we want to form a system of K elements taken from n elements such that $(k < n)$, we will choose the first element of this system from among n elements. Therefore, there are n possible cases for choosing the first element. If we fix the first element, there are only $(n - 1)$ possible cases left for choosing the second element, and so on until we reach the k^{th} element, which has $(n - k + 1)$ possibilities for selection. Therefore, the number of arrangements is equal to:

$$\begin{aligned} A_n^k &= n \times (n - 1) \times (n - 2) \times \dots \times (n - k + 1) \\ &= n \times (n - 1) \times (n - 2) \times \dots \times (n - k + 1) \left[\frac{(n - k + 1) \times (n - k) \times \dots \times 1}{(n - k + 1) \times (n - k) \times \dots \times 1} \right] \end{aligned}$$

Therefore:

$$A_n^k = \frac{n!}{(n - k)!}$$

The number of arrangements for r elements taken from n elements (A_n^r) is equal to:

EXAMPLE

If we take two candidates based on the points they earned in the competition and take into account the ranking, what is the number of arrangements for selecting two elements n from three:

$$A_n^k = \frac{n!}{(n - k)!} \Rightarrow A_3^2 = \frac{3!}{(3 - 2)!} = 6$$

CHAPTER ONE: COMBINATORIAL ANALYSIS

Practical exercise:

The archive contains 20 driver's licenses. Three licenses have been withdrawn from the total.

How many samples can be formed if:

- The withdrawal is with repetition.
- The withdrawal is without repetition.

Solution :



The number of samples in the case of repeated sampling is equal to:

$$n^r \Rightarrow n^3 = 20 \times 20 \times 20 = 8000$$

The number of samples in a draw without repetition is equal to:

$$A_n^k = \frac{n!}{(n-k)!} \Rightarrow A_{20}^3 = \frac{20!}{(20-3)!} = 20 \times 19 \times 18 = 6840$$

4. COMBINATIONS :

There is a third counting rule that is often useful, which allows us to calculate the number of results obtained when choosing r elements from n elements without taking order into account.

This counting rule is called combinations. A combination of r elements taken from among n elements is any subset of r elements, and we symbolize it as: C_n^r

4.1. Combination with repetition:

If the draw is with replacement, it is possible that the drawn item will appear again. In this case, we have a combination with repetition, and therefore the number of ways to draw partial sets is:

$$C_{n+p-1}^p = \frac{(n+p-1)!}{p!(n-1)!}$$

EXAMPLE

We want to draw a sample of 3 students from among 6 students, knowing that the sampling method is with replacement.

How many ways can this sample be drawn?

Solution:

$$C_{6+3-1}^3 = \frac{(6+3-1)!}{3!(6-1)!} = \frac{8!}{3!5!} = \frac{8 \times 7 \times 6 \times 5!}{3!5!} = 56$$

4.2. Combination without repetition:

The number of combinations r elements selected from n elements is equal to:

CHAPTER ONE: COMBINATORIAL ANALYSIS

$$C_n^r = \frac{n!}{(n-r)! \times r!}$$

EXAMPLE

The soccer coach wanted to form a team of 11 players, selected from among 20 players.

How many teams can be formed?

Solution:

The coach selects 11 players from among 20, and each team differs from the other in terms of players, so the ranking is not taken into consideration if the number of teams is equal to the number of combinations of elements from among 20 elements:

$$C_{20}^{11} = \frac{20!}{(20-11)! \times 11!} = 167960$$

5. PROPERTIES

Let n and p be two natural integers. So let n and p be two natural integers. Then we ;

- For all natural integers p and n ($n \geq 1, p \leq n$), $A_n^p = \frac{n!}{(n-p)!}$
- For all natural integers p and n ($n \geq 1, p \leq n$), $C_n^p = \frac{n!}{p!(n-p)!}$
- For all integers $n \geq 1$ $C_n^0 = C_n^n = 1$
- For all natural integers p and n ($n \geq 1, p \leq n$), $C_n^p = C_n^{n-p}$
- For all natural integers p and n ($n \geq 1, p \leq n$), $C_n^p + C_n^{p+1} = C_{n+1}^{p+1}$

Practical exercise:

An urn contains 8 red balls, 3 white balls, and 9 blue balls; we draw 3 balls at random from this urn in a single draw. What is the number of ways to draw

1. three red balls?
2. one ball of each color?
3. at least one of the balls is white?

Solution :



Drawing all at once eliminates the notion of order.

1) The number of ways to draw 3 red balls is the number of choices of 3 elements from 8; hence the result is :

$$C_8^3 = \frac{8!}{3! \times 5!} = 56$$

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2) The number of ways to draw a ball of each color is the number of choices of one element from 8 and one element from 3 and one element from 9; hence the result is:

$$C_8^1 \times C_3^1 \times C_9^1 = 8 \times 3 \times 9 = 216$$

3) The number of ways to draw at least one of the balls that is white is

$$C_3^1 \times C_{17}^2 + C_3^2 \times C_{17}^1 + C_3^3 = 3 \times \frac{17!}{2! \times 15!} + 3 \times 17 + 1 = 460$$

Or, the result is given directly by :

$$C_{20}^3 - C_{17}^3 = \frac{20!}{3! \times 17!} - \frac{17!}{3! \times 14!} = 460$$

6. NEWTON'S PAIRING FORMULA

Theorem:

Let $a, b \in \mathbb{R}$ and n be a positive integer, then:

$$(a + b)^n = \sum_{k=0}^n \binom{n}{k} a^{n-k} \cdot b^k$$

Or we can say :

For all real a, b and for any natural whole non-zero n , we have :

$$(a + b)^n = \sum_{k=0}^n C_n^k a^{n-k} \cdot b^k$$

In other words:

$$(a + b)^n = \binom{n}{0} a^n \cdot b^0 + \binom{n}{1} a^{n-1} \cdot b^1 + \dots + \binom{n}{k} a^{n-k} \cdot b^k + \dots + \binom{n}{n} a^0 \cdot b^n$$

The theorem is also true if a and b are complex numbers.

Proof. (By induction)

For :

$$\begin{aligned} n = 1, (a + b)^1 &= 1a + 1b = C_1^0 a + C_1^1 b \\ n = 2, (a + b)^2 &= 1a^2 + 2ab + b^2 = C_2^0 a^2 + C_2^1 ab + C_2^2 b^2 \end{aligned}$$

We assume it is true for n and we prove that it is true for $n + 1$;

According to property, and since $C_n^0 = C_{n+1}^0$ and $C_n^n = C_{n+1}^{n+1}$: So we can write:

$$\begin{aligned} (a + b)^{n+1} &= (a + b)(a + b)^n \\ &= (a + b)(C_n^0 a^n + C_n^1 a^{n-1} b + \dots + C_n^{n-1} a b^{n-1} + C_n^n b^n) \\ &= C_n^0 a^{n+1} + (C_n^0 + C_n^1) a^n b + \dots + (C_n^{n-1} + C_n^n) a b^n + C_n^n b^{n+1} \end{aligned}$$

CHAPTER ONE: COMBINATORIAL ANALYSIS

$$\begin{aligned} &= C_{n+1}^0 a^{n+1} + C_{n+1}^1 a^n b + \dots + C_{n+1}^n a b^n + C_{n+1}^{n+1} b^{n+1} \\ &= \sum_{k=0}^{n+1} C_{n+1}^k a^{n+1-k} b^k \end{aligned}$$

EXAMPLE

Develop $(a - b)^6$

Solution:

$$\begin{aligned} (a - b)^6 &= (a + (-b))^6 \\ &= a^6 + 6a^5(-b) + 15a^4(-b)^2 + 20a^3(-b)^3 + 15a^2(-b)^4 + 6a(-b)^5 + (-b)^6 \\ &= a^6 - 6a^5b + 15a^4b^2 - 20a^3b^3 + 15a^2b^4 - 6ab^5 + b^6 \end{aligned}$$

CHAPTER ONE: COMBINATORIAL ANALYSIS

SERIES NO. 01: COMBINATORIAL ANALYSIS

Exercise 01:

1- Calculate the following:

$$\frac{20!}{3! \times 5! \times 2!}, \quad \frac{300!}{3! \times 297!}, \quad (4 \times 3)!, \quad 4! \times 3!, \quad (4 + 3)!, \quad 4! + 3!$$

2- Simplify the expressions below (n is a positive integer):

$$\frac{n!}{(n-1)!}, \quad \frac{n!}{(n-2)!}, \quad \frac{(n+1)!}{n!}, \quad \frac{(n+1)!}{(n-1)!}$$

Exercise 02:

Let us have the following words: MATHS, STATISTICS.

- How many words can be formed (no matter the meaning) of the letters of the two words in each case?

Exercise 03:

06 workers were invited to attend a meeting at a company to form a workers' representative committee.

1. How can they sit in a row with six seats?
2. How can they form a committee of 03 workers?
3. How can they form a committee of 03 workers containing a President, Rapporteur and Secretary General?

Exercise 04:

Someone wanted to create a password for their e-mail using the numbers, how many words can be created and made up of:

1. Three digits with the possibility of redundancy?
2. Four digits without duplication?
3. Two even and two odd numbers with the possibility of repetition?

Exercise 05:

What is the number of groups of six people that can be formed with 4 men and 6 women if we want them to necessarily contain:

1. exactly 2 men.
2. at least 2 men.

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Exercise 06:

A sports team consists of 8 players in red uniforms, 3 players in white uniforms, and 9 players in blue uniforms; we randomly select 3 players from this team in one draw. What is the number of ways to take:

1. Three players in red uniforms?
2. One player of each color?
3. At least one player in white uniform?

Exercise 07:

What is the count of possibilities:

1. To choose 6 questions from 10
2. For seating 10 people on a chair with 4 places.
3. To form a word of 5 letters knowing that all the letters are different.
4. To form a 5-letter word knowing that there are two identical letters among them.
5. To form a 5-letter word knowing that 4 letters are identical in pairs.

Exercise 08: Find n where:

$$3C_{n+1}^3 = 7C_n^2$$

$$A_{n+1}^3 = A_n^4$$

Exercise 09: Calculate the following expressions:

$$(2a+3)^4, (2a+3b)^5,$$

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ANSWER TO SERIES NUMBER 1

Exercise 01:

1- Calculate the following:

$$\frac{20!}{3! \times 5! \times 2!} = 1.69 \times 10^{15}$$

$$\frac{300!}{3! \times 297!} = 4455100$$

$$(4 \times 3)! = 12! = 479\,001\,600$$

$$4! \times 3! = 144$$

$$(4 + 3)! = 7! = 5040$$

$$4! + 3! = 24 + 6 = 30$$

2- Simplify the expressions (n is a positive integer):

$$\frac{n!}{(n-1)!} = \frac{n(n-1)!}{(n-1)!} = n$$

$$\frac{n!}{(n-2)!} = \frac{n(n-1)(n-2)!}{(n-2)!} = n(n-1) = n^2 - n$$

$$\frac{(n+1)!}{n!} = \frac{(n+1)n!}{n!} = n+1$$

$$\frac{(n+1)!}{(n-1)!} = \frac{(n+1) \times n \times (n-1)!}{(n-1)!} = (n+1)n = n^2 + n$$

Exercise 02:

1- Number of words that can be formed from the word "MATHS" (no matter the meaning)

We note that we are dealing with:

all from all, order is important, there is no repetition, so we are dealing with a permutation without repetition:

$$P_n^n = n! ; P_5^5 = 5! = 5 \times 4 \times 3 \times 2 \times 1 = 120$$

2- Number of words that can be formed from the word "STATISTICS" (no matter the meaning)

We note that we are dealing with:

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all from all, order is important, there is repetition, so we are dealing with a permutation with repetition:

$$P_n^{n_1, n_2, \dots, n_k} = \frac{n!}{n_1! \times n_2! \times \dots \times n_k!} \Rightarrow P_{10}^{3, 3, 1, 2, 1} = \frac{10!}{3! \times 3! \times 1! \times 2! \times 1!} = 50400$$

Exercise 03:

1) How can they sit in a row with six seats?

We note that we are dealing with:

all from all, order is important, there is no repetition, so we are dealing with a permutation without repetition:

$$P_n^n = n! ; P_6^6 = 6! = 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 720$$

2) How can they form a committee of 03 workers?

We note that we are dealing with:

part from all, order is not important, there is no repetition, so we are dealing with a Combination without repetition:

$$C_n^k = \frac{n!}{(n-k)! \times k!} \Rightarrow C_6^3 = \frac{6!}{(6-3)! \times 3!} = \frac{6 \times 5 \times 4 \times 3!}{3! \times 3!} = \frac{6 \times 5 \times 4}{3!} = 20$$

3) How can they form a committee of 03 workers?

We note that we are dealing with:

part from all, order is important (President, Rapporteur and Secretary General), there is no repetition, so we are dealing with a arrangement without repetition:

$$A_n^k = \frac{n!}{(n-k)!} \Rightarrow A_6^3 = \frac{6!}{(6-3)!} = \frac{6 \times 5 \times 4 \times 3!}{3!} = 6 \times 5 \times 4 = 120$$

Exercise 04:

1-Create a password consisting of 03 digits with repetition:

We note that we are dealing with:

part from all (3 number from 10), order is important, there is repetition, so we are dealing with a arrangement with repetition:

$$A_n^k = n^k \Rightarrow A_{10}^3 = 10^3 = 1000$$

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2- Four numbers without repetition:

We note that we are dealing with:

part from all, order is important, there is no repetition, so we are dealing with a arrangement without repetition:

$$A_n^k = \frac{n!}{(n-k)!} \Rightarrow A_{10}^4 = \frac{10!}{(10-4)!} = \frac{10 \times 9 \times 8 \times 7 \times 6!}{6!} = 10 \times 9 \times 8 \times 7 = 5040$$

3- Two even numbers and two odd numbers (with the possibility of repetition)

We have even numbers :2, 4, 6, 8, 10 5 even numbers.

We have odd numbers :1, 3, 5, 7, 9 5 odd numbers.

The conditions are the same as in the first case, in addition to the condition of two even numbers and two odd numbers, because we must take two numbers from 5 even numbers and two numbers from 5 odd numbers, and the letter “**and**” here means multiplication, so we use the rule of arrangement with repetition, i.e.:

$$A_5^2 \times A_5^2 = 5^2 \times 5^2 = 625$$

Exercise 05:

1-The number of groups required is the number of selections of 2 men from 4 and 4 women from 6, and the result is:

$$C_4^2 \times C_6^4 = \frac{4!}{2! \times 2!} \times \frac{6!}{4! \times 2!} = 90$$

2-At least two men. These are combinations without repetition, and the result is:

$$C_4^2 \times C_6^4 + C_4^3 \times C_6^3 + C_4^4 \times C_6^2 = 90 + 4 \times \frac{6!}{3! \times 3!} + \frac{6!}{2! \times 4!} = 185$$

Exercise 06:

1- **The number of ways to select 3 players in red uniforms** is to select 3 players from 8, so the result is:

$$C_8^3 = \frac{8!}{3! \times 5!} = 56$$

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2– The number of ways to select a player from each uniform (each color): is the number of options to select one player from 8 players in red uniforms, one player from 3 players in white uniforms, and one player from 9 players in blue uniforms. Therefore, the result is:

$$C_8^1 \times C_3^1 \times C_9^1 = 8 \times 3 \times 9 = 216$$

3– The number of ways to select at least one player wearing white is:

$$C_3^1 \times C_{17}^2 + C_3^2 \times C_{17}^1 + C_3^3 = 3 \times \frac{17!}{2! \times 15!} + 3 \times 17 + 1 = 460$$

Or the result can be given directly as follows:

$$C_{20}^3 - C_{17}^3 = \frac{20!}{3! \times 17!} - \frac{17!}{3! \times 14!} = 460$$

Exercise 07:

1–The number of possibilities for choosing 6 questions out of 10 is:

$$C_{10}^6 = \frac{10!}{6! (10 - 6)!} = 210$$

2–The number of possibilities for seating 10 people on a 4-seat bench is:

$$A_{10}^4 = 10 \times 9 \times 8 \times 7 = 5040$$

3–The number of possibilities for forming a 5-letter word, given that each letter is different. (These words do not need to have any meaning.)

$$P_5 = 5! = 5 \times 4 \times 3 \times 2 \times 1 = 120$$

4– The number of possibilities for forming a 5-letter word, knowing that two of the letters are identical, is:

$$P_5^{2,1,1,1} = \frac{5!}{2! \times 1! \times 1! \times 1!} = 60$$

5– The number of possibilities for forming a 5-letter word, given that 4 letters are identical in pairs, is:

$$P_5^{2,2,1} = \frac{5!}{2! \times 2! \times 1!} = 30$$

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Exercise 08:

$$3C_{n+1}^3 = 7C_n^2 \Rightarrow 3 \frac{(n+1)!}{3!(n-2)!} = 7 \frac{n!}{2!(n-2)!} \Rightarrow 3 \frac{(n+1)n!}{3 \times 2!n!} = 7 \frac{(n-2)!}{2!(n-2)!} \Rightarrow n+1 = 7 \\ \Rightarrow n = 6$$

$$A_{n+1}^3 = A_n^4 \Rightarrow (n+1)(n)(n-1) = n(n-1)(n-2)(n-3) \Rightarrow (n+1) = n^2 - 5n + 6 \\ \Leftrightarrow (n-1)(n-5) \Rightarrow n = 1, n = 5$$

Exercise 09:

$$(2a+3)^4 = \sum_{p=0}^{p=4} C_4^p (2a)^{4-p} b^p \\ = C_4^0 (2a)^4 + C_4^1 (2a)^3 (3) + C_4^2 (2a)^2 (3)^2 + C_4^3 (2a)(3)^3 + C_4^4 (3)^4 \\ = (2a)^4 + 4 \times (2a)^3 (3) + 6 \times (2a)^2 (3)^2 + 4 \times (2a)b^3 + (3)^4 \\ = 2^4 \times a^4 + 4 \times 2^3 \times a^3 (3) + 6 \times 2^2 \times a^2 (3)^2 + 4 \times 2 \times a \times 3^3 + 3^4 \\ = 16a^4 + 96a^3 + 216a^2 + 216a + 81.$$

$$(2a+3b)^5 = \sum_{p=0}^{p=5} C_5^p (2a)^{5-p} (3b)^p \\ = C_5^0 (2a)^5 + C_5^1 (2a)^4 (3b) + C_5^2 (2a)^3 (3b)^2 + C_5^3 (2a)^2 (3b)^3 + C_5^4 (2a)(3b)^4 + C_5^5 (3b)^5 \\ = 2^5 .a^5 + 5 \times 2^4 .a^4 \times 3.b + 10 \times 2^3 .a^3 \times 3^2 .b^2 + 10 \times 2^2 .a^2 \times 3^3 .b^3 + 5 \times 2 .a \times 3^4 .b^4 + 3^5 .b^5 \\ = 32a^5 + 240a^4b + 720a^3b^2 + 1080a^2b^3 + 810ab^4 + 243b^5$$

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SUGGESTED EXERCISES FOR SOLUTION:

Exercise 01:

Simplify the expressions below :

$$\frac{n!}{(n-1)!}, \frac{(n+3)!}{(n-1)!}, \frac{(n-1)!(n-3)!}{(n-4)!(n+2)!}$$

Exercise 02:

How many committees can be formed from representatives of professors if they are selected randomly according to their rank, such that three assistant professors, two lecturers, and one professor are selected, given that the number of assistant professors, lecturers, and professors are 20, 10, and 5, respectively?

Exercise 03:

How many times can a maximum of 5 players be selected from a team consisting of 11 players?

Exercise 04:

One of the clerks has 16 books, including 5 books with the same title and 7 others with the same title but different from the first. The rest have different titles.

Find the number of different ways to organize these books on the library shelf.

Exercise 05:

A class of 30 students, 12 girls and 18 boys, must elect a committee composed of a president, a vice-president, and a secretary.

1. How many committees can we form?
2. How many committees can we form knowing that the position of secretary must be held by a girl?
3. What is the number of committees including student X?
4. What is the number of committees for which the president is a boy and the secretary is a girl?

Exercise 06:

We call an anagram of the word MAI any word (whether it makes sense or not) formed from the three letters M, A, and I. How many anagrams are there of the word MAI?

2. How many anagrams can be formed using the letters of the words INTERNATIONAL, THEOREME, and ANANAS?

Exercise 07:

A multinational company decides to launch dog toothpaste. The name of this new essential product must consist of three letters.

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1. How many names can we form with all the letters of the alphabet?
2. How many names can we form containing one consonant and two vowels?
3. How many names can we form consisting of one consonant and two different vowels?

Exercise 08:

In a batch of 20 manufactured parts, 4 are bad. In how many different ways can we select 4 in the following cases:

1. all 4 pieces are good.
2. At least one of them is bad.
3. At least two are bad.

Exercise 09:

Let a and b be two real numbers.

1. Expand

$$(a + b)^4, (a + b)^5, (a + b)^2$$

2. Expand

$$(a + 1)^6, (a - 1)^5$$

CHAPTER TWO : GENERALITIES OF NUMERICAL SEQUENCES

1. GENERAL DEFINITIONS ABOUT SEQUENCES

1.1. Definition of a sequence

We call a sequence (u_n) any function that assigns to each natural number n the real number u_n .

And knowledge in the following form:

$$u_n : \mathbb{N} \rightarrow \mathbb{R}$$

$$n \mapsto u(n) = u_n$$

is called the general term of the sequence u , which is also noted $(u_n)_{n \in \mathbb{N}}$ or more simply (u_n) .

Remark 2.1. A sequence (u_n) can be defined in two ways, in explicit form or in recursive form.

1.2. Explicit definition:

Definition:

A sequence (u_n) is said to be explicit if it is possible to calculate u_n directly from n . We then denote $u_n = g(n)$ where g is a function defined on $\mathbb{N}$ (and most often on $\mathbb{R}^+$ as well).

EXAMPLE

For all $n \in \mathbb{N}$, we have:

$$a \quad u_n = (-1)^n \cos\left(\frac{n\pi}{2}\right), u_n = \sin n$$

1.3. Sequence defined by recurrence:

A sequence is defined recursively if the term u_{n+1} can be defined from:

$$u_n : u_{n+1} = f(u_n) \text{ with } f \text{ being a function usually defined on } \mathbb{R}.$$

We define the sequence $(u_n)_{n \in \mathbb{N}}$ as follows:

$$\begin{cases} u_0 = a, \\ u_{n+1} = f(u_n) \end{cases}$$

EXAMPLE

We define the sequence $(u_n)_{n \in \mathbb{N}}$ as follows:

$$\begin{cases} u_0 = 2, \\ u_{n+1} = 3u_n + 2 \quad \text{for all } n \in \mathbb{N}^* \end{cases}$$

In other words, we call a real number sequence u every function that assigns to each natural number n , greater than or equal to a given natural number n_0 , the number $u(n)$.

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Practical exercise:

We consider the following sequences u, v and w defined on $\mathbb{N}$ as:

$$u_n = 3n + 1, \quad v_n = \frac{n}{n+1}, \quad w_n = -n^2 + 2n - 1$$

Calculate the first five terms of each of these sequences?

Solution :



$$u_0 = 3 \times 0 + 1 = 1,$$

$$u_1 = 3 \times 1 + 1 = 4,$$

$$u_2 = 3 \times 2 + 1 = 7,$$

$$u_3 = 3 \times 3 + 1 = 10,$$

$$u_4 = 3 \times 4 + 1 = 13$$

We notice that the sequence u_n is an arithmetic sequence and its base is $r = 3$

$$v_0 = \frac{0}{0+1} = 0,$$

$$v_1 = \frac{1}{1+1} = \frac{1}{2},$$

$$v_2 = \frac{2}{2+1} = \frac{2}{3},$$

$$v_3 = \frac{3}{3+1} = \frac{3}{4},$$

$$v_4 = \frac{4}{4+1} = \frac{4}{5}$$

We notice that the sequence $w_n = -n^2 + 2n - 1 = -(n^2 - 2n + 1)$
so that $w_n = -(n - 1)^2$

$$w_0 = -(0 - 1)^2 = -1,$$

$$w_1 = -(1 - 1)^2 = 0,$$

$$w_2 = -(2 - 1)^2 = -1,$$

$$w_3 = -(3 - 1)^2 = -4,$$

$$w_4 = -(4 - 1)^2 = -9$$

1.4. Terms:

- The sequence u is denoted by the symbol $(u_n)_{n \geq n_0}$ if the sequence u is defined for n greater than or equal to n_0 .

It is also denoted by the symbol $(u_n)_{n \in \mathbb{N}}$ or (u_n) if the sequence u is defined on $\mathbb{N}$.

- u_n is the term whose index is n , also called the general term of the sequence u .
- u_{n_0} is the first term of the sequence u if it is defined for $n \geq n_0$.
- u_0 is the first term of the sequence u if it is defined on $\mathbb{N}$.

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And knowledge is either:

- In general terms, such as: $v_n = \frac{n^3-3n}{7}$
- With its first term w_{n_0} and the relation $w_{n+1} = f(w_n)$ (recursive sequence), such as:

$$\begin{cases} w_0 = -1 \\ w_{n+1} = 2w_n - 3 \end{cases}$$

EXAMPLE

- Sequence (u_n) where: $u_n = 4n^2 - 6$ known on $\mathbb{N}$.
- Sequence w where: $w_n = \frac{7-3n}{n-4}$ Known for $n \geq 5$ and we write $n \geq 5$.

1.5. Direction change of sequence:

1.5.1. strictly Increasing sequence

We say that the sequence (u_n) is strictly increasing if:

$$\forall n \in \mathbb{N}: u_{n+1} > u_n \text{ or } (\forall n \in \mathbb{N}: u_{n+1} - u_n > 0)$$

EXAMPLE

1-We consider the sequence u_n such that: $u_n = n + 1; \forall n \in \mathbb{N}$

Note that for all $n \in \mathbb{N}$, we have :

$$u_{n+1} - u_n = (n + 2) - (n + 1) = 1 > 0$$

Therefore, u_n is strictly increasing.

2- We consider the sequence u_n such that: $u_n = 1 - \frac{1}{n}; \forall n \in \mathbb{N}^*$

Note that for all $n \in \mathbb{N}^*$, we have :

$$\begin{aligned} u_{n+1} - u_n &= \left(1 - \frac{1}{n+1}\right) - \left(1 - \frac{1}{n}\right) \\ &= -\frac{1}{n+1} + \frac{1}{n} \\ &= \frac{1}{n(n+1)} > 0. \end{aligned}$$

Therefore, u_n is strictly increasing.

1.5.2. Increasing sequence

We say that the sequence (u_n) is increasing if:

$$\forall n \in \mathbb{N}: u_{n+1} \geq u_n \text{ or } (\forall n \in \mathbb{N}: u_{n+1} - u_n \geq 0)$$

EXAMPLE

We consider the sequence u_n such that: $u_n = n^2 - 5n; \forall n \geq 2$

Note that for all $n \geq 2$, we have :

$$\begin{aligned}
 u_{n+1} - u_n &= ((n+1)^2 - 5(n+1)) - (n^2 - 5n) \\
 &= (n^2 + 2n + 1 - 5n - 5) - (n^2 - 5n) \\
 &= n^2 - 3n - 4 - n^2 + 5n \\
 &= 2n - 4 \geq 0
 \end{aligned}$$

Therefore u_n is increasing.

1.5.3. strictly Decreasing sequence :

We say that the sequence (u_n) is strictly decreasing if:

$$\forall n \in \mathbb{N}: u_{n+1} < u_n \text{ or } (\forall n \in \mathbb{N}: u_{n+1} - u_n < 0)$$

EXAMPLE

We consider the sequence u_n such that: $u_n = 1 + \frac{1}{n}; \forall n \in \mathbb{N}^*$

Note that for all $n \in \mathbb{N}^*$, we have :

$$\begin{aligned}
 u_{n+1} - u_n &= \left(1 + \frac{1}{n+1}\right) - \left(1 + \frac{1}{n}\right) \\
 &= -\frac{1}{n+1} - \frac{1}{n} \\
 &= \frac{-1}{n(n+1)} < 0.
 \end{aligned}$$

Therefore, u_n is strictly increasing.

1.5.4. Decreasing sequence:

We say that the sequence (u_n) is decreasing if:

$$\forall n \in \mathbb{N}: u_{n+1} \leq u_n \text{ or } (\forall n \in \mathbb{N}: u_{n+1} - u_n \leq 0)$$

EXAMPLE

We consider the sequence u_n such that: $u_n = 5n - n^2; \forall n \geq 2$

Note that for all $n \geq 2$, we have :

$$u_{n+1} - u_n = 4 - 2n \leq 0$$

Therefore, u_n is strictly decreasing.

1.5.5. Constant sequence:

We say that the sequence (u_n) is constant if:

$$\exists n_0 \in \mathbb{N} \text{ such us } \forall n \geq n_0, u_n = u_{n_0} \text{ or } \forall n \in \mathbb{N}: u_{n+1} = u_n \text{ (All boundaries are equal).}$$

CHAPTER TWO : GENERALITIES OF NUMERICAL SEQUENCES

a) Monotonous sequence

We say that the sequence (u_n) is monotonic on the domain I of N if it is either increasing on the domain I of N or decreasing on the domain I of N .



There are sequences that are neither increasing nor decreasing; we say that they are non-monotonic.

For example: The sequence (u_n) defined by its general term $(u_n) = -1^n$

b) Method:

To study the **Direction change of sequence** (u_n) , you can:

1. We examine the signal $u_{n+1} - u_n$ for every natural number n : or
2. We compare between : $\frac{u_{n+1}}{u_n}$ and 1.
3. If we find a function f such that for every natural number n : $u_n = f(n)$, we study the changes in the function f .

EXAMPLE

We apply method 1 to study the direction of change of the sequence:

$$u_n = \frac{n^3 - n}{5}.$$

We have:

$$u_{n+1} - u_n = \frac{(n+1)^3 - (n+1)}{5} - \frac{n^3 - n}{5} = \frac{2n}{5} \geq 0$$

And the sequence (u_n) is increasing.

We apply method 2 to study the direction of change of the sequence:

$$u_n = \frac{1}{n+1}$$

We have:

$$\frac{u_{n+1}}{u_n} = \frac{\frac{1}{n+2}}{\frac{1}{n+1}} = \frac{n+1}{n+2}$$

For every natural number n : $n+1 < n+2$ And from it : $\frac{u_{n+1}}{u_n} < 1$

Therefore, (v_n) is a strictly decreasing sequence for every natural number n .

We apply method 3 The sequence (u_n) is defined by the formula

$u_n = f(n)$ where:

$$f(x) = x^2 + 2x$$

The function is defined and differentiable on R , and we have:

$$\hat{f}(x) = 2x + 2$$

After studying the signal, we find that $\hat{f}(x) > 0$ on the domain $] -1, +\infty[$

And from it, the function f is strictly increasing on the domain $[0, +\infty[$

Therefore, the sequence (u_n) is strictly increasing.

1.6. Finite sequences:

-To say that the sequence $(u_n)_{n \in \mathbb{N}}$ is limited from above means that there exists a real number M such that $\forall n \in \mathbb{N}, u_n \leq M$, We say that M is a sharp element from above.

-To say that the sequence $(u_n)_{n \in \mathbb{N}}$ is limited from below means that there exists a real number m such that $\forall n \in \mathbb{N}, u_n \geq m$, We say that m is a sharp element from below.

- To say that the sequence $(u_n)_{n \in \mathbb{N}}$ is finite if it is limited from above and from below, i.e.:

$$\forall n \in \mathbb{N}, m \leq u_n \leq M, \exists M, m \in \mathbb{R}$$

EXAMPLE

1-Sequence $u_n = 5 - \frac{1}{n}$ (where: $n \geq 1$)

The sequence (u_n) is limited from above and has 5 sharp elements above.

The sequence (u_n) is limited from below and has 4 sharp elements below.

So the sequence (u_n) is finite , i.e., $\forall n \in \mathbb{N}^*$, then $4 \leq u_n \leq 5$.

2- The sequence $u_n = -\frac{2}{n}$ (where: $n > 0$)

is limited from above with -2 and limited from below with 0, i.e:

$\forall n \in \mathbb{N}^*$, then $0 \leq v_n \leq -2$.

1.7. Convergent sequences

We say about a sequence $(u_n)_{n \in \mathbb{N}}$ is convergent if there exists a real number l such that:

$$\forall \varepsilon > 0, \exists n_0 \in \mathbb{N} / n \geq n_0 \Rightarrow |u_n - l| < \varepsilon$$

In this case, we say that the sequence (u_n) converges to the real number l , and we write:

$$\lim_{n \rightarrow +\infty} u_n = l \text{ or } \lim u_n = l \text{ or } u_n \xrightarrow{n \rightarrow +\infty} l$$

If (u_n) is a non-convergent sequence, then it is divergent (its end is infinite or does not exist).

In this case, we say that the sequence (u_n) diverges to $+\infty$ such that:

$$\forall A > 0, \exists n_0 \in \mathbb{N} / n \geq n_0 \Rightarrow u_n > A$$

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Or diverge toward $-\infty$ such that:

$$\forall A > 0, \exists n_0 \in \mathbb{N} / n \geq n_0 \Rightarrow u_n < -A$$

And we write:

$$u_n \xrightarrow{n \rightarrow +\infty} \pm \infty \text{ or } \lim u_n = \pm \infty \text{ or } \lim_{n \rightarrow +\infty} u_n = \pm \infty$$

Or we could say:

Suppose that u_n is a sequence. We say that $\lim_{n \rightarrow \infty} u_n = l$ if for every $A > 0$ there is an $N > 0$ so that whenever $n > N$, $|u_n - l| < A$. If $\lim_{n \rightarrow \infty} u_n = l$ we say that the sequence converges, otherwise it diverges.

EXAMPLE

The sequence (u_n) defined by: $u_n = \frac{5}{n}$ Converges to zero because:

$$\lim_{n \rightarrow +\infty} u_n = 0$$

The sequence (v_n) defined by: $v_n = n^2 + 1$ is divergent because:

$$\lim_{n \rightarrow +\infty} v_n = +\infty.$$

Reminder

Let the sequence (u_n) be defined as follows: $u_n = f(n)$, where the function is defined on the domain $[\alpha, +\infty[$.

Where α is a real number, if $\lim_{x \rightarrow \infty} f(x) = l$, then $\lim_{n \rightarrow \infty} u_n = l$.

EXAMPLE

Let the sequence (u_n) defined on $\mathbb{N}$ be as follows: $u_n = \frac{2n+3}{4n+1}$

-Set the end of the sequence (u_n) .

Solution:

The sequence (u_n) of the form $u_n = f(n)$ where f is a function defined on $[0, +\infty[$ as follows:

$$f(n) = \frac{2x+3}{4x+1}$$

We have : $\lim_{x \rightarrow \infty} f(x) = \frac{1}{2}$, so $\lim_{n \rightarrow \infty} u_n = \frac{1}{2}$, Then the sequence (u_n) is convergent.

Result

- If (u_n) is an increasing sequence limited from above, then it is convergent.

-If (u_n) is a decreasing sequence and limited from below, then it is convergent.

EXAMPLE

The sequence $v_n = \frac{5}{n}$ is decreasing and limited from below by 0, so it is convergent.

1.8. Adjacent sequences

We say that the sequences (v_n) and (u_n) are adjacent if:

One is increasing and the other is decreasing, and the difference between them is zero.

$$\left(\lim_{n \rightarrow +\infty} (u_n - v_n) = 0 \right).$$

EXAMPLE

Let (u_n) and (v_n) be numerical sequences such that:

$$v_n = \frac{2}{n+1} \text{ and } u_n = \frac{-1}{n+2}$$

Are the sequences (u_n) and (v_n) adjacent?

Solution:

Study of the direction of change of the sequences (u_n) and (v_n) :

direction of change of sequence (u_n) :

we have:

$$\forall n \in \mathbb{N}; u_{n+1} - u_n = \frac{-1}{n+3} - \frac{-1}{n+2} = \frac{1}{(n+3)(n+2)} > 0$$

Then the sequence (u_n) is increasing for every natural number n .

direction of change of sequence (v_n) :

we have:

$$\forall n \in \mathbb{N}; v_{n+1} - v_n = \frac{2}{n+2} - \frac{2}{n+1} = \frac{-2}{(n+1)(n+2)} < 0$$

Then the sequence (v_n) is decreasing for every natural number n .

Calculation of $\lim_{n \rightarrow +\infty} (u_n - v_n) = 0$

we have:

$$\lim_{n \rightarrow +\infty} (u_n - v_n) = \lim_{n \rightarrow +\infty} \left(\frac{2}{n+1} - \frac{-1}{n+2} \right) = \lim_{n \rightarrow +\infty} \left(\frac{3n+5}{n^2+3n+2} \right) = 0$$

Since (u_n) is increasing, (v_n) is decreasing, and $\lim_{n \rightarrow +\infty} (u_n - v_n) = 0$, then (u_n) and (v_n) are adjacent.

CHAPTER TWO : GENERALITIES OF NUMERICAL SEQUENCES

2. ARITHMETIC SEQUENCES:

Identification We call an arithmetic sequence with a base r Each sequence $(u_n)_{n \in \mathbb{N}}$

verification : $\forall n \in \mathbb{N}: u_{n+1} = u_n + r$

2.1. The general term of the arithmetic sequence

is given by:

$u_n = u_0 + n r$ If the first term of the sequence is u_0 .

$u_n = u_1 + (n - 1) r$ If the first term of the sequence is: u_1

EXAMPLE

Does the number 203 belong to the arithmetic sequence 3, 7, 11, ...?

Solution

Here $u_1 = 3$ and $r = 4$, so $u_n = 3 + (n - 1)4 = 4n - 1$

We set $4n - 1 = 203$ and find that $n = 51$ Hence, 203 is the 51st term of the sequence.

EXAMPLE

Find the formula for the n^{th} term of the arithmetic sequence

1. 2, 5, 8 ...

2. 107, 98, 89

Solution:

1- Hence $u_1 = 2$ and $r = 3$, so

$$u_n = 2 + (n - 1)3 = 3n - 1$$

2- Hence $u_1 = 107$ and $r = -9$, so

$$u_n = 107 + (n - 1)(-9) = 116 - 9n$$

2.2. The sum of the terms of an arithmetic sequence:

if the first term of the sequence is u_0 Given by:

$$S_n = u_0 + u_1 + u_2 + \cdots \cdots \cdots + u_n = \frac{n+1}{2} (u_0 + u_n)$$

if the first term of the sequence is u_1 Given by:

$$S_n = u_1 + u_2 + \cdots \cdots \cdots + u_n = \frac{n}{2} (u_1 + u_n)$$

CHAPTER TWO : GENERALITIES OF NUMERICAL SEQUENCES

And we can write it in general as follows:

$$S_n = \frac{\text{Number of terms}}{2} (\text{first term} + \text{last term})$$

where:

Number of terms = last term guide - First limit guide + 1.

EXAMPLE

Find the formula for the sum of the first n terms of the arithmetic sequence

1. 2, 5, 8 ...

2. 107, 98, 89

Solution:

1- Here $u_1 = 2$ and $r = 3$, so

$$S_n = \frac{n}{2} [2u_1 + (n-1)r]$$

$$S_n = \frac{n}{2} [4 + (n-1)3] = \frac{n}{2} 3n + 1$$

2- Here $u_1 = 107$ and $r = -9$, so

$$S_n = \frac{n}{2} [2u_1 + (n-1)r]$$

$$S_n = \frac{n}{2} [214 + (n-1)(-9)] = \frac{n}{2} 223 - 9n$$

2.3. Direction of change of arithmetic sequence

Let it be u_n Arithmetic sequence with base r

We know that: $u_{n+1} = u_n + r$ And from it: $u_{n+1} - u_n = r$ So we conclude that:

- (u_n) Increasing if: $r > 0$
- (u_n) Decreasing if: $r < 0$
- (u_n) Fixed if: $r = 0$

Property:

to be a and b and c In this order, the consecutive terms of an arithmetic sequence are:

$$a + c = 2b$$

It is called b arithmetic mean of two numbers a and c .

CHAPTER TWO : GENERALITIES OF NUMERICAL SEQUENCES

Practical exercise:

Consider the numerical sequence defined as follows:

$$\forall n \in \mathbb{N}: u_n = 3n + 5$$

1. Prove that (u_n) An arithmetic sequence and determine its first base.
2. Calculate u_3 and u_{11}
3. Calculate the sum of: $u_3 + u_4 + \dots + u_{11}$

Solution :



1. We have:

$$u_{n+1} - u_n = 3(n+1) + 5 - 3n = 3$$

$$\text{And } u_0 = 3 + (0) + 5 = 5$$

permission (u_n) Arithmetic sequence with base $r = 3$ And its first term

$$u_0 = 5$$

2. We have:

$$u_3 = 3 + (3) + 5 = 14 \text{ and } u_{11} = 3 + (11) + 5 = 38$$

3.

$$u_3 + u_4 + \dots + u_{11} = (11 - 3 + 1) \frac{u_{11} + u_3}{2} = 9 \frac{52}{2} = 234$$

CHAPTER TWO : GENERALITIES OF NUMERICAL SEQUENCES

3. GEOMETRIC SEQUENCE:

3.1. Definition of geometric sequence and its laws:

We call a geometric sequence based on q any sequence (u_n) that satisfies the following:

$$\forall n \in \mathbb{N}: u_{n+1} = u_n \times q$$

3.2. The general term of a geometric sequence:

(u_n) A geometric sequence whose first term u_0 and base q are not zero:

$$\begin{cases} u_1 = u_0 \times q \\ u_2 = u_1 \times q = (u_0 \times q) \times q = u_0 \times q^2 \\ u_3 = u_2 \times q = (u_0 \times q^2) \times q = u_0 \times q^3 \end{cases} \Rightarrow \forall n \in \mathbb{N}: u_n = u_0 \times q^n$$

In general:

(u_n) is a geometric sequence whose first term u_m and base q are non zero:

$$\forall n \in \mathbb{N}: u_n = u_m \times q^{n-m}$$

The proof:

$$\forall n \in \mathbb{N}; \forall m \in \mathbb{N}: \begin{cases} u_n = u_0 \times q^n \dots \dots \dots (1) \\ u_m = u_0 \times q^m \dots \dots \dots (2) \end{cases}$$

Dividing (1) by (2), we find:

$$\frac{u_n}{u_m} = \frac{u_0 \times q^n}{u_0 \times q^m} = \frac{q^n}{q^m} = q^n \times q^{-m} = q^{n-m}$$

Meaning that :

$$\frac{u_n}{u_m} = q^{n-m} \Rightarrow u_n = u_m \times q^{n-m}$$

EXAMPLE

Find the formula for the n^{th} term of the geometric sequence

1. 2, 6, 18, ...
2. 486, 162, 54,

Solution:

1) Hence $u_1 = 2$ and $q = 3$, so

$$u_n = 2 \times 3^{n-1}$$

2) Hence $u_1 = 486$ and $q = \frac{1}{3}$, so

$$u_n = 486 \times \frac{1^{n-1}}{3}$$

EXAMPLE

Does the number 6072 belong to the geometric sequence 3, -6, 12, -24, 48, ...?

Solution

Here $u_1 = 3$ and $q = -2$, so

$$u_n = 3 \times (-2)^{n-1}$$

We set $3 \times (-2)^{n-1} = 6072$ this give $(-2)^{n-1} = 2024$

But 2024 is not a power of 2, and so 6072 does not belong to the sequence.

3.3. Sum of terms of a geometric sequence:

Let S_n be the sum of the terms of a geometric sequence with base q , first term u_0 , and last term u_n :

$$S_n = u_0 + u_1 + u_2 + \cdots \dots \dots + u_n = u_0 \times \left(\frac{1 - q^{n-0+1}}{1 - q} \right) = u_0 \times \left(\frac{1 - q^{n+1}}{1 - q} \right)$$

Let S_n be the sum of the terms of a geometric sequence with base q , first term u_1 , and last term u_n :

$$S_n = u_1 + u_2 + u_3 + \cdots \dots \dots + u_n = u_1 \times \left(\frac{1 - q^{n-1+1}}{1 - q} \right) = u_1 \times \left(\frac{1 - q^n}{1 - q} \right)$$

In general:

Let S_n be the sum of the terms of a geometric sequence with base q , first term u_p , and last term u_n :

$$S_n = u_p + u_{p+1} + u_{p+2} + \cdots \dots \dots + u_n = u_p \times \left(\frac{1 - q^{n-p+1}}{1 - q} \right)$$

EXAMPLE

Find the formula for the sum of the first n terms of the geometric sequence

1. 2, 6, 18, ...
2. 486, 162, 54,

Solution:

$u_1 = 2$ and $q = 3$, so

$$S_n = \frac{2(3^n - 1)}{3 - 1} = 3^n - 1$$

Here $u_1 = 486$ and $q = \frac{1}{3}$, so

$$S_n = \frac{486(1 - \frac{1^n}{3})}{1 - \frac{1}{3}} = 729(1 - \frac{1}{3})$$

3.4. The direction of change of a geometric sequence:

Let (u_n) be a geometric sequence with its first term u_0 and its base q :

$$\forall n \in \mathbb{N} : \begin{cases} u_n = u_0 \times q^n \\ u_{n+1} = u_0 \times q^{n+1} \end{cases} \Rightarrow u_{n+1} - u_n = (u_0 \times q^{n+1}) - (u_0 \times q^n)$$

$$u_{n+1} - u_n = (u_0 \times q^n \times q) - (u_0 \times q^n) = u_0 \times q^n \times (q - 1)$$

$$\forall n \in \mathbb{N} : u_{n+1} - u_n = u_0 \times (q - 1) \times q^n$$



The sign of the difference $u_{n+1} - u_n$ relates to the first term u_0 and the base q .

If it is:

- $q > 1$ and $U_0 > 0$ We say that (u_n) is an increasing sequence.
- $q > 1$ and $U_0 < 0$ We say that (u_n) is a decreasing sequence
- $0 < q < 1$ and $U_0 > 0$ We say that (u_n) is a decreasing sequence
- $0 < q < 1$ and $U_0 < 0$ We say that (u_n) is an increasing sequence.
- $q = 1$ We say that (u_n) is a constant sequence and all its terms are equal and equal to the first term u_0 .
- $q < 0$ and u_0 (Positive or negative) we say that (u_n) is sometimes positive and sometimes negative, so it is an unordered sequence.

3.5. The limit of a geometric sequence:

(u_n) is a geometric sequence with its first term u_0 and its base q :

We know that the general term of a geometric sequence with the first term u_0 and the common ratio q is:

$$u_n = u_0 \times q^n$$

so:

$$\lim_{n \rightarrow +\infty} u_n = \lim_{n \rightarrow +\infty} (u_0 \times q^n) = u_0 \times \lim_{n \rightarrow +\infty} q^n$$

If it is:

- $q < -1$ We say that (u_n) does not have a limit because $\lim_{n \rightarrow +\infty} u_n = +\infty$ or $\lim_{n \rightarrow +\infty} u_n = -\infty$ Therefore, we say that (u_n) is a divergent sequence.
- $q = -1$ We say that (u_n) does not have a limit because $\lim_{n \rightarrow +\infty} u_n = +u_0$ or $\lim_{n \rightarrow +\infty} u_n = -u_0$ Therefore, we say that (u_n) is a divergent sequence.
- $-1 < q < 0 \Rightarrow \lim_{n \rightarrow +\infty} u_n = 0$ So (u_n) is a convergent sequence.
- $q = 0 \Rightarrow \lim_{n \rightarrow +\infty} u_n = 0$ So (u_n) is a convergent sequence
- $0 < q < 1 \Rightarrow \lim_{n \rightarrow +\infty} u_n = 0$ So (u_n) is a convergent sequence
- $q = 1 \Rightarrow \lim_{n \rightarrow +\infty} u_n = u_0$ So (u_n) is a convergent sequence
- $q > 1$ and $u_0 > 0 \Rightarrow \lim_{n \rightarrow +\infty} u_n = +\infty$ So (u_n) is a divergent sequence
- $q > 1$ and $u_0 < 0 \Rightarrow \lim_{n \rightarrow +\infty} u_n = -\infty$ So (u_n) is a divergent sequence

EXAMPLE

Find the limiting sum for the geometric series

1- $1 + \frac{1}{3} + \frac{1}{9} + \dots$

2- $8 - 6 + \frac{9}{2} + \dots$

Solution:

1- Here $u_1 = 1$ and $q = \frac{1}{3}$, so the limiting sum exists and is equal to

$$s_{\infty} = \frac{1}{1 - \frac{1}{3}} = \frac{3}{2}$$

2- Here $u_1 = 8$ and $q = \frac{3}{4}$, so the limiting sum exists and is equal to

$$s_{\infty} = \frac{1}{1 + \frac{3}{4}} = \frac{32}{7}$$

CHAPTER TWO : GENERALITIES OF NUMERICAL SEQUENCES



- The sequence (u_n) converges to the number m if and only if $\lim_{n \rightarrow +\infty} u_n = m$. And if it is not convergent, it is divergent.
- The divergent sequence is the sequence that has no end or has an infinite end ($\lim_{n \rightarrow +\infty} u_n = \pm\infty$).
- The geometric sequence converges if and only if the base belongs to the domain $q \in]-1, 1]$.

Theory:

- ✓ Every increasing geometric sequence that is limited from above is convergent.
- ✓ Every decreasing geometric sequence that is limited from below is convergent.

Property

Let it be a and b and c In this order, the consecutive terms of geometric sequence are:

$$b^2 = ac$$

It is called b geometric mean of two numbers a and c

Practice Exercise:

We consider the numerical sequence: $(u_n)_{n \in \mathbb{N}}$ defined as follows:

$$\begin{cases} u_0 = -2, \\ u_{n+1} = \frac{2}{3}u_n - 1. \end{cases}$$

We put: $v_n = u_n + 3$

1- Prove that the sequence $(u_n)_{n \in \mathbb{N}}$ is geometric sequence.

2-Deduce formula of (u_n) and (v_n) Indicative n , then conclude $\lim_{n \rightarrow +\infty} u_n$

3- We put : $S_n = v_0 + v_1 + \dots + v_n$ Write S_n Indicative n

solution:



1-We have:

$$\frac{v_{n+1}}{v_n} = \frac{u_{n+1} + 3}{u_n + 3} = \frac{\frac{2}{3}u_n - 1 + 3}{u_n + 3} = \frac{\frac{2}{3}u_n + 2}{u_n + 3} = \frac{2}{3} \left(\frac{u_n + 3}{u_n + 3} \right) = \frac{2}{3}$$

And from it: (v_n) is geometric sequence with base $q = \frac{2}{3}$ and it's first term is: $v_0 = u_0 + 3 = -2 + 3 = 1$

2-Deduce formula of (u_n) and (v_n) Indicative n

formula of (v_n) :

$$v_n = v_0 q^n = \left(\frac{2}{3}\right)^n$$

formula of (u_n)

$$u_n = v_n - 3 = \left(\frac{2}{3}\right)^n - 3$$

conclude $\lim_{n \rightarrow +\infty} u_n$

$$\lim_{n \rightarrow +\infty} u_n = \lim_{n \rightarrow +\infty} \left(\frac{2}{3}\right)^n - 3 = -3$$

3- calculate : $S_n = v_0 + v_1 + \dots + v_n$

$$S_n = v_0 + v_1 + \dots + v_n = v_0 \left(\frac{1 - \left(\frac{2}{3}\right)^{n+1}}{1 - \frac{2}{3}} \right) = 3 \left(1 - \left(\frac{2}{3}\right)^{n+1} \right)$$

4. APPLICATIONS TO FINANCE:

One of the many applications of sequences and series occurs in financial mathematics.

Here we will briefly discuss compound interest and superannuation.

4.1. Compound interest

Compound interest was discussed in the TIMES module *Consumer arithmetic* (Year 9). We invest an amount $\$P$ at an interest rate q paid at the end of a prescribed interval for a period of n such intervals. Interest rates are usually quoted as percentages, and so an interest rate of 3% means that $q = 0.03$.

Let us here assume the time interval is one year. Hence, after one year, the investment has the value $u_1 + qu_1 = u_1(1 + q)$.

After the end of two years, the investment has the value

$$u_1(1 + q) + qu_1(1 + q) = u_1(1 + q)^2.$$

We can see from this that, after n years, the investment will be worth $u_1(1 + q)^n$.

These amounts form a geometric sequence with common ratio $1 + q$, and the n^{th} term is

$$u_1(1 + q)^n.$$

EXAMPLE

If \$1000 is invested at 4% p.a. compounded yearly, what is the value of the investment after ten years?

Solution

Here $u_1 = 1000$ and $q = 0.04$ and $n = 10$. Thus, after ten years, the investment is worth

$$1000(1 + 0.04)^{10} \approx 1480.24 \$$$

CHAPTER TWO : GENERALITIES OF NUMERICAL SEQUENCES

Depreciation is closely related to compound interest. When a company, for example, buys a car for work-related purposes, it is able to claim the depreciation in the value of the car over time as a tax deduction.

If the car is initially worth $\$u_1$ and is depreciated at a rate of q per annum, then the value of the car after one year is: $u_1 - qu_1 = u_1(1 - q)$. After the end of two years, the car has value $u_1(1 - q) + qu_1(1 - q) = u_1(1 - q)^2$. We can see from this that, after n years, the car will be worth $u_1(1 - q)^n$

EXAMPLE

What is the value of a car after ten years, if it is initially worth \$30000 and is depreciated at 6% p.a.?

Solution:

Here $u_1 = 30000$ and $q = 0.06$ and $n = 10$. Thus, after ten years, the car is worth

$$30000(1 - 0.06)^{10} \approx 16158 \$$$

4.2. Superannuation

Superannuation is a way of saving for retirement. Money is regularly invested over a long period of time and (compound) interest is paid. Suppose I invest in a superannuation scheme for 30 years which pays 6% per annum. I put \$3000 each year into the scheme, and (for the sake of simplicity) we will suppose that the interest is added yearly.

Thus, the first \$3000 will be invested for the full 30 years at 6% per annum. Hence, it will accrue to $\$3000 \times 1.06^{30}$. The next \$3000, deposited at the beginning of the second year, will be invested for 29 years and so will accrue to $\$3000 \times 1.06^{29}$, and so on. Writing the amounts from smallest to largest, the total value of my investment after 30 years is

$$A = 3000 \times 1.06 + 3000 \times 1.06^2 + \cdots + 3000 \times 1.06^{30}.$$

This is a geometric series with first term $u_1 = 3000 \times 1.06$ and common ratio $q = 1.06$; the number of terms is $n = 30$. Using the formula for the sum of a finite geometric series, the final value of the investment is $A \approx \$251405$.

CHAPTER TWO : GENERALITIES OF NUMERICAL SEQUENCES

This example illustrates both the value of regular saving and the power of compound interest. The \$90000 invested becomes roughly \$251400 over 30 years.

4.3.Simple interest savings status application

A customer deposited an amount of 6000 DZD in the bank with simple interest for several years, meaning that at the end of each year the bank gives interest of 8% To increase his savings every year by a fixed amount equal to 8%From the initial amount.

- calculate u_1, u_2, u_3
- Verify that for every natural number n , We have : $u_{n+1} = u_n + 480$
- Write u_n Indicativen
- How many years do have he to wait for the initial amount to triple?

Series No. 02: NUMERICAL SEQUENCES

Exercise 01:

Let the sequence u_n be defined as follows:

$$\begin{cases} u_0 = \frac{3}{2}, \\ u_{n+1} = (u_n - 1)^2 + 1. \end{cases}$$

- 1-Prove that for every $n \in \mathbb{N}$, $1 < u_n < 2$.
- 2- Prove that the sequence $(u_n)_{n \in \mathbb{N}}$ is increasing.
- 3 - Deduce that the sequence $(u_n)_{n \in \mathbb{N}}$ is convergent, and determine its limit ($l = ?$)

Exercise 02:

We consider the numerical sequence $(u_n)_{n \in \mathbb{N}}$ defined by :

$$\begin{cases} u_0 = 3 \\ u_{n+1} = \frac{5u_n - 4}{u_n + 1} \end{cases}$$

- 1- Prove that: $\forall n \in \mathbb{N} : u_n > 2$
- 2- study monotony. $(u_n)_{n \in \mathbb{N}}$
- 3- Deduce the convergence of the sequence $(u_n)_{n \in \mathbb{N}}$

Exercise 03:

We consider The two sequences (u_n) and (v_n) The identifiers By: $u_0 = 12$ and $v_0 = 1$, and for, every natural number n :

$$v_{n+1} = \frac{u_n + 3v_n}{4} \quad \text{and} \quad u_{n+1} = \frac{u_n + 2v_n}{3}$$

- 1-for, every natural number n We put $w_n = u_n - v_n$, prove That the sequence w_n is geometric and determine its first term and base.
- 2- Study the direction of change of the two sequences (u_n) and (v_n) then conclude that they are adjacent.

Exercise 04:

Let ; the arithmetic sequence (u_n) , with first term u_1 :

- 1- Given that $u_7 = 7u_{19}$ and $u_{31} = 25$, find the base of the sequence (u_n) , Then deduce the expression u_n in terms of n .
the last terms of the sequence (u_n) is 150.
- 2- Determine the number of terms in the sequence (u_n)
- 3- Find the sum of the last 20 terms of the sequence (u_n) .

Exercise 05:

CHAPTER TWO : GENERALITIES OF NUMERICAL SEQUENCES

(u_n) is a numerical sequence defined by:

$$\begin{cases} u_0 = \frac{1}{2} \\ u_{n+1} = \frac{1}{2 - u_n} \end{cases}$$

We Consider the sequence (v_n) defined by: $v_n = \frac{1}{1 - u_n}$

1- Calculate the terms: v_0 , v_1 and v_2 ..

2- Prove that the sequence (v_n) is arithmetic by determining its base.

3- Write v_n and then u_n in terms of n .

Exercise 06:

Let the sequence (u_n) be defined as follows: $u_0 = e$ and for, every natural number n : $u_{n+1} = \sqrt{u_n}$

1- Prove by regressing, for every natural number n : $u_n > 1$

2- Study the direction of change of the sequence (u_n) And concluded its convergence.

3- Let it be The sequence (v_n) Knowledge is as follows : $v_n = \ln(u_n)$.

a. prove that (v_n) is Geometric sequence, assignment required its basis and its first term.

b. Write v_n And u_n by implication, n then calculate the limit of the sequence (u_n) .

c. We put $S_n = v_0 + v_1 + \dots + v_n$ and $P_n = u_0 \times u_1 \times \dots \times u_n$:

calculate S_n then P_n Indicatively. n

Exercise 07:

Osama starts his new job on an annual salary of 18000 DZD . His contract promises a pay rise of 1800 DZD in each subsequent year until his salary reaches 36000 DZD. When the salary reaches 36000 DZD Osama will receive **no more** pay rises. Osama's salary first reaches the maximum salary of 36000 DZD in year N .

a) Determine the value of N .

b) Find Osama's total salary earnings during the first N years of his employment.

Obama starts his new job at the same time as Osama on an annual salary of A DZD .

His contract promises a pay rise of 1000 DZD in each subsequent year until his salary reaches 36000 DZD. When the salary reaches 36000 DZD Obama will receive **no more** pay rises. Obama's salary first reaches the maximum salary of 36000 DZD in year 15.

c) Find the year when both Osama and Obama have the same annual salary.

d) Calculate the difference in the total salary earnings between Osama and Obama in the first 15 years of their employment.

CHAPTER TWO : GENERALITIES OF NUMERICAL SEQUENCES

يبدأ أسامة عمله الجديد براتب سنوي قدره 18000 دينار جزائري. ينص عقده على زيادة في الراتب قدرها 1800 دينار جزائري في كل سنة تالية حتى يصل راتبه إلى 36000 دينار جزائري. عندما يصل الراتب إلى 36000 دينار جزائري، لن يحصل أسامة على أي زيادة أخرى في الراتب. يصل راتب أسامة لأول مرة إلى الحد الأقصى البالغ 36000 دينار جزائري في السنة N ، إذا اعتبرنا أن u_1 هو راتب عامه الأول في العمل.

(أ) حدد قيمة N .

(ب) أوجد إجمالي راتب أسامة خلال السنوات N الأولى من عمله.

يبدأ أواماً عمله الجديد في نفس الوقت الذي بدأ فيه أسامة براتب سنوي قدره A دينار جزائري.

يضمن عقده زيادة في الراتب قدرها 1000 دينار جزائري في كل سنة تالية حتى يصل راتبه إلى 36000 دينار جزائري. عندما يصل الراتب إلى 36000 دينار جزائري، لن يحصل أواماً على أي زيادة أخرى في الراتب. يصل راتب أواماً لأول مرة إلى الحد الأقصى للراتب البالغ 36000 دينار جزائري في السنة 15.

(ج) أوجد السنة التي يكون فيها راتب أسامة وأواماً متساويين.

(د) احسب الفرق في إجمالي دخل الراتب بين أسامة وأواماً في أول 15 سنة من عملهما.

Exercise 08:

A person deposited 11,000 DZD in a bank in 2000, earning 6% compound interest per annum.

Let us consider the deposited amount as u_0 and the number u_n as the new balance after n years.

1- Calculate the amount earned in 2001, 2002, and 2003.

2- Find the relation between u_n and u_{n+1}

3- Conclude the nature of the sequence $(u_n)_{n \in \mathbb{N}}$, and write u_n in terms of n .

أودع شخص مبلغاً قدره 11000 د.ج. بإحدى البنوك عام 2000 بحيث حصل له فائدة سنوية مركبة قدرها 6%.

إذا اعتبرنا أن المبلغ المودع هو u_0 ونعتبر العدد u_n الرصيد الجديد بعد n سنوات.

1- أحسب المبلغ المحصل عليه عام 2001، 2002، 2003.

2- أوجد العلاقة بين u_n و u_{n+1}

3- استنتج طبيعة المتتالية $(u_n)_{n \in \mathbb{N}}$ ، وكتابة u_n بدلالة n

ANSWER TO SERIES NUMBER 2

Exercise 01:

1-Prove that for every $n \in \mathbb{N}$, $1 < u_n < 2$.

-Let's assume that:

$$(P_n): 1 < u_n < 2.$$

Using the proof by regression we prove that (P_n) is true for every: $n \in \mathbb{N}$

Proof:

For: $n = 0$, we assume that : $1 < u_0 = \frac{3}{2} < 2$.

Heredity:

Let us assume that the proposition (P_k) is true for all $1 \leq k \leq n$, and prove that $P(n+1)$ is also true ($1 < u_{n+1} < 2$).

Since $1 < u_n < 2$ (recursion hypothesis).

Therefore, $0 < u_n - 1 < 1$, hence $0 < (u_n - 1)^2 < 1$

Then $1 < (u_n - 1)^2 + 1 < 2$.

That is mean: $P(n+1)$ is also true.

Conclusion:

By the principle of recurrence, we deduce that (P_n) is true for all $n \in \mathbb{N}$. That is mean :

$$1 < u_n < 2.$$

2- Prove that the sequence $(u_n)_{n \in \mathbb{N}}$ is strictly increasing.

Let $n \in \mathbb{N}$. Calculate $u_{n+1} - u_n$:

We have :

$$\begin{aligned} u_{n+1} - u_n &= (u_n - 1)^2 + 1 - u_n \\ &= u_n^2 - 2u_n + 1 + 1 - u_n \\ &= u_n^2 - 3u_n + 2 \\ &= (u_n - 1)(u_n - 2). \end{aligned}$$

According to the first question, $u_n - 1 > 0$ and $u_n - 2 < 0$.

Therefore, $u_{n+1} - u_n < 0$

Hence, the sequence $(u_n)_{n \in \mathbb{N}}$ is strictly decreasing.

3 - Deduce that the sequence $(u_n)_{n \in \mathbb{N}}$ is convergent, and determine its limit ($l = ?$)

-Since the sequence $(u_n)_{n \in \mathbb{N}}$ is limited from below by 1 and strictly decreasing, so

$(u_n)_{n \in \mathbb{N}}$ is convergent to l such that l verifies $l = (l - 1)^2 + 1$.

Therefore, $l^2 + 3l + 2 = 0$.

So $l = 1$ or $l = 2$.

Since the first term $u_0 = \frac{3}{2}$ and $(u_n)_{n \in \mathbb{N}}$ is decreasing, then $l = 1$.

CHAPTER TWO : GENERALITIES OF NUMERICAL SEQUENCES

Exercise 02:

We consider the numerical sequence $(u_n)_{n \in \mathbb{N}}$ defined by :

$$\begin{cases} u_0 = 3 \\ u_{n+1} = \frac{5u_n - 4}{u_n + 1} \end{cases}$$

1- To prove that: $\forall n \in \mathbb{N} : u_n > 2$, we use the proof by regression:

$$(P_n): \forall n \in \mathbb{N} : u_n > 2.$$

Using proof by regression, we prove that (P_n) is true for all: $n \in \mathbb{N}$

2- We verify the validity of (P_0) , that is, the validity of the relation for: $n = 0$

We have: $u_0 = 3 > 2$ and thus: (P_0) is verified (true).

3- We assume that (P_n) is true and prove the truth of (P_{n+1}) .

We have:

$$u_{n+1} = \frac{5u_n - 4}{u_n + 1} = \frac{5u_n - 4 + 5 - 5}{u_n + 1} = \frac{u_n + 5}{u_n + 1} - \frac{9}{u_n + 1} = 5 - \frac{9}{u_n + 1}$$

And we have :

$$u_n > 2 \Rightarrow u_n + 1 > 3 \Rightarrow \frac{1}{u_n + 1} < \frac{1}{3} \Rightarrow \frac{9}{u_n + 1} < 3 \Rightarrow \frac{-9}{u_n + 1} > -3 \Rightarrow 5 - \frac{9}{u_n + 1} > 2$$

And from it: $u_{n+1} > 2$ so: (P_{n+1}) is correct

We conclude that: $\forall n \in \mathbb{N} : u_n > 2$.

2- study monotony. $(u_n)_{n \in \mathbb{N}}$

$$\begin{aligned} u_{n+1} - u_n &= \frac{5u_n - 4}{u_n + 1} - u_n = \frac{(5u_n - 4) - u_n(u_n + 1)}{u_n + 1} = \frac{5u_n - 4 - u_n^2 - u_n}{u_n + 1} \\ &= \frac{-u_n^2 + 4u_n - 4}{u_n + 1} = \frac{-(u_n^2 - 4u_n + 4)}{u_n + 1} = \frac{-(u_n - 2)^2}{u_n + 1} \end{aligned}$$

We have: $u_{n+1} > 0$ and $-(u_n - 2)^2 < 0$, so $u_{n+1} - u_n < 0$ and from it: $(u_n)_{n \in \mathbb{N}}$ is strictly decreasing.

3- Deduce the convergence of the sequence $(u_n)_{n \in \mathbb{N}}$

We have u_n limited from below by the number 2, and it is strictly decreasing, and therefore u_n is convergent.

Exercise 03:

1 -prove That the sequence w_n is geometric :

$$\begin{aligned} w_{n+1} &= u_{n+1} - v_{n+1} = \frac{u_n + 2v_n}{3} - \frac{u_n + 3v_n}{4} \\ w_{n+1} &= \frac{4u_n + 8v_n - 3u_n - 9v_n}{12} = \frac{u_n - v_n}{12} \end{aligned}$$

CHAPTER TWO : GENERALITIES OF NUMERICAL SEQUENCES

So :

$$w_{n+1} = \frac{1}{12} w_n$$

So the sequence w_n is a geometric sequence with a base $q = \frac{1}{12}$ and the first term $w_0 = u_0 - v_0 = 11$

For every natural number n :

$$w_n = 11 \left(\frac{1}{12} \right)^n$$

2-Study the direction of change of the two sequences (u_n) and (v_n) :

$$u_{n+1} - u_n = \frac{u_n + 2v_n}{3} - u_n = \frac{u_n + 2v_n - 3u_n}{3} = \frac{-2(u_n - v_n)}{3} = \frac{-2}{3} w_n$$

Then: $u_{n+1} - u_n = \frac{-22}{3} \left(\frac{1}{12} \right)^n$ Therefore, for every natural number n : $u_{n+1} - u_n < 0$

And the sequence (u_n) is decreasing on N .

$$v_{n+1} - v_n = \frac{u_n + 3v_n}{4} - v_n = \frac{u_n + 3v_n - 4v_n}{4} = \frac{(u_n - v_n)}{4} = \frac{1}{4} w_n$$

So: $v_{n+1} - v_n = \frac{11}{4} \left(\frac{1}{12} \right)^n$ and thus for every natural number n : $v_{n+1} - v_n > 0$

And the sequence (v_n) is increasing on N .

We know that: $\lim_{n \rightarrow +\infty} w_n = 0$ and that: $w_n = u_n - v_n$, thus: $\lim_{n \rightarrow +\infty} (u_n - v_n) = 0$

We conclude that (u_n) and (v_n) are convergent.

Exercise 04:

1-find the common difference (base) of the sequence (u_n) :

we have :

$$\begin{array}{lcl}
 u_n = u_1 + (n-1)r & & \\
 \begin{array}{l}
 u_7 = 7u_1 \\
 \Rightarrow u_1 + 6r = 7(u_1 + 18r) \\
 \Rightarrow u_1 + 6r = 7u_1 + 126r \\
 \Rightarrow 6u_1 = -120r \\
 u_1 = -20r
 \end{array} & \swarrow \quad \searrow & \begin{array}{l}
 u_{31} = 25 \\
 \Rightarrow u_1 + 30r = 25 \\
 \Rightarrow u_1 = 25 - 30r
 \end{array} \\
 & & -20r = 25 - 30r \\
 & & 10r = 25 \\
 & & r = \frac{5}{2}
 \end{array}$$

the expression u_n in terms of n :

we have :

$$u_1 = -20r \Rightarrow u_1 = -20 \left(\frac{5}{2} \right) \Rightarrow u_1 = -50$$

CHAPTER TWO : GENERALITIES OF NUMERICAL SEQUENCES

So that : $u_n = -50 + (n - 1)\frac{5}{2}$

2- Determine the number of terms in the sequence (u_n) :

$$\begin{aligned}u_n &= u_1 + (n - 1)r \\ \Rightarrow 150 &= -50 + (n - 1)\frac{5}{2} \\ \Rightarrow 200 &= (n - 1)\frac{5}{2} \\ \Rightarrow 200\frac{2}{5} &= (n - 1) \\ \Rightarrow (n - 1) &= 80 \\ \text{So : } n &= 81\end{aligned}$$

3- Find the sum of the last 20 terms of the sequence (u_n) :

$$\begin{array}{c} \text{20 terms} \\ \overbrace{150 + 147.5 + 145 + \dots\dots\dots} \\ S_n = \frac{n}{2}[2u_1 + (n - 1)r]\end{array}$$

We have : $u_{n+1} - u_n = r \Rightarrow u_n = u_{n+1} - r$ so that :

$$\begin{aligned}S_n &= \frac{20}{2}\left[2(150) - (19)\frac{5}{2}\right] \\ S_n &= 10\left[300 - (19)\frac{5}{2}\right] = 3000 - 475 = 2525\end{aligned}$$

Exercise 05:

Calculate the terms: v_0 , v_1 and v_2 .. :

$$\begin{aligned}u_1 &= \frac{1}{2 - u_0} = \frac{1}{2 - \frac{1}{2}} = \frac{2}{3} & v_0 &= \frac{1}{1 - u_0} = \frac{1}{1 - \frac{1}{2}} = 2 \\ u_2 &= \frac{1}{2 - u_1} = \frac{1}{2 - \frac{2}{3}} = \frac{3}{4} & v_1 &= \frac{1}{1 - u_1} = \frac{1}{1 - \frac{2}{3}} = 3 \\ & & v_2 &= \frac{1}{1 - u_2} = \frac{1}{1 - \frac{3}{4}} = 4\end{aligned}$$

Prove that the sequence (v_n) is arithmetic

We know that: $v_{n+1} - v_n = r$

We have: $v_n = \frac{1}{1 - u_n}$

So

$$v_{n+1} - v_n = \frac{1}{1 - u_{n+1}} - \frac{1}{1 - u_n}$$

$$\frac{1}{1 - \frac{1}{2-u_n}} - \frac{1}{1 - u_n} = \frac{1}{\frac{1-u_n}{2-u_n}} - \frac{1}{1 - u_n} = \frac{2 - u_n}{1 - u_n} - \frac{1}{1 - u_n} = \frac{2 - u_n - 1}{1 - u_n} = \frac{1 - u_n}{1 - u_n} = 1$$

The sequence (v_n) is arithmetic with base $r = 1$.

Writing v_n in terms of n .

$$v_n = v_0 + nr = 2 + n$$

Writing u_n in terms of n .

$$v_n = \frac{1}{1 - u_n} \Rightarrow u_n = 1 - \frac{1}{v_n} = 1 - \frac{1}{2 + n} \Rightarrow u_n = \frac{1 + n}{2 + n}$$

Exercise 06:

1-Prove by regressing, for every natural number n : $u_n > 1$

We have: the property is true for $n = 0$

We assume $u_n > 1$ that and from it $\sqrt{u_n} > 1$ i.e. $u_{n+1} > 1$ and therefore, according to the principle of regression inference, for every natural number n , $u_n > 1$

2- Study the direction of change of the sequence (u_n)

Since $u_n > 1$, then $u_n^2 > u_n$, and thus $u_n > \sqrt{u_n}$. Therefore, the sequence (u_n) is decreasing.

Convergence conclusion:

Since the sequence (u_n) is decreasing and limited from below by the number 1, it is convergent.

3-prove that (v_n) is Geometric sequence, assignment required its basis and its first term.

Let n be a natural number.

$$v_{n+1} = \ln u_{n+1} = \ln \sqrt{u_n} = \ln u_n^{\frac{1}{2}} = \frac{1}{2} \ln u_n = \frac{1}{2} v_n$$

So, (v_n) is a geometric sequence with a base of $\frac{1}{2}$ and the first term:

$$v_0 = \ln u_0 = \ln e = 1$$

Writing u_n and v_n in terms of n :

$$u_n = e^{v_n} = e^{\left(\frac{1}{2}\right)^n}; \quad v_n = v_0 \left(\frac{1}{2}\right)^n = \left(\frac{1}{2}\right)^n$$

calculate the limit of the sequence (u_n) .

We have : $\lim_{n \rightarrow +\infty} \left(\frac{1}{2}\right)^n = 0$: so : $\lim_{n \rightarrow +\infty} u_n = \lim_{n \rightarrow +\infty} e^{\left(\frac{1}{2}\right)^n} = 1$

calculate S_n then P_n Indicatively.

$$S_n = v_0 + v_1 + \dots + v_n = v_0 \left(\frac{1 - \left(\frac{1}{2}\right)^{n+1}}{1 - \frac{1}{2}} \right) = 2 \left(1 - \left(\frac{1}{2}\right)^{n+1} \right)$$

$$P_n = u_0 \times u_1 \times \dots \times u_n = e^{v_0} \times e^{v_1} \times \dots \times e^{v_n} = e^{v_0 + v_1 + \dots + v_n}$$

$$P_n = e^{2 \left(1 - \left(\frac{1}{2}\right)^{n+1} \right)}$$

Exercise 07:

We have : $u_1 = 18000$

$$u_2 = 18000 + 1800 = 19800$$

$$u_2 = u_1 + 1800$$

$$u_2 = u_1 + (2 - 1) \times 1800$$

$$u_3 = 19800 + 1800 = 21600$$

$$u_3 = u_2 + 1800$$

$$u_3 = u_1 + 2 \times 1800$$

$$u_3 = u_1 + (3 - 1) \times 1800$$

.
.
.
.

$$u_n = u_1 + (n - 1) \times 1800$$

Thus, the sequence u_n is arithmetic with the first term $u_1 = 18000$ and the base $r = 1800$.

a) Determine the value of N .

N can be calculated as follows:

$$u_n = u_1 + (n - 1) \times r$$

$$36000 = 18000 + (n - 1) \times 1800$$

$$18000 = (n - 1) \times 1800$$

$$10 = n - 1$$

$$n = 11$$

The number of years is: 11 years.

b) Find Osama's total salary earnings during the first N years of his employment.

$$S_n = \frac{n}{2} [u_1 + u_n]$$

$$S_{11} = \frac{11}{2} [18000 + 36000]$$

$$S_{11} = \frac{11}{2} [54000] = 11 \times 27000 = 297000$$

CHAPTER TWO : GENERALITIES OF NUMERICAL SEQUENCES

c) Find the year when both Osama and Obama have the same annual salary.

We have:

$$\begin{aligned}u_n &= u_1 + (n - 1) \times r \\36000 &= A + (15 - 1) \times 1000 \\36000 &= A + 14000 \\A &= 22000\end{aligned}$$

So that the two salaries are equal, we have:

$$\begin{aligned}18000 + (n - 1) \times 1800 &= 22000 + (n - 1) \times 1000 \\1800 \times (n - 1) - 1000 \times (n - 1) &= 4000 \\800 \times (n - 1) &= 4000 \\n - 1 &= 5 \\n &= 6\end{aligned}$$

So Osama and Obama's salary is equal in the sixth year of work.

d) Calculate the difference in the total salary earnings between Osama and Obama in the first 15 years of their employment

Find Obama's total salary over 15 years.

$$\begin{aligned}S_n &= \frac{n}{2} [u_1 + u_n] \\S_{15} &= \frac{15}{2} [22000 + 36000] \\S_{15} &= \frac{15}{2} [58000] = 15 \times 29000 = 435000\end{aligned}$$

Find Osama's total salary over 15 years.

$$297000 + (4 \times 36000) = 297000 + 144000 = 441000$$

The difference between the two salaries : $441000 - 435000 = 6000 \text{ DZD}$

Exercise 08:

Since it is compound interest, then:

1- Calculate the amount earned in 2001, 2002, and 2003.

:2001

$$u_1 = u_0 + 0.06 \times u_0 = 11000 \times 1.06 = 11660$$

:2002

$$u_2 = u_1 + 0.06 \times u_1 = u_1 \times 1.06 = 11660 \times 1.06 = 12359.6$$

:2003

$$u_3 = u_2 + 0.06 \times u_2 = u_2 \times 1.06 = 12359.6 \times 1.06 = 13101.176$$

2- Find the relation between u_n and u_{n+1}

$$u_{n+1} = u_n + 0.06 \times u_n$$

CHAPTER TWO : GENERALITIES OF NUMERICAL SEQUENCES

$$u_{n+1} = u_n \times (1 + 0.06)$$

$$u_{n+1} = u_n \times (1.06)$$

3–Conclude the nature of the sequence $(u_n)_{n \in \mathbb{N}}$,

so, we can say that the sequence $(u_n)_{n \in \mathbb{N}}$ is geometric with its first term: $u_0 = 11000$, and its base: $q = 1.06$.

write u_n in terms of n

$$u_n = 11000 \times (1.06)^n$$

CHAPTER THREE: LOGARITHMIC AND EXPONENTIAL FUNCTIONS

1. GENERALITIES ABOUT NUMERICAL FUNCTIONS OF A REAL VARIABLE

At the beginning of this chapter, we present some general definitions related to numerical functions with a real variable.

Definition 1:

A function is a rule which maps a number to another unique number.

In other words, if we start off with an input, and we apply the function, we get an output.

For example, we might have a function that added 3 to any number. So if we apply this function to the number 2, we get the number 5. If we apply this function to the number 8, we get the number 11. If we apply this function to the number x , we get the number $x + 3$. We can show this mathematically by writing :

$$f(x) = x + 3.$$

The number x that we use for the input of the function is called the argument of the function. So if we choose an argument of 2, we get:

$$f(2) = 2 + 3 = 5.$$

If we choose an argument of 8, we get:

$$f(8) = 8 + 3 = 11.$$

If we choose an argument of -6, we get:

$$f(-6) = -6 + 3 = -3.$$

If we choose an argument of z , we get:

$$f(z) = z + 3.$$

If we choose an argument of x^2 , we get:

$$f(x^2) = x^2 + 3.$$

At first sight, it seems that we can pick any number we choose for the argument. However, that is not the case, as we shall see later. But because we do have some choice in the number we can pick, we call the argument the independent variable. The output of the function, e.g. $f(x)$, $f(5)$, etc. depends upon the argument, and so this is called the dependent variable.

Definition2:

A function with a real variable is an application of a domain I towards R ($I \subseteq \mathbb{R}$), we usually denote the function by f , and the set of functions by $\mathcal{F}(I, \mathbb{R})$

The group $D_f = \{x \in \mathbb{R}; f(x)\}$ It is the function definition set. f

The group $C_f = \{(x, f(x)); x \in D_f\}$ It is a statement of the function. f

Definition 3:

The function is every application of f defined from I towards $\mathbb{R}$ ($f: I \rightarrow \mathbb{R}$), Where I is a part (domain) of $\mathbb{R}$ and we call I the domain of definition of the function f .

EXAMPLE

1- $f_1(x) = \sqrt{(x+4)}$;

$D_{f_1} = \{x/x+4 \geq, x \in [-4, +\infty[\}$ where:

$x+4 \geq 0 \Rightarrow x \geq -4 \Rightarrow x \in [-4, +\infty[$

2- $f_2(x) = \frac{1}{x+1}$;

$D_{f_1} = \{x/x+1 \neq 0, x \in]-\infty, 1[\cup]-1, +\infty[\}$ where:

$x+1 \neq 0 \Rightarrow x \neq -1 \Rightarrow x \in \mathbb{R} - \{-1\} \Rightarrow x \in]-\infty, 1[\cup]-1, +\infty[$

1.1. Plotting the graph of a function:

If we have a function given by a formula, we can try to plot its graph. Suppose.

EXAMPLE

: we have a function f defined by:

$f(x) = 3x^2 - 4$.

The argument of the function (the independent variable) is x , and the output (the dependent variable) is $3x^2 - 4$. So we can calculate the output of the function for different arguments:

$f(0) = 3 \times 0^2 - 4 = -4$

$f(1) = 3 \times 1^2 - 4 = -1$

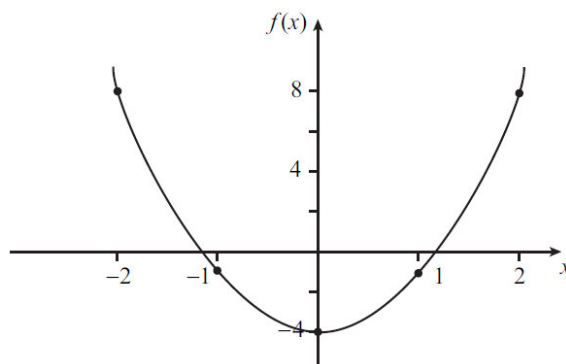
$f(2) = 3 \times 2^2 - 4 = 8$

$f(-1) = 3 \times (-1)^2 - 4 = -1$

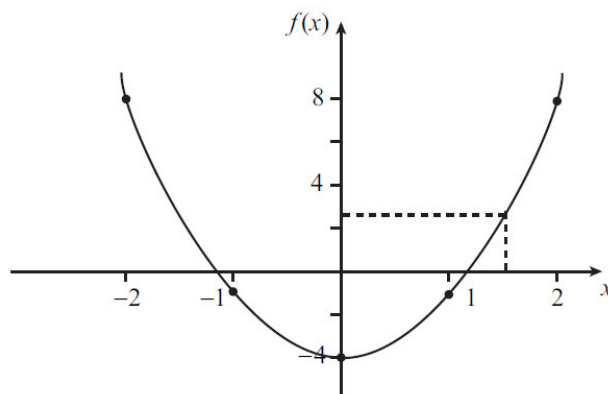
$f(-2) = 3 \times (-2)^2 - 4 = 8$.

We can put this information into a table to help us plot the graph of the function.

x	-2	-1	0	1	2
$f(x)$	8	-1	-4	-1	8



we can also use the graph for values of x which are not in our table. If we have an argument of 1.5, we follow the line up to the graph, and then across to the vertical axis. The result is a number between 2 and 3.



If we want to calculate this number exactly, we can substitute 1.5 into the formula:

$$f(1.5) = 2.75$$

1.2..Algebraic operations on numerical functions

Let f and g be two functions defined from I towards R as follows:

$$\begin{array}{ll} f: I \rightarrow R & g: I \rightarrow R \\ x \rightarrow f(x) & x \rightarrow g(x) \end{array}$$

We define the sum function $(g + f): I \rightarrow R$ by the relation:

$$(g + f)(x) = g(x) + f(x), \forall x \in I$$

We define the multiplication function $(gf): I \rightarrow R$ by the relation:

$$(gf)(x) = g(x)f(x), \forall x \in I$$

We define the division function $(\frac{f}{g}): I \rightarrow R$ where the dependent $g \neq 0$ by the relation:

$$(\frac{f}{g})(x) = \frac{f(x)}{g(x)}, \forall x \in I$$

We define the function multiplying the function by a real number $(\alpha f): D \rightarrow R$ where $\alpha \in R$ by the relation:

$$(\alpha f)(x) = \alpha f(x), \forall x \in I$$

1.3..composite function :

Given two functions f and g , the **composite function**, denoted by $f \circ g$ (read as “ f composed with g ”), is defined by:

$$(f \circ g) = f(g(x))$$

The domain of $f \circ g$ is the set of all numbers x in the domain of g such that $g(x)$ is in the domain of f .

Example 1:

EXAMPLE

Suppose that $g(x) = 2x + 3$ and $f(x) = x^2$, find: $f \circ g$

Solution:

$$f \circ g = f(g(x)) = (2x + 3)^2$$

Example 2:

EXAMPLE

Suppose that $f(x) = 2x^2 - 3$ and $g(x) = 4x$

find: 1- $(f \circ g)(1)$

2- $(g \circ f)(1)$

3- $(f \circ f)(-2)$

4- $(g \circ g)(-1)$

Solution:

$$1- f \circ g = f(g(x)) = 2(4x)^2 - 3 \Rightarrow (f \circ g)(1) = f(g(1)) = f(4) = 2.4^2 - 3 = 29$$

$$2- (g \circ f)(1) = g(f(1)) = g(-1) = 4.(-1) = -4$$

$$3- (f \circ f)(-2) = f(f(-2)) = f(5) = 2.5^2 - 3 = 47$$

$$4- (g \circ g)(-1) = g(g(-1)) = g(-4) = 4.(-4) = -16$$

1.4..Even, odd and periodic functions:

A Function can be classified as Even, Odd or Neither. This classification can be determined graphically or algebraically.

1.4.1.Even Function:

$y = f(x)$ is Even when, for each x in the domain of $f(x)$, $f(-x) = f(x)$

Or; we say about a function $f: I \rightarrow \mathbb{R}$ that it is even if: $\forall x \in I, -x \in I : f(x) = f(-x)$

CHAPTER THREE: LOGARITHMIC AND EXPONENTIAL FUNCTIONS

1.4.2. Odd Function:

$y = f(x)$ is Odd when, for each x in the domain of $f(x)$, $f(-x) = -f(x)$

Or; we say about a function $f: I \rightarrow \mathbb{R}$ that it is odd if: $\forall x \in I, -x \in I : f(x) = -f(x)$

EXAMPLE

1) $f(x) = x^2 + 4$

$f(-x) = (-x)^2 + 4$

$f(-x) = x^2 + 4$

$f(-x) = f(x)$

f Is even function

2) $f(x) = x^3 - 2x$

$f(-x) = (-x)^3 - 2(-x)$

$f(-x) = -x^3 + 2x$

$f(-x) = -(x^3 - 2x)$

$= -f(x)$

f Is odd function

3) $f(x) = x^2 - 3x + 4$

$f(-x) = (-x)^2$

$-3(-x) + 4$

$f(-x) = x^2 + 3x + 4$

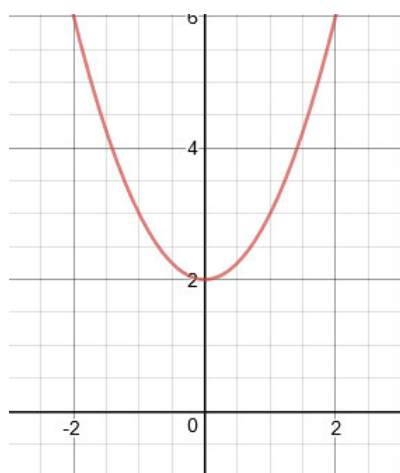
$f(-x) \neq f(x)$

Neither

Graphical Interpretation

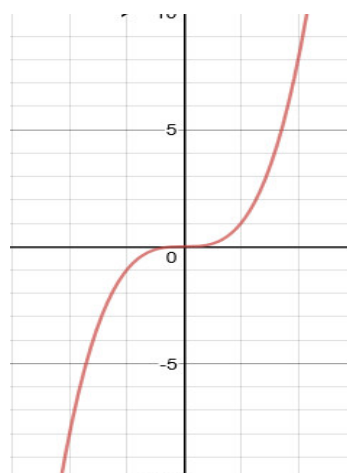
Even Function:

Have a graph that is symmetric with respect to the **Y-Axis**.



Odd Function:

Have a graph that is symmetric with respect to the **Origin**.



1.4.3. Periodic function:

A function $f(x)$ is periodic if the function values repeat at regular intervals of the independent variable T . The regular interval is referred to as the period.

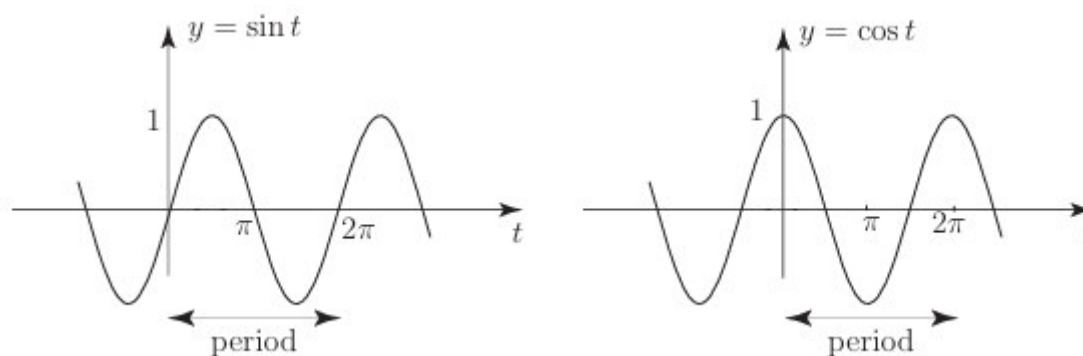
We say about f it is **periodic** and its role $\tau > 0$: if it was:

$$\forall x \in R, f(x + T) = f(x)$$

Such as: functions ***sin x*** and ***cos x*** They are periodic functions whose period is:

$$T = 2\pi$$

because: $\sin(x + 2\pi) = \sin x$, $\cos(x + 2\pi) = \cos x$, As shown by the following figure:



1.5. Finite functions :

Let be the function f defined from I towards R as follow:

$$f: I \rightarrow \mathbb{R}$$

$$x \rightarrow f(x)$$

We then say that :

f is Constant function if : $\forall x \in I, \exists \alpha \in R: f(x) = \alpha$

f is Limited from above if : $\forall x \in I, \exists M \in R: f(x) \leq M$

f is Limited from Bellow if : $\forall x \in I, \exists m \in R: f(x) \geq m$

f is Finite if : $\forall x \in I, \exists M \in R: |f(x)| \leq M$

EXAMPLE

Function $f(x) = \cos(x)$ is Limited from above because: $|\cos(x)| \leq 1$

Function $f(x) = \sin(x)$ is Limited from Bellow because: $\sin(x) \geq -1$

1.6. Increasing and decreasing functions (monotonicity)

1.6.1. Monotonicity about a point

Let a function f be defined on an open interval containing the point $x = a$. We have a set of four standard terms to describe the increasing–decreasing behaviour of the function in a sufficiently small neighbourhood around $x = a$. They are as follows:

Increasing: $\forall x, y \in I, x \leq y \Rightarrow f(x) \leq f(y)$

Strictly increasing: $\forall x, y \in I, x < y \Rightarrow f(x) < f(y)$

Decreasing: $\forall x, y \in I, x \geq y \Rightarrow f(x) \geq f(y)$

Strictly decreasing: $\forall x, y \in I, x > y \Rightarrow f(x) > f(y)$

1.6.2. Increasing:

A function $f(x)$ is said to be increasing about the point $x = a$ if $f(a - h) \leq f(a) \leq f(a + h)$, where h is a small positive arbitrary number.

Derivatives can be used to determine if a function is increasing, on an interval:

$f(x)$ is increasing if derivative $f'(x) > 0$.

EXAMPLE

Examine the behaviour of the function $f(x) = x^3 - 3x + 2$ at the points $x = 2$.

Solution:

At the point $x = 2, f'(2) = 9 > 0 \Rightarrow f(x)$ is increasing at $x = 2$.

1.6.3. Strictly increasing:

A function $f(x)$ is said to be strictly increasing about the point $x = a$ if:

$f(a - h) < f(a) < f(a + h)$, where h is a small positive arbitrary number.

EXAMPLE

Function $\sin(x)$ is strictly monotonic on each interval

$[-\pi/2 + k\pi, \pi/2 + k\pi]$, $k = 0, \pm 1, \pm 2, \pm 3, \dots$ It is increasing on $[-\pi/2 + k\pi, \pi/2 + k\pi]$, $k = 0, \pm 2, \pm 4, \pm 6, \dots$

1.6.4. Decreasing:

A function $f(x)$ is said to be decreasing about the point $x = a$ if $f(a - h) \geq f(a) \geq f(a + h)$, where h is a small positive arbitrary number.

Derivatives can be used to determine if a function is decreasing on an interval:

$f(x)$ is decreasing if derivative $f'(x) < 0$

EXAMPLE

Examine the behaviour of the function $f(x) = x^3 - 3x + 2$ at the points $x = 0$.

Solution:

At the point $x = 0$, $f'(0) = -3 < 0 \Rightarrow f(x)$ is decreasing at $x = 0$

1.6.5. Strictly decreasing:

A function $f(x)$ is said to be strictly decreasing about the point $x = a$ if:

$f(a - h) > f(a) > f(a + h)$, where h is a small positive arbitrary number.

EXAMPLE

Examine the behaviour of the function $f(x) = \frac{1}{x+1}$ at the point $x = 0$.

Solution:

We have $f(x) = \frac{1}{x+1}$

$$f(0) = 1$$

$$f(0 - h) = \frac{1}{1-h} > 1$$

$$f(0 + h) = \frac{1}{1+h} < 1$$

Since, $f(0 - h) > f(0) > f(0 + h)$, where h is a small positive arbitrary number, f is strictly decreasing at $x = 0$.

Practical exercise 1:

Find where $f(x) = x^2 - 5x + 1$ is increasing and where it is decreasing.



Consider $f'(x) = 2x - 5$

$f'(x) > 0$ if $x > 5/2$,

$f'(x) < 0$ if $x < 5/2$.

By Thm. B: $f(x)$ is increasing for $x > \frac{5}{2}$

$f(x)$ is decreasing for $x < \frac{5}{2}$

1.7. The endings Limits

In this section, we will cover the definitions and some properties related to the limits of numerical functions of a real variable.

Definition:

The limits are defined as the value that the function approaches as it goes to an x value.

1.7.1. Function defined at the point

Definition:

We call a definition at the point x_0 in R any open interval centered at x_0 . Any interval $]x_0 - \alpha, x_0 + \alpha[$ where $\alpha > 0$. We say that the function f is defined at the point x_0 and at x_0 itself if and only if there exists a number $\alpha > 0$ such that $]x_0 - \alpha, x_0 + \alpha[\subset D_f$, where D_f is the domain of the function f .

And we say that the function f is defined at the point x_0 and except for x_0 if and only if there exists a number $\alpha > 0$ such that: $]x_0 - \alpha, x_0[\cup]x_0, x_0 + \alpha[\subset D_f$

EXAMPLE

$$f(x) = \frac{1}{x-1}$$

is defined at the point 1 except for 1.

1.7.2. One-Sided Limits: General Definition

One-sided limits are differentiated as right-hand limits (when the limit approaches from the right) and left-hand limits (when the limit approaches from the left) whereas ordinary limits are sometimes referred to as two-sided limits. Right-hand limits approach the specified point from positive infinity. Left hand limits approach this point from negative infinity.

We say that f has a limit at x_0 if and only if $l_1 = l_2 = l$

And written as: $\lim_{x \rightarrow x_0} f(x) = l$

1.7.3. The right-handed limit:

Let $f: I \subset \mathbb{R} \rightarrow \mathbb{R}$ and let x_0 be an interior point of I , and f is defined on I except for x_0 at most.

We say that f has a right-hand limit at x_0 if there exists a unique real number l_2 that satisfies the following:

$$\forall \varepsilon > 0, \exists \delta > 0, \forall x \in D_f, 0 \leq x - x_0 < \delta \Rightarrow |f(x) - l_2| < \varepsilon$$

And written as: $\lim_{x \rightarrow x_0^+} f(x) = l_2$

1.7.4. The left-handed limit:

Let $f: I \subset \mathbb{R} \rightarrow \mathbb{R}$ and let x_0 be an interior point of I , and f is defined on I except for x_0 at most.

We say that f has a left-hand limit at x_0 if there exists a unique real number l_1 that satisfies the following:

$$\forall \varepsilon > 0, \exists \delta > 0, \forall x \in D_f, -\delta \leq x - x_0 < 0 \Rightarrow |f(x) - l_1| < \varepsilon$$

And written as: $\lim_{x \rightarrow x_0^-} f(x) = l_1$

EXAMPLE

Prove that: $\lim_{x \rightarrow 1} (x^2 + 1) = 2$

Solution:

Prove that: $\lim_{x \rightarrow 1} (x^2 + 1) = 2$

According to the definition:

$$\forall \varepsilon > 0, \exists \delta(\varepsilon) > 0, \forall x \in \mathbb{R} : |x - 1| < \delta \Rightarrow |(x^2 + 1) - 2| < \varepsilon$$

Suppose that

$$|x - 1| < \delta \text{ so } |x + 1| = |(x - 1) + 2| < \delta + 2$$

And from it

$$|x - 1| < \delta \Rightarrow |(x^2 + 1) - 2| = |x^2 - 1| = |(x - 1)(x + 1)| < \delta(2 + \delta) < \varepsilon$$

By solving the inequality

$\delta^2 + 2\delta - \varepsilon < 0$ We find that:

$$\delta = -1 + \sqrt{1 + \varepsilon} \text{ Achieves definition.}$$



When does a limit not exist?

- A general limit does not exist if the left- and right-hand limits aren't equal (resulting in a discontinuity in the function).
- A general limit does not exist wherever a function increases or decreases infinitely ("without bound") as it approaches a given x-value.
- A general limit does not exist in the cases of infinite oscillation when approaching a fixed point.

1.7.5. Unfinished endings:

1- We say that $\lim_{x \rightarrow +\infty} (x) = l$ If and only if the following definition is satisfied:

$$\forall \varepsilon > 0, \exists K(\varepsilon) > 0, \forall x \geq K \Rightarrow |f(x) - l| < \varepsilon.$$

2- We say that $\lim_{x \rightarrow -\infty} (x) = l$ If and only if the following definition is satisfied:

$$\forall \varepsilon > 0, \exists K(\varepsilon) < 0, \forall x \leq K \Rightarrow |f(x) - l| < \varepsilon.$$

3- We say that $\lim_{x \rightarrow x_0} f(x) = +\infty$ If and only if the following definition is satisfied:

$$\forall A > 0, \exists \delta(A) > 0, \forall x, |x - x_0| \leq \delta \Rightarrow f(x) \geq A.$$

4- We say that $\lim_{x \rightarrow x_0} f(x) = -\infty$ If and only if the following definition is satisfied:

$$\forall A < 0, \exists \delta(A) > 0, \forall x, |x - x_0| \leq \delta \Rightarrow f(x) \leq A.$$

5-We say that $\lim_{x \rightarrow +\infty} (x) = +\infty$ If and only if the following definition is satisfied:

$$\forall A > 0, \exists K(A) > 0, \forall x \geq K \Rightarrow f(x) \geq A.$$

6-We say that $\lim_{x \rightarrow +\infty} (x) = -\infty$ If and only if the following definition is satisfied:

$$\forall A < 0, \exists K(A) > 0, \forall x \geq K \Rightarrow f(x) \leq A.$$

7-We say that $\lim_{x \rightarrow -\infty} (x) = +\infty$ If and only if the following definition is satisfied:

$$\forall A > 0, \exists K(A) < 0, \forall x \leq K \Rightarrow f(x) \geq A.$$

8-We say that $\lim_{x \rightarrow -\infty} (x) = -\infty$ If and only if the following definition is satisfied:

$$\forall A < 0, \exists K(A) < 0, \forall x \leq K \Rightarrow f(x) \leq A.$$



If $\lim_{x \rightarrow \infty} f(x) = \infty$ and g is a limited function from below

$\forall x \in D_f, g(x) \geq K$, then :

$$\lim_{x \rightarrow \infty} (f + g)(x) = \infty$$

1.7.6. Properties of limits :

Theory

If it is $\lim_{x \rightarrow x_0} g(x) = b$ and $\lim_{x \rightarrow x_0} f(x) = a$ where $x_0 = \pm\infty$ or $x_0 \in R$ and $a, b \in R$ Then:

- $\lim_{x \rightarrow x_0} (\alpha f(x) + \beta g(x)) = \alpha a + \beta b$ For all $\alpha, \beta \in R$.
- $\lim_{x \rightarrow x_0} (f(x)g(x)) = ab$
- If it is $b \neq 0$ then $\lim_{x \rightarrow x_0} \frac{f(x)}{g(x)} = \frac{a}{b}$.

Special cases:

- 1) $\alpha = \beta = 1$ We find that $\lim_{x \rightarrow x_0} (f(x) + g(x)) = +a$.
- 2) $\alpha = 1$ and $\beta = -1$ We find that $\lim_{x \rightarrow x_0} (f(x) - g(x)) = -a$,
- 3) $\alpha \in R$ and $\beta = 0$ We find that $\lim_{x \rightarrow x_0} (\alpha f(x)) = \alpha a$,

- 4) 4) If we put $f(x) = g(x)$ then $\lim_{x \rightarrow x_0} (f(x))^2 = a^2$, More generally, we have : $\forall n \in \mathbb{N}$,

$$\lim_{x \rightarrow x_0} (f(x))^n = a^n$$
- 5) If $f(x) \leq g(x)$ in the neighborhood of the point x_0 , then $a \leq b$, and in particular, if $f(x) \leq K$ in the neighborhood of the point x_0 , then $a \leq K$.

1.8. Limits: Numeric Solutions

There are a many better (and more accurate) ways to find the value of the limit than graphing or plugging in numbers that get closer and closer to the value of interest. These solution methods fall under three categories: substitution, factoring, and the conjugate method. But first things first, let's discuss some of the general rules for limits.

1.8.1. Limit Rules

Here are some of the general limit rules with: $\lim_{x \rightarrow x_0} f(x) = L$ and $\lim_{x \rightarrow x_0} g(x) = M$

a–Sum Rule: The limit of the sum of two functions is the sum of their limits

$$\lim_{x \rightarrow x_0} (f(x) + g(x)) = \lim_{x \rightarrow x_0} f(x) + \lim_{x \rightarrow x_0} g(x) = L + M$$

b–Difference Rule: The limit of the difference of two functions is the difference of their limits

$$\lim_{x \rightarrow x_0} (f(x) - g(x)) = \lim_{x \rightarrow x_0} f(x) - \lim_{x \rightarrow x_0} g(x) = L - M$$

c–Product Rule: The limit of a product of two functions is the product of their limits

$$\lim_{x \rightarrow x_0} (f(x) \times g(x)) = \lim_{x \rightarrow x_0} f(x) \times \lim_{x \rightarrow x_0} g(x) = L \times M$$

d–Constant Multiple Rule: The limits of a constant times a function is the constant times the limit of the function

$$\lim_{x \rightarrow x_0} (k * f(x)) = k \lim_{x \rightarrow x_0} f(x) = k * L$$

e–Quotient Rule: The limit of a quotient of two functions is the quotient of their limits, provided the limit of the denominator is not zero

$$\lim_{x \rightarrow x_0} \left(\frac{f(x)}{g(x)} \right) = \frac{\lim_{x \rightarrow x_0} f(x)}{\lim_{x \rightarrow x_0} g(x)} = \frac{L}{M}, \quad M \neq 0.$$

EXAMPLE

Find the following limits using the above limit rules:

$$\begin{aligned} \lim_{x \rightarrow 3} (x^3 + 4x^2 - 3x) &= \lim_{x \rightarrow 3} x^3 + \lim_{x \rightarrow 3} 4x^2 - \lim_{x \rightarrow 3} 3x \\ &= 3^3 + 4 * 3^2 - 3 * 3 = 27 + 36 - 9 = 54 \\ \lim_{x \rightarrow 2} \frac{x^4 + x^2}{x^2 + 5} &= \frac{\lim_{x \rightarrow 2} x^4 + \lim_{x \rightarrow 2} x^2}{\lim_{x \rightarrow 2} (x^2 + 5)} = \frac{2^4 + 2^2}{2^2 + 5} = \frac{8 + 4}{4 + 5} = \frac{12}{9} \end{aligned}$$

NOTE

If it is $\lim_{x \rightarrow x_0} g(x) = M$ and $\lim_{x \rightarrow x_0} f(x) = L$ then :

1) $a \neq \pm\infty$ and $b = \pm\infty$ So it will be:

$$\lim_{x \rightarrow x_0} (f(x) + g(x)) = \pm\infty \text{ and } \lim_{x \rightarrow x_0} \frac{f(x)}{g(x)} = 0$$

2) $a \neq \pm\infty$ and $b = \pm\infty$ and $a \neq 0$ So it will be:

$$\lim_{x \rightarrow x_0} (f(x)g(x)) = \pm\infty \text{ and } \lim_{x \rightarrow x_0} \frac{g(x)}{f(x)} = \pm\infty \text{ With consideration of signal } a.$$

3) $a = +\infty$ and $b = +\infty$ So it will be:

$$\lim_{x \rightarrow x_0} (f(x) + g(x)) = +\infty \text{ and } \lim_{x \rightarrow x_0} (f(x)g(x)) = +\infty ,$$

4) $a = -\infty$ and $b = -\infty$ So it will be:

$$\lim_{x \rightarrow x_0} (f(x) + g(x)) = -\infty \text{ and } \lim_{x \rightarrow x_0} (f(x)g(x)) = +\infty ,$$

5) $a = +\infty$ and $b = -\infty$ So it will be: $\lim_{x \rightarrow x_0} (f(x)g(x)) = -\infty$

1.8.2.Limits of Rational Functions:

A rational function is a function that can be written as the ratio of two algebraic expressions. If a function is considered rational and the denominator is not zero.

a-Substitution Method: the limit can be found by substitution. This can be seen in the example below :

$$\lim_{x \rightarrow 2} \frac{x^4 + x^2 - 3}{x^2 + 5} = \frac{2^4 + 2^2 - 3}{2^2 + 5} = \frac{8 + 4 - 3}{4 + 5} = \frac{9}{9} = 1$$

This can be defined more formally as: If $f(x)$ and $g(x)$ are algebraic expressions and $g(x) \neq 0$, then:

$$\lim_{x \rightarrow x_0} \frac{f(x)}{g(x)} = \frac{f(x_0)}{g(x_0)}$$

b-Factoring Method

Consider the function : $f(x) = \frac{x^2-9}{x+3}$

How would you find the limit of f as x approaches -3 ?

If you try to use substitution to find the limit, world-ending paradoxes ensue:

$$\lim_{x \rightarrow -3} \frac{x^2 - 9}{x + 3} = \frac{(-3)^2 - 9}{(-3) + 3} = \frac{0}{0}$$

But fear not, this answer just tells us that we must use a different method to find the limit, because the function likely has a “hole” at the given x value. Therefore, the factoring method can be tried. To start this method, the numerator and denominator must be factored (in this case the denominator is “factored” already).

$$\lim_{x \rightarrow -3} \frac{(x + 3)(x - 3)}{x + 3}$$

The factor $(x + 3)$ can be canceled to get the much simpler limit expression of that can easily be evaluated via substitution:

$$\lim_{x \rightarrow -3} (x - 3) = -3 - 3 = -6$$

Therefore, the result of the limit can be found, with the understanding that there is a “hole” in the graph at $x = -3$. Hence,

$$\lim_{x \rightarrow -3} \frac{x^2 - 9}{x + 3} = -6$$

c-Conjugate Method :

The conjugate of a binomial expression (i.e. an expression with two terms, you can tell this because of the Latin root bi- meaning two) is the same expression with opposite middle signs. For example, the conjugate of $(\sqrt{x} - 5)$ is $(\sqrt{x} + 5)$. This is really useful if you have a radical in your limit. This is because the product of two conjugates containing radicals will, itself, contain no radical expressions. See below:

$$(\sqrt{x^2} - 5)(\sqrt{x} + 5) = \sqrt{x} + 5\sqrt{x} - 5\sqrt{x} - 25 = x - 25$$

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You should use the conjugate method whenever you have a limit problem containing radicals for which substitution does not work.

EXAMPLE

Evaluate :

$$\lim_{x \rightarrow 5} \frac{\sqrt{x+11}-4}{x-5}$$

Solution:

First try the substitution method:

$$\lim_{x \rightarrow 5} \frac{\sqrt{x+11}-4}{x-5} = \frac{\sqrt{5+11}-4}{5-5} = \frac{0}{0}$$

Well, another hole in the universe, or at least the graph. Indicating that you'll need another method to find the limit since the function probably has a hole at $x = 5$. To start, multiply both the numerator and denominator by the conjugate of the radical expression ($\sqrt{x+11} + 4$):

$$\begin{aligned} \lim_{x \rightarrow 5} \frac{\sqrt{x+11}-4}{x-5} &\times \frac{(\sqrt{x+11}+4)}{(\sqrt{x+11}+4)} \\ &= \lim_{x \rightarrow 5} \frac{(x+11)-16}{(x-5)(\sqrt{x+11}+4)} \\ &= \lim_{x \rightarrow 5} \frac{x-5}{(x-5)(\sqrt{x+11}+4)} \end{aligned}$$

Cancel the $(x-5)$ factor in the numerator and denominator.

$$\lim_{x \rightarrow 5} \frac{1}{(\sqrt{x+11}+4)} = \frac{1}{(\sqrt{5+11}+4)} = \frac{1}{8}$$

1.8.3.Limits: indeterminate form

Previously, when we found that the result of a limit doing straight substitution yielded we used factoring or conjugation to be able to solve the problem. What happens when neither of those methods prove useful? You become very grateful for the 17th-century French mathematician Guillaume de L'Hôpital. L'Hôpital was the man that derived a method of solving these types of equations, known as indeterminate forms. This method, known as L'Hôpital's Rule, is formally defined below.

1.9.Formal Definition: L'Hôpital's Rule

If the limit $\lim_{x \rightarrow x_0} \frac{f(x)}{g(x)}$ results in one of the following forms:

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$$\frac{\infty}{\infty}, \frac{0}{0}, \infty - \infty, 0 \times \infty, 1^\infty, 0^0, \infty^0,$$

And $\lim_{x \rightarrow x_0} \frac{f'(x)}{g'(x)}$ exists and $g'(x) \neq 0$, then:

$$\lim_{x \rightarrow x_0} \frac{f(x)}{g(x)} = \lim_{x \rightarrow x_0} \frac{f'(x)}{g'(x)}$$

EXAMPLE

Find the limit $\lim_{x \rightarrow 0} \frac{e^x - 1}{x}$

$$\lim_{x \rightarrow 0} \frac{e^x - 1}{x} = \frac{0}{0}$$

Using L'Hôpital's Rule:

$$\lim_{x \rightarrow 0} \frac{e^x - 1}{x} = \lim_{x \rightarrow 0} \frac{e^x}{1} = \frac{1}{1} = 1$$

Practical exercise:

1-Find the limit $\lim_{x \rightarrow \infty} \frac{x^2}{2^x}$

2-Find the limit $\lim_{x \rightarrow 2} \left(\frac{4}{x^2 - 4} - \frac{1}{x - 2} \right)$

3-Find the limit $\lim_{x \rightarrow 1} x^{\frac{1}{1-x}}$

4-Find the limit $\lim_{x \rightarrow 0^+} (x * \ln x)$

$$\lim_{x \rightarrow \infty} \frac{x^2}{2^x} = \frac{\infty}{\infty}$$

Using L'Hôpital's Rule:

$$\lim_{x \rightarrow \infty} \frac{x^2}{2^x} = \lim_{x \rightarrow \infty} \frac{2x}{2^x \ln 2} = \frac{2}{\ln 2} \lim_{x \rightarrow \infty} \frac{x}{2^x} = \frac{\infty}{\infty}$$

Using L'Hôpital's Rule again:

$$\frac{2}{\ln 2} \lim_{x \rightarrow \infty} \frac{x}{2^x} = \frac{2}{\ln 2} \lim_{x \rightarrow \infty} \frac{1}{2^x \ln 2} = \frac{2}{(\ln 2)^2} \lim_{x \rightarrow \infty} \frac{1}{2^x} = \frac{2}{(\ln 2)^2} \times 0 = 0$$

$$\lim_{x \rightarrow 2} \left(\frac{4}{x^2 - 4} - \frac{1}{x - 2} \right) = \infty - \infty$$

Using L'Hôpital's Rule:

$$\lim_{x \rightarrow 2} \left(\frac{4}{x^2 - 4} - \frac{1}{x - 2} \right) = \lim_{x \rightarrow 2} \frac{4 - (x + 2)}{(x - 2)(x + 2)} = \lim_{x \rightarrow 2} \frac{2 - x}{x^2 - 4} = \lim_{x \rightarrow 2} \frac{-1}{2x} = -\frac{1}{4}$$

$$\lim_{x \rightarrow 1} x^{\frac{1}{1-x}} = 1^\infty$$

$$\text{Let } y = x^{\frac{1}{1-x}}. \text{ Then } \ln y = \ln x^{\frac{1}{1-x}} = \frac{\ln x}{1-x}$$

Using L'Hôpital's Rule:

$$\lim_{x \rightarrow 1} \ln y = \lim_{x \rightarrow 1} \frac{\ln x}{1-x} = \lim_{x \rightarrow 1} \frac{\frac{1}{x}}{-1} = -\lim_{x \rightarrow 1} \frac{1}{x} = -1$$

Therefore

$$\lim_{x \rightarrow 1} \ln y = -1, \quad \lim_{x \rightarrow 1} x^{\frac{1}{1-x}} = \lim_{x \rightarrow 1} y = e^{-1} = \frac{1}{e}$$

$$\lim_{x \rightarrow 0} (x * \ln x) = 0 \times \infty$$

Using L'Hôpital's Rule:

$$\lim_{x \rightarrow 0^+} (x * \ln x) = \lim_{x \rightarrow 0^+} \frac{\ln x}{\frac{1}{x}} = \lim_{x \rightarrow 0^+} \frac{\frac{1}{x}}{-\frac{1}{x^2}} = \lim_{x \rightarrow 0^+} -x = 0$$

1.10. Formal Definition of Continuity

From this information, a more formal definition can be found. Continuity, at a point a , is defined when the limit of the function from the left equals the limit from the right and this value is also equal to the value of the function. Using notation, for all points a where ,

$$\lim_{x \rightarrow x_0^-} f(x) = \lim_{x \rightarrow x_0^+} f(x) = f(x_0).$$

1.10.1. The function is said to be continuous.

Definition

Let $f: I \subset \mathbb{R} \rightarrow \mathbb{R}$ be a real function, $x_0 \in I$. We say that f is continuous at the point x_0 if the following holds:

$$\lim_{x \rightarrow x_0} f(x) = f(x_0)$$

1.10.2. Continue on the right (left) of the point.

- We say that f is continuous to the right of x_0 if the following holds: $\lim_{x \rightarrow x_0^+} f(x) = f(x_0)$
- We say that f is continuous to the left of x_0 if the following holds: $\lim_{x \rightarrow x_0^-} f(x) = f(x_0)$
- We say that f is continuous at x_0 if and only if f is continuous to the right of x_0 and to the left of x_0 at the same time.



If f is continuous at every point x_0 of I , we say that f is continuous on the set I .

EXAMPLE

1-The function $f(x) = x + 2$ is continuous on the entire set R because if $x_0 \in R$ is any point in R , we have:

$$\lim_{x \rightarrow x_0} f(x) = \lim_{x \rightarrow x_0} (x + 2) = x_0 + 2 = f(x_0)$$

2-Let the function g be defined as follows:

$$g(x) = \begin{cases} x^2; & x > 1 \\ 1; & x = 1 \\ x + 2; & x < 1 \end{cases}$$

We have : $\lim_{x \rightarrow 1^+} g(x) = 1 = g(1)$ And thus g is continuous to the right of 1,

And we have $\lim_{x \rightarrow 1^-} g(x) = 3 \neq g(1)$, hence g is not continuous to the left of 1, so g is not continuous at 1.

NOTE

For f to be continuous at x_0 , three conditions must be met:

- That f is defined at x_0 ,
- That $\lim_{x \rightarrow x_0} f(x)$ exists and is unique,
- To be $\lim_{x \rightarrow x_0} f(x) = f(x_0)$.

Therefore, the function f is discontinuous at x_0 if none of the three previous conditions is satisfied.

1.10.4. Definition function continuous on an interval:

A function f is continuous on an interval if it is continuous at every number in the interval. (If f is defined only on one side of an endpoint of the interval, we understand continuous at the endpoint to mean continuous from the right or continuous from the left at the endpoint as appropriate.)

EXAMPLE

The function $f(x) = x$ is continuous on R

The constant function on a continuous domain on the same domain

Function f defined by:

$$f: R \rightarrow R \quad x \rightarrow f(x) = \begin{cases} 1, & x \in Q \\ 0, & x \notin Q \end{cases}$$

Is not continuous at all points of R .

1.10.5. Properties of continuous functions :

Theory 1:

If $f, g: I \rightarrow R$ are continuous functions at the point $x_0 \in I$, then:

1-All of the following functions $\left((x_0) \neq 0, \frac{f}{g}\right), (f \cdot g), (f - g), (f + g)$ are continuous at the point x_0 .

2-If $f \circ g$ is defined, and f is continuous at the point $x_0 = g(x_0)$, then $f \circ g$ is continuous at the point x_0 .

Theory 2:

If $f, g: I \rightarrow R$ are functions, and f is continuous at the point y_0 where $\lim_{x \rightarrow x_0} g(x) = y_0$, then:

$$\lim_{x \rightarrow x_0} (f \circ g)(x) = f(y_0) .$$

Theory 3:

If $f, g: I \rightarrow R$ are functions, and f is continuous at the point y_0 where $\lim_{x \rightarrow \pm\infty} g(x) = y_0$, then:

$$\lim_{x \rightarrow \pm\infty} (f \circ g)(x) = f(y_0) .$$

Theory 4:

If $f, g: I \rightarrow R$ are functions, and f is continuous $\lim_{x \rightarrow \pm\infty} f(x) = l$ and $\lim_{x \rightarrow x_0} g(x) = \pm \infty$, then:

$$\lim_{x \rightarrow x_0} (f \circ g)(x) = l$$

1.10.6. Intermediate Value Theorem

Suppose that $f(x)$ is continuous on the closed interval $[ab]$ and let α be any number between $f(a)$ and $f(b)$, ($f(a) \neq f(b)$), then there exists a number c in the interval (ab) with:

$$f(c) = \alpha .$$

1.10.7. The Mean Value Theorem:

If $f: I \rightarrow R$ is a continuous function on the closed and limited domain $I = [a, b]$, then:

$$\forall \alpha \in [\inf f(x), \sup f(x)] \Rightarrow \exists c \in [a, b] : f(c) = \alpha$$

EXAMPLE

If we consider the function $f(x) = x^2 - 1$, on the interval $[0, 2]$, we see that: $f(0) = 1 < 0$ and $f(2) = 3 > 0$.

Therefore the intermediate value theorem says that there must be some number, c , between 0 and 2 with $f(c) = 0$. The graph of $f(x)$ crosses the x axis at the point where $x = c$. What is the value of c in this case?

$$f(x) = x^2 - 1 = 0; \text{ if } x^2 = 1$$

This is true if $x = \pm 1$, therefore for $c = 1 \in [0, 2]$, we have $f(c) = 0$.

1.10.8. Continuation extension:

Definition: If a function is defined on a domain

$$\lim_{x \rightarrow x_0} f(x) = l \text{ and } I \setminus \{x_0\}$$

function g is defined as follows:

$$g(x) = \begin{cases} f(x), & x \neq x_0 \\ l, & x = x_0 \end{cases}$$

It is an extension of the continuity of f

EXAMPLE

Let it be

$$f(x) = \frac{\sin x}{x}, x \neq 0$$

We have:

$$\lim_{x \rightarrow 0} \frac{\sin(x)}{x} = 1$$

By applying the L'Hôpital rule and from there g is a continuation of f where :

$$g(x) = \begin{cases} \frac{\sin x}{x}, & x \neq 0 \\ 1, & x = 0 \end{cases}$$

2. DEFINITION OF THE NATURAL LOGARITHM FUNCTION

One solid approach to defining and understanding logarithms begins with a study of the natural logarithm function defined as an integral through the Fundamental Theorem of Calculus. While this approach may seem indirect, it enables us to derive quickly the familiar properties of logarithmic and exponential functions. We use calculus for the very definition of the logarithmic functions. The natural logarithm of a positive number x , written as $\ln x$, is the value of an integral.

$$\ln x = \int_1^x \frac{1}{t} dt, x > 0$$

2.1. DEFINITION The Natural Logarithm Function x

$$\ln(1) = 0, \forall x \in]0, +\infty[, (\ln x)' = \frac{1}{x}$$

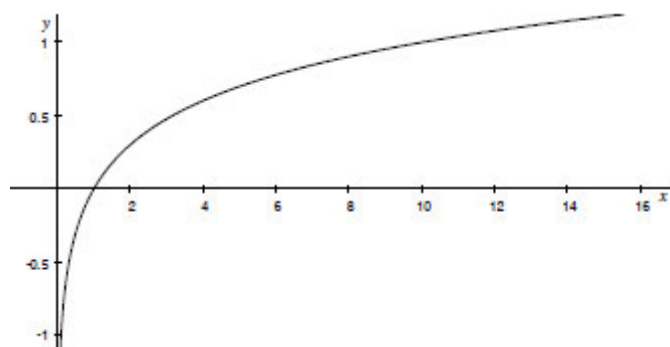
The function $\ln$ is a continuous and strictly increasing on $]0, +\infty[$.

CHAPTER THREE: LOGARITHMIC AND EXPONENTIAL FUNCTIONS

2.2. Graph

A logarithmic function is a function that possesses the general form $y = \ln(x)$. Logarithmic functions increase infinitely but at a decreasing rate, as x increases. The function has an asymptote at the vertical axis, meaning it approaches, but never touches, the y-axis. This is because you cannot take a logarithm of a negative number, and hence the graph only exists for positive values of x .

Consider the graph of a logarithmic function below,



2.3. Rules:

The below are rules and examples for using Natural Logarithm.

Rule	Example
$\ln(xy) = \ln x + \ln y$	$\ln(4 \sin x) = \ln 4 + \ln \sin x$
$\ln\left(\frac{1}{x}\right) = -\ln x$	$\ln\frac{1}{8} = -\ln 8$
$\ln\left(\frac{x}{y}\right) = \ln x - \ln y$	$\ln\frac{x+1}{2x-3} = \ln(x+1) - \ln(2x-3)$
$\ln(x^\alpha) = \alpha \ln x$	$\ln \sqrt[3]{x+1} = \ln(x+1)^{\frac{1}{3}} = \frac{1}{3} \ln(x+1)$

2.4. Some Limits of the logarithmic function:

$\lim_{x \rightarrow 0^+} \ln(x) = -\infty$	$\lim_{x \rightarrow +\infty} \ln(x) = +\infty$
$\lim_{x \rightarrow 0} \frac{\ln(x+1)}{x} = 1$	$\lim_{x \rightarrow +\infty} \frac{\ln(x+1)}{x} = 0$
$\lim_{x \rightarrow +\infty} \frac{\ln(x)}{x^\alpha} = 0^+, \alpha > 0$	$\lim_{x \rightarrow 0} \frac{\ln(\alpha x + 1)}{x} = \alpha$

$$\lim_{x \rightarrow 0} \frac{\alpha x}{\ln(\alpha x + 1)} = 1$$

$$\lim_{x \rightarrow 0} x^\alpha \ln(x) = 0, \alpha > 0$$

2.5. Solving logarithmic equations and inequalities:

By solving examples about:

2.5.1. logarithmic equations :

Solve each of the following logarithmic equations.

equations:	Solution
$\ln(x + 3) = 2$	$\Rightarrow e^{\ln(x+3)} = e^2$ $\Rightarrow x + 3 = e^2$ $\Rightarrow x = e^2 - 3$ $\Rightarrow x \approx 4.39$
$2\ln(1 - y) + 1 = 3$	$\Rightarrow 2\ln(1 - y) = 2$ $\Rightarrow \ln(1 - y) = 1$ $\Rightarrow e^{\ln(1-y)} = e^1$ $\Rightarrow 1 - y = e$ $\Rightarrow 1 - e = y$ $\Rightarrow y \approx -1.72$
$\ln(2w + 1) = 1 + \ln(w - 1)$	$\Rightarrow \ln(2w + 1) - \ln(w - 1) = 1$ $\Rightarrow \ln\left(\frac{2w + 1}{w - 1}\right) = e^1$ $\Rightarrow \frac{2w + 1}{w - 1} = e$ $\Rightarrow 2w + 1 = ew - e$ $\Rightarrow 1 + e = ew - 2w$ $\Rightarrow 1 + e = w(e - 2)$ $\Rightarrow w = \frac{1 + e}{e - 2}$ $\Rightarrow w \approx 5.18$

2.5.2. logarithmic inequalities:

Solve each of the following logarithmic inequalities.

inequalities:	Solution
$\ln x \leq -2$	$\Rightarrow e^{\ln x} \leq e^{-2}$ $\Rightarrow x \leq \frac{1}{e^2}$
$\ln(x + 5) > 5$	$\Rightarrow e^{\ln(x+5)} > e^5$ $\Rightarrow x + 5 > e^5$ $\Rightarrow x > e^5 - 5$

3. EXPONENTIAL FUNCTIONS AND THEIR GRAPHS

Exponential functions are functions like $f(x) = e^x$ where the base e is a fixed number and the index is given different values. When $e > 1$, the function increases rapidly as x increases, and when $e < 1$ the function decreases rapidly as x increases

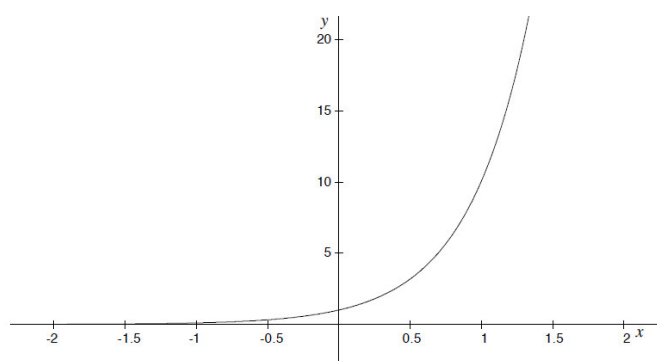
$$e : \mathbb{R} \rightarrow \mathbb{R}^+$$

$$x \mapsto e^x$$

3.1. Graph

An exponential function is a function that possesses the general form of $y = e^x$ where “ e ” is any constant greater than one. The graph of an exponential function will increase at an increasing rate as x increases, and it will also have an **asymptote** at the x -axis, that is, it will approach the x -axis but never touch it.

Consider the graph of an exponential function below,



3.2. Rules:

The below are rules and examples for using exponentials.

Rule	Example
$e^0 = 1$	$2e^0 + 3 = 5$
$\forall x \in \mathbb{R}, \quad \ln(e^x) = x$	$\ln(e^{2x+3}) = 2x + 3$
$\forall x \in \mathbb{R}^+, \quad e^{\ln(x)} = x$	$e^{\ln(x^2+5x-2)} = x^2 + 5x - 2$
$\forall (x, y) \in \mathbb{R} \times \mathbb{R}, \quad e^{x+y} = e^x \times e^y$	$e^5 + e^2 = e^7$
$\forall (x, y) \in \mathbb{R} \times \mathbb{R}, \quad e^{x-y} = \frac{e^x}{e^y}$	$\frac{e^{5x-2}}{e^{x^2}} = e^{5x-2-x^2}$
$\forall x \in \mathbb{R}, \quad e^{-x} = \frac{1}{e^x}$	$e^{-4} = \frac{1}{e^4}$

$a^x = e^{x \ln(a)}$	$5^x = e^{x \ln(5)}$
$(e^{f(x)})' = f'(x)e^{f(x)}$	$(e^{2x^2+5x})' = (4x+5)e^{2x^2+5x}$

3.3. Some Limits of an exponential function:

$\lim_{x \rightarrow -\infty} e^x = 0$	$\lim_{x \rightarrow +\infty} e^x = +\infty$
$\lim_{x \rightarrow +\infty} \frac{e^x}{x^\alpha} = +\infty, \alpha \in \mathbb{Z}$	$\lim_{x \rightarrow 0} \frac{e^x - 1}{x} = 1$

3.4. Solving exponential equations and inequalities

By solving examples about:

3.4.1. Exponential equations:

Solve each of the following exponential equations.

equations:	Solution
$e^{x+1} = 17$	$e^{x+1} = 17 \Rightarrow \ln(e^{x+1}) = \ln 17$ $\Rightarrow x+1 = \ln 17$ $\Rightarrow x = -1 + \ln 17$ $\Rightarrow x \approx 1.83$
$3e^y + 2e^{-y} = 7$	$\Rightarrow 3e^y + \frac{2}{e^y} = 7$ $\Rightarrow 3a + \frac{2}{a} = 7, \quad (a = e^y)$ $\Rightarrow 3a^2 + 2 = 7a$ $\Rightarrow 3a^2 - 7a + 2 = 0$ $\Rightarrow (3a-1)(a-2) = 0$ $\Rightarrow a = \begin{cases} a_1 = 2 \\ a_2 = \frac{1}{3} \end{cases} \Rightarrow e^y = \begin{cases} 2 \\ \frac{1}{3} \end{cases}$ $\Rightarrow y = \begin{cases} \ln 2 \\ \ln \frac{1}{3} = -\ln 3 \end{cases}$
$e^{2x} + 15 = 8e^x$	$a^2 + 15 = 8a, \quad (a = e^x)$ $a^2 - 8a + 15 = 0$ $\Rightarrow (a-3)(a-5) = 0$ $\Rightarrow a = \begin{cases} a_1 = 3 \\ a_2 = 5 \end{cases} \Rightarrow e^x = \begin{cases} 3 \\ 5 \end{cases}$ $\Rightarrow x = \begin{cases} \ln 3 \\ \ln 5 \end{cases}$

3.4.2. exponential inequalities:

Solve each of the following exponential inequalities.

inequalities:	Solution
$e^{4-3x} > 20$	$\Rightarrow \ln(e^{4-3x}) > \ln 20$ $\Rightarrow 4 - 3x > \ln 20$ $\Rightarrow 4 - \ln 20 > 3x$ $x > \frac{1}{3}(4 - \ln 20)$ $\Rightarrow x > 0,335$
$e^x > 4$	$\Rightarrow \ln(e^x) > \ln 4$ $\Rightarrow x > 2 \ln 2$

Practical exercise:

1- A cup of coffee is cooling down in a room.

The temperature $T^\circ\text{C}$ of the coffee, t minutes after it was made is modelled by

$$T = 20 + 50e^{\frac{-t}{15}}, \quad t \geq 0$$

- State the temperature of the coffee when it was first made.
- Find the temperature of the coffee, after 30 minutes.
- Calculate, to the nearest minute, the value of t when the temperature of the coffee has reached 35°C .



a) $T = 20 + 50e^{\frac{-t}{15}}$

$t = 0$

$$T = 20 + 50e^0$$

$$\Rightarrow T = 20 + 50 \Rightarrow T = 70^\circ\text{C}$$

b) $t = 30$

$$T = 20 + 50e^{-2} \Rightarrow T \approx 26.8^\circ\text{C}$$

c) $t = 35$

$$35 = 20 + 50e^{\frac{-t}{15}}$$

$$\Rightarrow 15 = 50e^{\frac{-1}{15}t} \Rightarrow \frac{3}{10} = e^{\frac{-1}{15}t}$$

$$\Rightarrow \frac{10}{3} = e^{\frac{1}{15}t} \Rightarrow \ln\left(\frac{10}{3}\right) = \frac{1}{15}t$$

$$\Rightarrow t = 15 \ln\left(\frac{10}{3}\right)$$

$$\Rightarrow t \approx 18$$

SERIES NUMBER 3

Exercise 01:

1- Create a definition set for each function in the following cases:

$$f_1(x) = \sqrt{(x+2)} + \frac{1}{x+1}; \quad f_2(x) = -\frac{x}{2} + \ln \left| \frac{x-1}{x} \right|; \quad f_3(x) = e^{\frac{1}{x^2}}; \quad f_4(x) = 2x - |x| \ln(x^2)$$

$$f_5(x) = \frac{2|x-1|}{x^2+3x+2}; \quad f_6(x) = \begin{cases} \frac{3(x+1)}{\sqrt{x+2}}, & x \leq 1 \\ \frac{x^2}{2x-1}, & x > 1 \end{cases}; \quad f_7(x) =$$

$$\begin{cases} x + \frac{1}{2} + \frac{1}{x^2+1}, & x \leq 0 \\ \frac{x}{x^2-1} + \frac{3}{2}, & x > 0 \end{cases}$$

2- Determine the type of the following functions (odd or even).

$$f(x) = 5x^3 - 2x; \quad f(x) = x^2 + \sqrt{2x^2 + 61}; \quad f(x) = \frac{x^3}{\sqrt{x^2+5}}; \quad f(x) = \frac{3}{\sqrt{x^4+2}}$$

Exercise 02: Compute the limits of the following functions:

$$\lim_{x \rightarrow 2} \frac{x-2}{\sqrt{x^2-4}}; \quad \lim_{x \rightarrow 0} \frac{\sin x}{\tan x}; \quad \lim_{x \rightarrow 3} \frac{\sqrt{x^2-9}}{x-3}; \quad \lim_{x \rightarrow 0} \frac{e^{2x}-1}{3x^2+x}; \quad \lim_{x \rightarrow 1} \frac{x \ln x - x + 1}{(x-1) \ln x};$$

$$\lim_{x \rightarrow \infty} \frac{\sqrt{x^2+1}}{x+1}$$

Exercise 03 : Study the continuity of the following functions:

$$g(x) = \begin{cases} (x+1)e^{x^2}, & x \geq -1 \\ (x+1)^2 \cos\left(\frac{3}{x+1}\right), & x < -1 \end{cases}; \quad f(x) = \begin{cases} \ln(1+x), & x \geq 0 \\ x^4 \sin\left(\frac{5}{x}\right), & x < 0 \end{cases}$$

Exercise 04 :

1- Find "a" values so that f and g are continuous

$$g(x) = \begin{cases} \frac{\sqrt{x+2}-2}{x-2}, & x > 2 \\ a + e^{x-2}, & x \leq 2 \end{cases}; \quad f(x) = \begin{cases} \frac{\ln(x+1)}{x}, & x < 0 \\ x^5 + 2 + a, & x \geq 0 \end{cases}$$

2-Is it possible to continue extending the f function where:

$$f(x) = \frac{\sqrt{x-1}-2}{x-5}, \quad x \neq 5$$

ANSWER TO SERIES NUMBER 3

Exercise 01:

1- Find the definition set of the following functions:

$$f_1(x) = \sqrt{(x+2)} + \frac{1}{x+1}$$

$$D_{f_1} = \{x / x+2 \geq 0 \wedge x+1 \neq 0\};$$

$$x+2 \geq 0 \Rightarrow x \geq -2 \Rightarrow x \in [-2, +\infty[$$

$$x+1 \neq 0 \Rightarrow x \neq -1 \Rightarrow x \in \mathbb{R} - \{-1\} \Rightarrow x \in]-\infty, -1[\cup]-1, +\infty[$$

$$D_{f_1} = \{x / x \in]-1, +\infty[\}$$

$$f_2(x) = -\frac{x}{2} + \ln \left| \frac{x-1}{x} \right|$$

$$D_{f_2} = \{x / |x-1| > 0 \wedge |x| > 0\}$$

$$|x-1| > 0 \Rightarrow \begin{cases} x-1 > 0 \Rightarrow x > +1 \\ -x+1 > 0 \Rightarrow -x > -1 \Rightarrow x < +1 \end{cases}$$

$$x \in]-\infty, 1[\cup]1, +\infty[$$

$$|x| > 0 \Rightarrow \begin{cases} x > 0 \\ -x > 0 \Rightarrow x < 0 \end{cases}$$

$$\Rightarrow x \in]-\infty, 0[\cup]0, +\infty[$$

$$D_{f_2} = \{x / x \in \mathbb{R} - \{0, 1\}\} = \{x / x \in]-\infty, 0[\cup]0, 1[\cup]1, +\infty[\}$$

$$f_3(x) = e^{\frac{1}{x^2}}$$

$$D_{f_3} = \{x / x^2 \neq 0\}$$

$$x^2 \neq 0 \Rightarrow x \neq 0 \Rightarrow x \in \mathbb{R} - \{0\}$$

$$D_{f_3} = \{x / x \in]-\infty, 0[\cup]0, +\infty[\}$$

$$f_4(x) = 2x - |x| \ln(x^2)$$

$$D_{f_4} = \{x / x^2 > 0\}$$

$$x^2 > 0 \Rightarrow x \in \mathbb{R} - \{0\}$$

$$D_{f_4} = \{x / x \in]-\infty, 0[\cup]0, +\infty[\}$$

$$f_5(x) = \frac{2|x-1|}{x^2+3x+2}$$

$$D_{f_5} = \{x / x^2 + 3x + 2 \neq 0\}$$

$$x^2 + 3x + 2 = 0 \Rightarrow \sqrt{\Delta} = 1 \Rightarrow x_1 = -1, x_2 = -2$$

$$D_{f_5} = \{x / x \in \mathbb{R} - \{-2, -1\}\} = \{x / x \in]-\infty, -2[\cup]-2, -1[\cup]-1, +\infty[\}$$

$$f_6(x) = \begin{cases} \frac{3(x+1)}{\sqrt{x+2}} & , \quad x \leq 1 \\ \frac{x^2}{2x-1} & , \quad x > 1 \end{cases};$$

$$D_{f_6} = \{x / (x + 2 > 0) \cap (x \leq 1)\} \cup \{x / 2x - 1 \neq 0 \cap x > 1\}$$

$$\begin{cases} x + 2 > 0 \Rightarrow x > -2 \\ x \leq 1 \end{cases}$$

$$x \in]-2, +\infty[\text{ and }]-\infty, 1] \Rightarrow x \in]-2, 1]$$

and we have :

$$2x - 1 \neq 0 \Rightarrow x \neq \frac{1}{2}$$

$$x \in]-\infty, \frac{1}{2}[\cup]\frac{1}{2}, +\infty[$$

$$x > 1 \Rightarrow x \in]1, +\infty[\text{ and we have } \Rightarrow x \in]1, +\infty[$$

So Union is:

$$D_{f_6} = \{x / x \in]-2, 1] \cup]1, +\infty[\}$$

$$D_{f_6} = \{x / x \in]-2, +\infty[\}$$

$$f_7(x) = \begin{cases} x + \frac{1}{2} + \frac{1}{x^2+1} & , \quad x \leq 0 \\ \frac{x}{x^2-1} + \frac{3}{2} & , \quad x > 0 \end{cases}$$

$$D_{f_7} = \{x / (x^2 + 1 \neq 0) \cap (2 \leq 1)\} \cup \{x / x^2 - 1 \neq 0 \cap x > 1\}$$

$$\begin{cases} x^2 + 1 = 0 \Rightarrow x^2 = -1 \\ \text{no solution in } R \\ \text{and } x \leq 0 \end{cases}$$

$$x \in]-\infty, 0]$$

and we have :

$$x^2 - 1 = 0 \Rightarrow x^2 = 1$$

$$x \in]-\infty, -1[\cup]-1, 1[\cup]1, +\infty[$$

$$x > 1 \Rightarrow x \in]1, +\infty[\text{ and we have } \Rightarrow x \in]1, +\infty[$$

and $x > 0$

$$x \in]0, 1] \cup]1, +\infty[$$

So Union is:

$$D_{f_7} = \{x / x \in]-\infty, 1] \cup]1, +\infty[\}$$

1-Determine the type of the following functions (odd or even):

CHAPTER THREE: LOGARITHMIC AND EXPONENTIAL FUNCTIONS

We say that the function is odd if : $\forall x \in X, f(-x) = -f(x)$

We say that the function is even if : $\forall x \in X, f(-x) = f(x)$

$$1- f(x) = 5x^3 - 2x$$

$$f(-x) = 5(-x)^3 - 2(-x) = -5x^3 + 2x \Rightarrow -(5x^3 - 2x) = -f(x)$$

And hence the function is odd.

$$2- f(x) = x^2 + \sqrt{2x^2 + 61}$$

$$f(-x) = (-x)^2 + \sqrt{2(-x)^2 + 61} = x^2 + \sqrt{2x^2 + 61} = f(x)$$

And hence the function is even.

$$3- f(x) = \frac{x^3}{\sqrt{x^2+5}}$$

$$f(-x) = \frac{(-x)^3}{\sqrt{(-x)^2+5}} = \frac{-x^3}{\sqrt{x^2+5}} = -f(x)$$

And hence the function is odd.

$$4- f(x) = \frac{3}{\sqrt{x^4+2}}$$

$$f(-x) = \frac{3}{\sqrt{(-x)^4+2}} = \frac{3}{\sqrt{x^4+2}} = f(x)$$

And hence the function is even.

Exercise 2:

$$\lim_{x \rightarrow 2} \frac{x-2}{\sqrt{x^2-4}} = \frac{0}{0}$$

By using L'Hôpital's rule, we find:

$$\begin{aligned} \lim_{x \rightarrow 2} \frac{x-2}{\sqrt{x^2-4}} &= \lim_{x \rightarrow 2} \frac{(x-2)'}{(\sqrt{x^2-4})'} \\ &= \lim_{x \rightarrow 2} \frac{\frac{1}{2x}}{\frac{2x}{2\sqrt{x^2-4}}} = \lim_{x \rightarrow 2} \frac{\sqrt{x^2-4}}{2x} = \frac{0}{4} = 0 \end{aligned}$$

$$\lim_{x \rightarrow 0} \frac{\sin x}{\tan x} = \frac{0}{0}$$

So we have : $\tan x = \frac{\sin x}{\cos x}$, then:

$$\lim_{x \rightarrow 0} \frac{\sin x}{\tan x} = \lim_{x \rightarrow 0} \frac{\sin x}{\frac{\sin x}{\cos x}}$$

$$= \lim_{x \rightarrow 0} \frac{\sin x}{\sin x} \cos x = \lim_{x \rightarrow 0} \cos x = \cos(0) = 1$$

$$\lim_{x \rightarrow 3} \frac{\sqrt{x^2-9}}{x-3} = \frac{0}{0}$$

Case of non-appointment and therefore:

$$\begin{aligned} \lim_{x \rightarrow 3} \frac{\sqrt{x^2-9}}{x-3} &= \lim_{x \rightarrow 3} \left(\frac{\sqrt{x^2-9}}{x-3} \times \frac{\sqrt{x^2-9}}{\sqrt{x^2-9}} \right) \\ &= \lim_{x \rightarrow 3} \frac{\sqrt{x^2-9} \times \sqrt{x^2-9}}{(x-3) \times \sqrt{x^2-9}} = \lim_{x \rightarrow 3} \frac{x^2-9}{(x-3) \times \sqrt{x^2-9}} \\ &= \lim_{x \rightarrow 3} \frac{(x-3)(x+3)}{(x-3) \times \sqrt{x^2-9}} = \lim_{x \rightarrow 3} \frac{(x+3)}{\sqrt{x^2-9}} \\ &= \lim_{x \rightarrow 3} \left[(x+3) \frac{1}{\sqrt{x^2-9}} \right] = 6 + \infty = +\infty \end{aligned}$$

$$\lim_{x \rightarrow 0} \frac{e^{2x}-1}{3x^2+x} = \frac{0}{0}$$

By using L'Hôpital's rule, we find:

$$\lim_{x \rightarrow 0} \frac{(e^{2x}-1)'}{(3x^2+x)'} = \lim_{x \rightarrow 0} \frac{2e^{2x}}{6x+1} = \frac{2e^{2(0)}}{6(0)+1} = 2$$

$$\lim_{x \rightarrow 1} \frac{x \ln x - x + 1}{(x-1) \ln x} = \frac{0}{0}$$

By using L'Hôpital's rule, we find:

$$\begin{aligned} \lim_{x \rightarrow 1} \frac{x \ln x - x + 1}{(x-1) \ln x} &= \lim_{x \rightarrow 1} \frac{(x \ln x - x + 1)'}{((x-1) \ln x)'} = \lim_{x \rightarrow 1} \frac{\ln x - x \frac{1}{x} - 1}{\ln x + \frac{1}{x}(x-1)} = \lim_{x \rightarrow 1} \frac{\ln 1 - 1 - 1}{\ln 1 + 1(1-1)} \\ &= \frac{0}{0} \end{aligned}$$

By using L'Hôpital's rule again, we find:

$$\lim_{x \rightarrow 1} \frac{(\ln x)'}{(\ln x + (1+x^{-1}))'} = \lim_{x \rightarrow 1} \frac{\frac{1}{x}}{\frac{1}{x} + x^{-2}} = \frac{\frac{1}{1}}{\frac{1}{1} + 1^{-2}} = \frac{1}{2}$$

$$\lim_{x \rightarrow \infty} \frac{\sqrt{x^2+1}}{x+1} = \frac{+\infty}{+\infty}$$

Case of non-appointment and therefore:

$$\lim_{x \rightarrow \infty} \frac{\sqrt{x^2 + 1}}{x + 1} = \lim_{x \rightarrow \infty} \frac{\sqrt{x^2 \left(1 + \frac{1}{x^2}\right)}}{x \left(1 + \frac{1}{x}\right)} = \lim_{x \rightarrow \infty} \frac{x \sqrt{\left(1 + \frac{1}{x^2}\right)}}{x \left(1 + \frac{1}{x}\right)} = \lim_{x \rightarrow \infty} \frac{1 + \frac{1}{x^2}}{1 + \frac{1}{x}} = 1$$

Exercise 3:

Study the continuity of the following functions:

$$1-f(x) = \begin{cases} \ln(1+x) & , \quad x \geq 0 \\ x^4 \sin\left(\frac{5}{x}\right) & , \quad x < 0 \end{cases}$$

The function f is continuous at x_0 if: $\lim_{x \rightarrow x_0^-} f(x) = \lim_{x \rightarrow x_0^+} f(x) = f(x_0)$

$$\lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0^+} \ln(1+x) = \ln(1) = 0$$

$$\lim_{x \rightarrow 0^-} f(x) = \lim_{x \rightarrow 0^-} x^4 \sin\left(\frac{5}{x}\right) = 0$$

Because according to our multiplication property:

$$\begin{cases} -1 \leq \sin \frac{5}{x} \leq 1 \\ \lim_{x \rightarrow 0} x^4 = 0 \end{cases}$$

The first is bounded and the second approaches zero, so the product approaches zero, and we have:

$$f(0) = \ln(1+0) = \ln(1) = 0$$

$$\lim_{x \rightarrow 0^-} f(x) = \lim_{x \rightarrow 0^+} f(x) = f(0) = 0$$

Therefore, the function f is continuous at "0".

$$2-g(x) = \begin{cases} (x+1)e^{x^2} & , \quad x \geq -1 \\ (x+1)^2 \cos\left(\frac{3}{x+1}\right) & , \quad x < -1 \end{cases}$$

The function g is continuous at x_0 if: $\lim_{x \rightarrow x_0^-} g(x) = \lim_{x \rightarrow x_0^+} g(x) = g(x_0)$

$$\lim_{x \rightarrow -1^+} g(x) = \lim_{x \rightarrow -1^+} (x+1)e^{x^2} = (0)e^1 = 0$$

$$\lim_{x \rightarrow -1^-} g(x) = \lim_{x \rightarrow -1^-} (x+1)^2 \cos\left(\frac{3}{x+1}\right) = 0$$

Because according to our multiplication property:

$$\begin{cases} -1 \leq \cos\left(\frac{3}{x+1}\right) \leq 1 \\ \lim_{x \rightarrow -1} (x+1)^2 = 0 \end{cases}$$

The first is bounded and the second approaches zero, so the product approaches zero, and we have:

$$g(-1) = (-1 + 1)e^1 = 0$$

$$\lim_{x \rightarrow -1^-} g(x) = \lim_{x \rightarrow -1^+} g(x) = g(-1) = 0$$

Therefore, the function g is continuous at -1 .

Exercise 4:

Find the value of a so that the functions f and g are continuous:

$$1- f(x) = \begin{cases} \frac{\ln(x+1)}{x} & , \quad x < 0 \\ x^5 + 2 + a & , \quad x \geq 0 \end{cases}$$

$$\lim_{x \rightarrow 0^-} f(x) = \lim_{x \rightarrow 0^-} \frac{\ln(x+1)}{x} = \frac{0}{0}$$

By applying L'Hôpital's rule, we find:

$$\lim_{x \rightarrow 0^-} f(x) = \lim_{x \rightarrow 0^-} \frac{\frac{1}{x+1}}{1} = \lim_{x \rightarrow 0^+} \frac{1}{x+1} = 1$$

$$\lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0^+} x^5 + 2 + a = 2 + a$$

$$\lim_{x \rightarrow 0^-} f(x) = \lim_{x \rightarrow 0^+} f(x) = 2 + a = 1 \Rightarrow a = -1$$

$$2- g(x) = \begin{cases} \frac{\sqrt{x+2}-2}{x-2} & , \quad x > 2 \\ a + e^{x-2} & , \quad x \leq 2 \end{cases}$$

$$\lim_{x \rightarrow 2^+} g(x) = \lim_{x \rightarrow 2^+} \frac{\sqrt{x+2}-2}{x-2} = \frac{0}{0}$$

By applying L'Hôpital's rule, we find:

$$\lim_{x \rightarrow 2^+} \frac{\frac{1}{2\sqrt{x+2}}}{1} = \lim_{x \rightarrow 2^+} \frac{1}{2\sqrt{x+2}} = \frac{1}{4}$$

$$\lim_{x \rightarrow 2^-} (a + e^{x-2}) = a + e^0 = a + 1$$

$$\lim_{x \rightarrow 2^-} g(x) = \lim_{x \rightarrow 2^+} g(x) = a + 1 = \frac{1}{4} \Rightarrow a = -\frac{3}{4}$$

2-Can the function be extended continuously?

$$f(x) = \frac{\sqrt{x-1}-2}{x-5} \quad , \quad x \neq 5$$

We calculate the limit at : $x_0 = 5$

$$\lim_{x \rightarrow 5} f(x) = \lim_{x \rightarrow 5} \frac{\sqrt{x-1} - 2}{x-5} = \frac{0}{0}$$

By using L'Hôpital's rule:

$$\lim_{x \rightarrow 5} \frac{\sqrt{x-1} - 2}{x-5} = \lim_{x \rightarrow 5} \frac{\frac{1}{2\sqrt{x-1}}}{1} = \frac{1}{4}$$

We have: $\lim_{x \rightarrow 5} f(x) = l$ if $f(x)$ is extendable continuously at the point “5” and we write:

$$f(x) = \begin{cases} \frac{\sqrt{x-1} - 2}{x-5} & , \quad x \neq 5 \\ \frac{1}{4} & , \quad x = 5 \end{cases}$$

CHAPTER FOUR: DERIVATIVES

CHAPTER FOUR: DERIVATIVES

1. GENERALITIES ABOUT DERIVATIVES:

1.1. Definitions: We will present a set of definitions for differentiation as follows:

Definition 1:

Let $f: [a, b] \rightarrow \mathbb{R}$ be a function of a real variable, and let $x_0 \in [a, b]$. The rate of increase of the function f at the point x_0 is the function $T_{x_0}: [a, b] \setminus \{x_0\} \rightarrow \mathbb{R}$ defined by:

$$T_{x_0}(x) = \frac{f(x) - f(x_0)}{x - x_0}.$$

By setting $h = x - x_0$, the rate of increase of the function f at the point x_0 is written as:

$$T_{x_0}(h) = \frac{f(x_0 + h) - f(x_0)}{h}.$$

The real number $T_{x_0}(h)$ is the slope of the secant line that connects the two points: $(x_0, f(x_0))$ and $(x_0 + h, f(x_0 + h))$

EXAMPLE

Let the function f be set to R by $f(x) = x^2$.

Let's calculate the number derived from the function f into any real number a .

For $h \neq 0$:

$$\frac{f(a+h) - f(a)}{h} = \frac{(a+h)^2 - a^2}{h} = \frac{a^2 + 2ah + h^2 - a^2}{h} = 2a + h$$

$$\text{Or: } \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h} = \lim_{h \rightarrow 0} 2a + h = 2a$$

For any number a , the number derived from the function f equal to $2a$ is associated.

So we set it to R a function, denoted f' whose expression is:

$$f'(x) = 2x.$$

This function is called the derivative function of f .

Definition 2:

Let the function $f: [a, b] \rightarrow \mathbb{R}$, and let $x_0 \in [a, b]$. We define the derivative of f at the point x_0 as the limit (if it exists) of the increment ratio of the function f at the point x_0 as x approaches x_0 . In this case, we say that f is differentiable at the point x_0 , and we call the limit the derivative of the function f at the point x_0 and denote it by:

$$f'(x_0) = \lim_{x \rightarrow x_0} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{h \rightarrow 0} \frac{f(x_0 + h) - f(x_0)}{h}.$$

CHAPTER FOUR: DERIVATIVES

We say that the function f is differentiable on the interval $[a, b]$ if it is differentiable at every point x in the interval $[a, b]$. In this case, the function defined by $x \mapsto f'(x)$ is called the derivative function of f , and we denote it by f' .



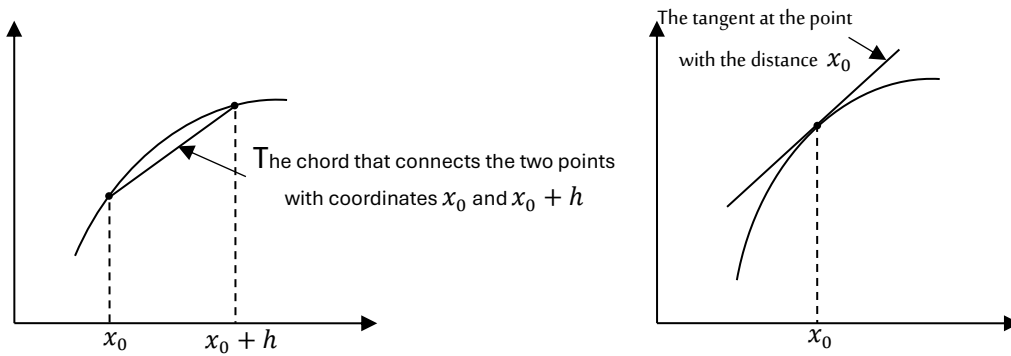
If $x_0 = a$, then the previous limit is a right-hand limit (with larger values), that is:

$$f'(a) = \lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a} = \lim_{x \rightarrow a^+} \frac{f(x) - f(a)}{x - a}.$$

If $x_0 = b$, then the previous limit is a left-hand limit (with smaller values), that is :

$$f'(b) = \lim_{x \rightarrow b} \frac{f(x) - f(b)}{x - b} = \lim_{x \rightarrow b^-} \frac{f(x) - f(b)}{x - b}.$$

Figure 1: Geometric interpretation of the derivative



Definition 3:

Let the function $f: [a, b] \rightarrow \mathbb{R}$, and let $x_0 \in]a, b[$.

– We say that f is differentiable to the right of x_0 if the limit:

$$\lim_{x \rightarrow x_0^+} \frac{f(x) - f(x_0)}{x - x_0}$$

It exists and is called the right-hand derivative of the function f at the point x_0 , denoted by:

$$f'_d(x_0) = \lim_{x \rightarrow x_0^+} \frac{f(x) - f(x_0)}{x - x_0}.$$

– We say that f is differentiable to the left of x_0 if the limit:

$$\lim_{x \rightarrow x_0^-} \frac{f(x) - f(x_0)}{x - x_0}$$

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It exists, and it is called the left-hand derivative of the function f at the point x_0 , denoted by:

$$f'_g(x_0) = \lim_{x \rightarrow x_0} \frac{f(x) - f(x_0)}{x - x_0}.$$



If the derivatives $f'_d(x_0)$ and $f'_g(x_0)$ exist and $f'_d(x_0) = f'_g(x_0)$, then the function f is differentiable at x_0 and we have:

$$f'(x_0) = f'_d(x_0) = f'_g(x_0).$$

EXAMPLE

1-Let the function defined on $\mathbb{R}$ be $f(x) = x^2$ we have :

$$\lim_{x \rightarrow 1} \frac{f(x) - f(1)}{x - 1} = \lim_{x \rightarrow 1} \frac{x^2 - 1}{x - 1} = 2.$$

Thus, f is differentiable at the point $x_0 = 1$ and $f'(1) = 2$.

2-Let the function defined on $\mathbb{R}^+$ be $f(x) = \sqrt{x}$. We have:

$$\lim_{x \rightarrow 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \rightarrow 0} \frac{\sqrt{x} - 0}{x} = \lim_{x \rightarrow 0} \frac{1}{\sqrt{x}} = +\infty.$$

Thus, f is not differentiable at the point $x_0 = 0$.

3- Let the function defined on $\mathbb{R}$ be $f(x) = |x|$. We have:

$$f'_d(0) = \lim_{x \rightarrow 0} \frac{x - 0}{x} = 1$$

and

$$f'_g(0) = \lim_{x \rightarrow 0} \frac{-x - 0}{x} = -1.$$

$f'_g(0)$ and $f'_d(0)$ exist, hence f is differentiable to the right and to the left of the point $x_0 = 0$, but $f'_d(0) \neq f'_g(0)$, therefore f is not differentiable at the point $x_0 = 0$.

CHAPTER FOUR: DERIVATIVES

Table 1: Derivatives of Common Functions

Function f	derivative function f'	Ensemble de définition de f
(Real constant) a	0	$\mathbb{R}$
x	1	$\mathbb{R}$
$x^n, n \in \mathbb{N}$	nx^{n-1}	$\mathbb{R}$
$\frac{1}{x^n}$	$-\frac{n}{x^{n+1}}$	$\mathbb{R}/\{0\}$
$\sqrt{x}$	$\frac{1}{2\sqrt{x}}$	$[0, +\infty[$
$\frac{1}{x}$	$-\frac{1}{x^2}$	$\mathbb{R}/\{0\}$
$\cos x$	$-\sin x$	$\mathbb{R}$
$\sin x$	$\cos x$	$\mathbb{R}$
$\ln x$	$\frac{1}{x}$	$]0, +\infty[$
e^x	e^x	$\mathbb{R}$

1.2. Rules of derivation:

Theory 1:

Let $f: [a, b] \rightarrow \mathbb{R}$ and $g: [a, b] \rightarrow \mathbb{R}$ be two real functions defined on the interval $[a, b]$ and differentiable at the point $x_0 \in [a, b]$.

1-The function $f + g$ is differentiable at x_0 and we have:

$$(f + g)'(x_0) = (f' + g')(x_0) = f'(x_0) + g'(x_0).$$

2-The function $f \times g$ is differentiable at x_0 and we have:

$$(f \times g)'(x_0) = (f' \times g + f \times g')(x_0) = f'(x_0)g(x_0) + f(x_0)g'(x_0).$$

3- If $g(x_0) \neq 0$, then the function $\frac{f}{g}$ is differentiable at x_0 and we have:

$$\left(\frac{f}{g}\right)'(x_0) = \left(\frac{f' \times g - f \times g'}{g^2}\right)(x_0) = \frac{f'(x_0) \times g(x_0) - f(x_0) \times g'(x_0)}{(g(x_0))^2}.$$

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Theory 2:

Let $f: [a, b] \rightarrow [c, d]$ be a function defined on $[a, b]$ and differentiable at $x_0 \in [a, b]$, and let $g: [c, d] \rightarrow \mathbb{R}$ be a function defined on $[c, d]$ and differentiable at $f(x_0) \in [c, d]$. Then the function $g \circ f: [a, b] \rightarrow \mathbb{R}$ is differentiable at x_0 , and we have:

$$(g \circ f)'(x_0) = f'(x_0) \times g'(f(x_0)).$$

Table 2: Derivation Rules

Function	derivative function f'	Example
af	af'	$f(x) = 4x^3 \Rightarrow f'(x) = 12x^2$
$f + g$	$f' + g'$	$f(x) = x + x^2 \Rightarrow f'(x) = 1 + 2x$
$f \times g$	$f'g + fg'$	$f(x) = (3x^2 + 4x)(5x - 1) \Rightarrow f'(x)$ $= [(6x + 4)(5x - 1)] + [(5)(3x^2 + 4x)]$ $= 45x^2 + 34x - 4$
$\frac{1}{f}$	$\frac{-f'}{f^2}$	$f(x) = \frac{1}{2x^2 + 5x} \Rightarrow f'(x) = \frac{-4x - 5}{(2x^2 + 5x)^2}$
$\frac{f}{g}$	$\frac{f'g - fg'}{g^2}$	$f(x) = \frac{6x - 5}{x^3 - 2x^2 - 1}$ $\Rightarrow f'(x) = \frac{6(x^3 - 2x^2 - 1) - (6x - 5)(3x^2 - 4x)}{(x^3 - 2x^2 - 1)^2}$ $= \frac{-12x^3 + 272x^2 - 20x - 6}{(x^3 - 2x^2 - 1)^2}$
$(f)^n$	$nf'f^{n-1}$	$f(x) = (3x^2 + x - 1)^3$ $\Rightarrow f'(x) = 3(6x + 1)(3x^2 + x - 1)$
$\sqrt{f}$	$\frac{f'}{2\sqrt{f}}$	$f(x) = 3x^2 + 4\sqrt{x} \Rightarrow f'(x) = 6x + \frac{2}{\sqrt{x}}$
$\ln f$	$\frac{f'}{f}$	$f(x) = \ln(3x^2 + 5x) \Rightarrow f'(x) = \frac{6x + 5}{3x^2 + 5x}$
e^f	$f'e^f$	$f(x) = e^{2x^3 + x^2}$ $\Rightarrow f'(x) = (6x^2 + 2x)e^{2x^3 + x^2}$

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1.3. Properties of differentiation:

Theory 3:

Let $f: [a, b] \rightarrow \mathbb{R}$ be a real function defined on the interval $[a, b]$. If f is differentiable at the point $x_0 \in [a, b]$, then it is continuous at that point, but the reverse is not true.

Theory 4:

Let $f: [a, b] \rightarrow [c, d]$ be a real-valued function defined, continuous, and monotonic on the interval $[a, b]$. If f is differentiable at the point $x_0 \in [a, b]$ and $f'(x_0) \neq 0$, then the inverse function $f^{-1}: [c, d] \rightarrow [a, b]$ is differentiable at the point $y_0 = f(x_0)$ and we have:

$$(f^{-1})'(y_0) = \frac{1}{f'(x_0)}$$

1.4. Rolle's theorem :

Let f be: $f: [a, b] \rightarrow \mathbb{R}$ a defined and continuous real function on the domain $[a, b]$. If f is derivable on the open domain $]a, b[$ and achieves $f(a) = f(b)$ then there exists $c \in]a, b[$ achieves $f'(c) = 0$.

EXAMPLE

Verify Rolle's Theorem for the function : $f(x) = 2x^2 + 20x - 3$ on $[4, 6]$.

Solution:

1- $f(x)$ is continuous on $[4, 6]$?

Yes, $f(x)$ is polynomial so it's defined on $\mathbb{R}$.

2- $f(x)$ is differentiable on $[4, 6]$?

Yes, because $f'(x)$ is also a polynomial which is defined on $\mathbb{R}$.

3- $f(4) = f(6)$?

$$f(4) = 2(4)^2 - 20(4) - 3 = -45$$

$$f(6) = 2(6)^2 - 20(6) - 3 = -45$$

So Rolle's theorem guarantees at least one place where $f'(x) = 0$ between $x = 4$ and $x = 6$

$$\text{Check: } f'(x) = 4x - 20$$

$$f'(c) = 4c - 20 = 0$$

$$c = 5$$

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Practical exercise:

Check requirement of Rolle's Theorem for the function : $f(x) = 1 - x^{2/3}$ on $[-1,1]$.



1- $f(x)$ is continuous on $[-1,1]$?

Yes, $f(x)$ is defined for all x between -1 and 1

$$x = -1,$$

$$f(-1) = 1 - (-1)^{2/3} = 1 - [(-1)^{1/3}]^2 = 1 - (-1)^2 = 0$$

2- $f(x)$ is differentiable on $[-1,1]$?

No , because

$$f'(x) = -\frac{2}{3}x^{-1/3} = \frac{-2}{3\sqrt[3]{x}}.$$

Clearly undefined at $x = 0$

3- $f(-1) = f(1)$?

$$f(-1) = 1 - (-1)^{2/3} = 0$$

$$f(1) = 1 - (1)^{2/3} = 0$$

So Rolle's theorem can not be used to guarantee the existence of horizontal tangent between $x = -1$ and $x = 1$

1.5.Lagrange's mean Value (The theory of finite increments):

Let f be : $f: [a, b] \rightarrow \mathbb{R}$ a defined and continuous real function on the domain $[a, b]$. If f is derivable on the open domain $]a, b[$ so:

$$\exists c \in]a, b[: f'(c) = \frac{f(b) - f(a)}{b - a}.$$

EXAMPLE

Verify Lagrange's mean Value Theorem for the function

$$f(x) = 3x^2 - 5x + 1 \text{ defined in the interval } [2,5].$$

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Solution:

1- Since $f(x)$ is a polynomial, it is a continuous function in $[2,5]$

2- $f'(x) = 6x - 5$ exists in $(2,5)$.

Thus the conditions of Lagrange's mean Value Theorem is satisfied. To verify mean Value Theorem, we have to show that there exists at-least one point c in the open interval $]2,5[$, such that:

$$f'(c) = \frac{f(5) - f(2)}{5 - 2}$$

Now :

$$f(5) = 3 \times 5^2 - 5 \times 5 + 1 = 51$$

$$f(2) = 3 \times 2^2 - 5 \times 2 + 1 = 3$$

$$6c - 5 = \frac{51-3}{3}$$

$$\Rightarrow 6c - 5 = 16 \Rightarrow c = \frac{21}{6}$$

$$\Rightarrow c = \frac{7}{2} = 3\frac{1}{2} \in [2,5] \text{ Hence mean Value Theorem is verified.}$$

Practical exercise:

Verify Lagrange's mean Value Theorem for the function $f(x) = \sqrt{x^2 - x}$ in the interval $[1,4]$



1- $f(x)$ is an algebraic function, so it is a continuous in $[1,4]$

2- $f'(x) = \frac{2x-1}{2\sqrt{x^2-x}}$ exists in $(1,4)$

Therefore all the conditions of Lagrange's mean Value Theorem is satisfied. To verify **finite increments** Theorem, we have to show that there exists at least one point c in the interval $(1,4)$ such that

$$f'(c) = \frac{f(4) - f(1)}{4 - 1}$$

$$\text{Now : } f(4) = \sqrt{4^2 - 4} = 2\sqrt{3}$$

$$\text{And : } f(1) = \sqrt{1^2 - 1} = 0$$

$$f'(c) = \frac{2c - 1}{2\sqrt{c^2 - c}} = \frac{2\sqrt{3}}{3}$$

$$\Rightarrow \frac{4c^2 - 4c + 1}{4(c^2 - c)} = \frac{4}{3}$$

$$\Rightarrow 16c^2 - 16c = 12c^2 - 12c + 3$$

$$\Rightarrow 4c^2 - 4c - 3 = 0$$

$$\Rightarrow 4c^2 - 6c + 2c - 3 = 0$$

$$\Rightarrow 2c(2c - 3) + (2c - 3) = 0$$

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$$\Rightarrow (2c - 3)(2c + 1) = 0$$

$$\Rightarrow c = \frac{3}{2} \text{ or } c = -\frac{1}{2}$$

$$c = \frac{3}{2} \text{ lies in } [1, 4]$$

Hence mean Value Theorem is verified.

1.6. Application of derivation to the concept of maximum and minimum stationary point

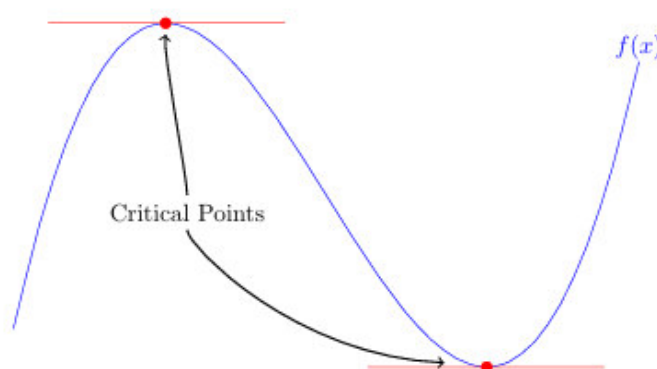
An important goal of life is to maximize nice quantities and minimize unpleasant ones. Extremizing quantities is also a most important principle which nature follows: laws in physics like Newton's law, or the Maxwell equations describing light, or equations describing matter can be based on the principle of extremization. The most important intuitive insight is that at maxima or minima of a function f , the tangent to the graph must be horizontal. This leads to the following notion for differentiable functions:

A point x_0 is a critical point of f if $f'(x_0) = 0$

Definition:

A critical point (or stationary point) of $f(x)$ is a point $(a, f(a))$ such that $f'(a) = 0$

Recall that, geometrically, these are points on the graph of $f(x)$ who have a horizontal tangent line, i.e. a constant tangent line.



EXAMPLE

Find all critical points of $f(x) = x^3 - 3x^2 - 9x + 5$.

We see that the derivative is $f'(x) = 3x^2 - 6x - 9$. We need to solve $f'(x) = 0$.

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$$f'(x) = 3x^2 - 6x - 9 = 3(x^2 - 2x - 3) = 3(x + 1)(x - 3) =$$

$$0 \Rightarrow x = -1, x = 3$$

Thus the critical points of $f(x)$ are:

$$(-1, f(-1)) = (-1, 10) \text{ and } (3, f(3)) = (3, -22).$$

You may notice, particularly from the graph, that the critical points seem to coincide with the peaks of the graph. These is almost true. In fact we have the following definition:

Suppose $(a, f(a))$ is a critical point of $f(x)$. Then,

$$(a, f(a)) \text{ is a local minimum} \Leftrightarrow f''(a) > 0$$

$$(a, f(a)) \text{ is a local maximum} \Leftrightarrow f''(a) < 0$$

$$(a, f(a)) \text{ is a point of inflection} \Leftrightarrow f''(a) = 0$$

We can think of this definition as a test to identify the local maximum and local minimum points of a function $f(x)$.

If $f'(a) < 0$ or $f'(a) > 0$ then we have a local maximum or minimum, respectively, and if $f'(a) = 0$ then we know nothing. Different cases of $f'(a) = 0$ will be explored later.

EXAMPLE

Find all local extrema of $f(x) = x^3 - 3x^2 - 9x + 5$

on the previous Example we found $f'(x) = 3x^2 - 6x - 9$ and that the critical points occur at $(-1, 10)$ and $(3, -22)$. To apply the second derivative test we must first compute the second derivative:

$$f''(x) = 6x - 6$$

Then we simply take the points $x = -1$ and $x = 3$ and plug them into $f''(x)$.

$x = -1$	$x = 3$
$f''(-1) = 6(-1) - 6 = -6 - 6 = -12 < 0$ $\Rightarrow$ local maximum	$f''(3) = 6(3) - 6 = 18 - 6 = 12 > 0$ $\Rightarrow$ local minimum

So the function $f(x)$ has a local maximum at the point $(-1, 10)$ and a local minimum at the point $(3, -22)$.

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1.7. Critical Point Theory:

Let f be $f: [a, b] \rightarrow \mathbb{R}$ defined and derivable on the open domain $]a, b[$. Every point is called $c \in]a, b[$ which achieves $f'(c) = 0$ is Critical point.

- If $f''(c) > 0$ so c is Minimum critical point.
- If $f''(c) < 0$ so c is Maximum critical point.
- If $f''(c) = 0$ so c is Inflection point.

EXAMPLE

Find all local extrema of:

$$y = x + \frac{1}{x}$$

Solution:

first we calculate the critical point(s) of y . Differentiating gives $y' = 1 - x^{-2}$.

Solving $y' = 0$ gives

$$y' = 1 - x^{-2} = 0 \Rightarrow 1 = x^{-2} \Rightarrow 1 = \frac{1}{x^2} \Rightarrow 1 = x^2 \Rightarrow x = -1 \text{ or } x = 1$$

Next we find the second derivative:

$$y'' = 2x^{-3} = \frac{2}{x^3}$$

Finally we plug both $x = -1$ and $x = 1$ into y''

$$\begin{aligned} x &= -1 \\ y''(-1) &= \frac{2}{(-1)^3} = \frac{2}{-1} = -2 < 0 \\ &\Rightarrow \text{local maximum} \end{aligned}$$

$$\begin{aligned} x &= 1 \\ y''(1) &= \frac{2}{(1)^3} = \frac{2}{1} = 2 > 0 \\ &\Rightarrow \text{local minimum} \end{aligned}$$

Thus y has a local maximum at the point $(-1, y(-1)) = (-1, -2)$ and a local minimum at the point $(1, y(1)) = (1, 2)$.

CHAPTER FOUR: DERIVATIVES

2.SUCCESSIVE DIFFERENTIATION AND LEIBNITZ'S THEOREM:

2.1.Introduction

Successive Differentiation is the process of differentiating a given function successively times and the results of such differentiation are called successive derivatives. The higher order differential coefficients are of utmost importance in scientific and engineering applications.

Let be $f(x)$ a differentiable function and let its successive derivatives be denoted by .

$$f'(x), f''(x), \dots, f^{(n)}(x)$$

2.2.Common notations of higher order Derivatives of $y = f(x)$

1st Derivative: $f'(x)$ or y' or y_1 or $\frac{dy}{dx}$ or Dy

2nd Derivative: $f''(x)$ or y'' or y_2 or $\frac{d^2y}{dx^2}$ or D^2y

$\vdots$

n^{th} Derivative: $f^{(n)}(x)$ or $y^{(n)}$ or y_n or $\frac{d^ny}{dx^n}$ or D^ny

Very basic finite difference formulas in numerical analysis approximates the derivative $f'(x)$ using a sequence $h_n > 0$ such that $h_n \rightarrow 0$. The two basic formulas are

$$\frac{f(x+h_n)-f(x)}{h_n} \rightarrow f'(x) \text{ and } \frac{f(x+h_n)-f(x-h_n)}{2h_n} \rightarrow f'(x)$$

Let be I the domain of and $f: I \rightarrow \mathbb{R}$. The derivative of the degree n of the function f that we encode by the symbol is defined by regression :

$$\text{In order to } n = 0 : \quad f^{(0)} = f \text{ and : } \quad (f^{(n)})' = f^{(n+1)}$$

We say that f a function is differentiable up to the order n at $\alpha \in I$ if $f^{(n-1)}$ it is differentiable at α .

And we write: $(f^{(n-1)})'(\alpha) = f^{(n)}(\alpha)$. We also say that a function f accepts a derivative of the order n on I if it accepts the derivative n time on I . That is, if $f^{(n)}(x)$ is known to every, $x \in I$

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2.3. Calculation of n^{th} Derivatives :

1. n^{th} Derivative of e^{ax}

$$\text{Let: } y = e^{ax}$$

$$y_1 = ae^{ax}$$

$$y_2 = a^2 e^{ax}$$

$$\vdots$$

$$y_n = a^n e^{ax}$$

2. n^{th} Derivative of $(ax + b)^m$, m is a^+ , ve integer greater than n

$$\text{Let } y = (ax + b)^m$$

$$y_1 = m \cdot a(ax + b)^{m-1}$$

$$y_2 = m(m-1) a^2 (ax + b)^{m-2}$$

$$\vdots$$

$$y_n = m(m-1) \dots (m-n+1) a^n (ax + b)^{m-n}$$

$$= \frac{m!}{(m-n)!} a^n (ax + b)^{m-n}$$

3. n^{th} Derivative of $y = \log(ax + b)$

$$\text{Let } y = \log(ax + b)$$

$$y_1 = \frac{a}{(ax+b)}$$

$$y_2 = \frac{-a^2}{(ax+b)^2}$$

$$y_3 = \frac{2!-a^3}{(ax+b)^3}$$

$$\vdots$$

$$y_n = (-1)^{n-1} \frac{(n-1)!-a^n}{(ax+b)^n}$$

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4. n^{th} Derivative of $y = \sin(ax + b)$

$$\text{Let } y = \sin(ax + b)$$

$$y_1 = a \cos(ax + b) = a \sin\left(ax + b + \frac{\pi}{2}\right)$$

$$y_2 = a^2 \cos\left(ax + b + \frac{\pi}{2}\right) = a^2 \sin\left(ax + b + \frac{\pi}{2}\right)$$

$$\vdots$$

$$y_n = a^n \sin\left(ax + b + \frac{n\pi}{2}\right)$$

Similarly if $y = \cos(ax + b)$

$$y_n = a^n \cos\left(ax + b + \frac{n\pi}{2}\right)$$

Table 3 : Summary of Results

Function	n^{th} Derivative
$y = e^{ax}$	$y_n = a^n e^{ax}$
$y = (ax + b)^m$	$y_n = \begin{cases} \frac{m!}{(m-n)!} a^n (ax + b)^{m-n}, & m > 0, m \geq n \\ 0, & m > 0, m < n \\ n! a^n, & m = n \\ \frac{(-1)^n n! a^n}{(ax + b)^{n+1}}, & m = -1 \end{cases}$
$y = \log(ax + b)$	$y_n = (-1)^{n-1} \frac{(n-1)! - a^n}{(ax + b)^n}$
$y = \sin(ax + b)$	$y_n = a^n \sin\left(ax + b + \frac{n\pi}{2}\right)$
$y = \cos(ax + b)$	$y_n = a^n \cos\left(ax + b + \frac{n\pi}{2}\right)$
$y = e^{ax} \sin(bx + c)$	$y_n = e^{ax} (a^2 + b^2)^{\frac{n}{2}} \sin\left(bx + c + n \tan^{-1} \frac{b}{a}\right)$
$y = e^{ax} \cos(bx + c)$	$y_n = e^{ax} (a^2 + b^2)^{\frac{n}{2}} \cos\left(bx + c + n \tan^{-1} \frac{b}{a}\right)$

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EXAMPLE

Find the n^{th} derivative of

$$\frac{1}{1 - 5x + 6x^2}$$

Solution:

1: Let

$$y = \frac{1}{1 - 5x + 6x^2}$$

Resolving into partial fractions

$$\begin{aligned} y &= \frac{1}{1 - 5x + 6x^2} = \frac{1}{(1 - 3x)(1 - 2x)} = \frac{3}{1 - 3x} - \frac{2}{1 - 2x} \\ y_n &= \frac{3(-3)^n(-1)^nn!}{(1 - 3x)^{n+1}} - \frac{2(-2)^n(-1)^nn!}{(1 - 2x)^{n+1}} \\ \Rightarrow y_n &= (-1)^{n+1}n! \left[\left(\frac{3}{1 - 3x} \right)^{n+1} - \left(\frac{2}{1 - 2x} \right)^{n+1} \right] \end{aligned}$$

Example 2 :

EXAMPLE

Find the n^{th} derivative of $\sin 6x \cos 4x$

Solution: Let $y = \sin 6x \cos 4x$

$$y = \frac{1}{2}(\sin 10x \cos 2x)$$

$$y_n = \frac{1}{2} \left[10^n \sin \left(10x + \frac{n\pi}{2} \right) + 2^n \cos \left(2x + \frac{n\pi}{2} \right) \right]$$

3. LEIBNITZ'S THEOREM

If u and v are functions of x such that their n^{th} derivatives exist, then the n^{th} derivative of their product is given by

$$(u v)_n = u_n v + n_{c_1} u_{n-1} v_1 + n_{c_2} u_{n-2} v_2 + \cdots + n_{c_r} u_{n-r} v_r + \cdots + u v_n$$

Where u_r and v_r represent r^{th} derivatives of u and v respectively.

EXAMPLE

Find the n^{th} derivative of $x \log x$

Solution:

Let $u = \log x$ and $v = x$

$$\text{Then } u_n = (-1)^{n-1} \frac{(n-1)!}{x^n} \quad \text{and } u_{n-1} = (-1)^{n-2} \frac{(n-2)!}{x^{n-1}}$$

By Leibnitz's theorem, we have

$$\begin{aligned} (uv)_n &= u_n v + n_{c_1} u_{n-1} v_1 + n_{c_2} u_{n-2} v_2 + \cdots + n_{c_r} u_{n-r} v_r + \cdots + u v_n \\ \Rightarrow (x \log x)_n &= (-1)^{n-1} \frac{(n-1)!}{x^n} x + n(-1)^{n-2} \frac{(n-2)!}{x^{n-1}} + 0 \\ &= (-1)^{n-1} \frac{(n-1)!}{x^{n-1}} + n(-1)^{n-2} \frac{(n-2)!}{x^{n-1}} \\ &= (-1)^{n-2} \frac{(n-2)!}{x^{n-1}} [-(n-1) + n] \\ &= (-1)^{n-2} \frac{(n-2)!}{x^{n-1}} \end{aligned}$$

EXAMPLE

Find the n^{th} derivative of $x^2 e^{3x} \sin 4x$

Solution:

Let $u = e^{3x} \sin 4x$ and $v = x^2$

$$\text{Then : } u_n = e^{3x} 25^{\pi/2} \sin \left(4x + n \tan^{-1} \frac{4}{3} \right)$$

$$= e^{3x} 5^n \sin \left(4x + n \tan^{-1} \frac{4}{3} \right)$$

By Leibnitz's theorem, we have;

$$\begin{aligned} (uv)_n &= u_n v + n_{c_1} u_{n-1} v_1 + n_{c_2} u_{n-2} v_2 + \cdots + n_{c_r} u_{n-r} v_r + \cdots + u v_n \\ \Rightarrow (x^2 e^{3x} \sin 4x)_n &= x^2 e^{3x} 5^n \sin \left(4x + n \tan^{-1} \frac{4}{3} \right) + 2nx e^{3x} 5^{n-1} \sin \left(4x + (n-1) \tan^{-1} \frac{4}{3} \right) \\ &\quad + n(n-1) e^{3x} 5^{n-2} \sin \left(4x + (n-2) \tan^{-1} \frac{4}{3} \right) + 0 \\ &= e^{3x} 5^n \left[x^2 \sin \left(4x + \tan^{-1} \frac{4}{3} \right) + \frac{2nx}{5} \sin \left(4x + (n-1) \tan^{-1} \frac{4}{3} \right) \right. \\ &\quad \left. + \frac{n(n-1)}{25} \sin \left(4x + (n-2) \tan^{-1} \frac{4}{3} \right) \right] \end{aligned}$$

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Or we can write:

Let f and g be two functions n times differentiable on I , then fg is n times differentiable on I and we have:

$$\forall x \in I, \quad (f \times g)^{(n)}(x) = \sum_{k=0}^n C_k^n f^k(x) g^{(n-k)}(x)$$

With :

$$C_n^k = \frac{n!}{k!(n-k)!}$$

EXAMPLE

Find the 4th derivative of $h(x) = x^2(x+1)^5$

Solution:

By Leibnitz's theorem, we have

$$(f \times g)^{(n)}(x) = \sum_{k=0}^n C_k^n f^k(x) g^{(n-k)}(x)$$

Then :

$$f(x) = x^2 \Rightarrow$$

$$f_{(x)}^{(1)} = 2x$$

$$f_{(x)}^{(2)} = 2$$

$$f_{(x)}^{(3)} = 0$$

$$g(x) = (x+1)^5 \Rightarrow$$

$$g^{(1)}(x) = 5(x+1)^4$$

$$g^{(2)}(x) = 5 \times 4(x+1)^3$$

$$g^{(3)}(x) = 5 \times 4 \times 3(x+1)^2$$

So that :

$$h^{(n)}(x) = C_n^0 f_{(x)}^{(0)} \cdot g^{(n)}(x) + C_n^1 f_{(x)}^{(1)}(x) g^{(n-1)}(x) + C_n^2 f_{(x)}^{(2)}(x) g^{(n-2)}(x) + 0 \dots + 0$$

$$h^{(4)}(x) = C_n^0 f_{(x)}^{(0)} \cdot g^{(4)}(x) + C_4^1 f_{(x)}^{(1)}(x) g^{(3)}(x) + C_4^2 f_{(x)}^{(2)}(x) g^{(2)}(x) + 0 \dots + 0$$

$$\begin{aligned} h^{(4)}(x) &= x^2 120(x+1) + 4 \cdot (2x)(5 \times 4 \times 3)(x+1)^2 + 6(2)(20)(x+1)^3 \\ &= 120x^2(x+1) + 480x(x+1)^2 + 240(x+1)^3 \end{aligned}$$

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4. TAYLOR'S FORMULAS

4.1. Taylor's formula with Lagrange remainder:

Let $x_0 \in [a, b]$ and $f : [a, b] \rightarrow \mathbb{R}$ be a function that checks:

1. $f \in C^n$ on $[a, b]$.
2. $f^{(n)}$ is differentiable on $]a, b[$.

then, $\forall x \in [a, b]$ (with $x \neq x_0$), $\exists c \in [a, b]$ such that:

$$f(x) = f(x_0) + \frac{f^{(1)}(x_0)}{1!} (x - x_0) + \frac{f^{(2)}(x_0)}{2!} (x - x_0)^2 + \dots + \frac{f^{(n)}(x_0)}{n!} (x - x_0)^n + \frac{f^{(n+1)}(c)}{(n+1)!} (x - x_0)^{n+1}$$

This expression is the Taylor formula of order n with the Lagrange remainder

$$R_n(x, x_0) = \frac{f^{(n+1)}(c)}{(n+1)!} (x - x_0)^{n+1}$$

4.2. Taylor Mac-Laurin formula:

If we set $x_0 = 0$ in the Taylor-Lagrange formula, we obtain:

$\exists \theta \in]0, 1[$ such that:

$$f(x) = f(0) + \frac{f^{(1)}(0)}{1!} (x) + \frac{f^{(2)}(0)}{2!} (x)^2 + \dots + \frac{f^{(n)}(0)}{n!} (x)^n + \frac{f^{(n+1)}(\theta x)}{(n+1)!} (x)^{n+1}$$

This is Taylor Mac-Laurin's formula

Example:

Show that for every $x \in \mathbb{R}_+$:

$$x - \frac{x^2}{2} \leq \ln(1+x) \leq x - \frac{x^2}{2} + \frac{x^3}{3}$$

Let $x \geq 0$, Applying the Taylor Mac-Laurin formula of order 2 to the function:

$f(x) = \ln(1+x)$, we find:

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$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3(1+\theta x)^3} \quad / \theta \in]0,1[$$

Since $x \geq 0$ then,

$$\begin{aligned} x - \frac{x^2}{2} &\leq x - \frac{x^2}{2} + \frac{x^3}{3(1+\theta x)^3} \\ \Rightarrow x - \frac{x^2}{2} &\leq \ln(1+x) \dots\dots (1) \end{aligned}$$

On the other hand

$$\begin{aligned} \frac{x^3}{3(1+\theta x)^3} &\leq \frac{x^3}{3} \\ \Rightarrow x - \frac{x^2}{2} + \frac{x^3}{3(1+\theta x)^3} &\leq x - \frac{x^2}{2} + \frac{x^3}{3} \\ \Rightarrow \ln(1+x) &\leq x - \frac{x^2}{2} + \frac{x^3}{3} \dots\dots (2) \end{aligned}$$

From (1) and (2) we get:

$$x - \frac{x^2}{2} \leq \ln(1+x) \leq x - \frac{x^2}{2} + \frac{x^3}{3}$$

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SERIES N° 4: DERIVATIVES

Exercise 01:

Let the function f defined be as follows:

$$f(x) = \begin{cases} 1 + x\sqrt{x} & : \text{ si } x \geq 0 \\ 1 + \ln(1 + X^2) & : \text{ si } x < 0 \end{cases}$$

1. Identify the domain of the function. f
2. study the differentiability of the function f , is it f' continuous?
3. Apply the mean value theorem of the function f to the domain if possible. $[-1 ; 1]$

Exercise 02:

Find the derivatives of order 2 of the following functions:

$$f(x) = \sqrt{x^2 - 4} \quad ; \quad g(x) = 1/2 - x; \quad h(x) = x^3(x + 1)^2$$

Exercise 03:

Find the critical values of the following functions:

$$f(x) = (x - 1)^2(x + 1) \quad ; \quad g(x) = x/x^2 + 1$$

Exercise 04:

Let the defined function f be as follows:

$$f: x \rightarrow [x; x + 1] \quad [x; x + 1] \subset \mathbb{R}_+^* \\ f: x \rightarrow \ln x$$

Using the mean value theorem, show that:

$$\frac{1}{x+1} < \ln(x+1) - \ln x < \frac{1}{x}$$

Exercise 05:

Apply the mean value theorem to the function if possible:

$$f(x) = 1 - \ln(x + 1) \quad x \in [0; 1]$$

Apply Rolle's theorem to the functions if possible:

$$f(x) = x^3 - 4x \quad x \in [-2; 0] \\ g(x) = |x| \quad x \in [-1; 1]$$

Exercise 06:

Find the derivatives n of the following functions:

$$f_1(x) = x^4 + x^3 \quad ; f_2(x) = xe^ax \quad ; f_3(x) = x^2 \ln(x)$$

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ANSWER TO SERIES NUMBER 4

Exercise 1:

1- Determine the domain of the function f :

$$f(x) = \begin{cases} 1 + x\sqrt{x} & : \text{si } x \geq 0 \\ 1 + \ln(1 + X^2) & : \text{si } x < 0 \end{cases} \Rightarrow \begin{matrix} h(x) \\ g(x) \end{matrix}$$

The domain of the function f is:

$$D_f = D_h \cup D_g = \mathbb{R}$$

2-study the differentiability of the function f , is it f' continuous?

We have: $(1 + x\sqrt{x})$ is differentiable on the domain $\mathbb{R}^+$.

And $[1 + \ln(1 + X^2)]$ is differentiable on the domain $\mathbb{R}_-^*$

We study differentiability at the point $x_0 = 0$

$$\begin{aligned} \lim_{x \rightarrow 0^+} \frac{f(x) - f(0)}{x - 0} &= \lim_{x \rightarrow 0^+} \frac{1 + x\sqrt{x} - 1}{x} = \lim_{x \rightarrow 0^+} \sqrt{x} = 0 \\ \lim_{x \rightarrow 0^-} \frac{f(x) - f(0)}{x - 0} &= \lim_{x \rightarrow 0^-} \frac{1 + \ln(1 + X^2) - 1}{x} = \frac{0}{0} \end{aligned}$$

By using L'Hôpital's rule, we find:

$$\lim_{x \rightarrow 0^-} \frac{\ln(1 + X^2)}{x} = \lim_{x \rightarrow 0^-} \frac{2x}{1 + X^2} = 0$$

We have: $f'_d(0) = f'_g(0) = 0$ so it is differentiable at $x_0 = 0$ and therefore differentiable on $\mathbb{R}$

Study of the continuity of the function f' :

$$f'(x) = \begin{cases} \sqrt{x} + \frac{x}{2\sqrt{x}} & ; x > 0 \\ \frac{2x}{1 + X^2} & ; x < 0 \\ 0 & ; x = 0 \end{cases}$$

We have : $f'(0) = 0$

$$\begin{aligned} \lim_{x \rightarrow 0^-} f'(x) &= \lim_{x \rightarrow 0^-} \frac{2x}{1 + X^2} = 0 \\ \lim_{x \rightarrow 0^+} f'(x) &= \lim_{x \rightarrow 0^+} \sqrt{x} + \frac{x}{2\sqrt{x}} \end{aligned}$$

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By using L'Hôpital's rule, we find:

$$\lim_{x \rightarrow 0^+} \frac{1}{\frac{2}{2\sqrt{x}}} = \lim_{x \rightarrow 0^+} \sqrt{x} = 0$$

Thus, the function f' is continuous on $\mathbb{R}$.

3-Application of the Mean Value Theorem for the function f on the interval $[-1; 1]$:

We have the function f differentiable on R , so it is continuous on R (property – but the reverse is not true).

If f is continuous and differentiable on R , then it is continuous on the closed interval $[-1, 1]$ and differentiable on the open interval $] -1; 1[$ (according to the theorem).

And According to the Mean Value Theorem:

$$\exists c \in] -1; 1[: f'(c) = \frac{f(1) - f(-1)}{1 - (-1)} ; \left[\frac{f(b) - f(a)}{b - a} \right]$$
$$f'(c) = \frac{2 - [1 + \ln(2)]}{2} = \frac{1 - \ln(2)}{2}$$

And by calculating $f'(c)$ we find

$$f'(c) = \sqrt{c} + \frac{c}{2\sqrt{c}} = \frac{2c + c}{2\sqrt{c}} = \frac{3c}{2\sqrt{c}} = \frac{3}{2}\sqrt{c}$$

And thus we have:

$$\frac{3}{2}\sqrt{c} = \frac{1 - \ln(2)}{2} \Rightarrow \sqrt{c} = \frac{1 - \ln(2)}{3} \Rightarrow c = \left(\frac{1 - \ln(2)}{3} \right)^2$$

Exercise 2:

1-Determining the second-order derivatives of functions:

$$f(x) = \sqrt{x^2 - 4}$$

$$f_{(x)}^{(1)} = \frac{2x}{2\sqrt{x^2 - 4}} = \frac{x}{\sqrt{x^2 - 4}}$$
$$f_{(x)}^{(2)} = \frac{1(\sqrt{x^2 - 4}) - \left(\frac{x}{\sqrt{x^2 - 4}}\right)x}{(\sqrt{x^2 - 4})^2} = \frac{\frac{x^2 - 4 - x^2}{\sqrt{x^2 - 4}}}{x^2 - 4} = \frac{-4}{(x^2 - 4)(\sqrt{x^2 - 4})}$$

$$g(x) = 1/2 - x$$

$$g_{(x)}^{(1)} = \frac{0(2 - x) - (-1)(1)}{(2 - x)^2} = \frac{1}{(2 - x)^2}$$

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$$g_{(x)}^{(2)} = \frac{0(2-x)^2 - [2(-1)(2-x)](1)}{((2-x)^2)^2} = \frac{2(2-x)}{(2-x)^4} = \frac{2}{(2-x)^3}$$

$$h(x) = x^3(x+1)^2$$

$$h_{(x)}^{(1)} = 3x^2(x+1)^2 + [2(x+1)(1)](x^3) = (x+1)[3x^2(x+1) + 2x^3]$$

$$h_{(x)}^{(2)} = (1)[3x^2(x+1) + 2x^3] + [6x(x+1) + 3x^2 + 6x^2](x+1)$$

$$h_{(x)}^{(2)} = x^2[3(x+1) + 2x] + 3x[2(x+1) + 3x](x+1)$$

$$h_{(x)}^{(2)} = x^2[5x+3] + 3x[5x+2](x+1)$$

Exercise 03:

1-Finding the critical values of functions:

$$f(x) = (x-1)^2(x+1)$$

$$D_f = \mathbb{R}$$

$$f'(x) = (x-1)^2 + (x+1)2(x-1) = (x-1)[(x-1) + 2(x+1)] = (x-1)(3x+1)$$

Accordingly, the critical values are:

$$f'(c) = 0 \Rightarrow \begin{cases} c-1=0 \\ \text{or} \\ 3c+1=0 \end{cases} \Rightarrow \begin{cases} c_1=1 \\ \text{or} \\ c_2=-\frac{1}{3} \end{cases} \in D_f$$

And we calculate:

$$f''(x) = (3x+1) + 3(x-1) = 6x-2 \Rightarrow f''(c) = 6c-2$$

Then :

$$f''(c_1) \Rightarrow f''(1) = 4 > 0 \Rightarrow \text{Minimum value}$$

$$f''(c_2) \Rightarrow f''\left(-\frac{1}{3}\right) = -4 < 0 \Rightarrow \text{Maximum value}$$

$$g(x) = \frac{x}{x^2+1}$$

$$D_g = \mathbb{R}$$

$$g'(x) = \frac{-x^2+1}{(x^2+1)^2}$$

Accordingly, the critical values are:

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$$g'(c) = 0 \Rightarrow -c^2 + 1 = 0 \Rightarrow c^2 = 1 \Rightarrow \begin{cases} c_1 = 1 \\ or \\ c_2 = -1 \end{cases} \in D_g$$

And we calculate :

$$g''(x) = \frac{-2x(x^2 + 1)^2 - 4x(x^2 + 1)(-x^2 + 1)}{(x^2 + 1)^4} = \frac{(x^2 + 1)(-2x^3 - 2x + 4x^3 - 4x)}{(x^2 + 1)^4} \\ = \frac{2x^3 - 6x}{(x^2 + 1)^3}$$

Then :

$$g''(c) = \frac{2c^3 - 6c}{(c^2 + 1)^3}$$

$$g''(c_1) \Rightarrow g''(1) = \frac{-4}{8} < 0 \Rightarrow \text{Maximum value}$$

$$g''(c_2) \Rightarrow g''(-1) = \frac{4}{8} > 0 \Rightarrow \text{Minimum value}$$

Exercise 4:

We have the function f differentiable and continuous on R_+^* , hence:

$$f(x) = \ln(x) \quad , \quad x \in [x, x + 1] \subset R_+^*$$

f is continuous on the closed interval $[x, x + 1]$

And differentiable on the open interval $]x, x + 1[$ (according to the theorem).

And according to the mean value theorem:

$$\exists c \in]x, x + 1[: f'(c) = \frac{f(x + 1) - f(x)}{x + 1 - x} = f(x + 1) - f(x)$$

$$f'(c) = \ln(x + 1) - \ln(x)$$

And by calculating $f'(c)$ we find:

$$f'(c) = \frac{1}{c}$$

And it becomes:

$$\frac{1}{c} = \ln(x + 1) - \ln(x) \dots (1)$$

And we have :

$$x \in]x, x + 1[\Rightarrow x < c < x + 1$$

$$\frac{1}{x + 1} < \frac{1}{c} < \frac{1}{x} \dots (2)$$

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We substitute (1) in (2) and find:

$$\frac{1}{x+1} < \ln(x+1) - \ln(x) < \frac{1}{x}$$

Exercise 5:

1-Application of the mean value theorem on the function:

$$f(x) = 1 - \ln(x+1) \quad x \in [0; 1]$$

$$D_f = \{x / 1+x > 0\} = \{x / x > -1\} = \{x/x \in]-1, +\infty[\}$$

f is continuous and differentiable on the interval $] -1, +\infty[$ then:

f is continuous on the closed interval $[0,1]$

And differentiable on the open interval $]0,1[$ (according to the theorem).

And according to the mean value theorem:

$$\exists c \in]0,1[: f'(c) = \frac{f(1) - f(0)}{1 - 0} = f(1) - f(0) \Rightarrow f'(c) = 1 - \ln(2) - 1$$

And we have :

$$f'(x) = \frac{-1}{1+x} \Rightarrow f'(c) = \frac{-1}{1+c}$$

So :

$$\frac{-1}{1+c} = 1 - \ln(2) - 1 \Rightarrow \frac{1}{1+c} = \ln(2)$$

$$\Leftrightarrow \ln(2) + c \ln(2) = 1$$

$$\Leftrightarrow c = \frac{1 - \ln(2)}{\ln(2)} \Leftrightarrow c = 0.44 \in]0,1[$$

2- Applying Rolle's Theorem to the function:

$$f(x) = x^3 - 4x \quad x \in [-2; 0]$$

f is continuous and differentiable on R , then:

f is continuous on the closed interval $[-2, 0]$

And differentiable on the open interval $] -2, 0[$

And $f(-2) = f(0) = 0$ (according to the theorem) where:

$$f(-2) = (-2)^3 - 4(-2) = -8 + 8 = 0$$

and

$$f(0) = (0)^3 - 4(0) = 0 - 0 = 0$$

CHAPTER FOUR: DERIVATIVES

according to the theorem we have:

$$\exists c \in]-2, 0[: f'(c) = 0$$

$$f'(c) = 3c^2 - 4 = 0 \Rightarrow 3c^2 = 4 \Rightarrow c^2 = \frac{4}{3} \Leftrightarrow \begin{cases} c_1 = \frac{2}{\sqrt{3}} \Rightarrow \text{Rejected solution} \\ \text{or} \\ c_2 = -\frac{2}{\sqrt{3}} \Rightarrow c_2 \in]-2, 0[\end{cases}$$

$$g(x) = |x| \quad x \in [-1; 1]$$

And it can be written in the form:

$$g(x) = |x| = \begin{cases} x & x \geq 0 \\ -x & x < 0 \end{cases}$$

Rolle's theorem cannot be applied to the interval $[-1; 1]$ because:

$$f(-1) \neq f(1)$$

Exercise 06:

1-Determining the n^{th} derivative of functions:

$$f_1(x) = x^4 + x^3$$

The function is differentiable, and therefore: $f_1^{(n)} = 0$

$$f_2(x) = xe^{ax}$$

Finding the n^{th} derivative of multiplying of two functions using Leibniz's rule (leibniz formula)

$$(f \times g)^{(n)}(x) = \sum_{k=0}^n C_k^n f^{(k)}(x) g^{(n-k)}(x)$$

We have:

$$f(x) = x \Rightarrow f_{(x)}^{(2)} = 0$$

$$g(x) = e^{ax} \Rightarrow g^{(n)}(x) = a^n e^{ax}$$

then :

$$\begin{aligned} f_2^{(n)}(x) &= C_n^0 f_{(x)}^{(0)} \cdot g^{(n)}(x) + C_n^1 f^{(1)}(x) g^{(n-1)}(x) \\ &= 1 \cdot x \cdot a^n e^{ax} + n \cdot 1 \cdot a^{n-1} e^{ax} \\ &= x a^n e^{ax} + n a^n a^{n-1} e^{ax} \\ f_2^{(n)}(x) &= a^n e^{ax} (x + n a^{-1}) = a^n e^{ax} \left(x + \frac{n}{a} \right) \end{aligned}$$

CHAPTER FOUR: DERIVATIVES

$$f_3(x) = x^2 \ln(x)$$

Finding the N-derivative of multiplying two functions using Leibniz's rule:

$$(f \times g)^{(n)}(x) = \sum_{k=0}^n C_k^n f^{(k)}(x) g^{(n-k)}(x)$$

We have:

$$\left. \begin{array}{l} f(x) = x^2 \Rightarrow \\ f_{(x)}^{(1)} = 2x \\ f_{(x)}^{(2)} = 2 \\ f_{(x)}^{(3)} = 0 \end{array} \right| \begin{array}{l} g(x) = \ln x \Rightarrow \\ g^{(1)}(x) = \frac{1}{x} = x^{-1} \\ g^{(2)}(x) = -1 \cdot x^{-2} \\ g^{(3)}(x) = +1 \times 2 \cdot x^{-3} \\ g^{(4)}(x) = -1 \times 2 \times 3 \cdot x^{-4} \\ . \\ . \\ g^{(n)}(x) = (-1)^{n+1} (n-1)! x^{-n} \end{array}$$

Then:

$$f_3^{(n)}(x) = C_n^0 f_{(x)}^{(0)} \cdot g^{(n)}(x) + C_n^1 f^{(1)}(x) g^{(n-1)}(x) + C_n^2 f^{(2)}(x) g^{(n-2)}(x) + 0 \dots + 0$$

$$\begin{aligned} f_3^{(n)}(x) &= x^2 (-1)^{n+1} (n-1)! x^{-n} + n \cdot 2x (-1)^n (n-2)! x^{-(n-1)} \\ &\quad + \frac{1}{2} n(n-1) \cdot 2 (-1)^{n-1} (n-3)! x^{-(n-2)} \end{aligned}$$

$$\begin{aligned} f_3^{(n)}(x) &= (-1)^{n+1} (n-1)! x^{-n+2} + 2n (-1)^n (n-2)! x^{-n+2} \\ &\quad + n(n-1) (-1)^{n-1} (n-3)! x^{-n+2} \end{aligned}$$

CHAPTER FIVE: ORIGINAL FUNCTIONS AND INTEGRATION

CHAPTER FIVE: ORIGINAL FUNCTIONS AND INTEGRATION

1. ANTIDERIVATIVES (PRIMITIVES) AND INTEGRALS

First, let's recall the definition of an antiderivative (a primitive).

1.1. Definition.

A primitive of a function f , defined on an interval $]a, b[$, is any function

F derivable on $]a, b[$, whose derivative coincides with f on $]a, b[$.

$$F'(x) = f(x), \forall x \in]a, b[$$

Any continuous function on an interval I has a primitive on this interval.



If F is a primitive of the function $f : I \rightarrow R$, any primitive G of f on I is of the form

$$\begin{aligned} G : I &\rightarrow R \\ x &\rightarrow G(x) = F(x) + c, \end{aligned}$$

where c is a real constant.

Two primitives of the same function differ by one real constant, whose derivative is zero.

Proof. Let H be a function defined as follows:

$$H(x) = G(x) - F(x), \quad \forall x \in I$$

where F and G are antiderivatives of the function f . Let $a, b \in I$ such that $a < b$. Since F and G are antiderivatives of f , then

$$H'(x) = G'(x) - F'(x) = f(x) - f(x) = 0$$

for every $x \in I$. Since the function H is differentiable, it is continuous. From the mean value theorem on $[a; b]$, there is a number $c \in (a; b)$ such that

$$H'(c) = \frac{H(b) - H(a)}{b - a}$$

Since $H'(x) = 0$ on I , then $H'(c) = 0$. This implies $H(a) = H(b)$ and this means H is a constant function.

EXAMPLE

Find the general form of the antiderivatives of $f(x) = 6x^5$.

Solution:

If $F(x) = x^6$, then $F'(x) = 6x^5$. The function $F(x) = x^6 + c$ is the general antiderivative of f .

CHAPTER FIVE: ORIGINAL FUNCTIONS AND INTEGRATION

Table1: The list of basic Antiderivatives (primitives) rules

$f(x)$	$F(x)$	I
0	$c, c \in R$	R
a	$ax + c$	R
x	$\frac{1}{2}x^2 + c$	R
x^2	$\frac{1}{3}x^3 + c$	R
$\frac{1}{x^2}$	$-\frac{1}{x} + c$	R^*
$\frac{1}{\sqrt{x}}$	$2\sqrt{x} + c$	$]0, +\infty[$
x^n	$\frac{1}{n+1} x^{n+1} + c$	R
$\frac{1}{x^n}$	$\frac{1}{(n-1)x^{n-1}} + c$	R^*
$\frac{1}{x}$	$\ln x + c$	$]0, +\infty[$
$\frac{1}{x}$	$\ln x + c$	R^*
e^x	$e^x + c$	R

1.2. Indefinite Integrals:

if the function $F(x) + c$ is an antiderivative of $f(x)$, then there exist no antiderivatives in different forms for the function $f(x)$. This leads us to define the indefinite integral.

Definition :

Let f be a continuous function on an interval I . The indefinite integral of f is the general antiderivative of f on I :

$$\int f(x) dx = F(x) + c$$

The function f is called the integrand, the symbol $\int$ is the integral sign, x is called the variable of the integration and c is the constant of the integration.

CHAPTER FIVE: ORIGINAL FUNCTIONS AND INTEGRATION

Now, by using the previous definition, the general antiderivatives in follow Example are:

EXAMPLE

$$\begin{aligned}\int (2x + 3)dx &= x^2 + 3x + c \\ \int (\cos x + 1)dx &= \sin x + x + c\end{aligned}$$

We can now work out how to evaluate some integrals. To do that, we should remember differentiation rules of some functions. **Basic Integration Rules**

Rule 1: Power of x.

$$\frac{d}{dx}x^{n+1} = (n+1)x^n, \text{ so } \int (n+1)x^n dx = x^{n+1} + c$$

Generally, for $n \neq -1$,

$$\int x^n dx = \frac{x^{n+1}}{n+1} + c$$

In words, to integrate the function x^n , we add 1 to the power and divide the function by $n+1$.

If $n = -1$, we have a special case

$$\int 1 dx = x + c$$

Rule 2: Trigonometric functions.

$$\frac{d}{dx} \sin x = \cos x, \text{ so } \int \cos x dx = \sin x + c$$

$$\frac{d}{dx} \cos x = -\sin x, \text{ so } \int -\sin x dx = \cos x + c$$

Therefore, $\int \sin x dx = -\cos x + c$

The other trigonometric functions with the previous rules are listed in the following table:

Table2: The list of basic integration rules.

Derivative	Indefinite Integral
$\frac{d}{dx}(x) = 1$	$\int 1 dx = x + c$
$\frac{d}{dx}\left(\frac{x^{n+1}}{n+1}\right) = x^n, n \neq -1$	$\int x^n dx = \frac{x^{n+1}}{n+1} + c$

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$\frac{d}{dx}(\sin x) = \cos x$	$\int \cos x \, dx = \sin x + c$
$\frac{d}{dx}(-\cos x) = \sin x$	$\int \sin x \, dx = -\cos x + c$
$\frac{d}{dx}(\tan x) = \sec^2 x$	$\int \sec^2 x \, dx = \tan x + c$
$\frac{d}{dx}(-\cot x) = \csc^2 x$	$\int \csc^2 x \, dx = -\cot x + c$
$\frac{d}{dx}(\sec x) = \sec x \tan x$	$\int \sec x \tan x \, dx = \sec x + c$
$\frac{d}{dx}(-\csc x) = \csc x \cot x$	$\int \csc x \cot x \, dx = -\csc x + c$

Example:

EXAMPLE

Evaluate the integral.

$$\int x^{-3} dx$$

$$\int \frac{1}{\cos^2 x} dx$$

Solution:

$$\int x^{-3} dx = \frac{x^{-3+1}}{-3+1} + c = -\frac{1}{x^2} + c$$

$$\int \frac{1}{\cos^2 x} dx = \int \sec^2 x \, dx = \tan x + c$$

1.3. Properties of the Indefinite Integral :

Assume f and g have antiderivatives on an interval I , then:

$\frac{d}{dx} \int f(x) \, dx = f(x)$
$\int a f(x) \, dx = a \int f(x) \, dx, a \in \mathbb{R}$
$\int (f(x) \pm g(x)) \, dx = \int f(x) \, dx \pm \int g(x) \, dx$

$$\int f f' dx = \frac{1}{2} f^2 + c$$

$$\int f' f^n dx = \frac{1}{n+1} f^{n+1} + c, n \neq -1$$

$$\int \frac{f'}{f^2} dx = -\frac{1}{f} + c$$

$$\int \frac{f'}{f^n} dx = \frac{-1}{(n-1) f^{n-1}} + c$$

$$\int \frac{f'}{\sqrt{f}} dx = 2\sqrt{f} + c$$

$$\int \frac{f'}{f} dx = \ln f(x) + c$$

$$\int f' e^f dx = e^{f(x)} + c$$

Example 1 :

Evaluate the integral



$$\int (4x + 3) dx$$

$$\int (2 \sin x + 3 \cos x) dx$$

$$\frac{d}{dx} \int \sqrt{x+1} dx$$

Solution:

$$\int (4x + 3) dx = \frac{4}{2} x^2 + 3x + c = 2x^2 + 3x + c$$

$$\int (2 \sin x + 3 \cos x) dx = -2 \cos x + 3 \sin x + c$$

$$\frac{d}{dx} \int \sqrt{x+1} dx = \sqrt{x+1}$$

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Example 2 :

EXAMPLE

Solve the differential equation $f'(x) = x^3$ subject to the initial condition $f(0) = 1$

Solution:

$$\int f'(x) dx = \int x^3 dx \Rightarrow f(x) = \frac{1}{4}x^4 + c$$

If $x = 0$, then $f(0) = \frac{1}{4}(0)^4 + c = 1$ and this implies $c = 1$.

Hence, the solution of the differential equation is $f(x) = \frac{1}{4}x^4 + 1$

1.4. Integration By Substitution

The integration by substitution (known as u-substitution) is a technique for solving some composite functions. The method is based on changing the variable of the integration to obtain a simple indefinite integral. The following theorem shows how the substitution technique work.

Theorem :

Let g be a differentiable function on an interval I where the derivative is continuous. Let f be continuous on the interval J contains the range of the function g . If F is an antiderivative of the function f on J , then

$$\int f(g(x)) g'(x) dx = F(g(x)) + c, \quad x \in I.$$

EXAMPLE

Evaluate the integral $\int 2x (x^2 + 1)^3 dx$

Solution:

We can use the previous theorem as follows:

Let $f(x) = x^3$ and $g(x) = x^2 + 1$, then $f(g(x)) = (x^2 + 1)^3$. since $g'(x) = 2x$, then from previous Theorem, we have:

$$\int 2x (x^2 + 1)^3 dx = \frac{(x^2 + 1)^4}{4} + c$$

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We can end with the same solution by using the five steps of the substitution method given below.

Steps of the integration by substitution:

Step 1: Choose a new variable u .

Step 2: Determine the value of du .

Step 3: Make the substitution i.e., eliminate all occurrences of x in the integral by making the entire integral is in terms of u .

Step 4: Evaluate the new integral.

Step 5: Return the evaluation to the initial variable x .

In previous Example, let $u = x^2 + 1$, then $du = 2x dx$. By substituting that into the original integral, we have

$$\int u^3 du = \frac{u^4}{4} + c$$

Now, by returning the evaluation to the initial variable x , we have:

$$\int 2x (x^2 + 1)^3 dx = \frac{(x^2 + 1)^4}{4} + c$$

1.5. Integration by Parts

we describe one more method of integration, that is found quite useful in integrating products of functions. If u and v are any two differentiable functions of a single variable x (say). Then, by the product rule of differentiation, we have:

$$\frac{d}{dx}(uv) = u \frac{dv}{dx} + v \frac{du}{dx}$$

Integrating both sides, we get

$$uv = \int u \frac{dv}{dx} dx + \int v \frac{du}{dx} dx$$

$$\text{Or } \int u \frac{dv}{dx} dx = uv - \int v \frac{du}{dx} dx \dots \dots (1)$$

Let $u = f(x)$ and $\frac{dv}{dx} = g(x)$ then

$$\frac{du}{dx} = f'(x) \quad \text{and} \quad v = \int g(x) dx$$

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Therefore, expression (1) can be rewritten as:

$$\int f(x)g(x) dx = f(x) \int g(x) dx - \int \left[\int g(x) dx \right] f'(x) dx$$

i.e.,

$$\int f(x)g(x) dx = f(x) \int g(x) dx - \int \left[f'(x) \int g(x) dx \right] dx$$

If we take f as the first function and g as the second function, then this formula may be stated as follows:

“The integral of the product of two functions = (first function) × (integral of the second function) – Integral of [(differential coefficient of the first function) × (integral of the second function)]”

Example1 :



find $\int xe^x dx$

Solution

Take first function as x and second function as e^x . The integral of the second function is e^x

Therefore, $\int xe^x dx = xe^x - \int 1 \cdot e^x dx = xe^x - e^x + c$

Example 2:



find $\int x^2 \ln x dx$

Solution:

Take first function as $\ln x$ and second function as x^2 . The integral of the second function is $\frac{1}{3}x^3$

Therefore,

$$\begin{aligned} \int x^2 \ln x dx &= \frac{1}{3}x^3 \ln x - \int \frac{1}{3}x^3 \cdot \frac{1}{x} dx \\ &= \frac{1}{3}x^3 \ln x - \frac{1}{3} \int x^2 dx = \frac{1}{3}x^3 \ln x - \frac{1}{3} \cdot \frac{x^3}{3} + c = \frac{1}{3}x^3 \ln x - \frac{1}{9}x^3 + c \end{aligned}$$

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2. THE DEFINITE INTEGRALS

In this section, we shall study what is called definite integral of a function. The definite integral has a unique value. A definite integral is denoted by $\int_a^b f(x)$, where a is called the lower limit of the integral and b is called the upper limit of the integral. The definite integral is introduced either as the limit of a sum or if it has an anti derivative F in the interval $[a, b]$, then its value is the difference between the values of F at the end points, i.e., $F(b) - F(a)$.

2.1. First fundamental theorem of integral calculus

Theorem 1

Let f be a continuous function on the closed interval $[a, b]$ and let $A(x)$ be the area function. Then $A'(x) = f(x)$, for all $x \in [a, b]$.

2.2. Second fundamental theorem of integral calculus

We state below an important theorem which enables us to evaluate definite integrals by making use of anti derivative.

Theorem 2

Let f be continuous function defined on the closed interval $[a, b]$ and F be an anti derivative of f . Then

$$\int_a^b f(x) dx = [F(x)]_a^b = F(b) - F(a)$$

Steps for calculating $\int_a^b f(x) dx$:

1-Find the indefinite integral $\int f(x)dx$. Let this be $F(x)$. There is no need to keep integration constant C because if we consider $F(x) + C$ instead of $F(x)$, we get

$$\int_a^b f(x) dx = [F(x) + c]_a^b = [F(b) + c] - [F(a) + c] = F(b) - F(a)$$

Thus, the arbitrary constant disappears in evaluating the value of the definite integral.

2- Evaluate $F(b) - F(a) = [F(x)]_a^b$, which is the value of $\int_a^b f(x) dx$

We now consider some examples

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Examples: Evaluate the following integrals:

$$(1) \int_{-1}^2 x^2 dx$$

$$(2) \int_{-1}^4 (6x + 4) dx$$

Solution:

$$\begin{aligned}(1) &= \int_{-1}^2 x^2 = \left[\frac{x^3}{3} \right]_{-1}^2 \\ &= \frac{2^3}{3} - \frac{(-1)^3}{3} = 3\end{aligned}$$

$$\begin{aligned}(2) &= \int_{-1}^4 (6x + 4) dx = \left[\frac{6x^2}{2} + 4x \right]_{-1}^4 = [3x^2 + 4x]_{-1}^4 \\ &= [3(4)^2 + 4(4)] - [3(-1) + 4(-1)] = 64 - (-1) = 65\end{aligned}$$

2.3. Some Properties of Definite Integrals

We list below some important properties of definite integrals. These will be useful in evaluating the definite integrals more easily

$$\int_a^b f(x) dx = - \int_b^a f(x) dx.$$

$$\int_a^a f(x) = 0.$$

$$\int_a^b f(x) dx = \int_a^c f(x) dx + \int_c^b f(x) dx.$$

$$\int_a^b \alpha f(x) dx = \alpha \int_a^b f(x) dx.$$

$$\int_a^b (f + g)(x) dx = \int_a^b f(x) dx + \int_a^b g(x) dx.$$

$$\int_a^b f(x) dx = \int_a^b f(t) dx.$$

$$\int_a^b f(x) dx = \int_a^b f(a + b - x) dx.$$

$$\int_0^a f(x) dx = \int_0^a f(a - x) dx.$$

$$\int_0^{2a} f(x) dx = \int_0^a f(x) dx + \int_0^a f(2a - x) dx$$

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$$\int_0^{2a} f(x)dx = 2 \int_0^a f(x) dx \text{ if } f(2a - x) = f(x)$$

and 0 if $f(2a - x) = -f(x)$

$$\int_{-a}^a f(x) = 2 \int_0^a f(x) dx \text{ if } f \text{ is an even function, i.e., if } f(-x) = f(x).$$

$$\int_{-a}^a f(x) = 0 \text{ if } f \text{ is an odd function, i.e., if } f(-x) = -f(x).$$

3. Applications of integration in economics

– Economic applications of Definite integration: There are several economic and administrative applications, the most important of which are as follows:

A. Consumer surplus.

B. Producer surplus.

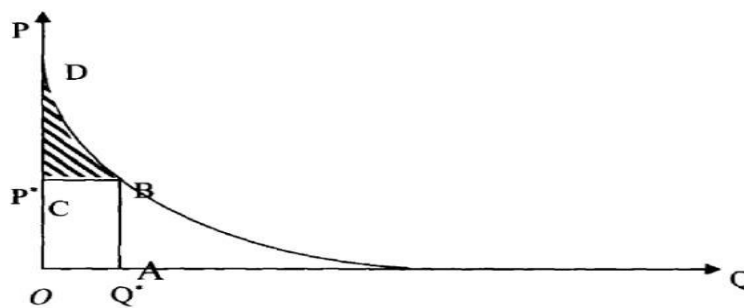
3.1.Consumer's Surplus :If the demand function is as follows : $P = f(Q)$

Where:

P :The price of the item.

Q :The required quantity of the item.

The following figure illustrates the difference in prices that consumers are willing to pay for different quantities of a commodity.



Source: Eid Ahmed Abu Bakr, 2013 – 1434, Page: 345

The area OABC represents the total amount of money paid by consumers to obtain quantity Q^* , which is P^*Q^* . When $Q^* = Q$, the price $P^* = P$.

CHAPTER FIVE: ORIGINAL FUNCTIONS AND INTEGRATION

The shaded part BCD represents the advantages that the consumer receives when paying the fixed price P^* , this is called the consumer surplus symbolized by the symbol CS can be found by the following relation:

Consumer surplus = what the consumer should have paid – what the consumer actually paid.

= Integration of the demand function from zero to Q^* – the product of the equilibrium price multiplied by the equilibrium quantity.

$$CS = \int_0^{Q^*} f(Q) dQ - P^* Q^*$$

EXAMPLE

If the demand curve for a commodity is : $P = 250 - 2Q$

Assuming that the equilibrium price in the market for this commodity is equal to 150

Required: Create the consumer's surplus.

Solution:

Since the price in equilibrium is equal to 150 by substituting in the equation of the demand curve, we get the equilibrium quantity as follows:

$$P = 250 - 2Q$$

$$150 = 250 - 2Q$$

$$2Q = 100$$

$$Q = \frac{100}{2} = 50$$

Now calculate the consumer surplus

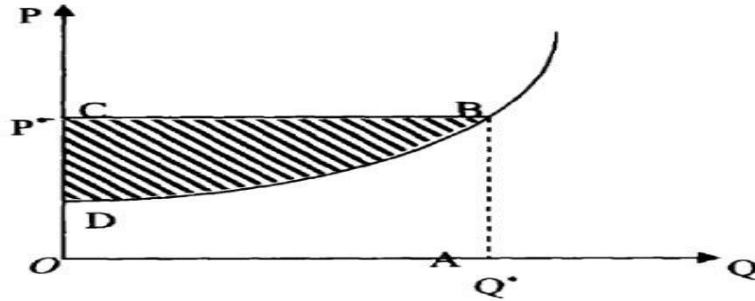
Consumer surplus = the integration of the demand function from zero to Q^* – the product of the equilibrium price multiplied by the equilibrium quantity.

$$\begin{aligned} CS &= \int_0^{Q^*} f(Q) dQ - P^* Q^* \\ &= \int_0^{50} (250 - 2Q) dQ - (150 \times 50) \\ &= \left[250Q - \frac{2Q^2}{2} \right]_0^{50} - 7500 \\ &= [250(50) - (50)^2 - (0)] - 7500 \\ &= [12500 - 2500] - 7500 \\ &= 10000 - 7500 = 2500 \end{aligned}$$

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3.2.Producer's Surplus :

If the supply function of a commodity is $P = g(Q)$ on the following image, the following figure shows the difference in prices at which producers want to display different quantities of goods.



Source: Eid Ahmed Abu Bakr, 2013 – 1434, Page: 349

The area $OABC$ represents the total amount of what the seller receives and is equal to P^*Q^* , this is when $Q^* = Q$ the price $P^* = P$ assumes that all goods are sold.

The shaded part BCD represents the advantages that the seller receives when selling the commodity at the specified price P^* , and this is called **the surplus of the product**, symbolized by the symbol PS , which can be found through the following relationship:

Surplus product = what the product actually got – what he wanted to get.

=The product of the equilibrium price multiplied by the equilibrium quantity – the integration of the width function from zero to Q^*

$$PS = P^*Q^* - \int_0^{Q^*} g(Q) dQ$$

EXAMPLE

If the supply curve of a commodity is determined by the relation :

$$P = 3Q^2 + 150$$

Assuming the price in equilibrium is 297

Required: Find the surplus product.

Solution:

We compensate by price to find the quantity

$$297 = 3Q^2 + 150$$

$$297 - 150 = 3Q^2$$

$$147 = 3Q^2$$

$$Q = \sqrt{49} = 7$$

That is, the equilibrium quantity = 7, the equilibrium price = 297.

Surplus product = what the product actually got – what he wanted to get.

$$\begin{aligned} PS &= P^*Q^* - \int_0^{Q^*} g(Q) dQ \\ &= 297 \times 7 - \int_0^7 (3Q^2 + 150) dQ \\ &= 2079 - \left[\frac{3Q^3}{3} - 150Q \right]_0^7 \\ &= 2079 - [(7)^3 + 150(7) - (0)] \\ &= 2079 - 1393 = 686 \end{aligned}$$

SERIES N° 5: INTEGRATION

Exercise 01:

Evaluate the following integral:

- ❖ $\int x^3 dx$
- ❖ $\int \sqrt[3]{x^2} dx$
- ❖ $\int (2x + 5)(x + 1) dx$
- ❖ $\int \frac{x-2}{(x^2-4x+5)} dx$
- ❖ $\int \sin(4x) dx$
- ❖ $\int \frac{x^3-1}{x^2} dx$

Exercise 02:

Solve the differential equation

1. $f'(x) = 6x^2 + x - 5$ subject to the initial condition $f(0) = 2$
2. $f''(x) = 5 \cos x + 2 \sin x$ subject to the initial conditions $f(0) = 3$ and $f'(0) = 4$
3. $f'(x) = 3x^2 - 2x + 1$ subject to the initial condition $f(2) = 3$
4. $f'(x) = ax - 5$ subject to the initial condition $f(-1) = 9$ and $f(2) = -3$

Exercise 03:

1- Evaluate the integral By Substitution formulae

- $\int \frac{x^2-1}{(x^3-3x+1)^6} dx$
- $\int \sqrt{2x-5} dx$
- $\int \frac{dx}{x \ln x}$
- $\int \frac{xdx}{\sqrt{9-4x^2}}$

2- Find the following integration By using the formulae for integration by part

- $\int e^{3x} \cdot 3 dx$
- $\int x^3 e^{2x} dx$

Exercise 04: Evaluate the following integrals:

- $I = \int_2^3 x^2 dx$,
- $I = \int_4^9 \frac{\sqrt{x}}{\left(30-x^{\frac{3}{2}}\right)^2} dx$
- $I = \int_{-1}^1 5x^4 \sqrt{x^5+1} dx$
- $I = \int_{-1}^2 |x^3 - x| dx$

ANSWER TO SERIES NUMBER 5

Exercise 01:

1- Evaluate the following integral:

$$\int x^3 dx$$

$$\Rightarrow f(x) = \frac{1}{4}x^4 + c$$

$$\int \sqrt[3]{x^2} dx$$

$$\Rightarrow x^{\frac{2}{3}} dx = \frac{3}{5}x^{\frac{5}{3}} + c$$

$$\int (2x + 5)(x + 1) dx$$

$$\Rightarrow \int (2x^2 + 2x + 5x + 5) dx$$

$$= \int (2x^2 + 7x + 5) dx = \int 2x^2 dx + \int 7x dx + \int 5 dx = \frac{2x^3}{3} + \frac{7x^2}{2} + 5x$$

$$\int \frac{x-2}{(x^2-4x+5)} dx$$

$$\Rightarrow \frac{1}{2} \int (x^2 - 4x + 5)^{-2} (2x - 4) dx = \frac{1}{2} \frac{(x^2 - 4x + 5)^{-1}}{-1} + c$$

$$\int \sin(4x) dx$$

$$\Rightarrow -\frac{1}{4} \cos(4x) + c$$

$$\int \frac{x^3 - 1}{x^2} dx$$

$$\Rightarrow \int x dx - \int x^{-2} dx = \frac{x^2}{2} + \frac{1}{x} + c$$

Exercise 02:

1-Solve the differential equation $f'(x) = 6x^2 + x - 5$ subject to the initial condition $f(0) = 2$

$$\int f'(x) dx = \int 6x^2 + x - 5 dx \Rightarrow f(x) = 2x^3 + \frac{1}{4}x^2 - 5x + c$$

Use the condition $f(0) = 2$ by substituting $x = 0$ into the function $f(x)$. We have

$$f(0) = 0 + 0 - 0 + c \Rightarrow c = 2$$

Therefore, the solution of the differential equation is $f(x) = 2x^3 + \frac{1}{4}x^2 - 5x + 2$

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2-Solve the differential equation $f''(x) = 5 \cos x + 2 \sin x$ subject to the initial conditions $f(0) = 3$ and $f'(0) = 4$

$$\int f''(x) dx = \int 5 \cos x + 2 \sin x dx \Rightarrow f'(x) = 5 \sin x - 2 \cos x + c$$

Using the condition $f'(0) = 4$ gives

$$f'(0) = 0 - 2 + c = 4 \Rightarrow c = 6$$

Hence, $f'(x) = 5 \sin x - 2 \cos x + 6$. Now, again

$$\begin{aligned} \int f'(x) dx &= \int (5 \sin x - 2 \cos x + 6) dx \\ \Rightarrow f(x) &= -5 \cos x - 2 \sin x + 6x + c \end{aligned}$$

Use the condition $f(0) = 3$ by substituting $x = 0$ into $f(x)$. We obtain

$$f(0) = -5 - 0 + 0 + c = 3 \Rightarrow c = 8$$

Hence, the solution of the differential equation is $f(x) = -5 \cos x - 2 \sin x + 6x + 8$

3-Solve the differential equation $f'(x) = 3x^2 - 2x + 1$ subject to the initial condition $f(2) = 3$

$$\int f'(x) dx = \int 3x^2 - 2x + 1 dx \Rightarrow f(x) = \frac{3x^3}{3} - \frac{2x^2}{2} + x + c = x^3 - x^2 + x + c$$

Use the condition $f(2) = 3$ by substituting $x = 2$ into the function $f(x)$. We have

$$f(2) = 8 - 4 + 2 + c \Rightarrow 3 = 6 + c \Rightarrow c = 3 - 6 = -3$$

Therefore, the solution of the differential equation is $f(x) = x^3 - x^2 + x - 3$

4-Solve the differential equation $f'(x) = ax - 5$ subject to the initial condition $f(-1) = 9$ and $f(2) = -3$

$$\int f'(x) dx = \int ax - 5 dx \Rightarrow f(x) = \frac{ax^2}{2} - 5x + c$$

Using the condition $f'(-1) = 9$ gives

$$\begin{aligned} f'(-1) &= \frac{a(-1)^2}{2} - 5(-1) + c = 9 \Rightarrow \frac{a}{2} + 5 + c = 9 \\ \Rightarrow \frac{a}{2} + c &= 4 \Rightarrow a + 2c = 8 \dots \dots (1) \end{aligned}$$

Now, again, Using the condition $f'(2) = -3$ gives

$$f'(2) = \frac{a(2)^2}{2} - 5(2) + c = -3 \Rightarrow \frac{4a}{2} - 10 + c = -3$$

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$$\Rightarrow 2a - 10 + c = -3 \Rightarrow 2a + c = 7$$

$$(2a + c = 7) \times (2) = 4a + 2c = 14 \dots \dots (2)$$

By Subtracting the first equation from the second, we get:

$$3a = 6 \Rightarrow a = 2$$

We substitute $a = 2$ in Equation 1

$$2 + 2c = 8 \Rightarrow 2c = 6 \Rightarrow c = 3$$

Therefore, the solution of the differential equation is $f(x) = x^2 - 5x + 3$

Exercise 03:

1-Evaluate the integral By Substitution formulae

$$\int \frac{x^2 - 1}{(x^3 - 3x + 1)^6} dx$$

Let $u = x^3 - 3x + 1$, then $du = 3(x^2 - 1) dx$. By substitution, we have:

$$\frac{1}{3} \int u^{-6} du = \frac{1}{3} \frac{1}{-5u^{-5}} + c = \frac{-1}{15(x^3 - 3x + 1)^5} + c$$

$$\int \sqrt{2x - 5} dx$$

Let $u = 2x - 5$, then $du = 2 dx$. By substitution, we have:

$$\frac{1}{2} \int u^{\frac{1}{2}} du = \frac{1}{2} \frac{(2x-5)^{3/2}}{3/2} + c = \frac{(2x-5)^{3/2}}{3} + c = \frac{1}{3} \sqrt{(2x-5)^3} + c$$

$$\int \frac{dx}{x \ln x}$$

The fact that dx/x is the differential of $\ln x$ suggests the substitution $u = \ln x$, so $du = dx/x$ and

$$\int \frac{dx}{x \ln x} = \int \frac{du}{u} = \ln u = \ln(\ln x) + c$$

$$\int \frac{xdx}{\sqrt{9-4x^2}}$$

Here the fact that the x in the numerator is essentially the derivative of the expression $9 - 4x^2$ inside the radical suggests the substitution $u = 9 - 4x^2$. Then $du = -8x dx$, and

$$\int \frac{xdx}{\sqrt{9-4x^2}} = -\frac{1}{8} \int \frac{du}{\sqrt{u}} = -\frac{1}{8} \int u^{-1/2} du = -\frac{1}{8} \frac{u^{1/2}}{\frac{1}{2}} = -\frac{1}{4} \sqrt{u} = -\frac{1}{4} \sqrt{9-4x^2} + c$$

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2-Find the following integration By using the formulae for integration by part

$$\int e^{3x} \cdot 3 \, dx$$

Take first function as 3 and second function as e^{3x} . The integral of the second function is $\frac{1}{3}e^{3x}$

Therefore, $\int e^{3x} \cdot 3 \, dx = 3 \cdot \frac{1}{3}e^{3x} - \int \frac{1}{3}e^{3x}(0)dx = e^{3x} + c$

$$\int x^3 e^{2x} \, dx$$

$$\int x^3 e^{2x} \, dx = x^2 \frac{e^{2x}}{2} - \int \frac{2xe^{2x}}{2} \, dx$$

$$= \frac{1}{2}x^2 e^{2x} - \int x e^{2x} \, dx$$

$$= \frac{1}{2}x^2 e^{2x} - \left[\frac{x e^{2x}}{2} - \int 1 \cdot \frac{e^{2x}}{2} \, dx \right]$$

$$= \frac{1}{2}x^2 e^{2x} - \left[\frac{1}{2}x e^{2x} - \frac{1}{4}e^{2x} \right] + c$$

$$= \frac{1}{2}x^2 e^{2x} - \frac{1}{2}x e^{2x} - \frac{1}{4}e^{2x} + c$$

$$\frac{1}{2}e^{2x} - \left[x^2 - x + \frac{1}{2} \right] + c$$

Exercise 04:

$$I = \int_2^3 x^2 \, dx$$

$$I = \int_2^3 x^2 \, dx = \left[\frac{x^3}{3} \right]_2^3 = F(3) - F(2) = \frac{27}{3} - \frac{8}{3} = \frac{19}{3}$$

$$I = \int_4^9 \frac{\sqrt{x}}{(30 - x^{\frac{3}{2}})^2} \, dx$$

Let $I = \int_4^9 \frac{\sqrt{x}}{(30 - x^{\frac{3}{2}})^2} \, dx$ We first find the anti derivative of the integrand

Put $30 - x^{\frac{3}{2}} = t$ then $-\frac{3}{2}\sqrt{x}dx = dt$ or $\sqrt{x}dx = -\frac{2}{3}dt$; Thus,

$$\int \frac{\sqrt{x}}{(30 - x^{\frac{3}{2}})^2} \, dx = -\frac{2}{3} \int \frac{dt}{t^2} = \frac{2}{3} \left[\frac{1}{t} \right] = \frac{2}{3} \left[\frac{1}{(30 - x^{\frac{3}{2}})} \right] = F(x)$$

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Therefore, by the second fundamental theorem of calculus, we have

$$I = F(9) - F(4) = \frac{2}{3} \left[\frac{1}{(30 - x^{\frac{3}{2}})} \right]_2^9 = \frac{2}{3} \left[\frac{1}{(30 - 27)} - \frac{1}{(30 - 8)} \right] = \frac{2}{3} \left[\frac{1}{3} - \frac{1}{22} \right] = \frac{19}{99}$$

$$I = \int_{-1}^1 5x^4 \sqrt{x^5 + 1} dx$$

Put $t = x^5 + 1$, then $dt = 5x^4 dx$

Therefore,

$$\int 5x^4 \sqrt{x^5 + 1} dx = \int \sqrt{t} dt = \frac{2}{3} t^{\frac{3}{2}} = \frac{2}{3} (x^5 + 1)^{\frac{3}{2}}$$

Hence

$$\begin{aligned} I &= \int_{-1}^1 5x^4 \sqrt{x^5 + 1} dx = \frac{2}{3} \left[(x^5 + 1)^{\frac{3}{2}} \right]_{-1}^1 \\ &= \frac{2}{3} \left[(1^5 + 1)^{\frac{3}{2}} - ((-1)^5 + 1)^{\frac{3}{2}} \right] \\ &= \frac{2}{3} \left[2^{\frac{3}{2}} - 0^{\frac{3}{2}} \right] = \frac{2}{3} (2\sqrt{2}) = \frac{4\sqrt{2}}{3} \\ I &= \int_{-1}^2 |x^3 - x| dx \end{aligned}$$

We note that $x^3 - x \geq 0$ on $[-1, 0]$ and $x^3 - x \leq 0$ on $[0, 1]$ and that $x^3 - x \geq 0$ on $[1, 2]$.

So we write

$$\begin{aligned} \int_{-1}^2 |x^3 - x| dx &= \int_{-1}^0 (x^3 - x) dx + \int_0^1 -(x^3 - x) dx + \int_1^2 (x^3 - x) dx \\ &= \int_{-1}^0 (x^3 - x) dx + \int_0^1 (x - x^3) dx + \int_1^2 (x^3 - x) dx \\ &= \left[\frac{x^4}{4} - \frac{x^2}{2} \right]_{-1}^0 + \left[\frac{x^2}{2} - \frac{x^4}{4} \right]_0^1 + \left[\frac{x^4}{4} - \frac{x^2}{2} \right]_1^2 \\ &= -\left(\frac{1}{4} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{4}\right) + (4 - 2) - \left(\frac{1}{4} - \frac{1}{2}\right) \\ &= -\frac{1}{4} + \frac{1}{2} + \frac{1}{2} - \frac{1}{4} + 2 - \frac{1}{4} + \frac{1}{2} = \frac{3}{2} - \frac{3}{4} + 2 = \frac{11}{4} \end{aligned}$$

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