People's Democratic Republic of Algeria
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Faculty of **Matter Sciences and Computer Science**
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DIFFERENTIALS EQUATIONS

Courses and corrected exercises
Intended for L3 Mathematics students

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University year: **2025 - 2026**

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General Introduction

The course **Differential Equations**, offered in **Semester 05** within the **Fundamental Teaching Unit**, constitutes a core component of the undergraduate mathematics curriculum. Differential equations represent one of the most powerful tools in mathematics, providing a natural framework for modeling change and describing the evolution of systems over time. They arise in a wide variety of scientific fields, including physics, engineering, biology, economics, and social sciences, which makes their study essential for both theoretical development and practical applications.

The primary objective of this course is to introduce students to the **fundamental concepts, methods, and theorems** governing the theory of **ordinary differential equations (ODEs)**. Emphasis is placed on the *qualitative study* of solutions rather than on computational techniques alone. The course aims to develop a rigorous understanding of the behavior of solutions, the conditions ensuring their existence and uniqueness, and their dependence on initial conditions. These notions form the theoretical foundation of modern analysis and are crucial for advanced studies in mathematics and applied sciences.

The course begins with an in-depth study of **first-order differential equations**, which serve as the cornerstone of the theory. Fundamental results are established in a rigorous analytical setting, including local and global existence theorems and uniqueness results. Particular attention is devoted to the continuous dependence of solutions on initial data, a key concept for understanding the stability and predictability of mathematical models. Through this study, students gain insight into how small variations in initial conditions can influence the behavior of solutions.

The course then extends to **higher-order differential equations**, demonstrating how they can be systematically reduced to systems of first-order equations. This approach provides a unified framework that allows the effective use of tools from linear algebra. By studying differential equations through the lens of systems, students develop a deeper understanding of their structural and geometric properties.

A major part of the course is dedicated to **linear systems of differential equations**. The theory of the *matrix exponential* is introduced as a fundamental method for solving linear systems, along with the study of second-order systems and the notion of the resolvent. These concepts are essential for analyzing the dynamics of linear systems and constitute a foundation for further studies in areas such as control theory, numerical analysis, and mathematical modeling. Throughout this chapter, the strong interaction between linear algebra and differential equations is emphasized.

In the final chapter, the course introduces the basic **notions of stability**, which play a central role in the qualitative theory of differential equations. Stability analysis allows students to understand the long-term behavior of solutions and to determine whether a system approaches equilibrium, exhibits oscillatory behavior, or becomes unstable. These ideas are fundamental in the study of dynamical systems and are of great importance in real-world applications where robustness and reliability are critical.

This course relies on prior knowledge of **Real Analysis, Linear Algebra, and Topology**, which provide the analytical and conceptual tools necessary for a rigorous treatment of differential equations. By integrating these prerequisite subjects, the course strengthens students' mathematical maturity and coherence in analysis.

With a workload corresponding to **6 credits** and a **coefficient of 4**, this course is designed to offer a solid theoretical foundation while preparing students for advanced studies in differential equations, dynamical systems, and applied mathematics. Upon successful completion, students will have acquired the essential analytical skills and conceptual understanding required to model, analyze, and interpret complex phenomena using ordinary differential equations.

Chapter 1

First-order equation

1.1 Fundamental Results

1.1.1 First-order ODE (Ordinary Differential Equation).

Definition 1.1.

A differential equation is a relation of the form

$$F(t, y, y', ..., y^{(n)}) = 0 \tag{1.1}$$

between a variable t, a function y, and its successive derivatives up to order n. [17]

Example 1.1.1.

(1) $yy'' = t - t^2y'$.

(2) $x' - tx + t^2 = 1$.

Definition 1.2.

The order of a differential equation is defined as the order of its highest derivative.[18]

Example 1.1.2.

(1) $yy'' = t - t^2y'$ is a second-order differential equation.

(2) $x' - tx + t^2 = 1$is a first-order differential equation.

Definition 1.3.

A solution (or integral) of a differential equation is any function $y = y(t)$ defined on an interval $I \subset \mathbb{R}$, possessing successive derivatives $y', y'', \ldots, y^{(n)}$ and satisfying relation (1.1).[17]

Example 1.1.3.

For the differential equation $y''+y=0$; (a second-order differential equation). The function $y_1 = \sin x$ is a solution, since $y_1 = \sin x$ is defined on $I = \mathbb{R}$, and $y_1' = \cos x$, $y_1'' = -\sin x$, hence $y''+y=0$. Similarly, $y_2 = \cos x$ is also a solution of the given equation, and likewise $y_3 = \sin x+\cos x$. In general, $y = a\sin x+b\cos x$ is also a solution, where a and b are two real constants. This last one is called the *general solution, while the others* $y_1, y_2, \ldots, y_n$ are called *particular solution*

Remark 1.1.1.

1. To solve or integrate a differential equation means to find all the solutions of this differential equation.
2. The graph of the general solution **(resp. particular solution)** is called the integral curve.

1.1.2 First-order differential equation

Definition 1.4.

A first-order differential equation is any relation of the form:

$$y' = F(t, y) \tag{1.2}$$

1.1.3 Separable differential equations

Definition 1.5.

A separable differential equation is any equation of the form:[17]

$$f(y)y' = g(t) \tag{1.3}$$

where f and g are two numerical functions defined and continuous on intervals to be specified.

Remark 1.1.2.

Equation (1.3) can also be written in the form:

$$f(y)dy = g(t)dt.$$

The solutions of equation (1.3) are defined by

$$\int f(y)dy = \int g(t)dt + c.$$

Proof 1.1.1.

By substituting $Y' = \dfrac{dy}{dt}$ into (1.3), we obtain $f(y)\dfrac{dy}{dt} = g(t)$ or $f(y)\,dy = g(t)\,dt$.

$$f(y)dy = g(t)dt \tag{1.4}$$

We integrate (1.4) term by term in order to obtain the general solution of equation (1.3) in the form: ...

$$\int f(y)dy = \int g(x)dx + c.$$

where **c** is an arbitrary constant.

Example 1.1.4.

Solve on $\mathbb{R}_+^*$ the equation:

$$t^2y' - y^2 = 0$$

It is obvious that $y = 0$ is a solution.
For $y \neq 0$, we can separate the variables since $y' = \dfrac{dy}{dt}$, we obtain:

$$\begin{aligned} t^2y' - y^2 = 0 \Longrightarrow & t^2\frac{dy}{dt} = y^2\frac{dy}{y^2} = \frac{dt}{t^2} \\ & \Longleftrightarrow \frac{-1}{y} = \frac{-1}{t} + c, \quad c \in \mathbb{R} \\ & \Longleftrightarrow \frac{1}{y} = \frac{1-ct}{t} \\ & \Longleftrightarrow y = \frac{t}{1-ct}, \quad c \in \mathbb{R}. \end{aligned}$$

Therefore, the set S of solutions is

$$S = \{y = 0 \text{ or } y = \frac{t}{1-ct}, c \in \mathbb{R}\}$$

Example 1.1.5.

Integrate the following equation

$$t^3y' = e^{3y}$$

We can separate the variables t and y; we obtain

$$e^{-3y}dy = \frac{dt}{t^3}.$$

By integrating, we obtain

$$\frac{e^{-3y}}{3} = \frac{1}{2t^2} + c,$$

$$e^{-3y} = 3\Big(\frac{1}{2t^2} + c\Big) \Longrightarrow -3y = ln\Big|3\Big(\frac{1}{2t^2} + c\Big)\Big|$$
$$\Longrightarrow y = \frac{-1}{3} ln\Big|\frac{3}{2t^2} + \tilde{c}\Big|, \quad \tilde{c} \in \mathbb{R}.$$

1.1.4 Homogeneous differential equations

Recall :

f is said to be homogeneous of degree n if it satisfies the identity

$$\forall (x, y) \in D_f, f(\lambda x, \lambda y) = \lambda^n = f(x, y).$$

Example 1.1.6.

We show that $f(x, y) = x^2 + y^2 - xy$ is a homogeneous function :

Indeed, for all $(x, y) \in \mathbb{R}^2$ and $\lambda \in \mathbb{R}$, we have :

$$\begin{aligned} f(\lambda x, \lambda y) &= (\lambda x)^2 + (\lambda y)^2 - (\lambda x)(\lambda y) \\ &= \lambda^2 x^2 + \lambda^2 y^2 - \lambda^2 xy \\ f(\lambda x, \lambda y) &= \lambda^2 (x^2 + y^2 - xy) \end{aligned}$$

Therefore, f is a homogeneous function of order 2.

Definition 1.6.

A differential equation of the form $y' = f(t, y)$ is called homogeneous if the function $f(t, y)$ is homogeneous of degree zero.[17]

Remark 1.1.3.

A homogeneous differential equation can always be written in the form $y' = \varphi(\frac{y}{t})$.

1.1.5 The solution of a homogeneous differential equation

Consider the homogeneous differential equation

$$y' = \varphi(\frac{y}{t}) \tag{1.5}$$

Let $\frac{y}{t} = u$. It follows that $y = ut \iff y' = u't + u$. Therefore

$$\begin{aligned} y' = \varphi(\frac{y}{t}) \quad &\iff u't + u = f(u) \\ &\iff tu' + u = f(u) \\ &\iff tu' = f(u) - u \iff \frac{u'}{f(u) - u} = \frac{1}{t} \\ &\iff \frac{du}{f(u) - u} = \frac{dt}{t} \\ &\iff \int \frac{du}{f(u) - u} = \ln|t| + c, \qquad c \in \mathbb{R}. \end{aligned}$$

Therefore, the solutions of equation (1.5) are given by

$$y = tu \quad et \quad t = ke^{\int \frac{du}{f(u) - u}}, \qquad k \in \mathbb{R}.$$

Example 1.1.7.

Solve the following equation

$$ty' = \sqrt{t^2 - y^2} + y \tag{1.6}$$

For $t \neq 0$, it follows that

$$\begin{aligned} (1.6) \iff y' &= \sqrt{\frac{t^2 - y^2}{t^2}} + \frac{y}{t} \\ \iff y' &= \sqrt{1 - (\frac{y}{t})^2} + \frac{y}{t}. \end{aligned} \tag{1.7}$$

Let $\frac{y}{t} = u$, thus $y' = tu' + u$. Then:

$$\begin{aligned} (1.6) &\iff tu' + u = \sqrt{1 - u^2} + u \\ &\iff tu' = \sqrt{1 - u^2} \\ &\iff \frac{u'}{\sqrt{1 - u^2}} = \frac{1}{t} \iff \frac{du}{\sqrt{1 - u^2}} = \frac{dt}{t} \\ &\iff \arcsin u = \ln|t| + c, \qquad c \in \mathbb{R} \\ &\iff u = \sin(\ln|t| + c) \\ &\iff y = t \sin(\ln|t| + c), \qquad c \in \mathbb{R} \end{aligned}$$

1.1.6 First-order linear differential equations

Definition 1.7.

The general form of a first-order linear equation is

$$a(t)y' + b(t)y = f(t) \tag{1.8}$$

where a, b, and f are continuous functions on an interval I $\subset \mathbb{R}$.

1. It is linear because the operator $L(y) = a(t)y' + b(t)y$ is linear.
2. If the function $f \equiv 0$ on I, the operator (2.5) is said to be homogeneous, or without a nonhomogeneous term.
3. The general solution is given by $y = y_h + y_p$, where y_h is the general solution of the homogeneous equation, and y_p is a particular solution of equation (2.5).[17]

1.1.7 Method of solution

Let y be the general solution and y_p a particular solution of the equation (2.5).

$$\begin{cases} ay' + by = f(t) \\ ay_p' + by_p = f(t) \end{cases} \tag{1.9}$$

Hence, by difference

$$a(y' - y_p') + b(y - y_p) = 0$$

ou

$$a(y - y_p)' + b(y - y_p) = 0$$

Then, $(y - y_p)$ represents the general solution, denoted y_h, of the homogeneous equation (without the nonhomogeneous term). Therefore,

$$y = y_h + y_p$$

Let us now look for the general solution of the homogeneous equation associated with (2.5).

If $a(t) \neq 0$ on I, then:

$$\begin{aligned} a(t)y' + b(t)y = 0 &\Longleftrightarrow \frac{y'}{y} = -\frac{b(t)}{a(t)} \\ &\Longleftrightarrow \ln|y| = -\int \frac{b(t)}{a(t)}dt \\ &\Longleftrightarrow y = ke^{-\int \frac{b(t)}{a(t)}}dt, \qquad k \in \mathbb{R}. \end{aligned}$$

Therefore, the general solution of the homogeneous equation is:

$$y_h = ke^{-\int \frac{b(t)}{a(t)}dt}, \qquad k \in \mathbb{R}.$$

It remains to determine y_p, a particular solution of equation (2.5). For this, we can use the method of variation of constants, which consists in seeking a particular solution in the form:

$$y_p = k(t)e^{-\int \frac{b(t)}{a(t)}dt},$$

Let us suppose that the constant k is a function of t. Then we determine $k(t)$ from the complete equation (2.5).

Example 1.1.8.

Integrate the following differential equation:

$$ty' + t + y = 0 \qquad pour \qquad t \neq 0, \tag{1.10}$$

We now solve the homogeneous equation associated with (1.10):

$$\begin{aligned} ty' + y = 0 &\Longleftrightarrow ty' = -y \\ &\Longleftrightarrow y = 0 \quad ou \quad \frac{y'}{y} = \frac{-1}{t} \\ &\Longleftrightarrow y = 0 \quad ou \quad \frac{dy}{y} = -\frac{dt}{t} \\ &\Longleftrightarrow y = 0 \quad ou \quad (\ln|y| = -\ln|t| + c, \qquad c \in \mathbb{R}) \\ &\Longleftrightarrow y = 0 \quad ou \quad y = \frac{k}{t}, \quad k \in \mathbb{R} \\ &\Longleftrightarrow y(t) = \frac{k}{t}, \quad k \in \mathbb{R}. \end{aligned}$$

Therefore, the general solution of the homogeneous equation is

$$y_h(t) = \frac{k}{t}, \qquad k \in \mathbb{R}.$$

We look for a particular solution of the form

$$y_p(t) = \frac{k(t)}{t}.$$

$$\begin{aligned}(1.10) &\iff t y_p' + y_p = -t \\ &\iff t\left(\frac{tk'(t) - k(t)}{t^2}\right) + \frac{k(t)}{t} = -t \\ &\iff k'(t) = -t \\ &\iff k(t) = -\frac{t^2}{2}.\end{aligned}$$

Thus, we can take

$$y_p(t) = -\frac{t}{2}$$

as a particular solution, and consequently the general solution of (1.10) is given by:

$$y(t) = \frac{k}{t} - \frac{t}{2}, \qquad k \in \mathbb{R}$$

$$y(t) = y_h(t) + y_p(t) = \frac{k}{t} - \frac{t}{2}, \quad k \in \mathbb{R}.$$

1.2 Nonlinear differential equations

1.2.1 Bernoulli differential equation

Definition 1.8.

We call a **Bernoulli equation** a first-order differential equation that can be written in the form:

$$y'(t) + a(t)y(t) = b(t)\, y^{\alpha}(t),$$

where $a(t)$ and $b(t)$ are continuous functions on an interval I, and $\alpha \in \mathbb{R}$.

$$a(t)y' + b(t)y = f(t)y^{\alpha}, \qquad \alpha \in \mathbb{R}^* - \{1\},$$

where a, b, and f are continuous functions on an interval I, and $a(t) \neq 0$ on I.

- **Note:** For $\alpha = 0$ and $\alpha = 1$, the equation reduces to a linear differential equation.[17]

1.2.2 Method of solution

By dividing by y^α, we obtain:

$$a(t)y'y^{-\alpha} + b(t)y^{1-\alpha} = f(t)$$

The change of function defined by $z = y^{1-\alpha}$ leads to a linear equation.

Indeed, $z' = (1-\alpha)y'y^{-\alpha}$

therefore $\dfrac{a(t)}{(1-\alpha)}z'(t) + b(t)z(t) = f(t)$

or again

$$a(t)Z' + (1-\alpha)b(t)Z = (1-\alpha)f(t).$$

Example 1.2.1.

Integrate the equation ($y \neq 0$)

$$y' - ty = ty^3 \tag{1.11}$$

Dividing (2.6) by y^3, we obtain:

$$\frac{y'}{y^3} - t\frac{1}{y^2} = t,$$

then we make the change of variable

$$z = \frac{1}{y^2} = y^{-2},$$

which gives

$$z' = -2y'y^{-3}.$$

By substituting into (2.6), we obtain a first–order linear differential equation.

$$\frac{y'}{y^3} - t\frac{1}{y^2} = t$$
$$-2y'y^{-3} + 2t\frac{1}{y^2} = 2t$$

$$z' + 2tz = 2t \tag{1.12}$$

Solving (1.12) yields:

$$z_H = ke^{-t^2}, \quad z_P = 1 \Longrightarrow z = 1 + ke^{-t^2}, \qquad k \in \mathbb{R}.$$

Therefore, the general solution of (2.6) is:

$$y^2 = \frac{1}{z} \Longrightarrow y = \pm\sqrt{\frac{1}{z}} \quad \text{where} \quad y = \pm\sqrt{\frac{1}{1+ce^{-t^2}}}.$$

1.2.3 Riccati differential equation

Definition 1.9.

A Riccati equation is called a first–order differential equation that can be written in the form:

$$y' = a(t)y^2 + b(t)y + c(t)$$

where a, b and c are continuous functions on an interval I, and $a(t) \neq 0$ on the interval I.[17]

1.2.4 Solution Method

• **Method 01:** (Transforming the Riccati equation into a Bernoulli equation)
Let the Riccati equation be:

$$a(t)y' + b(t)y + c(t)y^2 = f(t) \tag{1.13}$$

Let y_p be a particular solution of equation (1.13). Set $y = y_p + z$, hence $y' = y_p' + z'$. Thus, determining y reduces to determining z. Indeed:

$$\begin{aligned}
(1.13) &\Longleftrightarrow a(y_p' + z') + b(y_p + z) + c(y_p + z)^2 = f(t)\\
&\Longleftrightarrow ay_p' + az' + by_p + bz + c(y_p^2 + 2y_pz + z^2) = f(t)\\
&\Longleftrightarrow \underbrace{ay_p' + by_p + cy_p^2}_{=\text{f(t)}} + az' + (b + 2cy_p)z + cz^2 = f(t)\\
&\Longleftrightarrow az' + (b + 2cy_p)z + cz^2 = f(t).
\end{aligned}$$

By solving this last equation (a Bernoulli equation), we obtain z and consequently the solution y of the Riccati equation (1.13).

- **Method 02:** (Transforming the Riccati equation into a linear equation)

Let the Riccati equation be:

$$a(t)y' + b(t)y + c(t)y^2 = f(t)$$

Let y_p be a particular solution of equation (1.13). Set $y = y_p + \frac{1}{z}$, hence $y' = y_p' - \frac{z'}{z^2}$. Therefore, the determination of y reduces to determining z, indeed:

$$\begin{aligned}
(1.13) \iff & a\Big(y_p' - \frac{z'}{z^2}\Big) + b\Big(y_p + \frac{1}{z}\Big) + c\Big(y_p^2 + \frac{1}{z}\Big)^2 = f(t) \\
\iff & ay_p' - a\frac{z'}{z^2} + by_p + \frac{b}{z} + c\Big(y_p^2 + 2\frac{y_p}{z} + \frac{1}{z^2}\Big) = f(t) \\
\iff & \underbrace{ay_p' + by_p + cy_p^2}_{=\text{f(t)}} - a\frac{z'}{z^2} + \frac{b}{z} + 2c\frac{y_p}{z} + \frac{c}{z^2} = f(t) \\
\iff & -a\frac{z'}{z^2} + (b + 2cy_p) + \frac{c}{z^2} = 0 \\
\iff & az' - (b + 2cy_p) - c = 0.
\end{aligned}$$

By solving this last equation (first order), we obtain z and consequently the solution y of the Riccati equation (1.13).

Example 1.2.2.

Intégrer l'équation :

$$y' = y^2 - 2ty + t^2 + 1 \tag{1.14}$$

It is easy to verify that $y_p = t$ is a particular solution of (1.14).

By setting $y = t + z$, we have:

$$\begin{aligned}
y' = 1 + z' \iff & 1 + z' = (t + z)^2 - 2t(t + z) + t^2 + 1 \\
\iff & 1 + z' = 1 + z^2 \\
\iff & z' = z^2 \Longrightarrow \int \frac{dz}{z^2} = \int dt \Longrightarrow \frac{-1}{z} = t + c \\
\iff & \frac{1}{z} = -(t + c) \Longrightarrow z = \frac{-1}{t + c} \Longrightarrow y = \frac{-1}{t + c} + t,
\end{aligned}$$

Exercise 1.2.1. 1 Solve the following first-order differential equations:

1 $y^{'} = 3y$

2 $y^{'} = 2\frac{y}{x} - 1$

3 $y^{'} - ty = t$

4 $y^{'} + \frac{1}{\sqrt{t}}y = \frac{1}{\sqrt{t}}, t \succ 0$

5 $(t^2 - 1)y^{'} - y = t^2, t \in]1, +\infty[$

Exercise 1.2.2. 1 Solve the following first-order differential equations:

1 $\frac{dy}{dt} + \frac{y}{t} = 4$

2 $y^{'} \cos t + y \sin t = 1$

3 $\frac{dy}{dt} = (1 + y^2)e^t$

4 $dy + y \tan t = 0$

Exercise 1.2.3. 1. Solve the following first-order differential equations:

1 $y^{'} + y - 5e^{-t}y^5 = 4$

2 $y^{'} - y = ty^5$

3 $y^{'} + p(t)y + q(t)y^r = 0, r \in \mathbb{R}$

Exercise 1.2.4. 1. Solve the following first-order differential equations: .

1 $y^{'} - \frac{1}{t}y - y^2 = -9t^2, y_p = at, t \in]0, +\infty[$

2 $t^3y^{'} + y^2 + t^2y + 2t^4 = 0, y_p = -t^2.$

1.3 Existence and Uniqueness of Solutions

1.3.1 Maximal and global solutions

Let the O.D.E.

$$\frac{dy}{dt} = f(t, y),$$

where f is defined on the open set U of $I \times \mathbb{R}^n$; $U = I \times \mathbb{R}^n$; $f = I \times \Omega \longrightarrow \mathbb{R}$.

Extension of a solution

[5] Let y be a solution of (E) on $J \subset I$.

Let $\tilde{y}$ be a solution of (E) on $\tilde{J} \subset I$.

We say that $\tilde{y}$ is an **extension** of y if :

$$J \subset \tilde{J} \quad \text{and} \quad \tilde{y}_{|J} = y.$$

$\alpha)$ $J \subset \tilde{J}$.

$\beta)$ $\tilde{y}_{/J} = y \quad (\tilde{y}(t) = y(t), \quad \forall t \in J), \quad \tilde{y} = y$ on J.

Example 1.3.1.

Let us consider on $I = (0, +\infty)$ **the equation**

$$y'(t) = \frac{2}{t}y(t) \Longleftrightarrow \frac{dy}{dt} = \frac{2}{t}y \Longrightarrow \frac{dy}{y} = \frac{2}{t} \Longrightarrow \ln|y| = 2\ln t + c$$
$$\Longrightarrow |y| = kt^2 \Longrightarrow y = \pm kt^2 \Longrightarrow y = ct^2.$$

Let the solution be $y :]3, +\infty[\subset I \longmapsto \mathbb{R}$.

A solution defined by $y(t) = t^2$.

The solution $\tilde{y} :]2, +\infty[\longmapsto \mathbb{R}$ defined by :

$\tilde{y}(t) = t^2$ is an extension of y because :

- $J \subset \tilde{J} \quad (]3, +\infty[\subset]2, +\infty[)$.
- $y(t) = \tilde{y}(t), \forall t \in J =]3, +\infty[$.

1.3.2 Maximal and global solutions

[5] A solution $y : J \subset I \longmapsto \mathbb{R}$ is said to be **maximal** if it does not admit **any extension** $\tilde{y} : \tilde{J} \longmapsto \mathbb{R}$ with $J \subsetneq \tilde{J} \quad$ (J strictly included in $\tilde{J}$).

Example 1.3.2.

The function defined on $J = \mathbb{R}$ by $y(t) = e^{-4t}$ is a maximal solution of the differential equation $y' = -4y(t)$, since it is defined on $\mathbb{R}$ which is a maximal interval.

Theorem 1.3.1.

Let $\alpha, \beta \in \mathbb{R}$.

Let y be a solution of (E) defined on $]\alpha, +\infty[\quad (]-\infty, \beta[)$ if :

$$\lim_{t \overset{>}{\to} \alpha} y(t) \quad (\lim_{t \overset{<}{\to} \beta} y(t)) \quad doesnot \quad exist.$$

Then y is a **maximal** solution

Example 1.3.3.

The function $y : J =]-\infty, -1[\to \mathbb{R}$ defined by :

$$y(t) = \frac{1}{1+t}$$

is a maximal solution of $y'(t) = -y^2(t)$ because:

$$\lim_{t \overset{<}{\to} -1} \frac{1}{1+t} = -\infty.$$

Remark 1.3.1.

If y is a maximal solution on the interval I, this does not imply that y is the unique solution of (E) on I.

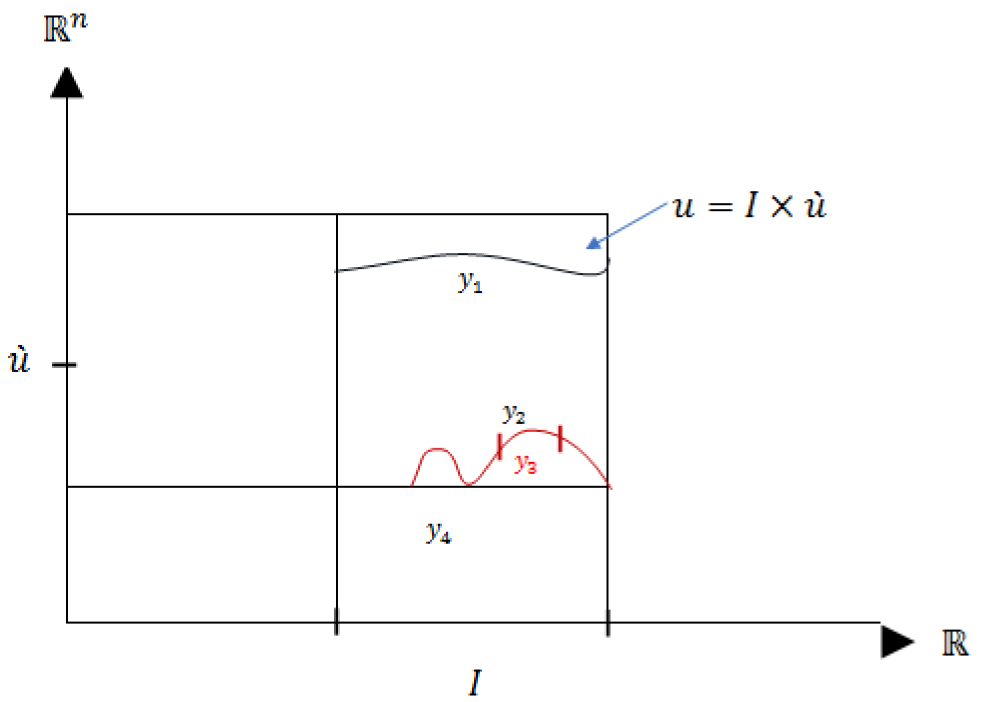
1.3.3 The global solution

[5] Let the ODE:

$$\frac{dy}{dt} = f(t, y),$$

where f is defined on the open set $U \subset \mathbb{R} \times \mathbb{R}^n$ such that $U = I \times \Omega$.

The function y is called a *global solution* if it is defined on the entire interval I.



- y_1 : **Global maximal solution**.
- y_3 : **Neither maximal nor global solution**
- y_2 :**Neither maximal nor global solution** • y_4 :**Maximal but not global solution**.

Remark 1.3.2.

Every global solution is maximal, but the converse is false.

Example 1.3.4.

We have: $\frac{dy}{dt} = y^2$ and $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ où $f(t, y) = y^2$ and $\begin{cases} I = \mathbb{R} \\ \Omega = \mathbb{R} \end{cases}$ The solutions are :

- $y_1(t) = 0, \quad \forall t \in \mathbb{R}$ **Global solution: it is defined** I.
- $y_2(t) = -\frac{1}{t}, \quad t \in]-\infty, 0[$. • $y_3(t) = -\frac{1}{t}, \quad t \in]0, +\infty[$.

y_2 et y_3 are maximal solutions.

Remark 1.3.3.

Every solution of (E) can be extended to a maximal solution.

1.3.4 Local and global existence, uniqueness

[5] A function of several variables is said to be of class C^k if it admits continuous partial derivatives up to order k.

- **For $k = 1$:**

Let f be a function defined on an open subset D of $\mathbb{R}^2$ with values in $\mathbb{R}$.
The function f is of class C^1 on D if $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$ exist and are continuous on D.

- **For $k = 2$:**

The function f is of class C^2 on D

$$\Longleftrightarrow \frac{\partial^2 f}{\partial x^2}, \quad \frac{\partial^2 f}{\partial x \partial y}, \quad \frac{\partial^2 f}{\partial y \partial x}, \quad \frac{\partial^2 f}{\partial y^2}$$

exist and are continuous on D.

Example 1.3.5.

We have : $f(x, y) = xe^{xy}$
$\dfrac{df}{dx}(x, y) = e^{xy} + y \cdot xe^{xy}$ continue on $\mathbb{R}^2$.
$\dfrac{df}{dy}(x, y) = x \cdot xe^{xy}$ continue on $\mathbb{R}^2$.
That is to say, f is of class C^2 on $\mathbb{R}^2$.

Let I be an open interval of $\mathbb{R}$, and Ω an open set in $\mathbb{R}^n$. The function f is defined on $I \times \Omega$.

Lemma 1.3.1. ***(Gronwall integral)***
Let φ be a continuous function from $[a, b]$ into $\mathbb{R}_+$ and $c \in [a, b]$. Suppose that there exist positive constants A and B such that:

$$\varphi(t) \leq A + B\left|\int_c^t \varphi(s) ds\right|, \qquad t \in [a, b]$$

Then:

$$\varphi(t) \leq Ae^{B|t-c|}, \qquad t \in [a, b].$$

Proof 1.3.1.

Let $t \in [a, b]$, and suppose that $t \geq c$. Define:

$$F(t) = A + B \int_c^t \varphi(s)\, ds.$$

Then:

$$F \in C^1 \text{ and } \varphi(t) \leq F(t) \text{ for } t \in [c, b].$$

We have:

$$\begin{aligned} F'(t) &= B\varphi(t), \\ \frac{d}{dt}\Big(e^{-Bt}F(t)\Big) &= -Be^{-Bt}F(t) + F'(t)e^{-Bt} \\ &= e^{-Bt}\Big(F'(t) - BF(t)\Big) \leq 0 \\ &= e^{-Bt}\Big(B\varphi(t) - BF(t)\Big) \leq 0, \end{aligned}$$

for $t \in [c, b]$.

Thus:

$$\begin{aligned} &\frac{d}{dt}\Big(e^{-Bt}F(t)\Big) \leq 0 \\ &\Rightarrow e^{-Bt}F(t) \leq e^{-Bc}F(c) = Ae^{-Bc} \quad (\text{since } F(c) = A). \end{aligned}$$

Hence:

$$e^{-Bt}\varphi(t) \leq e^{-Bt}F(t) \leq Ae^{-Bc},$$

and therefore:

$$\varphi(t) \leq Ae^{B(t-c)}.$$

In the same way, the result can be shown for $t \leq c$.

1.3.5 Regularity of solutions

Theorem 1.3.2.

[6] If $f : \Omega \subset \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ is of class C^k, then every solution of (E) is of class C^{k+1}.

Proof 1.3.2.

We proceed by induction on k.

$$P(n) : f \in C^n \Longrightarrow y \in C^{n+1}$$

By assumption, $y : I \to \mathbb{R}$ is differentiable (y' exists) and continuous (since $f(t, y)$ is continuous).

Thus, y is of class C^1.

- If the result is true for order "$k+1$", then y is at least of class C^k.

- Since f is of class C^k, it follows that $y' = f(t, y)$ is of class C^k (as the composition of C^k functions), hence y is of class C^{k+1}.

Proposition 1.3.1.

If f is continuous on a neighborhood of (t_0, y_0), then the Chauchy problem :

$$\begin{cases} \dot{y} = f(t, y) \\ y(t_0) = y_0 \end{cases}$$

is equivalent to the problem :

$$y(t) = y_0 + \int_{t_0}^{t} f(t, y(t))dt.$$

Proof 1.3.3.

- If $\dot{y} = f(t, y)$, then by integration we have :

$$\int_{y_0}^{y} dy = \int_{y_0}^{y} f(t, y(t))dt.$$

Therefore:

$$y - y_0 = \int_{y_0}^{y} \{(t, y(t))dt$$

and consequently:

$$y = y_0 + \int_{y_0}^{y} \{(t, y(t))dt.$$

Consversely, if :

$$y = y_0 + \int_{y_0}^{y} \{(t, y(t))dt,$$

then by differentiating, we find: $\dot{y} = \{(t, y)$.

Furthermore, for $t = t_0$, we have $y(t_0) = y_0$.

1.3.6 Cauchy–Lipschitz theorems

Definition 1.10. *[5]*

We say that f is **locally Lipschitz** with respect to its second variable if, for each $(t_0, y_0) \in \underbrace{I \times \Omega}_{\text{U}}$, there exists a neighborhood V of (t_0, y_0) such that the restriction $f_{/V}$ is Lipschitz with respect to the second variable.

Definition 1.11.

We say that f is **globally Lipschitz** with respect to the second variable if there exists $k > 0$ such that:

$$\|f(t, y_1) - f(t, y_2)\| \leq k\|y_1 - y_2\|, \quad \forall (t, y_1, y_2) \in I \times \Omega \times \Omega.$$

Example 1.3.6.

1. $f(t, y) = t^4 + 5y, \quad \forall t \in \mathbb{R}, \quad \forall (y_1, y_2) \in \mathbb{R}^2.$

$$|f(t, y_1) - f(t, y_2)| = |(t^4 + 5y_1) - (t^4 + 5y_2)| = 5|y_1 - y_2|.$$

Thus, the function f is globally Lipschitz with respect to the second variable y.

2. $g(t, y) = \cos(y)$. By the Mean Value Theorem, $\forall (y_1, y_2) \in \mathbb{R}^2$, $\forall t \in \mathbb{R}$, there exists $c \in]y_1, y_2[$ such that:

$$\cos(y_1) - \cos(y_2) = \cos'(c)(y_1 - y_2).$$

Hence:

$$\begin{aligned} |\cos(y_1) - \cos(y_2)| &= |-\sin(c)(y_1 - y_2)| \\ &= |\sin(c)|\,|y_1 - y_2| \\ &\leq |y_1 - y_2|. \end{aligned}$$

Therefore, g is globally Lipschitz with respect to the second variable.

3. $h(t, y) = \sqrt{y}$ is not locally Lipschitz in a neighborhood of 0. Indeed:

$$\lim_{y \to 0^+} \frac{h(t, y) - h(t, 0)}{y - 0} = \frac{\sqrt{y}}{y} = \frac{1}{\sqrt{y}} \longrightarrow_{y \to 0^+} +\infty.$$

Remark 1.3.4.

(1) If $f \in C^1(U)$, then f is locally Lipschitz.

(2) A Lipschitz mapping is locally Lipschitz and continuous.

1.3.7 Differential equation (Cauchy problem)

Definition 1.12 (Contraction)**.**

A **contraction** f on $A \subset E$ is a mapping

$$f : A \longrightarrow A$$

which is k-Lipschitz with $k \in [0, 1[$.

Remark 1.3.5.

$$f \text{ contraction } \implies f \text{ uniformly continuous } \implies f \text{ continuous.}$$

Theorem 1.3.3 (Banach fixed point)**.** *[11]*

Every contraction f of a non-empty closed subset of a Banach space has a unique fixed point (i.e., $f(x) = x$).

Proof 1.3.4.

Remark. There is uniqueness of the fixed point in case of existence.
Indeed, if x and y are two fixed points, then

$$\|x - y\| = \|f(x) - f(y)\| \leq k\|x - y\|.$$

Since $k \in [0, 1[$, necessarily $\|x - y\| = 0$, hence $x = y$.
To establish the existence of a fixed point, let us take $x_0 \in A$ and define the sequence

recursively by $x_{n+1} = f(x_n)$. We then have:

$$\begin{aligned}\|x_{n+1} - x_n\| &\leq k\|x_n - x_{n-1}\| \\ &\leq k^2\|x_{n-1} - x_{n-2}\| \\ &\vdots \\ &\leq k^n\|x_1 - x_0\|.\end{aligned}$$

From the triangle inequality, for $m < n$ we obtain:

$$\begin{aligned}\|x_n - x_m\| &\leq \|x_n - x_{n-1}\| + \|x_{n-1} - x_{n-2}\| + \cdots + \|x_{m+1} - x_m\| \\ &\leq \left(k^{n-1} + k^{n-2} + \cdots + k^m\right)\|x_1 - x_0\| \\ &\leq \frac{k^m}{1-k}\|x_1 - x_0\|.\end{aligned}$$

Thus (x_n) is a Cauchy sequence in A. Since A is a closed subset of the Banach space E, there exists $x^* \in A$ such that

$$\lim_{n\to\infty} x_n = x^*.$$

Finally, by continuity of f, we get $f(x^*) = x^*$, which is the unique fixed point.

Theorem 1.3.4.

If $f : \Omega \to \mathbb{R}^n$ is a function of class C^1 on an open set $\Omega \subset \mathbb{R} \times \mathbb{R}^n$, then for each $(t_0, y_0) \in \Omega$, there exists a **unique** solution of the equation $y' = f(t, y)$ with $y(t_0) = y_0$ on some open interval containing t_0.

Theorem 1.3.5.

If the function $f(t, y)$ is continuous in the domain U and has in U a bounded derivative $f_y(t, y)$, then through each point (t_0, y_0) of U passes one and only one integral curve $y = \varphi(t)$ of the equation:

$$y' = f(t, y) \text{ with } (\varphi(t_0) = y_0).$$

1.3.8 Existence and uniqueness of the solution satisfying an initial condition

Theorem 1.3.6. *[5]*

If the functions f and $\dfrac{df}{dy}$ are continuous in Ω and if (t_0, y_0) is a point of Ω, then there exists a unique solution φ, defined in a neighborhood of (t_0, y_0), which satisfies $\varphi(t_0) = y_0$.

Example 1.3.7.

Let the differential equation be:

$$y' = \alpha y \text{ with } \alpha \in \mathbb{R}.$$

We have:

$$\begin{cases} f(t,y) = \alpha y \\ \text{and} \\ \dfrac{\partial f}{\partial y}(t,y) = \alpha \end{cases}$$

The functions f and $\dfrac{\partial f}{\partial y}$ are continuous in the whole plane (t, y). The preceding theorem shows that there exists a unique solution passing through the point (t_0, y_0) of the plane. This solution is $\varphi(t) = y_0\, e^{\alpha(t-t_0)}$, which is the only solution passing through this point and it is defined for all t.

Example 1.3.8.

Let the differential equation be:

$$y' = f(t,y) = \frac{y}{t+1}$$

The functions f and $\dfrac{\partial f}{\partial y}$ are continuous in the whole plane (t, y), except on the line $t = -1$. The preceding theorem shows that there exists a unique solution passing through any point (t_0, y_0) of the plane (t, y), but it provides no information about the solutions passing through $(-1, y_0)$.

1.3.9 Global existence and uniqueness

Consider the following Cauchy problem:[6]

$$\begin{cases} y' = \psi(t, y), \\ y(t_0) = y_0 \end{cases} \qquad (PC)$$

Theorem 1.3.7 (Global existence and uniqueness)**.** *[8]*

We suppose that $U = E$ and $\psi \in C(I \times U)$ is a function **globally Lipschitz** with respect to y.

Then for every $y_0 \in U$, the Cauchy problem **(PC)** admits a unique solution. Moreover, every local solution is a restriction of this one.

Proof 1.3.5.

First suppose that the interval I is compact. We set

$$\varepsilon = C(I, E)$$

the set of continuous functions from I into E, endowed with the norm

$$\|y\|_{\varepsilon} = \max_{t \in I} e^{-2L|t-t_0|} \, \|y(t)\|_E,$$

where L is the Lipschitz constant of ψ.

It is clear that ε is a complete normed space (Banach space), since I is compact. We define the operator $\chi : \varepsilon \to \varepsilon$ by

$$(\chi y)(t) = y_0 + \int_{t_0}^{t} \psi(s, y(s)) \, ds, \quad t \in I.$$

It is clear that the operator χ maps ε into itself.

Suppose $t \geq t_0$. For all $y_1, y_2 \in \varepsilon$, we have:

$$\begin{aligned}
\|(\chi y_2)(t) - (\chi y_1)(t)\|_E &= \left\| y_0 + \int_{t_0}^{t} \psi(s, y_2(s))\, ds - y_0 - \int_{t_0}^{t} \psi(s, y_1(s))\, ds \right\|_E \\
&= \left\| \int_{t_0}^{t} \psi(s, y_2(s))\, ds - \int_{t_0}^{t} \psi(s, y_1(s))\, ds \right\|_E \\
&\leq \int_{t_0}^{t} \|\psi(s, y_2(s)) - \psi(s, y_1(s))\|_E\, ds \\
&\leq \int_{t_0}^{t} L\, \|y_2(s) - y_1(s)\|_E\, ds.
\end{aligned}$$

1.3.10 Theorems on global solutions

Theorem 1.3.8.

Let $f : I \times \mathbb{R}^n \to \mathbb{R}^n$ be a continuous function. Suppose there exists a continuous function $k : I \to \mathbb{R}$ such that for every $t \in I$ the map $y \mapsto f(t, y)$ is Lipschitz with constant $k(t)$. Then every maximal solution of the problem

$$y' = f(t, y)$$

is global (i.e. defined on the whole interval I).[8]

Example 1.3.9.

Every maximal solution of:

$$\begin{cases} y' = t\sqrt{t^2 + y^2}, \\ y(x_0) = y_0 \end{cases}$$

is global. Indeed:

1 For every $t \in \mathbb{R}$, the function

$$f(t, y) = t\sqrt{t^2 + y^2}$$

is continuous.

2 The function

$$y \longmapsto f(t, y) = t\sqrt{t^2 + y^2}$$

is Lipschitz with constant $k(t) = |t|$ (continuous on $\mathbb{R}$), because for all $y, z \in \mathbb{R}$, we have:

$$
\begin{aligned}
|f(t,y) - f(t,z)| &\leq |t| \frac{|y^2 - z^2|}{\sqrt{t^2 + y^2} + \sqrt{t^2 + z^2}} \\
&\leq |t| \frac{(|y| + |z|)\,|y - z|}{\sqrt{t^2 + y^2} + \sqrt{t^2 + z^2}} \\
&\leq |t|\,|y - z|.
\end{aligned} \tag{1.15}
$$

since

$$|t| \frac{|y - z|}{\sqrt{t^2 + y^2} + \sqrt{t^2 + z^2}} \leq 1.$$

From Theorem 1.3.10, every maximal solution of (2.1) is global.

Theorem 1.3.9.

Let $f : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ be a continuous function such that:

$$\|f(x, y)\| \leq \alpha(t) + \beta(t)\|y\|,$$

for α, β positive and continuous. Then, the solutions of the problem:

$$\begin{cases} y' = f(t, y), \quad t \in \mathbb{R}, \\ y(t_0) = y_0 \end{cases}$$

are global.

Theorem 1.3.10.

Let $f :]a, b[\times \mathbb{R}^n \to \mathbb{R}^n$ be a continuous and bounded function. Then every solution of:

$$\begin{cases} y' = f(t, y), \quad t \in \mathbb{R}, \\ y(t_0) = y_0 \end{cases}$$

is global.

Remark 1.3.6. *[7]*

The unique solution of the Cauchy problem can be constructed using the following method of successive approximations:

$$\begin{aligned} y_0(t) &= y_0, \\ y_1(t) &= \int_{t_0}^{t} f(s, y_0(s))\, ds + y_0, \\ y_2(t) &= \int_{t_0}^{t} f(s, y_1(s))\, ds + y_0, \\ &\vdots \\ y_n(t) &= \int_{t_0}^{t} f(s, y_{n-1}(s))\, ds + y_0. \end{aligned}$$

Exercise 1.3.1.

1. Show that the function f defined by $f(t, y) = t^2 + y^2$ is locally Lipschitz with respect to y on $\mathbb{R}$.

Proof. Let $f(t, y) = t^2 + y^2$. Fix an arbitrary point $(t_0, y_0) \in \mathbb{R} \times \mathbb{R}$. Choose $M > 0$ such that $|y_0| \leq M$ and consider the strip

$$S_M = \{(t, y) \in \mathbb{R} \times \mathbb{R} : \ |y| \leq M\}.$$

For any $t \in \mathbb{R}$ and any y, z with $|y| \leq M, \ |z| \leq M$, we have

$$|f(t, y) - f(t, z)| = |y^2 - z^2| = |y - z|\, |y + z|.$$

Since $|y + z| \leq |y| + |z| \leq 2M$, it follows that

$$|f(t, y) - f(t, z)| \leq 2M\, |y - z|.$$

Thus on the neighbourhood S_M the map $y \mapsto f(t, y)$ is Lipschitz with constant $K = 2M$. Because every point (t_0, y_0) admits such a neighbourhood, f is locally Lipschitz in y on $\mathbb{R}$.

Remark. f is not globally Lipschitz in y on $\mathbb{R}$ since the Lipschitz constant $2M$ increases with M and no single finite constant works for all $y \in \mathbb{R}$. □

Exercise 1.3.2.

Show that every maximal solution of

$$\begin{cases} y' = t\sqrt{t^2+y^2}, \\ y(t_0) = y_0, \end{cases}$$

is global.

Proof. We give two short proofs.

(I) By a variable Lipschitz constant. Define $f(t,y) = t\sqrt{t^2+y^2}$. For arbitrary $y, z \in \mathbb{R}$ and fixed t we have

$$\begin{aligned} |f(t,y) - f(t,z)| &= |t| \left| \sqrt{t^2+y^2} - \sqrt{t^2+z^2} \right| \\ &= |t| \frac{|y^2 - z^2|}{\sqrt{t^2+y^2} + \sqrt{t^2+z^2}} \le |t|\,|y-z|. \end{aligned}$$

Hence for each t the map $y \mapsto f(t,y)$ is Lipschitz with constant $k(t) = |t|$, and $k(t)$ is continuous on $\mathbb{R}$. Since f is continuous and the Lipschitz constant depends continuously on t, the standard continuation theorem (Grönwall/uniqueness + continuation) implies that any maximal solution cannot blow up in finite time and therefore is defined for all $t \in \mathbb{R}$. Thus every maximal solution is global.

(II) By a priori estimate (Grönwall argument). Let $y(t)$ be a solution on its maximal interval $I_{\max}$. Then

$$|y'(t)| = |t|\sqrt{t^2+y(t)^2} \le |t|(|t| + |y(t)|) = t^2 + |t|\,|y(t)|.$$

For t in any finite subinterval of $I_{\max}$ containing t_0 we get

$$|y(t)| \le |y_0| + \int_{t_0}^{t} \big(s^2 + |s|\,|y(s)|\big)\,ds.$$

Set $z(t) = |y(t)|$ and $C(t) = |y_0| + \int_{t_0}^{t} s^2\,ds$. Then

$$z(t) \le C(t) + \int_{t_0}^{t} |s|\,z(s)\,ds.$$

Applying Grönwall's inequality on any finite interval yields a finite bound for $z(t)$ on that interval. Hence the solution cannot blow up in finite time and can be extended past any finite endpoint of $I_{\max}$. Therefore $I_{\max} = \mathbb{R}$ and the maximal solution is global. $\square$

Exercise 1.3.3.

Consider the differential equation

$$(E_3) \qquad y' = (1 + \cos t)\, y - y^3.$$

Let $t_0, y_0 \in \mathbb{R}$. Prove existence and uniqueness of the maximal solution y of (E_3) satisfying $y(t_0) = y_0$, and show that this maximal solution is global.

1 Are the following functions Lipschitz in y?

$$f_1(t, y) = \ln(t^2 + y^2 + 1), \qquad f_2(t, y) = 2\sqrt{y}, \quad y \in [1, \infty).$$

2 Show that the function

$$\varphi(t) = \frac{1}{\sqrt{2(2-t)}}$$

defined on its natural domain is a maximal solution of the equation $y' = y^3$.

Proof. **(1) Lipschitz in y.**

For f_1. Compute the partial derivative with respect to y:

$$\frac{\partial f_1}{\partial y}(t, y) = \frac{2y}{t^2 + y^2 + 1}.$$

For every fixed (t, y) we have

$$\left|\frac{\partial f_1}{\partial y}(t, y)\right| = \frac{2|y|}{t^2 + y^2 + 1}.$$

For fixed t the function $g(y) = \dfrac{2|y|}{t^2 + y^2 + 1}$ attains its maximum at $|y| = \sqrt{t^2 + 1}$, and the maximal value is

$$\max_y g(y) = \frac{1}{\sqrt{t^2 + 1}} \le 1.$$

Hence for all $(t, y), (t, z)$ one has

$$|f_1(t, y) - f_1(t, z)| \le K\, |y - z|$$

with a Lipschitz constant $K = \sup_t \frac{1}{\sqrt{t^2+1}} = 1$. Thus f_1 is globally Lipschitz in y on $\mathbb{R} \times \mathbb{R}$ (with Lipschitz constant 1).

For f_2. On the domain $y \in [1, \infty)$ we have

$$\frac{d}{dy}(2\sqrt{y}) = \frac{1}{\sqrt{y}}.$$

On $[1, \infty)$ this derivative is bounded by 1. Therefore for all $y, z \geq 1$,

$$|f_2(t, y) - f_2(t, z)| \leq 1 \cdot |y - z|.$$

Hence f_2 is (globally) Lipschitz in y on the domain $y \geq 1$ (with Lipschitz constant 1).

(2) φ is a maximal solution of $y' = y^3$.

First note the natural domain of φ. The formula $\varphi(t) = \dfrac{1}{\sqrt{2(2-t)}}$ requires $2(2-t) > 0$, so the domain is

$$(-\infty, 2).$$

(At $t \to 2^-$ the denominator tends to 0^+ and $\varphi(t) \to +\infty$.)

Verification that φ is a solution. Differentiate:

$$\varphi(t) = \big(2(2-t)\big)^{-1/2} \quad \Longrightarrow \quad \varphi'(t) = \frac{d}{dt}\big(2(2-t)\big)^{-1/2} = (2(2-t))^{-3/2}.$$

But

$$\varphi(t)^3 = \big((2(2-t))^{-1/2}\big)^3 = (2(2-t))^{-3/2}.$$

Thus $\varphi'(t) = \varphi(t)^3$ for all $t \in (-\infty, 2)$, so φ is indeed a solution of $y' = y^3$ on $(-\infty, 2)$.

Maximality. Suppose by contradiction that φ could be extended to a solution $\tilde{\varphi}$ defined on a strictly larger interval $(-\infty, 2+\varepsilon)$ (or even defined at $t = 2$). Because $\varphi(t) \to +\infty$ as $t \to 2^-$, any such extension would have to assign a finite value at $t = 2$. But standard ODE continuation theory (continuation/uniqueness and the blow-up criterion) says that a solution of $y' = y^3$ can be continued past a time t^* if and only if its value remains finite as $t \to t^*$. Here the solution diverges to $+\infty$ when approaching $t = 2$, so no extension through $t = 2$ exists. Hence the interval $(-\infty, 2)$ is the maximal interval of existence for φ.

Therefore φ is a maximal solution of $y' = y^3$, defined on $(-\infty, 2)$. □

1 Study the Lipschitz property near 0 of the function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(y) = 3\sqrt{|y|}.$$

2 Let $a \geq 0$. Verify that the function $y : \mathbb{R} \to \mathbb{R}$ defined by

$$y(t) = \begin{cases} \frac{9}{4}(t-a)^2, & t > a, \\ 0, & t \leq a, \end{cases}$$

is a solution of the Cauchy problem $y' = 3\sqrt{|y(t)|}$ with $y(0) = 0$.

1.4 Dependence on Initial Conditions

In this section, we study the behavior of solutions when the initial conditions are perturbed. Consider the following Cauchy problem (PC):

$$\begin{cases} \frac{dX}{dt} = f(t, X), \\ X(t_0) = X_0, \end{cases} \qquad (t_0, X_0) \in I \times E.$$

We state the following theorem:

Theorem 1. Let $I = [a, b]$ be a bounded interval of $\mathbb{R}$. Let $f : I \times E \to \mathbb{R}^n$ be Lipschitz continuous with respect to the second variable, uniformly on I. Consider the Cauchy problem

$$\begin{cases} \frac{dY}{dt} = f(t, Y), \\ Y(t_0) = Y_0, \end{cases} \qquad Y_0 \in E \subset C(I, \mathbb{R}^n).$$

Define the function $\Phi : E \to C(I, \mathbb{R}^n)$ by

$$\Phi(X)(t) = X(t),$$

where $X(t)$ is the solution of (PC) with initial condition $X_0 = X$. Then Φ is continuous with respect to X on E.

Proof. We show that Φ is continuous with respect to the second variable on E. Let $X, Y \in E$, and let $X(t)$ and $Y(t)$ be the solutions of (PC) with initial conditions X and Y, respectively.

We have

$$X(t) = X + \int_{t_0}^{t} f(s, X(s))\, ds, \quad Y(t) = Y + \int_{t_0}^{t} f(s, Y(s))\, ds, \quad t \in I.$$

By definition of Φ, we have $\Phi(X)(t) = X(t)$ and $\Phi(Y)(t) = Y(t)$. Then, for all $t \in I$,

$$\Phi(X)(t) - \Phi(Y)(t) = X - Y + \int_{t_0}^{t} \big(f(s, X(s)) - f(s, Y(s))\big)\, ds.$$

Let $J = [t_0, t]$ if $t \geq t_0$ and $J = [t, t_0]$ if $t < t_0$. Define

$$\|X - Y\| = \max_{s \in I} \|X(s) - Y(s)\|.$$

Since $X(t_0) = X$ and $Y(t_0) = Y$, we get

$$\|\Phi(X)(t) - \Phi(Y)(t)\| \leq \|X - Y\| + \int_J \|f(s, X(s)) - f(s, Y(s))\|\, ds.$$

Using the Lipschitz property of f, we obtain

$$\|\Phi(X)(t) - \Phi(Y)(t)\| \leq \|X - Y\| + \int_J L\|X(s) - Y(s)\|\, ds,$$

where L is the Lipschitz constant. Applying Gronwall's inequality yields

$$\|\Phi(X)(t) - \Phi(Y)(t)\| \leq \|X - Y\| e^{L(b-a)}.$$

This shows that Φ is Lipschitz continuous with respect to the second variable, uniformly on $[a, b]$, with Lipschitz constant $e^{L(b-a)}$, which in particular implies continuity. □

1.5 Solved Exercises

Exercise 1.5.1. [See Correction 1.6.1]

Solve the linear differential equations with constant coefficients

1. $y'' - y' - 2y = e^{3t}$.
2. $y'' - y' - 2y = e^{2t}$.
3. $y'' - y' - 2y = te^{t}$.

4. $y'' - y' - 2y = 2t^2 + e^{2t}$.

5. $y'' - y' - 2y = \sin(2t)$.

Exercise 1.5.2. [See Correction 1.6.2]

We consider the differential equation (E_2):

$$t^2y'' - 3ty' + 4y = 0 \quad \text{on} \quad]0, +\infty[.$$

1 Determine a solution of the equation (E_2) of the form $y(t) = t^\alpha$ where $\alpha \in \mathbb{R}$.

2 Let $y(t) = t^\alpha z(t)$. What is the equation satisfied by z?

3 Deduce the solutions of (E_2) on $]0, +\infty[$.

Exercise 1.5.3. [See Correction 1.6.3]

We consider the differential equation (E_3):

$$ty'' - (t+1)y' + y = 0.$$

1 Determine a solution of the equation (E_3) of the form $y(t) = e^{\alpha t}$ where $\alpha \in \mathbb{R}$.

2 Let $y(t) = t^\alpha z(t)$. What is the equation satisfied by z?

3 Deduce the solutions of (E_3) on $]0, +\infty[$.

4 Determine the solutions satisfying $y(0) = 1$ and $y'(0) = 1$.

1.6 Solve the linear ODEs with constant coefficients

Solution of exercise 1.6.1. [Voir l'exercice 1.5.1]

1. Solve $y'' - y' - 2y = e^{3t}$.

   Characteristic equation: $r^2 - r - 2 = 0 \;\Rightarrow\; r_1 = 2,\; r_2 = -1$.

$$y_H(t) = C_1e^{2t} + C_2e^{-t}.$$

Seek $y_p(t) = Ae^{3t}$. Then $y_p' = 3Ae^{3t}$, $y_p'' = 9Ae^{3t}$, so

$$y_p'' - y_p' - 2y_p = (9A - 3A - 2A)e^{3t} = 4Ae^{3t} = e^{3t} \;\Rightarrow\; A = \tfrac{1}{4}.$$

Hence

$$\boxed{y(t) = C_1e^{2t} + C_2e^{-t} + \tfrac{1}{4}e^{3t}}.$$

2. Solve $y'' - y' - 2y = e^{2t}$.

   Same homogeneous solution: $y_H = C_1e^{2t} + C_2e^{-t}$. Since $r = 2$ is a simple root, try $y_p(t) = A\,t\,e^{2t}$. Then $y_p' = A(2t+1)e^{2t}$, $y_p'' = A(4t+4)e^{2t}$, hence

$$y_p'' - y_p' - 2y_p = A\big[(4t+4) - (2t+1) - 2t\big]e^{2t} = 3Ae^{2t} = e^{2t} \Rightarrow A = \tfrac{1}{3}.$$

   Thus

$$\boxed{y(t) = C_1e^{2t} + C_2e^{-t} + \tfrac{1}{3}\,t\,e^{2t}}.$$

3. Solve $y'' - y' - 2y = te^{t}$.

   Homogeneous part as before. Since $r = 1$ is not a root, try $y_p(t) = (At+B)e^{t}$. Compute $y_p' = (At + B + A)e^{t}$, $y_p'' = (At + B + 2A)e^{t}$, so

$$y_p'' - y_p' - 2y_p = \big[(-2A)t + (A - 2B)\big]e^{t} = te^{t}.$$

   Match coefficients: $-2A = 1 \Rightarrow A = -\tfrac{1}{2}$; and $A - 2B = 0 \Rightarrow B = -\tfrac{1}{4}$. Hence

$$\boxed{y(t) = C_1e^{2t} + C_2e^{-t} - \tfrac{1}{4}(2t+1)e^{t}}.$$

4. Solve $y'' - y' - 2y = 2t^2 + e^{2t}$.

   By superposition, find y_{p1} for $2t^2$ and y_{p2} for e^{2t}.

   For $2t^2$: try $y_{p1} = at^2+bt+c$. Then $y_{p1}''-y_{p1}'-2y_{p1} = (-2a)t^2+(-2a-2b)t+(2a-b-2c)$, so matching $2t^2$ gives $a = -1$, $b = 1$, $c = -\tfrac{3}{2}$:

$$y_{p1}(t) = -t^2 + t - \tfrac{3}{2}.$$

   For e^{2t}: since $r = 2$ is a simple root, try $y_{p2} = A\,t\,e^{2t}$. As in item 2, $y_{p2}'' - y_{p2}' - 2y_{p2} =$

$3Ae^{2t} = e^{2t}$, so $A = \frac{1}{3}$:

$$y_{p2}(t) = \tfrac{1}{3}\, t\, e^{2t}.$$

Thus

$$\boxed{y(t) = C_1 e^{2t} + C_2 e^{-t} - t^2 + t - \tfrac{3}{2} + \tfrac{1}{3}\, t\, e^{2t}}.$$

5. Solve $y'' - y' - 2y = \sin(2t)$.

   Homogeneous part as before. For a particular solution, try $y_p(t) = \alpha\cos(2t) + \beta\sin(2t)$. Then $y_p' = -2\alpha\sin(2t) + 2\beta\cos(2t)$, $y_p'' = -4\alpha\cos(2t) - 4\beta\sin(2t)$, so

$$y_p'' - y_p' - 2y_p = (-6\alpha - 2\beta)\cos(2t) + (2\alpha - 6\beta)\sin(2t).$$

   Match with $\sin(2t)$:

$$\begin{cases} -6\alpha - 2\beta = 0, \\ 2\alpha - 6\beta = 1, \end{cases} \quad \Rightarrow \quad \alpha = \tfrac{1}{20},\ \beta = -\tfrac{3}{20}.$$

   Therefore

$$\boxed{y(t) = C_1 e^{2t} + C_2 e^{-t} + \tfrac{1}{20}\cos(2t) - \tfrac{3}{20}\sin(2t)}.$$

Solution of exercise 1.6.2. [See the exercise 1.5.2]

1 Let $y(t) = t^\alpha$. Then

$$y' = \alpha t^{\alpha-1}, \qquad y'' = \alpha(\alpha-1)t^{\alpha-2}.$$

Substituting into (E_2) gives

$$t^2\alpha(\alpha-1)t^{\alpha-2} - 3t\alpha t^{\alpha-1} + 4t^\alpha = \big(\alpha(\alpha-1) - 3\alpha + 4\big)t^\alpha.$$

Hence the indicial polynomial is

$$\alpha^2 - 4\alpha + 4 = (\alpha-2)^2.$$

So $\alpha = 2$ (double root). Thus $y_0(t) = t^2$ is a solution.

2 Take $y(t) = t^2 z(t)$. Compute

$$y' = 2tz + t^2 z', \qquad y'' = 2z + 4tz' + t^2 z''.$$

Substitute into (E_2):

$$t^2(2z + 4tz' + t^2 z'') - 3t(2tz + t^2 z') + 4t^2 z = 0.$$

Simplifying yields

$$t^4 z'' + t^3 z' = 0.$$

Since $t > 0$, divide by t^3 to obtain

$$tz'' + z' = 0.$$

3 Let $u = z'$. Then $u' + \frac{1}{t}u = 0$. Solving this first-order linear ODE:

$$\frac{du}{u} = -\frac{dt}{t} \quad \Rightarrow \quad \ln|u| = -\ln t + C$$

hence

$$u(t) = \frac{C_1}{t}.$$

Integrating,

$$z(t) = C_1 \ln t + C_2, \qquad C_1, C_2 \in \mathbb{R}.$$

Therefore the general solution of (E_2) on $(0, \infty)$ is

$$\boxed{y(t) = t^2 z(t) = C_1 t^2 \ln t + C_2 t^2, \qquad C_1, C_2 \in \mathbb{R}\,.}$$

4 Remarks about the initial conditions at $t = 0$:

Determine the solutions that satisfy $\begin{cases} y(0) = 1 \\ y'(0) = 1 \end{cases}$

The first condition $y(0) = 1$ gives $\lambda + \mu = 1$.

The second condition $y'(0) = 1$ gives $-\lambda + \mu = 1$.

The solutions are:

$$y(t) = -\lambda(t + 1) + (\lambda + 1)e^t, \quad \lambda \in \mathbb{R}.$$

Solution of exercise 1.6.3. [See the exercise 1.5.3]

Chapter 2

Higher-order equations, first-order system

2.1 Second-order differential equation

The general form of a second-order differential equation is:

$$F(t, y, y', y'') = 0$$

or in the normal form: $Y'' = F(t, y, y')$.

The general solution "Y" usually depends on two parameters, λ and μ.

2.1.1 Second-order linear differential equations

Consider the second-order differential equation:

$$Y'' + a(t)Y' + b(t)Y = c(t) \tag{2.1}$$

1. If $c(t) = 0$, equation (2.1) is a homogeneous differential equation.

2. If

$$\begin{cases} a(t) = a \\ b(t) = b \end{cases}$$

   we obtain $Y'' + aY' + bY = c(t)$, which is a linear equation with constant coefficients.

Theorem 2.1.1.

The general solution of a second-order differential equation is given by:

$$S_G = S_H + S_P$$

S_H: Solution of the homogeneous equation.

S_P: A particular solution.

Definition 2.1.

Two solutions y_1 and y_2 of equation (2.1) are independent on an interval I if there is no real number k such that:

$$\text{for all } t \in I: \; y_2(t) = ky_1(t).$$

Remark 2.1.1.

The two functions y_1 and y_2 are independent, i.e., they are linearly independent in the sense of vector spaces.

Definition 2.2.

Let two functions be differentiable on the interval I.
y_1, y_2 are linearly independent if and only if the determinant

$$\begin{vmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{vmatrix}$$

is not identically zero.

Example 2.1.1.

The functions $\sin t$ and $\cos t$ are independent:

$$\begin{vmatrix} \sin t & \cos t \\ \cos t & -\sin t \end{vmatrix} = -\sin^2 t - \cos^2 t = -(\sin^2 t + \cos^2 t) = -1 \neq 0 \quad \forall t.$$

2.1.2 Second-order homogeneous linear equations

Case Where Two Independent Particular Solutions Are Known

If y_1, y_2 are two independent solutions of the equation:

$$y'' + a(t)y' + b(t)y = 0$$

the general solution is given by:

$$y = \lambda y_1 + \mu y_2,$$

where λ, μ are constants.

Case Where One Particular Solution Is Known

If a particular solution y_1 is known, we set the change of variable $y(t) = Y_1(t)v(t)$.

$y'(t) = Y_1'v + v'Y_1, \quad Y''(t) = Y_1''v + v'Y_1' + v''Y_1 + Y_1'v' = Y_1''v + 2Y_1'v' + v''Y_1.$

Substituting Y, Y', Y'' into the homogeneous equation, we obtain a second-order differential equation for the unknown v:

$$y_1 v'' + (2y_1' + a(t)y_1)v' = 0$$

We make the second change of variable "$w = v'$", giving:

$$\begin{aligned}
\frac{w'}{w} = \frac{v''}{v'} &= -2 \cdot \frac{y_1'}{y_1} - a(t) \\
\int \frac{dw}{w} &= -2 \int \frac{Y_1'}{Y_1} dt - \int a(t)dt \\
\ln w &= -2 \ln Y_1 - \int a(t)dt + c \\
w &= \frac{1}{Y_1^{-2}} \cdot e^{-\int a(t)dt} \cdot \lambda, \qquad \lambda \in \mathbb{R} \\
w(t) &= \lambda \cdot \frac{1}{Y_1^{-2}} \cdot e^{-\int a(t)dt}.
\end{aligned}$$

Hence:

$$v(t) = \int w(t)dt + \mu, \qquad \mu \in \mathbb{R}.$$

The general solution of the equation:

$$y'' + a(t)y' + b(t)y = 0$$

is therefore:

$$\begin{aligned}
y(t) &= y_1(t)v(t) \\
&= y_1(t)\Big(\int w(t)dt + \mu\Big) = y_1(t)\int w(t)dt + \mu y_1(t), \qquad \mu \in \mathbb{R}.
\end{aligned}$$

Consider the equation:

$$(t+1)y'' - (2t-1)y' + (t-2)y = 0$$

We can verify that $y_1(t) = e^t$ is a particular solution.

Let us seek the general solution in the form $y(t) = e^t v(t)$:

$$\begin{aligned} y'(t) &= e^t v + v' e^t \\ y''(t) &= e^t v + v' e^t + v'' e^t + e^t v' = e^t v + 2v' e^t + e^t v''. \end{aligned}$$

Substituting y, y', y'' into the homogeneous equation, we get:

$$\begin{aligned} (1+t)[e^t(v + 2v' + v'')] + (2t-1)[e^t(v+v')] + (t-2)e^t v &= 0 \\ e^t[(t+1)v + 2(1+t)v' + (1+t)v'' - (2t-1)v - (2t-1)v' + (t-2)v] &= 0 \\ e^t[(t+1)v'' + v'[2(t+1) - (2t-1)] + v[(t-1) - (2t-1) + (t-2)]] &= 0 \\ e^t[(t+1)v'' + 3v' + 0 \cdot v] &= 0, \end{aligned}$$

Since $e^t \neq 0$ for all $t \in \mathbb{R}$:

$$(t+1)v'' + 3v' = 0 \Longrightarrow \frac{v''}{v'} = \frac{-3}{1+t}, \qquad t \neq -1.$$

Let $\begin{cases} w = v' \\ w' = v'' \end{cases}$, we find:

$$\begin{aligned} \frac{w'}{w} &= \frac{-3}{1+t} \Longrightarrow \int \frac{dw}{w} = \int \frac{-3}{1+t} dt \\ \ln|w| &= -3\ln(1+t) + c \\ w &= e^c \frac{1}{(1+t)^3} \Longrightarrow w = \frac{\lambda}{(1+t)^3}, \quad \lambda \in \mathbb{R}. \end{aligned}$$

Hence:

$$\begin{aligned} v' &= \int \frac{\lambda}{(1+t)^3} dt = \lambda\Big(\frac{-1}{2} \cdot \frac{1}{(1+t)^2}\Big) + \mu \\ v(t) &= \frac{\lambda}{(1+t)^2} + \mu, \qquad \lambda, \mu \in \mathbb{R}. \end{aligned}$$

The general solution is given by:

$$y(t) = e^t v(t) = \lambda \frac{e^t}{(1+t)^2} + \mu e^t, \qquad \lambda, \mu \in \mathbb{R}.$$

2.2 Second-order non-homogeneous differential equation with non-constant coefficients

Consider the equation:

$$y'' + a(t)y' + b(t)y = c(t), \qquad c(t) \neq 0.$$

As for first-order linear equations.

Theorem 2.2.1.

The general solution of the non-homogeneous equation:

$$y'' + a(t)y' + b(t)y = c(t)$$

is equal to the sum of the general solution of the homogeneous equation and a particular solution of the non-homogeneous equation:

$$S_G = S_H + S_P$$

S_G: The general solution.

S_H: Solution of the homogeneous equation.

S_P: A particular solution.

2.2.1 Method of Variation of Parameters

The principle of this method is to consider λ and μ as functions of the variable t:
Suppose we seek the solution in the form:

$$y_p(t) = \lambda(t)y_1(t) + \mu(t)y_2(t).$$

Explanation:
Substituting this function into the non-homogeneous equation, after simplification, we get:

$$\begin{aligned}
&b(y = \lambda y_1 + \mu y_2) \\
&ay' = (\lambda' y_1 + y_1'\lambda + \mu' y_2 + y_2'\mu)a \\
&y'' = \lambda'' y_1 + y_1'\lambda' + y_1''\lambda + \lambda' y_1' + \mu'' y_2 + y_2'\mu' + y_2''\mu + \mu' y_2' \\
&\quad = \lambda'' y_1 + \lambda'(2y_1') + \lambda y_1'' + \mu'' y_2 + \mu'(2y_2') + \mu y_2''.
\end{aligned}$$

Thus:

$$\lambda''y_1 + \lambda'(2y_1') + \lambda y_1'' + \mu''y_2 + \mu'(2y_2') + \mu y_2'' + a\lambda' y_1 + ay_1'\lambda + a\mu' y_2 + ay_2'\mu + by_1\lambda + by_2\mu$$

$$= \lambda''y_1 + \mu''y_2 + \lambda'(2y_1' + ay_1) + \mu'(2y_2' + ay_2) + \lambda(y_1'' + ay_1' + by_1) + \mu(y_2'' + ay_2' + by_2) = c(t)$$

$$\lambda''y_1 + \mu''y_2 + \lambda'(2y_1' + ay_1) + \mu'(2y_2' + ay_2) = c(t).$$

Using the method of variation of constants such that:

$$\lambda' y_1 + \mu' y_2 = 0, \quad \lambda' y_1' + \mu' y_2' = c(t).$$

Solving this system, we obtain:

$$\lambda' = \frac{-c(t)y_2(t)}{y_1(t)y_2'(t) - y_1'(t)y_2(t)}, \quad \mu' = \frac{+c(t)y_1(t)}{y_1(t)y_2'(t) - y_1'(t)y_2(t)}.$$

Hence, the particular solution is given by:

$$y_p(t) = y_1(t)\int \frac{-c(t)Y_2}{Y_1Y_2' - Y_1'Y_2}dt + y_2(t)\int \frac{c(t)Y_1}{Y_1Y_2' - Y_1'Y_2}dt.$$

Consider the equation:

$$y'' - \frac{2}{t^2}y = te^t, \quad y = t^\alpha, \quad \alpha \in \mathbb{R}, \quad t \in]0, +\infty[$$

We first seek solutions for the homogeneous equation:

$$y'' - \frac{2}{t^2}y = 0 \text{ in the form } y(t) = t^\alpha, \quad \alpha \in \mathbb{R}.$$

We have:

$$y = t^\alpha \Longrightarrow \frac{2}{t^2}y = 2t^{\alpha-2},$$
$$y' = \alpha t^{\alpha-1},$$
$$y'' = \alpha(\alpha-1)t^{\alpha-2},$$

Summing these terms, we find:

$$y'' - \frac{2}{t^2}y = t^{\alpha-2}[\alpha(\alpha-1) - 2] = 0 \Longrightarrow \alpha(\alpha-1) - 2 = 0$$
$$\Longrightarrow \alpha^2 - \alpha - 2 = 0$$

Discriminant: $\Delta = 1 - 4(-2) = 9$, thus: $\alpha_1 = -1$, $\alpha_2 = 2$.

- For $\alpha_1 = -1 \Longrightarrow y_1(t) = \frac{1}{t}$.

- For $\alpha_2 = 2 \Longrightarrow y_2(t) = t^2$.

Hence, the general solution of the homogeneous equation is:

$$y_H(t) = \lambda y_1 + \mu y_2 = \lambda \cdot t^2 + \mu \cdot \frac{1}{t}, \qquad \lambda, \mu \in \mathbb{R}.$$

Assume the particular solution is given by:

$$y_p(t) = \frac{\lambda(t)}{t} + \mu(t)t^2, \quad \text{with derivatives } \lambda', \mu' \text{ satisfying:}$$

$$\begin{cases} \lambda' y_1 + \mu' y_2 = 0 \\ \lambda' y_1' + \mu' y_2' = te^t \end{cases}$$

Substituting $y_1 = 1/t$, $y_2 = t^2$:

$$\begin{cases} \lambda' \frac{1}{t} + \mu' t^2 = 0 \\ \lambda'(-\frac{1}{t^2}) + \mu'(2t) = te^t \end{cases}$$

Using Cramer's rule (since det $\neq 0$):

$$\begin{bmatrix} 1/t & t^2 \\ -1/t^2 & 2t \end{bmatrix} = 3 \neq 0$$

$$\lambda' = \frac{\begin{bmatrix} 0 & t^2 \\ te^t & 2t \end{bmatrix}}{3} = \frac{t^3 e^t}{3} \Longrightarrow \lambda(t) = \frac{1}{3}e^t(-t^3 + 3t^2 - 6t + 6),$$

$$\mu' = \frac{\begin{bmatrix} 1/t & 0 \\ -1/t^2 & te^t \end{bmatrix}}{3} = \frac{e^t}{3} \Longrightarrow \mu(t) = \frac{1}{3}e^t.$$

Hence:

$$\begin{aligned} S_p(t) &= \frac{1}{t}\left(\frac{1}{3}e^t(-t^3+3t^2-6t+6)\right) + t^2\left(\frac{1}{3}e^t\right) \\ &= \frac{1}{3}e^t(-t^2+3t-6+6/t+t^2) \\ &= \frac{1}{3}e^t(3t-6+6/t) \\ &= e^t(t-2+2/t) \end{aligned}$$

Thus, the general solution is:

$$S_G(t) = \lambda \cdot (1/t) + \mu \cdot t^2 + e^t(t-2+2/t).$$

2.3 Second-order linear equation with constant coefficients

A second-order linear differential equation with constant coefficients (2nd-order LDE with constant coefficients) is an equation of the form:

$$y'' + ay' + by = c(t)$$

where a, b are real constants, and $t \mapsto c(t)$ is a given continuous function on an interval $I \subset \mathbb{R}$.

We start by solving the associated homogeneous equation:

$$y'' + ay' + by = 0 \tag{2.2}$$

We look for solutions of the form:

$$y = e^{rt}, \qquad r \in \mathbb{R}.$$

Substituting into the homogeneous equation (2.2), we get:

$$(r^2 + ar + b)e^{rt} = 0, \quad \text{since } e^{rt} \neq 0, \forall t \in \mathbb{R}.$$

For a solution to exist, we must have:

$$r^2 + ar + b = 0.$$

This equation is called the **characteristic equation** associated with the homogeneous equation (2.2).

We have three cases:

1. If $a^2 - 4b > 0$, we find two distinct real roots r_1 and r_2. The general solution of the homogeneous equation is then:

$$y_H = \lambda e^{r_1 t} + \mu e^{r_2 t}, \qquad \lambda, \mu \in \mathbb{R}.$$

2. If $a^2 - 4b = 0$, we find a double real root r_0. The general solution of the homogeneous equation is:

$$y_H = (\lambda t + \mu)e^{r_0 t}, \qquad \lambda, \mu \in \mathbb{R}.$$

3. If $a^2 - 4b < 0$, we find two distinct complex conjugate roots of the form:

$$\begin{cases} r_1 = \alpha - i\beta \\ r_2 = \alpha + i\beta \end{cases}, \qquad \alpha, \beta \in \mathbb{R}.$$

   The general solution of the homogeneous equation is:

$$y_H = e^{\alpha t}(\lambda \cos(\beta t) + \mu \sin(\beta t)), \qquad \alpha, \beta, \lambda, \mu \in \mathbb{R}.$$

Example 2.3.1.

Consider the following equation:

$$y'' + 4y' + 3y = 0$$

The characteristic equation (C.E.) is: $\quad r^2 + 4r + 3 = 0.$

We have: $r_1 = -3$ and $r_2 = -1$, so the general solution is:

$$y = \lambda e^{-3t} + \mu e^{-t}, \qquad \lambda, \mu \in \mathbb{R}.$$

2.3.1 Search for a Particular Solution for Specific Second-Order Differential Equations

If $c(t) = p(t)$ where $p(t)$ is **a polynomial** of degree "n".

• Look for a solution $q(t)$ that is a **polynomial** of degree:

1. If $b \neq 0 \Longrightarrow q(t)$ of degree "n".
2. If $b = 0, \quad a \neq 0 \Longrightarrow q(t)$ of degree "$n+1$".
3. If $b = 0$ and $a = 0 \Longrightarrow q(t)$ of degree "$n+2$".

Example 2.3.2.

Consider the equation:

$$y'' - y' - 2y = 2t^2,$$

The characteristic equation (C.E.) is: $\quad r^2 - r - 2 = 0$ which admits two real roots:

$$r_1 = -1 \text{ and } r_2 = 2$$

Hence, the general solution is:

$$S_G = \lambda e^{-t} + \mu e^{2t}, \qquad \lambda, \mu \in \mathbb{R}.$$

We look for a particular solution in the form of a 2nd-degree polynomial:

$$y_p(t) = \alpha t^2 + \beta t + \gamma$$
$$y_p'(t) = 2\alpha t + \beta$$
$$y_p''(t) = 2\alpha$$

The equation becomes:

$$\begin{aligned} y_p'' - y_p' - 2y_p = 2t^2 &\Longleftrightarrow 2\alpha - (2\alpha t + \beta) - 2(\alpha t^2 + \beta t + \gamma) = 2t^2 \\ &\Longleftrightarrow (2\alpha - \beta - 2\gamma) + t(-2\alpha - 2\beta) + t^2(-2\alpha) = 2t^2 \end{aligned}$$

By identification, we have:

$$\begin{cases} 2\alpha - \beta - 2\gamma = 0 \\ -2\alpha - 2\beta = 0 \\ -2\alpha = 2 \end{cases} \Longleftrightarrow \begin{cases} \gamma = \dfrac{2\alpha - \beta}{2} = -\dfrac{3}{2} \\ \beta = 1 \\ \alpha = -1 \end{cases}$$

Hence, the general solution is:

$$y_G = \lambda e^{-t} + \mu e^{2t} + (-t^2 + t - \frac{3}{2}), \qquad \lambda, \mu \in \mathbb{R}.$$

2.4 Summary of the Possible Cases

Summarize the possible cases in the following table:

Second term $c(t)$	Particular solution $y_p(t)$
$c(t) = p_n(t)$ is a polynomial of degree "n"	$\bullet y_p(t) = q_n(t)$, if $b \neq 0$. $\bullet y_p(t) = q_{n+1}(t)$, if $b = 0$ and $a \neq 0$. $\bullet y_p(t) = q_{n+2}(t)$, if $b = 0$ and $a = 0$.
$c(t) = ke^{rt}$ with $r^2 + ar + b \neq 0$	$\bullet y_p(t) = \alpha e^{rt}$; r is not a root.
$c(t) = ke^{rt}$ with $r^2 + ar + b = 0$	$\bullet y_p = \alpha t e^{rt}$; r simple root. $\bullet y_p = \alpha t^2 e^{rt}$; r double root.
$c(t) = p_n(t)e^{rt}$, $r^2 + ar + b \neq 0$	$\bullet y_p(t) = q_n(t)e^{rt}$; $\deg(q) = n$.
$c(t) = p_n(t)e^{rt}$, $r^2 + ar + b = 0$	$\bullet y_p(t) = q_{n+1}(t)e^{rt}$; $\deg(q) = n + 1$.
$c(t) = p_n(t)e^{rt}$; $r = \frac{-a}{2}$ and $\deg(p) = n$	$\bullet y_p(t) = q_{n+2}(t)e^{rt}$; $\deg(q) = n + 2$.
$c(t) = d\cos(rt) + e\sin(rt)$	$\bullet y_p(t) = \alpha\cos(rt) + \beta\sin(rt)$.

Example 2.4.1.

Consider the equation:

$$y'' - y' - 2y = \sin(2t).$$

The general solution is:

$$S_G = \lambda e^{-t} + \mu e^{2t}, \quad \lambda, \mu \in \mathbb{R}.$$

The particular solution is:

$$y_p = \alpha\cos(2t) + \beta\sin(2t),$$

with $\alpha = \frac{1}{20}$ and $\beta = -\frac{3}{20}$. Hence, the general solution is:

$$y_G = \frac{1}{20}\cos(2t) - \frac{3}{20}\sin(2t) + \lambda e^{-t} + \mu e^{2t}, \quad \lambda, \mu \in \mathbb{R}.$$

Example 2.4.2.

Consider the equation:

$$y'' - y' - 2y = te^t \Longrightarrow y_H = \lambda e^{-t} + \mu e^{2t}, \quad \lambda, \mu \in \mathbb{R}. \tag{2.3}$$

Since 1 is not a root of the characteristic equation, we look for a particular solution of the form:

$$y_p = (\alpha t + \beta)e^t.$$

We have:

$$y_p' = (\alpha t + \alpha + \beta)e^t,$$
$$y_p'' = (\alpha t + 2\alpha + \beta)e^t.$$

Replacing into equation (2.3), we find:

$$\alpha = -\frac{1}{2}, \quad \beta = -\frac{1}{4}.$$

Thus, the general solution is:

$$y_G = -\frac{1}{4}(2t+1)e^t + \lambda e^{-t} + \mu e^{2t}, \quad \lambda, \mu \in \mathbb{R}.$$

Example 2.4.3.

Consider the equation:

$$y'' - y' - 2y = \sin(2t)$$

The general solution of the homogeneous equation is:

$$y_H = \lambda e^{-t} + \mu e^{2t}, \quad \lambda, \mu \in \mathbb{R}.$$

We look for a particular solution of the form:

$$y_p = \alpha \cos(2t) + \beta \sin(2t).$$

We have:

$$y_p' = -2\alpha \sin(2t) + 2\beta \cos(2t),$$
$$y_p'' = -4\alpha \cos(2t) - 4\beta \sin(2t).$$

Replacing in the equation, we get:

$$-2(3\alpha+\beta)\cos(2t)+(2\alpha-6\beta)\sin(2t)=\sin(2t)$$

so that

$$\begin{cases} 3\alpha+\beta=0 \\ 2\alpha-6\beta=1 \end{cases} \implies \alpha=\frac{1}{20}, \quad \beta=-\frac{3}{20}.$$

Thus, the general solution is:

$$y_G=\frac{1}{20}\cos(2t)-\frac{3}{20}\sin(2t)+\lambda e^{-t}+\mu e^{2t}, \quad \lambda,\mu\in\mathbb{R}.$$

2.5 Homogeneous linear equation with analytic coefficients

Consider the second-order homogeneous differential equation:

$$y''+a(t)y'+b(t)y=0. \tag{2.4}$$

Suppose that $a(t)$ and $b(t)$ are given by power series (i.e., series in non-negative integer powers of t):

$$a(t)=\sum_{k=0}^{+\infty}a_kt^k, \qquad b(t)=\sum_{k=0}^{+\infty}b_kt^k.$$

We look for a solution of equation (2.4) in the form of a power series:

$$y(t)=\sum_{k=0}^{+\infty}c_kt^k=c_0+c_1t+c_2t^2+\dots$$

Example 2.5.1.

Consider the second-order equation:

$$y''-ty'-2y=0. \tag{2.5}$$

We seek a solution as a power series:

$$y_1(t)=\sum_{k=0}^{+\infty}c_kt^k,$$

then

$$y_1'(t) = \sum_{k=1}^{+\infty} k c_k t^{k-1}, \qquad y_1''(t) = \sum_{k=2}^{+\infty} k(k-1) c_k t^{k-2}.$$

Substituting y_1, y_1', and y_1'' into equation (2.5) gives:

$$\sum_{k=2}^{+\infty} k(k-1)c_k t^{k-2} - t\sum_{k=1}^{+\infty} k c_k t^{k-1} - 2\sum_{k=0}^{+\infty} c_k t^k = 0$$

$$\Longrightarrow \sum_{k=2}^{+\infty} k(k-1)c_k t^{k-2} - \sum_{k=1}^{+\infty} k c_k t^{k} - 2\sum_{k=0}^{+\infty} c_k t^k = 0.$$

By changing the index in the first sum ($p = k-2 \Longrightarrow k = p+2$), we obtain:

$$2c_2 + \sum_{p=1}^{+\infty} [(p+2)(p+1)c_{p+2} - (p+2)c_p] t^p - 2c_0 - 2\sum_{p=1}^{+\infty} c_p t^p = 0$$

$$\Longrightarrow 2c_2 - 2c_0 + \sum_{p=1}^{+\infty} [(p+2)(p+1)c_{p+2} - (p+3)c_p] t^p = 0.$$

Setting the coefficients of all powers of t to zero allows us to recursively determine $c_0, c_1, \ldots$.

With initial conditions $y_1(0) = 1$ and $y_1'(0) = 0$, we find $c_0 = 1$ and $c_1 = 0$. Then we recursively obtain:

$$c_2 = 0, \quad c_3 = \frac{1}{2}c_1 = 0, \quad c_4 = \frac{1}{3}c_2 = \frac{1}{3}, \ldots$$

Thus:

$$y_1(t) = 1 + t^2 + \frac{1}{3}t^4 + \ldots$$

Similarly, taking a second solution of the form $y_2(t) = \sum_{k=0}^{+\infty} \alpha_k t^k$ with initial conditions

$$y_2(0) = 0, \quad y_2'(0) = 1,$$

we get $\alpha_0 = 0, \alpha_1 = 1$, and by substitution in (2.5):

$$\alpha_{k+2} = \frac{1}{k+1}\alpha_k, \quad k = 0, 1, 2, \ldots$$

$$\Longrightarrow \alpha_{2k} = 0, \quad \alpha_{2k+1} = \frac{1}{k! 2^k}.$$

Hence:

$$y_2(t) = \sum_{k=0}^{+\infty} \frac{1}{k!2^k} t^{2k+1} = t \sum_{k=0}^{+\infty} \frac{1}{k!} \left(\frac{t^2}{2}\right)^k = te^{t^2/2}.$$

The general solution of (2.5) is:

$$y(t) = \lambda y_1(t) + \mu y_2(t), \quad \lambda, \mu \in \mathbb{R}.$$

2.6 Equation of Order n and Differential Systems

Recall that an n-th order differential equation involves an unknown function $y(t)$, its derivatives up to order n, and the variable t, in the form:

$$y^{(n)} = F(t, y, y', \dots, y^{(n-1)}).$$

2.6.1 First-Order Differential System

Definition 2.3 (Canonical System)**.**

A system of ordinary differential equations:

$$F_k(t, y_1, y_1', \dots, y_1^{(k_1)}, y_2, y_2', \dots, y_2^{(k_2)}, \dots, y_n, y_n', \dots, y_n^{(k_n)}) = 0, \quad k = 1, 2, \dots, n, \tag{2.6}$$

is called a **canonical system**.

The **order** of the system (2.6) is the number

$$p = k_1 + k_2 + \cdots + k_n.$$

Integrating this system means determining the functions $y_1, y_2, \dots, y_n$ that satisfy the equations of the system (2.6).

Definition 2.4 (Normal System)**.**

Let $y_1, y_2, \dots, y_n$ be n differentiable functions of the variable t.

A **first-order differential system** is any system of differential equations of the form:

$$\begin{cases} \dfrac{dy_1}{dt} = F_1(t, y_1, \dots, y_n) \\ \dfrac{dy_2}{dt} = F_2(t, y_1, \dots, y_n) \\ \vdots \\ \dfrac{dy_n}{dt} = F_n(t, y_1, \dots, y_n) \end{cases}$$

Such a system can be written in vector form as:

$$y_i' = F_i(t, y), \quad i = 1, \dots, n \Longleftrightarrow y' = F(t, y), \quad y = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}, \quad y' = \begin{pmatrix} y_1' \\ \vdots \\ y_n' \end{pmatrix}, \quad F = \begin{pmatrix} F_1 \\ \vdots \\ F_n \end{pmatrix},$$

where F is continuous on a domain $D \subset \mathbb{R} \times \mathbb{R}^n$.

Theorem 2.6.1 (Existence and Uniqueness)**.**

Let the system $y' = F(t, y)$, where F is continuous on a domain $D \subset \mathbb{R} \times \mathbb{R}^n$ and has continuous partial derivatives with respect to y_i, $i = 1, \dots, n$, on D.
Then, for any $(t_0, y_0) \in D$, there exists a unique maximal solution $u(t)$ satisfying the initial condition

$$u(t_0) = y_0.$$

2.6.2 The solution of a system of n first-order equations

A **solution** of the system (S) is a set of real differentiable functions $y_1(t), \dots, y_n(t)$ defined on the same interval $I \subset \mathbb{R}$ and satisfying for all $t \in I$:

$$y_i' = F_i(t, y_1(t), y_2(t), \dots, y_n(t)), \quad i = 1, \dots, n.$$

Example 2.6.1.

Let a and b be real numbers.
The functions

$$\begin{cases} y_1(t) = a\cos t + b\sin t \\ y_2(t) = -a\sin t + b\cos t \end{cases}, \quad t \in \mathbb{R}$$

are solutions of the system:

$$\begin{cases} y_1' = y_2 \\ y_2' = -y_1 \end{cases}$$

since it is easy to verify that:

$$\begin{cases} y_1' = -a\sin t + b\cos t = y_2 \\ y_2' = -a\cos t - b\sin t = -y_1 \end{cases}$$

2.6.3 The Relationship Between an n-th Order Differential Equation and an n-th Order System

Let an n-th order differential equation be:

$$y^{(n)} = F(t, y, y', \dots, y^{(n-1)}).$$

By considering the successive derivatives $y, y', \dots, y^{(n-1)}$ as new unknown functions, we can rewrite it as a differential system:

$$\begin{cases} y_1(t) = y(t) \\ y_2(t) = y'(t) \\ \vdots \\ y_n(t) = y^{(n-1)}(t) \end{cases} \implies \begin{cases} y_1'(t) = y'(t) = y_2(t) \\ y_2'(t) = y''(t) = y_3(t) \\ \vdots \\ y_n'(t) = y^{(n)}(t) = F(t, y, y', \dots, y^{(n-1)}) \end{cases}$$

Example 2.6.2.

Consider the third-order equation:

$$y''' = 3y'' - y' + y.$$

It can be rewritten as the differential system:

$$y(t) = \begin{pmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{pmatrix} = \begin{pmatrix} y(t) \\ y'(t) \\ y''(t) \end{pmatrix} \implies y'(t) = \begin{pmatrix} y_1'(t) \\ y_2'(t) \\ y_3'(t) \end{pmatrix} = \begin{pmatrix} y'(t) \\ y''(t) \\ y'''(t) \end{pmatrix}$$

$$y'(t) = \begin{pmatrix} y_2(t) \\ y_3(t) \\ 3y_3(t) - y_2(t) + y_1(t) \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -1 & 3 \end{pmatrix} y(t)$$

2.6.4 Solved Exercises

Solve the following differential equations:

1. $y'' - 4y' + 3y = (2x+1)e^{-x}$

   Solution: The associated homogeneous equation is:

$$y_h'' - 4y_h' + 3y_h = 0$$

   with characteristic equation:

$$r^2 - 4r + 3 = 0 \implies r = 1, 3$$

   so

$$y_h = C_1 e^x + C_2 e^{3x}.$$

   For a particular solution y_p, we propose:

$$y_p = e^{-x}(Ax + B)$$

   Substituting and solving gives:

$$y_p = \frac{1}{8}(-2x - 5)e^{-x}.$$

   Therefore, the general solution is:

$$\boxed{y(x) = C_1 e^x + C_2 e^{3x} + \frac{-2x-5}{8} e^{-x}}$$

2. $y'' - 4y' + 3y = (2x+1)e^{x}$

   Solution: The homogeneous equation is the same as above, so

$$y_h = C_1 e^x + C_2 e^{3x}.$$

   Since e^x is already a solution of the homogeneous equation, we propose:

$$y_p = x(Ax + B)e^x$$

Substituting and solving gives:

$$y_p = \frac{1}{2}x(2x+5)e^x$$

General solution:

$$\boxed{y(x) = C_1e^x + C_2e^{3x} + \frac{1}{2}x(2x+5)e^x}$$

3. $y'' - 2y' + y = (x^2+1)e^x + e^{3x}$

   Solution: Homogeneous equation:

$$y_h'' - 2y_h' + y_h = 0, \quad r^2 - 2r + 1 = 0 \implies r = 1 \text{ double.}$$

   So

$$y_h = (C_1 + C_2x)e^x$$

   For the nonhomogeneous term $(x^2+1)e^x$, propose:

$$y_{p1} = (Ax^3 + Bx^2 + Cx + D)e^x$$

   After calculation:

$$y_{p1} = \frac{1}{6}x^3e^x$$

   For e^{3x}:

$$y_{p2} = Ae^{3x} \implies y_{p2} = \frac{1}{8}e^{3x}$$

   General solution:

$$\boxed{y(x) = (C_1 + C_2x)e^x + \frac{1}{6}x^3e^x + \frac{1}{8}e^{3x}}$$

4. $y'' - 4y' + 3y = x^2e^x + xe^{2x}\cos x$

   Solution: Homogeneous equation: $r^2 - 4r + 3 = 0 \implies r = 1, 3$

$$y_h = C_1e^x + C_2e^{3x}$$

   For x^2e^x, propose $y_{p1} = (Ax^3 + Bx^2 + Cx + D)e^x$ and solve for coefficients. For

$xe^{2x}\cos x$, propose $y_{p2} = e^{2x}((Ax+B)\cos x + (Cx+D)\sin x)$. General solution:

$$y(x) = C_1e^x + C_2e^{3x} + y_{p1} + y_{p2}$$

5. $y'' - 2y' + 5y = -4e^{-x}\cos x + 7e^{-x}\sin x - 4e^x \sin 2x$

Solution: Homogeneous equation: $r^2 - 2r + 5 = 0 \implies r = 1 \pm 2i$

$$y_h = e^x(C_1\cos 2x + C_2\sin 2x)$$

For the nonhomogeneous terms:

$$y_{p1} = e^{-x}(A\cos x + B\sin x), \quad y_{p2} = xe^x(C\cos 2x + D\sin 2x)$$

Solving for the coefficients gives y_{p1} and y_{p2}. General solution:

$$\boxed{y(x) = y_h + y_{p1} + y_{p2}}$$

Chapter 3

Linear differential system

Let I be a non-empty open interval of $\mathbb{R}$. For all $i, j = \overline{1, n}$, let a_{ij} and b_i be functions defined on I with values in $\mathbb{R}$.
Consider the system of n linear differential equations:

$$\begin{cases} y_1'(t) = a_{11}(t)y_1 + a_{12}(t)y_2 + \cdots + a_{1n}(t)y_n + b_1(t) \\ y_2'(t) = a_{21}(t)y_1 + a_{22}(t)y_2 + \cdots + a_{2n}(t)y_n + b_2(t) \\ \qquad\qquad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ y_n'(t) = a_{n1}(t)y_1 + a_{n2}(t)y_2 + \cdots + a_{nn}(t)y_n + b_n(t) \end{cases} \qquad (S)$$

The unknowns $y_1, y_2, \ldots, y_n$ are functions defined on I with values in $\mathbb{R}$.

Lemma 3.0.1.

The system (S) is equivalent to the system:

$$y'(t) = A(t)y(t) + B(t), \quad \forall t \in I \quad (S')$$

with

$$y : I \to \mathbb{R}^n, \quad t \mapsto y(t) = \begin{pmatrix} y_1(t) \\ \vdots \\ y_n(t) \end{pmatrix},$$

$$B : I \to \mathbb{R}^n, \quad t \mapsto B(t) = \begin{pmatrix} b_1(t) \\ \vdots \\ b_n(t) \end{pmatrix},$$

$$A : I \to M_n(\mathbb{R}), \quad t \mapsto A(t) = (a_{ij}(t))_{i,j=\overline{1,n}}.$$

[?]

Proof 3.0.1.

Indeed:

$$(S) \iff \begin{pmatrix} y_1'(t) \\ \vdots \\ y_n'(t) \end{pmatrix} = \begin{pmatrix} a_{11}(t)y_1(t) + a_{12}(t)y_2(t) + \cdots + a_{1n}(t)y_n(t) \\ a_{21}(t)y_1(t) + a_{22}(t)y_2(t) + \cdots + a_{2n}(t)y_n(t) \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ a_{n1}(t)y_1(t) + a_{n2}(t)y_2(t) + \cdots + a_{nn}(t)y_n(t) \end{pmatrix} + \begin{pmatrix} b_1(t) \\ \vdots \\ b_n(t) \end{pmatrix}$$

$$\iff y'(t) = A(t)y(t) + B(t)$$

Example 3.0.1.

Consider the system:

$$\begin{cases} y_1'(t) = t^2 y_1(t) - y_2(t) + 1 \\ y_2'(t) = \cos t\, y_1(t) - e^t y_2(t) \end{cases}, \quad t \in I = \mathbb{R}.$$

It can be written in the form:

$$y'(t) = A(t)y(t) + B(t), \quad \forall t \in I$$

with

$$y(t) = \begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix}, \quad A(t) = \begin{pmatrix} t^2 & -1 \\ \cos t & -e^t \end{pmatrix}, \quad B(t) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Lemma 3.0.2.

Every second-order equation can be written in the form:

$$y'(t) = A(t)y(t) + B(t).$$

Proof 3.0.2.

Consider a second-order equation:

$$y''(t) + a(t)y'(t) + b(t)y(t) = c(t)$$

and set

$$y_1(t) = y(t), \quad y_2(t) = y'(t)$$

then, for all $t \in I$:

$$\begin{cases} y_1'(t) = y_2(t) \\ y_2'(t) = y''(t) = -a(t)y_2(t) - b(t)y_1(t) + c(t) \end{cases}$$

which is equivalent to $y'(t) = A(t)y(t) + B(t)$, with

$$y(t) = \begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix}, \quad A(t) = \begin{pmatrix} 0 & 1 \\ -b(t) & -a(t) \end{pmatrix}, \quad B(t) = \begin{pmatrix} 0 \\ c(t) \end{pmatrix}, \quad \forall t \in I.$$

Definition 3.1.

The system (S') is called a first-order linear differential system with variable coefficients and a nonzero second member.

Remark 3.0.1.

If $B(t) = 0$ for all $t \in I$, then the system

$$y'(t) = A(t)y(t) \tag{3.1}$$

is called a homogeneous first-order linear differential system (without a second member). Equation (3.1) is called the homogeneous system associated with (S').

- Notation:

For simplicity, we write:

$$(S') = \begin{cases} y' = A(t)y & \text{(homogeneous)} \\ y' = A(t)y + B(t) & \text{(nonhomogeneous)} \end{cases}$$

3.0.1 The existence of a Cauchy problem

Theorem 3.0.1.

If A and B are continuous on the interval I, then for every $(t_0, y_0) \in I \times \mathbb{R}^n$, the system

$$\begin{cases} y' = A(t)y + B(t) \\ y(t_0) = y_0 \end{cases}$$

admits a unique global solution.

3.0.2 Homogeneous and non-homogeneous systems

Theorem 3.0.2.

The set of solutions of (3.1), denoted by S_H, is a vector space of dimension n.

Theorem 3.0.3.

[?] Let y_p be a particular solution of (E). Then the set of solutions of (E), denoted by S_E, is given by:

$$S_E = S_H + S_p,$$

where S_H is the set of solutions of the homogeneous system.[?]

3.1 Matrix exponential method

Let $A \in M_n(\mathbb{R})$. We define:

$$\begin{cases} A^k = \underbrace{A \cdots A}_{\text{k times}}, & \text{if } k \in \mathbb{N}^*, \\ A^0 = I_n \end{cases}$$

where $I_n = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \ddots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 \end{pmatrix}$ is the identity matrix.

Consider the series defined by:

$$\sum_{k=0}^{\infty} \frac{A^k}{k!} = \lim_{l \to +\infty} \left(I_n + \frac{A}{1!} + \cdots + \frac{A^l}{l!} \right)$$

with

$$\begin{cases} k! = 1 \cdot 2 \cdot \cdots \cdot k, & \text{if } k \in \mathbb{N}^*, \\ 0! = 1. \end{cases}$$

Theorem 3.1.1.

The series:

$$\sum_{k=0}^{\infty} \frac{A^k}{k!}$$

is convergent.

Proof 3.1.1.

We have

$$\left\| \frac{A^k}{k!} \right\| \leq \frac{\|A^k\|}{k!}.$$

Moreover, $\frac{A^k}{k!}$ represents the general term of a convergent numerical series, hence:

$$\sum_{k=0}^{\infty} \frac{A^k}{k!}$$

is normally convergent. Therefore, it is convergent.

Definition 3.2.

The exponential of a matrix A, denoted e^A, is defined as the series

$$e^A = \sum_{k=0}^{\infty} \frac{A^k}{k!}.$$

Example 3.1.1.

Consider:

$$e^{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}}, \quad e^{\begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix}}$$

We compute:

$$A_1^2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad A_2^2 = \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix}^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Thus, $A_1^2 = A_2^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, hence $A_1^n = A_2^n = 0$ for all $n \geq 2$.

$$e^{A_1} = \sum_{k=0}^{\infty} \frac{A_1^k}{k!} = \sum_{k=0}^{1} \frac{A_1^k}{k!} = \frac{A_1^0}{0!} + \frac{A_1}{1!}$$
$$= I_2 + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

Theorem 3.1.2.

Let $A, B \in M_n(\mathbb{R})$.

(1) $e^{0_n} = I_n$, where 0_n is the zero matrix.

(2) In general, $e^{A+B} \neq e^A \cdot e^B$, but if A and B commute, i.e., $AB = BA$, then

$$e^{A+B} = e^A \cdot e^B.$$

(3) e^A is invertible and $(e^A)^{-1} = e^{-A}$.

(4) The function

$$F : \mathbb{R} \to M_n(\mathbb{R}), \quad t \mapsto e^{tA}$$

is differentiable, and we have $(e^{tA})' = Ae^{tA}$ for all $t \in \mathbb{R}$.

- Indeed:

$$e^{0_n} = \sum_{k=0}^{\infty} \frac{(0_n)^k}{k!} = \frac{0_n^0}{0!} + \sum_{k=1}^{\infty} \frac{(0_n)^k}{k!} = I_n + 0_n = I_n.$$

- Also,

$$(e^{tA})' = \Big(\sum_{k=0}^{\infty} \frac{(tA)^k}{k!}\Big)' = \sum_{k=0}^{\infty} \Big(\frac{(tA)^k}{k!}\Big)'$$
$$= A \cdot \sum_{k=1}^{\infty} \frac{t^{k-1}A^{k-1}}{(k-1)!}.$$

Setting $p = k - 1$, we find:

$$(e^{tA})' = A \cdot \sum_{p=0}^{\infty} \frac{t^p A^p}{p!} = A \cdot e^{tA}.$$

Remark 3.1.1.

There exist matrices A and B such that

$$e^{A+B} \neq e^A \cdot e^B.$$

For example:

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}.$$

Then

$$A + B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

From the definition of the matrix exponential:

$$e^{A+B} = e^{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}} = \begin{pmatrix} e & 0 \\ 0 & 0 \end{pmatrix}.$$

Meanwhile,

$$e^A = \begin{pmatrix} e & e-1 \\ 0 & 1 \end{pmatrix}, \quad e^B = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \quad \Longrightarrow \quad e^A \cdot e^B = \begin{pmatrix} e & -1 \\ 0 & 1 \end{pmatrix}.$$

Thus, $e^{A+B} \neq e^A \cdot e^B$.

Lemma 3.1.1.

Let $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$. Then

$$e^{\begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix}} = \begin{bmatrix} e^{\lambda_1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & e^{\lambda_n} \end{bmatrix}.$$

Proof 3.1.2.

We have:

$$e^{\begin{bmatrix}\lambda_1 & \cdots & 0\\ \vdots & \ddots & \vdots\\ 0 & \cdots & \lambda_n\end{bmatrix}} = \sum_{k=0}^{\infty} \frac{\begin{bmatrix}\lambda_1 & \cdots & 0\\ \vdots & \ddots & \vdots\\ 0 & \cdots & \lambda_n\end{bmatrix}^k}{k!}.$$

But one can show by induction that for all $k \in \mathbb{N}$, we have:

$$\begin{bmatrix}\lambda_1 & \cdots & 0\\ \vdots & \ddots & \vdots\\ 0 & \cdots & \lambda_n\end{bmatrix}^k = \begin{bmatrix}\lambda_1^k & \cdots & 0\\ \vdots & \ddots & \vdots\\ 0 & \cdots & \lambda_n^k\end{bmatrix}.$$

Therefore:

$$\begin{aligned}
e^{\begin{bmatrix}\lambda_1 & \cdots & 0\\ \vdots & \ddots & \vdots\\ 0 & \cdots & \lambda_n\end{bmatrix}} &= \sum_{k=0}^{\infty} \frac{\begin{bmatrix}\lambda_1^k & \cdots & 0\\ \vdots & \ddots & \vdots\\ 0 & \cdots & \lambda_n^k\end{bmatrix}}{k!}\\
&= \lim_{l\to+\infty} \sum_{k=0}^{l} \begin{bmatrix}\frac{\lambda_1^k}{k!} & \cdots & 0\\ \vdots & \ddots & \vdots\\ 0 & \cdots & \frac{\lambda_n^k}{k!}\end{bmatrix}\\
&= \lim_{l\to+\infty} \begin{bmatrix}\sum_{k=0}^{l} \frac{\lambda_1^k}{k!} & \cdots & 0\\ \vdots & \ddots & \vdots\\ 0 & \cdots & \sum_{k=0}^{l} \frac{\lambda_n^k}{k!}\end{bmatrix}\\
&= \begin{bmatrix}\lim_{l\to+\infty} \sum_{k=0}^{l} \frac{\lambda_1^k}{k!} & \cdots & 0\\ \vdots & \ddots & \vdots\\ 0 & \cdots & \lim_{l\to+\infty} \sum_{k=0}^{l} \frac{\lambda_n^k}{k!}\end{bmatrix}\\
&= \begin{bmatrix}e^{\lambda_1} & \cdots & 0\\ \vdots & \ddots & \vdots\\ 0 & \cdots & e^{\lambda_n}\end{bmatrix}.
\end{aligned}$$

Example 3.1.2.

We have:

$$e^{\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}} = \begin{pmatrix} e^2 & 0 \\ 0 & e^3 \end{pmatrix}.$$

Also:

$$e^{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & -3 \end{pmatrix}} = \begin{pmatrix} e^1 & 0 & 0 \\ 0 & e^4 & 0 \\ 0 & 0 & e^{-3} \end{pmatrix}.$$

Theorem 3.1.3.

Let $A \in M_n(\mathbb{R})$.

1. If P is an invertible matrix, then

$$e^{PAP^{-1}} = P\, e^A\, P^{-1}.$$

2. For $\lambda \in \mathbb{R}$, we have

$$e^{\lambda I_n + A} = e^{\lambda} \cdot e^A.$$

Proof 3.1.3.

1. We compute:

$$e^{PAP^{-1}} = \sum_{k=0}^{\infty} \frac{(PAP^{-1})^k}{k!}.$$

By induction, one can show that for all $k \in \mathbb{N}$,

$$(PAP^{-1})^k = PA^kP^{-1}.$$

Thus,

$$\begin{aligned} e^{PAP^{-1}} &= \lim_{l\to+\infty} \sum_{k=0}^{l} \frac{PA^kP^{-1}}{k!} \\ &= \lim_{l\to+\infty} \Big[P\Big(\sum_{k=0}^{l} \frac{A^k}{k!}\Big)P^{-1}\Big] \\ &= P\Big(\lim_{l\to+\infty} \sum_{k=0}^{l} \frac{A^k}{k!}\Big)P^{-1} \\ &= P\, e^A\, P^{-1}. \end{aligned}$$

2. For $e^{\lambda I_n + A}$: since $(\lambda I_n)A = A(\lambda I_n)$, we know that λI_n and A commute. Thus,

$$e^{\lambda I_n + A} = e^{\lambda I_n} \cdot e^A.$$

Now,

$$\begin{aligned} e^{\lambda I_n} &= e^{\lambda\begin{pmatrix} 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{pmatrix}} = e^{\begin{pmatrix} \lambda & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda \end{pmatrix}} \\ &= \begin{pmatrix} e^\lambda & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & e^\lambda \end{pmatrix} = e^\lambda \begin{pmatrix} 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{pmatrix} = e^\lambda I_n. \end{aligned}$$

Therefore,

$$e^{\lambda I_n + A} = (e^\lambda I_n)e^A = e^\lambda (I_n e^A) = e^\lambda e^A.$$

Example 3.1.3.

Let us compute e^A where $A = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$. We have:

$$\det(A - \lambda I_2) = \left| \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \right| = \left| \begin{pmatrix} -\lambda & -1 \\ -1 & -\lambda \end{pmatrix} \right| = \lambda^2 - 1 = 0$$

which implies that $\lambda_1 = 1$ and $\lambda_2 = -1$ are the two distinct eigenvalues of A. Thus, A is diagonalizable:

$$A = P \cdot D \cdot P^{-1}, \quad D = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

where $P = (v_1, v_2)$, with v_1 and v_2 being the eigenvectors of A associated with λ_1 and λ_2, respectively.

The eigenspaces:

$$\begin{aligned} E_{\lambda_1} &= \{v_1 \in \mathbb{R}^2 \mid (A - \lambda_1 I)v_1 = 0\} = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 \middle| \begin{pmatrix} -1 & -1 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 0_{\mathbb{R}^2} \right\} \\ &= \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 \middle| -x - y = 0 \right\} = \left\{ \begin{pmatrix} x \\ -x \end{pmatrix} \in \mathbb{R}^2 \middle| x \in \mathbb{R} \right\} \\ &= \left\{ x \begin{pmatrix} 1 \\ -1 \end{pmatrix} \in \mathbb{R}^2 \middle| x \in \mathbb{R} \right\}. \end{aligned}$$

We take $v_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

Similarly,

$$\begin{aligned} E_{\lambda_2} &= \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 \middle| (A - \lambda_2 I) \begin{pmatrix} x \\ y \end{pmatrix} = 0_{\mathbb{R}^2} \right\} \\ &= \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 \middle| (A + I) \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\}. \end{aligned}$$

We take $v_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. Thus, the change of basis matrix is

$$P = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}.$$

Recall that if $ad - bc \neq 0$, then

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

So,

$$P^{-1} = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}.$$

Hence,

$$\begin{aligned} e^{A} &= e^{PDP^{-1}} = P\,e^{D}\,P^{-1} \\ &= \frac{1}{2}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} e^{1} & 0 \\ 0 & e^{-1} \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \\ &= \frac{1}{2}\begin{pmatrix} e^{-1}+e & e^{-1}-e \\ e^{-1}-e & e^{-1}+e \end{pmatrix}. \end{aligned}$$

Definition 3.3.

Let $N \in M_n(\mathbb{R})$. We say that N is a **nilpotent** matrix of index $m \in \mathbb{N}^*$ if $N^{m-1} \neq 0_n$ and $N^m = 0_n$.

Example 3.1.4.

The matrix

$$N = \begin{pmatrix} 3 & 9 & -9 \\ 2 & 0 & 0 \\ 3 & 3 & -3 \end{pmatrix}$$

is a nilpotent matrix of index $m = 3$. Indeed:

$$N^2 = \begin{pmatrix} 3 & 9 & -9 \\ 2 & 0 & 0 \\ 3 & 3 & -3 \end{pmatrix}^2 = \begin{pmatrix} 0 & 0 & 0 \\ 6 & 18 & -18 \\ 6 & 18 & 18 \end{pmatrix} \neq 0_3, \quad \text{and} \quad N^3 = N^2 \cdot N = 0_3.$$

Remark 3.1.2.

Every upper triangular matrix whose diagonal entries are all zero is nilpotent. For example, the matrix

$$N = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

is nilpotent because it is upper triangular with zeros on the diagonal.

Theorem 3.1.4.

Let N be a nilpotent matrix of index $m \in \mathbb{N}^*$. Then:

$$e^N = I_n + \frac{N}{1!} + \cdots + \frac{N^{m-1}}{(m-1)!}.$$

Proof 3.1.4.

We have:

$$e^N = \sum_{k=0}^{\infty} \frac{N^k}{k!} = I_n + \frac{N}{1!} + \frac{N^2}{2!} + \cdots + \frac{N^{m-1}}{(m-1)!} + \sum_{k=m}^{\infty} \frac{N^k}{k!}.$$

But since N is nilpotent of index m, for all $k \geq m$ we have $N^k = 0_n$. Indeed:

$$k \geq m \implies N^k = N^{(k-m)+m} = N^{k-m} \cdot N^m = N^{k-m} \cdot 0_n = 0_n.$$

Therefore,

$$e^N = I_n + \frac{N}{1!} + \cdots + \frac{N^{m-1}}{(m-1)!}.$$

Example 3.1.5.

Consider the matrix

$$N = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

It is nilpotent. Let us find its index m. We compute:

$$N^2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = 0_2,$$

so $m = 2$. Thus,

$$e^{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}} = I_2 + \frac{1}{1!} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

3.2 Second-Order Differential Systems

3.2.1 General Form

A second-order differential equation or system can be written as:

$$y''(t) = f(t, y(t), y'(t)), \quad y(t_0) = y_0, \quad y'(t_0) = y_1, \tag{3.2}$$

where $y(t) \in \mathbb{R}^n$ and $f : I \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is continuous.

3.2.2 Reduction to a First-Order System

To study existence and uniqueness, a second-order system can be reduced to a first-order system by setting:

$$z(t) = y'(t).$$

Then (3.2) becomes:

$$\begin{cases} y'(t) = z(t), \\ z'(t) = f(t, y(t), z(t)), \\ y(t_0) = y_0, \quad z(t_0) = y_1. \end{cases}$$

This is now a first-order system in the variable $X(t) = (y(t), z(t))^T \in \mathbb{R}^{2n}$.

3.2.3 Existence and Uniqueness

- If f is continuous in (t, y, z), there exists at least one local solution (Peano theorem).
- If f is Lipschitz continuous in (y, z) uniformly in t, the local solution is unique (Cauchy-Lipschitz theorem).

Example

Consider the second-order differential equation:

$$y'' + y = 0, \quad y(0) = 1, \; y'(0) = 0.$$

Reducing it to a first-order system:

$$\begin{cases} y_1' = y_2, \\ y_2' = -y_1, \\ y_1(0) = 1, \ y_2(0) = 0, \end{cases}$$

which has the unique global solution:

$$y(t) = \cos(t), \quad y'(t) = -\sin(t).$$

3.3 Resolvent of a Linear System

Consider the system:[?, ?]

$$y' = A(t)y.$$

Lemma 3.3.1.

If for all $t \in I$, we have:

$$R(t, t_0) = e^{(t-t_0)A}.$$

Proof 3.3.1.

From the definition of the resolvent, we have:

$$R(t, t_0) = e^{\int_{t_0}^{t} A(u)du} = e^{\int_{t_0}^{t} A\, du} = e^{A \int_{t_0}^{t} du} = e^{(t-t_0)A}.$$

Example 3.3.1.

The resolvent of the system $y' = Ay$ with

$$A = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}.$$

Thus:

$$R(t,t_0) = e^{(t-t_0)A}$$
$$= e^{(t-t_0)\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}} = e^{\begin{pmatrix} 2(t-t_0) & 0 \\ 0 & (t-t_0) \end{pmatrix}} = \begin{pmatrix} e^{2(t-t_0)} & 0 \\ 0 & e^{(t-t_0)} \end{pmatrix}.$$

Lemma 3.3.2.

[?, ?] The solution of the system $y' = Ay$ is given by:

$$\forall t \in \mathbb{R}: \quad y(t) = e^{tA}c,$$

with $c \in \mathbb{R}^n$ such that

$$c = \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}.$$

Proof 3.3.2.

Since $I = \mathbb{R}$, we have $0 \in I$. Thus, from the previous result, the solution of the homogeneous system is:

$$\forall t \in \mathbb{R}: \quad y(t) = R(t,0)c, \quad c \in \mathbb{R}^n.$$

But from the previous lemma:

$$R(t,0) = e^{(t-0)A} = e^{tA}.$$

Example 3.3.2.

The solution of the system

$$y' = \begin{pmatrix} 4 & 3 \\ 0 & 4 \end{pmatrix} y$$

is given by:

$$\forall t \in \mathbb{R}: \quad y(t) = e^{tA}c, \quad c \in \mathbb{R}^2.$$

Here,

$$A = \begin{pmatrix} 4 & 3 \\ 0 & 4 \end{pmatrix}, \quad \text{i.e. } y(t) = e^{tA}c,$$

but

$$e^{t\begin{pmatrix}4 & 3\\ 0 & 4\end{pmatrix}} = e^{\begin{pmatrix}4t & 3t\\ 0 & 4t\end{pmatrix}} = e^{4t\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}+\begin{pmatrix}0 & 3t\\ 0 & 0\end{pmatrix}} = e^{4tI_2+\begin{pmatrix}0 & 3t\\ 0 & 0\end{pmatrix}}.$$

We can show that

$$\begin{pmatrix}0 & 3t\\ 0 & 0\end{pmatrix}$$

is a nilpotent matrix of index $m = 2$. Therefore:

$$\begin{aligned} e^{tA} &= e^{4t}\cdot e^{\begin{pmatrix}0 & 3t\\ 0 & 0\end{pmatrix}} = e^{4t}\left[I_2 + \frac{1}{1!}\begin{pmatrix}0 & 3t\\ 0 & 0\end{pmatrix}\right] \\ &= e^{4t}\begin{pmatrix}1 & 3t\\ 0 & 1\end{pmatrix} = \begin{pmatrix}e^{4t} & 3te^{4t}\\ 0 & e^{4t}\end{pmatrix}. \end{aligned}$$

Thus:

$$\forall t \in \mathbb{R}: \quad y(t) = \begin{pmatrix}e^{4t} & 3te^{4t}\\ 0 & e^{4t}\end{pmatrix}\begin{pmatrix}c_1\\ c_2\end{pmatrix} = \begin{pmatrix}c_1e^{4t} + 3c_2te^{4t}\\ c_2e^{4t}\end{pmatrix}.$$

Lemma 3.3.3.

The solution of the following system:

$$\begin{cases} y' = Ay \\ y(t_0) = y_0 \end{cases}$$

is given by:

$$\forall t \in \mathbb{R}: \quad y(t) = e^{(t-t_0)A}y_0.$$

Example 3.3.3.

The solution of the system:

$$\begin{cases} y' = \begin{pmatrix}4 & 3\\ 0 & 4\end{pmatrix} y \\ y(1) = \begin{pmatrix}1\\ 0\end{pmatrix} \end{cases} \quad \text{is given by: } \forall t \in \mathbb{R}: y(t) = e^{(t-t_0)A}y_0.$$

With $A = \begin{pmatrix} 4 & 3 \\ 0 & 4 \end{pmatrix}$, $t_0 = 1$ and $y_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, i.e.:

$$\begin{aligned} y(t) = e^{(t-1)\begin{pmatrix} 4 & 3 \\ 0 & 4 \end{pmatrix}} &= e^{4(t-1)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 3(t-1) \\ 0 & 0 \end{pmatrix}} \\ &= e^{4(t-1)} \cdot e^{\begin{pmatrix} 0 & 3(t-1) \\ 0 & 0 \end{pmatrix}} \\ &= e^{4(t-1)} \left(I_2 + \begin{pmatrix} 0 & 3(t-1) \\ 0 & 0 \end{pmatrix} \right) \\ &= e^{4(t-1)} \begin{pmatrix} 1 & 3(t-1) \\ 0 & 1 \end{pmatrix} \end{aligned}$$

$$\begin{aligned} y(t) &= \begin{pmatrix} e^{4(t-1)} & 3(t-1)e^{4(t-1)} \\ 0 & e^{4(t-1)} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} e^{4(t-1)} \\ 0 \end{pmatrix} \end{aligned}$$

3.3.1 Non-homogeneous differential system

Theorem 3.3.1.

If the matrix A admits n linearly independent eigenvectors $v_1, v_2, \cdots, v_n$ associated with the real eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_n$, then the general solution of (H) is given by:

$$y(t) = c_1 v_1 e^{\lambda_1 t} + c_2 v_2 e^{\lambda_2 t} + \cdots + c_n v_n e^{\lambda_n t}, \quad \text{with } v_1, v_2, \cdots, v_n \in \mathbb{R}.$$

Example 3.3.4.

$$y' = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} y$$

Let us compute y_H:

$\lambda_1 = 1$ and $\lambda_2 = 2$ are the two distinct eigenvalues of $\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$.

$v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $v_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ are the eigenvectors of $\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ associated respectively with λ_1, λ_2.

$$y_H(t) = c_1 v_1 e^{\lambda_1 t} + c_2 v_2 e^{\lambda_2 t} = \begin{bmatrix} c_1 e^t \\ c_2 e^{2t} \end{bmatrix}.$$

- **Computation of y_p:**

Using the method of variation of constants, there exists a particular solution of the form:

$$y_p(t) = c_1 v_1 e^{\lambda_1 t} + c_2 v_2 e^{\lambda_2 t} + \cdots, \text{ with } \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}' = \left(v_1 e^{\lambda_1 t} \quad v_2 e^{\lambda_2 t}, \cdots, v_n e^{\lambda_n t} \right)^{-1} \cdot B(t).$$

Recall that if $\lambda_1, \lambda_2, \cdots, \lambda_n \in \mathbb{R}$, then:

$$\begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & 0 & \ddots & \vdots \\ 0 & \cdots & 0 & \lambda_n \end{bmatrix}^{-1} = \begin{bmatrix} \lambda_1^{-1} & 0 & \cdots & 0 \\ 0 & \lambda_2^{-1} & \cdots & 0 \\ \vdots & 0 & \ddots & \vdots \\ 0 & \cdots & 0 & \lambda_n^{-1} \end{bmatrix}$$

Example 3.3.5.

Solve the system: $y' = Ay + B$ where:

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix} \text{ and } B = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

The general solution of the system $y' = Ay + B$ is:

$y = y_H + y_p$, where y_H is the general solution of the associated homogeneous system $y' = Ay$, and y_p is a particular solution of the non-homogeneous system.

- **Computation of y_H:**

We have: $\lambda_1 = 1, \lambda_2 = 2, \lambda_3 = 4$ are the three distinct eigenvalues of A.

$v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 0 \\ 0 \\ 4 \end{bmatrix}$ are the eigenvectors of A associated respectively with $\lambda_1, \lambda_2, \lambda_3$. Thus:

$$\begin{aligned} y_H(t) &= c_1 v_1 e^{\lambda_1 t} + c_2 v_2 e^{\lambda_2 t} + c_3 v_3 e^{\lambda_3 t} \\ &= c_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} e^t + c_2 \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix} e^{2t} + c_3 \begin{bmatrix} 0 \\ 0 \\ 4 \end{bmatrix} e^{4t} = \begin{bmatrix} c_1 e^t \\ 2c_2 e^{2t} \\ 4c_3 e^{4t} \end{bmatrix}, \end{aligned}$$

for all $t \in \mathbb{R}$, with $c_1, c_2, c_3 \in \mathbb{R}$.

- **Computation of y_p:**

Using the method of variation of constants, there exists a particular solution of the form:

$$y_p(t) = c_1 v_1 e^{\lambda_1 t} + c_2 v_2 e^{\lambda_2 t} + c_3 v_3 e^{\lambda_3 t}, \text{ with}$$
$$\begin{bmatrix} c_1' \\ c_2' \\ c_3' \end{bmatrix} = \begin{pmatrix} v_1 e^{\lambda_1 t} & v_2 e^{\lambda_2 t} & v_3 e^{\lambda_3 t} \end{pmatrix}^{-1} \cdot B(t), \quad \forall t \in \mathbb{R}.$$

Then:

$$\begin{bmatrix} c_1' \\ c_2' \\ c_3' \end{bmatrix} = \begin{bmatrix} e^{-t} & 0 & 0 \\ 0 & 2e^{2t} & 0 \\ 0 & 0 & 4e^{4t} \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

Since

$$\begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & 0 & \ddots & \vdots \\ 0 & \cdots & 0 & \lambda_n \end{bmatrix}^{-1} = \begin{bmatrix} \lambda_1^{-1} & 0 & \cdots & 0 \\ 0 & \lambda_2^{-1} & \cdots & 0 \\ \vdots & 0 & \ddots & \vdots \\ 0 & \cdots & 0 & \lambda_n^{-1} \end{bmatrix},$$

we get

$$\begin{bmatrix} c_1' \\ c_2' \\ c_3' \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{1}{2} e^{-2t} \\ 0 \end{bmatrix}.$$

Thus $c_1' = c_3' = 0$ and $c_2' = \frac{1}{2} e^{-2t}$.

Taking integration constants suitably, one particular solution is:

$$y_p(t) = \begin{bmatrix} 0 \\ -\frac{1}{2} \\ 0 \end{bmatrix}.$$

Hence the full solution is:

$$y(t) = y_H(t) + y_p(t) = \begin{bmatrix} c_1 e^t \\ 2c_2 e^{2t} \\ 4c_3 e^{4t} \end{bmatrix} + \begin{bmatrix} 0 \\ -\frac{1}{2} \\ 0 \end{bmatrix} = \begin{bmatrix} c_1 e^t \\ 2c_2 e^{2t} - \frac{1}{2} \\ 4c_3 e^{4t} \end{bmatrix}.$$

- **Problem ($\mathbf{P}_1$):**

$$\begin{cases} y' = Ay \\ y(t_0) = y_0 \end{cases}$$

The solution of (P_1) is:

$$y(t) = e^{(t-t_0)A} \cdot y_0, \quad \forall t \in \mathbb{R}.$$

- **Problem ($\mathbf{P}_2$):**

$$\begin{cases} y' = Ay + B(t) \\ y(t_0) = y_0 \end{cases}$$

The solution of (P_2) is:

$$y(t) = e^{(t-t_0)A} \cdot y_0 + \int_{t_0}^{t} e^{(t-u)A} B(u)\, du, \quad \forall t \in I.$$

Example 3.3.6.

The solution of the system:

$$\begin{cases} y' = \begin{pmatrix} 4 & 3 \\ 0 & 4 \end{pmatrix} y \\ y(1) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \end{cases} \quad \text{is given by: } \forall t \in \mathbb{R} : y(t) = e^{(t-t_0)A} y_0.$$

With $A = \begin{pmatrix} 4 & 3 \\ 0 & 4 \end{pmatrix}$, $t_0 = 1$ and $y_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, that is:

$$\begin{aligned} y(t) &= e^{(t-1)\begin{pmatrix} 4 & 3 \\ 0 & 4 \end{pmatrix}} \\ &= e^{4(t-1)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 3(t-1) \\ 0 & 0 \end{pmatrix}} \\ &= e^{4(t-1)} \cdot e^{\begin{pmatrix} 0 & 3(t-1) \\ 0 & 0 \end{pmatrix}} \\ &= e^{4(t-1)} \left(I_2 + \begin{pmatrix} 0 & 3(t-1) \\ 0 & 0 \end{pmatrix} \right) \\ &= e^{4(t-1)} \begin{pmatrix} 1 & 3(t-1) \\ 0 & 1 \end{pmatrix}. \end{aligned}$$

$$\begin{aligned} y(t) &= \begin{pmatrix} e^{4(t-1)} & 3(t-1)e^{4(t-1)} \\ 0 & e^{4(t-1)} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} e^{4(t-1)} \\ 0 \end{pmatrix}. \end{aligned}$$

3.3.2 The solution of the homogeneous system (H)

Lemma 3.3.4.

Let $t, t_0 \in I$. Consider the function f_{t,t_0} defined from $\mathbb{R}^n$ to $\mathbb{R}^n$ by:

$$f_{t,t_0}(y_0) = y(t, t_0, y_0).$$

The mapping f_{t,t_0} is linear.

Definition 3.4.

The matrix associated with f_{t,t_0} is called the **resolvent matrix** of (H). It is denoted by:

$$R(t, t_0).$$

Theorem 3.3.2.

We have:

1. $\forall t, t_0 \in I; \quad R(t, t_0) \in M_n(\mathbb{R})$.
2. $\forall t, t_0 \in I; \quad R(t, t_0)y_0 = y(t, t_0, y_0)$.
3. $\forall t_0 \in I; \quad R(t_0, t_0) = I_n$, where I_n denotes the identity matrix.
4. $\forall t, s, r \in I; \quad R(t, s)R(s, r) = R(t, r)$.
5. $\forall t, s \in I; \quad R(t, s)$ is invertible and:

$$(R(t, s))^{-1} = R(s, t).$$

6. $\forall t, t_0 \in I; \quad \dfrac{d}{dt}R(t, t_0) = A(t) \cdot R(t, t_0)$.
7. $\forall t, t_0 \in I; \quad \dfrac{d}{dt}R(t, t_0) = -R(t_0, t)A(t)$.

The fundamental system of (H)

Let $y_1, y_2, \ldots, y_n \in F(I, \mathbb{R}^n)$.

Definition 3.5.

We say that $\{y_1, y_2, \ldots, y_n\}$ is a **fundamental system** of (H) if:

1. $y_1, y_2, \ldots, y_n$ are solutions of (H).

2. $y_1, y_2, \ldots, y_n$ are linearly independent, that is:

$$\forall \alpha_1, \ldots, \alpha_n \in \mathbb{R}, \quad (\alpha_1 y_1 + \alpha_2 y_2 + \cdots + \alpha_n y_n) = 0$$
$$\Longrightarrow \alpha_1 = \alpha_2 = \cdots = \alpha_n = 0.$$

Example 3.3.7.

For all $t \in \mathbb{R}$, let:

$$y_1(t) = \begin{bmatrix} t \\ 1 \end{bmatrix}, \quad y_2(t) = \begin{bmatrix} -1 \\ t \end{bmatrix}, \quad A(t) = \frac{1}{1+t^2}\begin{bmatrix} t & 1 \\ -1 & t \end{bmatrix}.$$

(1) Let us show that y_1 and y_2 are two solutions of $y' = A(t)y$. We have:

$$y_1'(t) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \text{ and}$$

$$A(t)y_1(t) = \frac{1}{1+t^2}\begin{bmatrix} t & 1 \\ -1 & t \end{bmatrix}\begin{bmatrix} t \\ 1 \end{bmatrix} = \frac{1}{1+t^2}\begin{bmatrix} t^2+1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

Thus:

$$y_1'(t) = A(t)y_1(t), \quad \forall t \in \mathbb{R}.$$

So y_1 is a solution of (H). Similarly, one shows that y_2 is also a solution of (H).

(2) Let us show that y_1 and y_2 are linearly independent. Suppose $\alpha, \beta \in \mathbb{R}$ such that $\alpha y_1 + \beta y_2 = 0$. Then:

$$\begin{aligned} \alpha y_1 + \beta y_2 = 0 &\Longrightarrow \alpha y_1(t) + \beta y_2(t) = 0, \quad \forall t \in \mathbb{R}, \\ &\Longrightarrow \alpha \begin{bmatrix} t \\ 1 \end{bmatrix} + \beta \begin{bmatrix} -1 \\ t \end{bmatrix} = 0, \quad \forall t \in \mathbb{R}, \\ &\Longrightarrow \begin{bmatrix} \alpha t - \beta \\ \alpha + \beta t \end{bmatrix} = 0, \quad \forall t \in \mathbb{R}. \end{aligned}$$

This represents infinitely many equations in the two unknowns α and β. It suffices to take $t = 0$, which gives $\alpha = \beta = 0$.

Hence, y_1 and y_2 are linearly independent.

Theorem 3.3.3.

Let $\{y_1, y_2, \ldots, y_n\}$ be a fundamental system of (H). Then:

$$\begin{aligned} S_H &= \Big[\{y_1, y_2, \ldots, y_n\}\Big] \\ &= \Big\{ y \in F(I, \mathbb{R}^n) \;\Big|\; y = \alpha_1 y_1 + \alpha_2 y_2 + \cdots + \alpha_n y_n,\ \alpha_1, \alpha_2, \ldots, \alpha_n \in \mathbb{R} \Big\}. \end{aligned}$$

Proof 3.3.3.

If $\{y_1, y_2, \ldots, y_n\}$ is a fundamental system of (H), then it is linearly independent. Since $\#\{y_1, y_2, \ldots, y_n\} = n = \dim S_H$, the set $\{y_1, y_2, \ldots, y_n\}$ is a basis of S_H.

Example 3.3.8.

For all $t \in \mathbb{R}$, let

$$y_1(t) = \begin{bmatrix} t \\ 1 \end{bmatrix}, \quad y_2(t) = \begin{bmatrix} -1 \\ t \end{bmatrix}, \quad A(t) = \frac{1}{1+t^2}\begin{bmatrix} t & 1 \\ -1 & t \end{bmatrix}.$$

Since $\{y_1, y_2\}$ is a fundamental system of $y' = A(t)y$, we have:

$$\begin{aligned} S_H &= [\{y_1, y_2\}] \\ &= \Big\{ y \in F(\mathbb{R}, \mathbb{R}^2) \mid y = \alpha_1 y_1 + \alpha_2 y_2,\ \alpha_1, \alpha_2 \in \mathbb{R} \Big\} \\ &= \Big\{ y \in F(\mathbb{R}, \mathbb{R}^2) \mid \forall t \in \mathbb{R},\ y = \alpha_1 y_1(t) + \alpha_2 y_2(t),\ \alpha_1, \alpha_2 \in \mathbb{R} \Big\} \\ &= \Big\{ y \in F(\mathbb{R}, \mathbb{R}^2) \mid \forall t \in \mathbb{R},\ y = \begin{bmatrix} \alpha_1 t - \alpha_2 \\ \alpha_1 + \alpha_2 t \end{bmatrix},\ \alpha_1, \alpha_2 \in \mathbb{R} \Big\}. \end{aligned}$$

Remark 3.3.1.

The general solution of (H) is:

$$y = c_1 y_1 + c_2 y_2 + \cdots + c_n y_n, \quad c_1, c_2, \ldots, c_n \in \mathbb{R}.$$

3.3.3 The fundamental matrix of (H)

Definition 3.6.

The matrix whose columns form a fundamental system of (H) is called a **fundamental matrix** of (H). That is, M is a fundamental matrix if:

$$M = (y_1, y_2, \ldots, y_n), \quad \{y_1, y_2, \ldots, y_n\} \text{ is a fundamental system of } (H).$$

Example 3.3.9.

For all $t \in \mathbb{R}$, let

$$y_1(t) = \begin{bmatrix} t \\ 1 \end{bmatrix}, \quad y_2(t) = \begin{bmatrix} -1 \\ t \end{bmatrix}, \quad A(t) = \frac{1}{1+t^2}\begin{bmatrix} t & 1 \\ -1 & t \end{bmatrix}.$$

Since $\{y_1, y_2\}$ is a fundamental system of $y' = A(t)y$ for $t \in \mathbb{R}$, we set:

$$M(t) = (y_1(t), y_2(t)) = \begin{bmatrix} t & -1 \\ 1 & t \end{bmatrix}.$$

Thus M is a fundamental matrix of $y' = A(t)y$.

Theorem 3.3.4.

Let M be a fundamental matrix of (H). Then:

1. For all $t \in \mathbb{R}$,

$$M'(t) = A(t)M(t).$$

2. The general solution of (H) is:

$$y = Mc, \quad c \in \mathbb{R}^n.$$

Proof 3.3.4.

If M is a fundamental matrix of (H), then:

1. We have:

$$\begin{aligned} M'(t) &= (y_1(t), y_2(t), \ldots, y_n(t))' \\ &= (y_1'(t), y_2'(t), \ldots, y_n'(t)) \\ &= (A(t)y_1(t), A(t)y_2(t), \ldots, A(t)y_n(t)) \\ &= A(t)(y_1(t), y_2(t), \ldots, y_n(t)) \\ &= A(t)M(t). \end{aligned}$$

2. And:

$$\begin{aligned} y(t) &= c_1y_1(t) + c_2y_2(t) + \cdots + c_ny_n(t) \\ &= (y_1(t), y_2(t), \ldots, y_n(t)) \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} \\ &= M(t)c, \quad c = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} \in \mathbb{R}^n. \end{aligned}$$

3.3.4 The Wronskian of a system of solutions of (H)

Let $y_1, y_2, \ldots, y_n \in S_H$ (the set of solutions of (H)).

Definition 3.7.

The **Wronskian** of $\{y_1, y_2, \ldots, y_n\}$, denoted W, is the determinant of the matrix whose columns are $y_1, y_2, \ldots, y_n$:

$$\forall t \in I, \quad W(t) := \det[y_1(t), y_2(t), \ldots, y_n(t)].$$

Theorem 3.3.5.

Let $y_1, y_2, \ldots, y_n \in S_H$. The following statements are equivalent:

1. $\forall t \in I, \quad W(t) \neq 0.$
2. $\exists t_0 \in I$ such that $W(t_0) \neq 0$.
3. $y_1, y_2, \ldots, y_n$ are linearly independent.

Proof 3.3.5.

1. (1) $\Rightarrow$ (2): Trivial since if $W(t) \neq 0$ for all t, then in particular at some t_0 we have $W(t_0) \neq 0$.
2. (2) $\Rightarrow$ (3): If $W(t_0) \neq 0$, then the vectors $(y_1(t_0), y_2(t_0), \ldots, y_n(t_0))$ are linearly independent.

3. (3) $\Rightarrow$ (1): Since $y_1, y_2, \dots, y_n$ are linearly independent solutions of (H), they remain linearly independent for all $t \in I$, hence $W(t) \neq 0$ for all $t \in I$.

3.3.5 The solution of the non-homogeneous system

Theorem 3.3.6.

Let $(t_0, y_0) \in I \times \mathbb{R}^n$. The solution of the system (E) is:

$$\forall t \in I, \quad y(t) = R(t, t_0)y_0 + \int_{t_0}^{t} R(t, u)B(u)\, du.$$

Proof 3.3.6.

Consider the function defined on I by:

$$Z(u) = R(t_0, u)y(u).$$

For $u \in I$, we have:

$$\begin{aligned} Z'(u) &= \frac{d}{du}\Big[R(t_0, u)y(u)\Big] \\ &= \frac{d}{du}(R(t_0, u)) \cdot y(u) + R(t_0, u)y'(u) \\ &= -R(t_0, u)A(u) \cdot y(u) + R(t_0, u)(A(u)y(u) + B(u)) \\ &= R(t_0, u)B(u). \end{aligned}$$

That is, $\forall u \in I; \quad Z'(u) = R(t_0, u)B(u)$, which implies:

$$\forall t \in I; \quad Z(t) = Z(t_0) + \int_{t_0}^{t} R(t_0, u)B(u)\, du$$

Thus,

$$\forall t \in I; \quad R(t_0, t)y(t) = R(t_0, t_0)y_0 + \int_{t_0}^{t} R(t_0, u)B(u)\, du$$

Therefore,

$$\begin{aligned} \forall t \in I; \quad y(t) &= R(t, t_0)I_n y(t_0) + \int_{t_0}^{t} R(t, t_0)R(t_0, u)B(u)\, du \\ &= R(t, t_0)y(t_0) + \int_{t_0}^{t} R(t, u)B(u)\, du. \end{aligned}$$

Hence the result.

Theorem 3.3.7.

Let $(t_0, y_0) \in I \times \mathbb{R}^n$. The solution of the system (H) is given by:

$$\forall t \in I; \quad y(t) = R(t, t_0)y_0.$$

Proof 3.3.7.

It is enough to apply the previous theorem with $B = 0$.

Computation of Y_H and Y_P for two non-homogeneous systems

1. Consider the system

$$Y' = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} Y + \begin{pmatrix} e^t \\ te^{2t} \end{pmatrix}.$$

Compute Y_H. The diagonal matrix has eigenvalues $\lambda_1 = 1,\ \lambda_2 = 2$ with eigenvectors

$$V_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad V_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Hence the homogeneous solution is

$$Y_H(t) = c_1 V_1 e^{\lambda_1 t} + c_2 V_2 e^{\lambda_2 t} = \begin{pmatrix} c_1 e^t \\ c_2 e^{2t} \end{pmatrix}, \qquad c_1, c_2 \in \mathbb{R}.$$

Compute Y_P. Using variation of constants, seek a particular solution of the form

$$Y_P(t) = c_1(t) V_1 e^{\lambda_1 t} + c_2(t) V_2 e^{\lambda_2 t},$$

with c_1, c_2 differentiable functions satisfying

$$\begin{pmatrix} c_1'(t) \\ c_2'(t) \end{pmatrix} = \left(V_1 e^{\lambda_1 t}\ V_2 e^{\lambda_2 t}\right)^{-1} \begin{pmatrix} e^t \\ te^{2t} \end{pmatrix} = \begin{pmatrix} e^{-t} & 0 \\ 0 & e^{-2t} \end{pmatrix} \begin{pmatrix} e^t \\ te^{2t} \end{pmatrix} = \begin{pmatrix} 1 \\ t \end{pmatrix}.$$

Thus $c_1'(t) = 1$ and $c_2'(t) = t$, so one antiderivative choice is

$$c_1(t) = t, \qquad c_2(t) = \tfrac{1}{2}t^2.$$

Then

$$Y_P(t) = t\begin{pmatrix}1\\0\end{pmatrix}e^t + \tfrac{1}{2}t^2\begin{pmatrix}0\\1\end{pmatrix}e^{2t} = \begin{pmatrix}te^t\\ \tfrac{1}{2}t^2e^{2t}\end{pmatrix}.$$

Therefore the general solution is

$$\boxed{Y(t) = Y_H(t) + Y_P(t) = \begin{pmatrix}(c_1+t)e^t\\ (c_2+\tfrac{1}{2}t^2)e^{2t}\end{pmatrix}, \qquad c_1, c_2 \in \mathbb{R}.}$$

2. Consider the initial-value problem

$$\begin{cases} Y' = \begin{pmatrix}1 & 1 & 0\\ 0 & 1 & 1\\ 0 & 0 & 1\end{pmatrix} Y + \begin{pmatrix}t\\1\\0\end{pmatrix}, \\ Y(0) = Y_0. \end{cases}$$

The variation-of-constants formula gives

$$Y(t) = e^{tA}Y_0 + \int_0^t e^{(t-u)A}B(u)\,du, \quad A = \begin{pmatrix}1 & 1 & 0\\ 0 & 1 & 1\\ 0 & 0 & 1\end{pmatrix}, \quad B(u) = \begin{pmatrix}u\\1\\0\end{pmatrix}.$$

To compute e^{tA}, write $A = I_3 + S$ where

$$S = \begin{pmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ 0 & 0 & 0\end{pmatrix}, \qquad S^3 = 0.$$

Then

$$e^{tA} = e^{t(I_3+S)} = e^te^{tS} = e^t\left(I_3 + tS + \tfrac{t^2}{2}S^2\right) = e^t\begin{pmatrix}1 & t & \tfrac{1}{2}t^2\\ 0 & 1 & t\\ 0 & 0 & 1\end{pmatrix}.$$

Hence

$$Y(t) = e^t \begin{pmatrix} 1 & t & \frac{1}{2}t^2 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix} Y_0 + \int_0^t e^{(t-u)} \begin{pmatrix} 1 & t-u & \frac{1}{2}(t-u)^2 \\ 0 & 1 & t-u \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} u \\ 1 \\ 0 \end{pmatrix} du.$$

One may evaluate the integral explicitly componentwise if a closed-form particular solution is desired; otherwise the integral representation above is the standard variation-of-constants solution. **Let us compute** Y_H : we have $\lambda_1 = 1$ and $\lambda_2 = 2$ which are the two distinct eigenvalues of $\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$. $V_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $V_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ are the eigenvectors. Thus:

$$\begin{aligned} Y_H(t) &= c_1 V_1 e^{\lambda_1 t} + c_2 V_2 e^{\lambda_2 t} \\ &= c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^t + c_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{2t} \\ &= \begin{pmatrix} c_1 e^t \\ c_2 e^{2t} \end{pmatrix}, \text{ with } c_1, c_2 \in \mathbb{R}. \end{aligned}$$

Let us compute Y_P : we use the method of variation of constants, there exists a particular solution of the form:

$$Y_P(t) = c_1(t) V_1 e^{\lambda_1 t} + c_2(t) V_2 e^{\lambda_2 t}$$

where $c_1(t), c_2(t)$ are two differentiable functions such that:

$$\begin{pmatrix} c_1' \\ c_2' \end{pmatrix} = \left(V_1 e^{\lambda_1 t} \; V_2 e^{\lambda_2 t} \right)^{-1} \cdot B(t) = \begin{pmatrix} e^t & 0 \\ 0 & e^{2t} \end{pmatrix}^{-1} \cdot \begin{pmatrix} e^t \\ te^{2t} \end{pmatrix} = \begin{pmatrix} e^{-t} & 0 \\ 0 & e^{-2t} \end{pmatrix} \begin{pmatrix} e^t \\ te^{2t} \end{pmatrix}$$

$$\begin{pmatrix} c_1' \\ c_2' \end{pmatrix} = \begin{pmatrix} 1 \\ t \end{pmatrix} \Longrightarrow \begin{cases} c_1(t) = t \\ c_2(t) = \dfrac{1}{2} t^2 \end{cases}. \text{ Thus}$$

$$
\begin{aligned}
Y_P(t) &= t\begin{pmatrix}1\\0\end{pmatrix}e^t+\frac{1}{2}t^2\begin{pmatrix}0\\1\end{pmatrix}e^{2t}\\
&= \begin{pmatrix}te^t\\0\end{pmatrix}+\begin{pmatrix}0\\ \frac{1}{2}t^2e^{2t}\end{pmatrix}\\
&= \begin{pmatrix}te^t\\ \frac{1}{2}t^2e^{2t}\end{pmatrix}
\end{aligned}
$$

Hence the general solution:

$$
Y(t)=\begin{pmatrix}c_1e^t\\c_2e^{2t}\end{pmatrix}+\begin{pmatrix}te^t\\ \frac{1}{2}t^2e^{2t}\end{pmatrix}.
$$

$$
\begin{cases}
Y'=\begin{pmatrix}1&1&0\\0&1&1\\0&0&1\end{pmatrix}Y+\begin{pmatrix}t\\1\\0\end{pmatrix}\\
Y(0)=Y_0.
\end{cases}
$$

We have the general solution:

$$
Y(t)=e^{tA}Y_0+\int_0^t e^{(t-u)A}\cdot B(u)\,du.
$$

Let us compute

$$
e^{tA}=e^{\begin{pmatrix}t&t&0\\0&t&t\\0&0&t\end{pmatrix}}=e^{tI_3+\begin{pmatrix}0&t&0\\0&0&t\\0&0&0\end{pmatrix}}=e^te^{\begin{pmatrix}0&t&0\\0&0&t\\0&0&0\end{pmatrix}}
$$

Let $N = \begin{pmatrix} 0 & t & 0 \\ 0 & 0 & t \\ 0 & 0 & 0 \end{pmatrix}$, which is a nilpotent matrix of order 3. We have:

$$\begin{aligned} e^N &= I_3 + \frac{N}{1!} + \frac{N^2}{2!} \\ &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & t & 0 \\ 0 & 0 & t \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & \frac{1}{2}t^2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 1 & t & \frac{1}{2}t^2 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix}. \end{aligned}$$

Therefore:

$$e^{tA} = e^t \begin{pmatrix} 1 & t & \frac{1}{2}t^2 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix}.$$

$$\begin{aligned} Y(t) &= e^{tA}Y_0 + \int_0^t e^{(t-u)A}B(u)\,du \\ &= e^t \begin{pmatrix} 1 & t & \frac{1}{2}t^2 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix} Y_0 + \int_0^t e^{(t-u)\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}} B(u)\,du \\ &= e^t \begin{pmatrix} 1 & t & \frac{1}{2}t^2 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix} Y_0 + \int_0^t e^{(t-u)} \begin{pmatrix} 1 & (t-u) & \frac{1}{2}(t-u)^2 \\ 0 & 1 & (t-u) \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} u \\ 1 \\ 0 \end{pmatrix} du. \end{aligned}$$

3.4 Solved Exercise

Exercise 3.4.1. [See the solution 3.4.1]

Compute the exponential of the following matrices:

$$A = \begin{pmatrix} 3 & 0 \\ 0 & 4 \end{pmatrix}, \quad B = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}, \quad C = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 4 & 3 \end{pmatrix}$$

Exercise 3.4.2. [See the solution 3.4.2]

Solve the following systems:

1.
$$\begin{cases} y_1' = y_1 + y_2, \\ y_2' = y_1 + y_2. \end{cases}$$

2.
$$Y' = \begin{pmatrix} 1 & 1 \\ 2 & 0 \end{pmatrix} Y.$$

3.
$$Y' = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} Y.$$

4.
$$\begin{cases} Y' = \begin{pmatrix} 0 & 4 \\ 1 & 0 \end{pmatrix} Y, \\ Y(t_0) = Y_0. \end{cases}$$

Exercise 3.4.3. [See the solution 3.4.3]

Résoudre les systèmes suivants :

1.
$$Y' = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} Y + \begin{pmatrix} e^{2t} \\ e^{t} \end{pmatrix}.$$

2.
$$\begin{cases} Y' = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} Y \\ Y(0) = Y_0. \end{cases}$$

3.
$$\begin{cases} Y' = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} Y + \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ Y(t_0) = Y_0. \end{cases}$$

4.
$$Y' = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} Y + \begin{pmatrix} e^t \\ te^{2t} \end{pmatrix}.$$

5.
$$Y' = \begin{pmatrix} 2 & 3 \\ 0 & 2 \end{pmatrix} Y + \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

6.
$$\begin{cases} Y' = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} Y + \begin{pmatrix} t \\ 1 \\ 0 \end{pmatrix} \\ Y(0) = Y_0. \end{cases}$$

3.4.1 Solutions

Solution of exercise 3.4.1. [See the exercise 3.4.1]

1. On a :
$$e^A = \begin{pmatrix} e^3 & 0 \\ 0 & e^4 \end{pmatrix} \quad \text{car} \quad A \text{ est une matrice diagonale.}$$

2. • we have :

$$e^{B}=e^{2I_2+\begin{pmatrix}0&4\\0&0\end{pmatrix}}=e^{2}\cdot e^{\begin{pmatrix}0&4\\0&0\end{pmatrix}}=e^{2}\left[\begin{pmatrix}1&0\\0&1\end{pmatrix}+\begin{pmatrix}0&4\\0&0\end{pmatrix}\right]$$
$$=e^{2}\cdot\begin{pmatrix}1&4\\0&1\end{pmatrix}$$
$$=\begin{pmatrix}e^{2}&4e^{2}\\0&e^{2}\end{pmatrix}$$

Hence $\begin{pmatrix}0&4\\0&0\end{pmatrix}$ It is an upper triangular matrix, therefore it is nilpotent

3. **Method 01:** $e^{\begin{pmatrix}2&0&0\\0&1&0\\0&4&3\end{pmatrix}}$, we have :

$$\det(C-\lambda I_3)=\det\begin{pmatrix}2-\lambda&0&0\\0&1-\lambda&0\\0&4&3-\lambda\end{pmatrix}=(2-\lambda)(1-\lambda)(3-\lambda).$$

$\lambda_1=2,\ \lambda_2=1$ et $\lambda_3=3$ are the three eigenvalues of the matrix C, then:

$$C=P\cdot D\cdot P^{-1}\text{ here }D=\begin{pmatrix}2&0&0\\0&1&0\\0&0&3\end{pmatrix}$$

Let us compute P, and since V_1,V_2,V_3 are the eigenvectors of C associated respectively with λ_1,λ_2 and λ_3, then:

$$CV_1=\lambda_1V_1,\ CV_2=\lambda_2V_2,\ CV_3=\lambda_3V_3.$$

- for $\lambda_1 = 2$:

$$(C-2I_3)V_1 = 0_{\mathbb{R}^3} \Longrightarrow \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 4 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Longrightarrow \begin{cases} y_1 = 0 \\ 4y_1 + z_1 = 0 \end{cases} \Longrightarrow \begin{cases} y_1 = 0 \\ z_1 = 0 \end{cases} \Longrightarrow \begin{pmatrix} x_1 \in \mathbb{R} \\ y_1 = 0 \\ z_1 = 0 \end{pmatrix} = x_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \ x_1 \in \mathbb{R}.$$

- for $\lambda_2 = 1$:

$$(C-I_3)V_2 = 0_{\mathbb{R}^3} \Longrightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 4 & 2 \end{pmatrix} \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Longrightarrow \begin{cases} x_2 = 0 \\ 4y_2 + 2z_2 = 0 \end{cases} \Longrightarrow \begin{cases} x_2 = 0 \\ z_2 = -2y_2 \end{cases} \Longrightarrow \begin{pmatrix} 0 \\ y_2 \\ -2y_2 \end{pmatrix} = y_2 \begin{pmatrix} 0 \\ 1 \\ -2 \end{pmatrix}, \ y_2 \in \mathbb{R}.$$

- for $\lambda_3 = 3$:

$$(C-3I_3)V_3 = 0_{\mathbb{R}^3} \Longrightarrow (C-3I_3) \begin{pmatrix} x_3 \\ y_3 \\ z_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Longrightarrow \begin{pmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 4 & 0 \end{pmatrix} \begin{pmatrix} x_3 \\ y_3 \\ z_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Longrightarrow \begin{cases} -x_3 = 0 \\ -2y_3 = 0 \\ 4y_3 = 0 \end{cases} \Longrightarrow \begin{pmatrix} x_3 \\ y_3 \\ z_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ z_3 \end{pmatrix} = z_3 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \ z_3 \in \mathbb{R}.$$

Therefore $P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{pmatrix}$ Let us calculate P^{-1}:

$$P^{-1} = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & k \end{pmatrix} \Longrightarrow P \cdot P^{-1} = I_3 \text{ alors :}$$

$$\begin{pmatrix}1&0&0\\0&1&0\\0&-2&1\end{pmatrix}=\begin{pmatrix}a&b&c\\d&e&f\\g&h&k\end{pmatrix}=\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}\Longrightarrow\begin{cases}a=1\\b=0\\c=0\\d=0\\e=1\\f=0\end{cases}\Longrightarrow\begin{cases}-2d+g=0\\-2e+h=0\\-2f+k=1\end{cases}$$

$$\Longrightarrow\begin{cases}g=0\\h=2\\k=1\end{cases}$$

such that :$-2d+g-2e+h-2f+k$.

$$P^{-1}=\begin{pmatrix}1&0&0\\0&1&0\\0&2&1\end{pmatrix}\text{ donc :}$$

$$\begin{aligned}e^C=e^{PDP^{-1}}&=P\cdot e^D\cdot P^{-1}\\&=\begin{pmatrix}1&0&0\\0&1&0\\0&-2&1\end{pmatrix}\begin{pmatrix}e^2&0&0\\0&e^1&0\\0&0&e^3\end{pmatrix}\begin{pmatrix}1&0&0\\0&1&0\\0&2&1\end{pmatrix}\\&=\begin{pmatrix}e^2&0&0\\0&e&0\\0&-2e&e^3\end{pmatrix}\begin{pmatrix}1&0&0\\0&1&0\\0&2&1\end{pmatrix}\\&=\begin{pmatrix}e^2&0&0\\0&e&0\\0&-2e+2e^3&e^3\end{pmatrix}.\end{aligned}$$

- **Method 2:** we have :

$C = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 4 & 3 \end{pmatrix} = \begin{pmatrix} C_1 & 0 \\ 0 & C_2 \end{pmatrix}$ is a block diagonal matrix with

$$C_1 = [2],\ C_2 = \begin{pmatrix} 1 & 0 \\ 4 & 3 \end{pmatrix},\ \text{alors :}\ \ e^C = \begin{pmatrix} e^2 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & e^{C_2} \end{pmatrix}$$

Let us calculate e^{C_2}. we have

$$\det(e^{C_2} - \lambda I_2) = \det \begin{pmatrix} 1-\lambda & 0 \\ 4 & 3-\lambda \end{pmatrix}$$

The factors $(1-\lambda)$ and $(3-\lambda)$ imply that $\lambda_1 = 1$ and $\lambda_2 = 3$ are the two distinct eigenvalues of C_2. Hence

$$C_2 = P\,D\,P^{-1} \qquad \text{with} \qquad D = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}.$$

Compute P:

V_1, V_2 are the distinct eigenvectors of C_2 associated respectively with λ_1 and λ_2.

Solution of exercise 3.4.2. [See the exercise3.4.2]

Solve the following systems

1.
$$\begin{cases} y_1' = y_1 + y_2, \\ y_2' = y_1 + y_2, \end{cases} \qquad \text{or} \qquad y' = Ay, \quad A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

Characteristic polynomial:

$$\det(A - \lambda I) = (1-\lambda)^2 - 1 = \lambda(\lambda - 2) = 0,$$

so $\lambda_1 = 0,\ \lambda_2 = 2$.

Eigenvectors:

$$\lambda_1 = 0: \quad (A - 0I)v = 0 \Rightarrow v_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \qquad \lambda_2 = 2: \quad (A - 2I)v = 0 \Rightarrow v_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

General solution:

$$\boxed{y(t) = c_1 v_1 e^{0\cdot t} + c_2 v_2 e^{2t} = c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{2t}, \quad c_1, c_2 \in \mathbb{R}.}$$

2.

$$Y' = A_1 Y, \qquad A_1 = \begin{pmatrix} 1 & 1 \\ 2 & 0 \end{pmatrix}.$$

Characteristic polynomial:

$$\det(A_1 - \lambda I) = -\lambda(1 - \lambda) - 2 = \lambda^2 + \lambda - 2 = (\lambda + 1)(\lambda - 2),$$

so $\lambda_1 = -1, \ \lambda_2 = 2$.

Eigenvectors:

$$\lambda_1 = -1: \ (A_1 + I)v = 0 \Rightarrow v_1 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}, \qquad \lambda_2 = 2: \ (A_1 - 2I)v = 0 \Rightarrow v_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

General solution:

$$\boxed{Y(t) = c_1 v_1 e^{-t} + c_2 v_2 e^{2t} = \begin{pmatrix} c_1 e^{-t} + c_2 e^{2t} \\ -2c_1 e^{-t} + c_2 e^{2t} \end{pmatrix}, \quad c_1, c_2 \in \mathbb{R}.}$$

(Alternatively, using the matrix-exponential method: if $A_1 = PDP^{-1}$ with $D = \operatorname{diag}(-1, 2)$ and $P = (v_1 \ v_2) = \begin{pmatrix} 1 & 1 \\ -2 & 1 \end{pmatrix}$, then $e^{tA_1} = Pe^{tD}P^{-1}$; one obtains the same solution.)

3.

$$Y' = A_2 Y, \qquad A_2 = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}.$$

Characteristic polynomial:

$$\det(A_2 - \lambda I) = (2-\lambda)^2 - 1 = (\lambda-1)(\lambda-3),$$

so $\lambda_1 = 1,\ \lambda_2 = 3.$

Eigenvectors:

$$\lambda_1 = 1:\ v_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \qquad \lambda_2 = 3:\ v_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

General solution:

$$\boxed{Y(t) = \alpha_1 v_1 e^t + \alpha_2 v_2 e^{3t} = \begin{pmatrix} \alpha_1 e^t + \alpha_2 e^{3t} \\ -\alpha_1 e^t + \alpha_2 e^{3t} \end{pmatrix},\ \alpha_1, \alpha_2 \in \mathbb{R}.}$$

4.

$$\begin{cases} Y' = A_3 Y, \\ Y(t_0) = Y_0, \end{cases} \qquad A_3 = \begin{pmatrix} 0 & 4 \\ 1 & 0 \end{pmatrix}.$$

Characteristic polynomial:

$$\det(A_3 - \lambda I) = \lambda^2 - 4 = (\lambda-2)(\lambda+2),$$

so $\lambda_1 = 2,\ \lambda_2 = -2.$

Eigenvectors:

$$\lambda_1 = 2:\ (A_3 - 2I)v = 0 \Rightarrow v_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \qquad \lambda_2 = -2:\ (A_3 + 2I)v = 0 \Rightarrow v_2 = \begin{pmatrix} -2 \\ 1 \end{pmatrix}.$$

Take $P = (v_1\ v_2) = \begin{pmatrix} 2 & -2 \\ 1 & 1 \end{pmatrix}$, $\det P = 4$, and $P^{-1} = \frac{1}{4}\begin{pmatrix} 1 & 2 \\ -1 & 2 \end{pmatrix}$. Then

$$e^{(t-t_0)A_3} = P \begin{pmatrix} e^{2(t-t_0)} & 0 \\ 0 & e^{-2(t-t_0)} \end{pmatrix} P^{-1}.$$

A convenient closed form (equivalent to the above) uses hyperbolic functions, since

$A_3^2 = 4I$:

$$\boxed{e^{(t-t_0)A_3} = \begin{pmatrix} \cosh(2(t-t_0)) & 2\sinh(2(t-t_0)) \\ \frac{1}{2}\sinh(2(t-t_0)) & \cosh(2(t-t_0)) \end{pmatrix},}$$

so the solution with initial condition is

$$\boxed{Y(t) = e^{(t-t_0)A_3}\,Y_0, \qquad \forall t \in \mathbb{R}.}$$

Solution of exercise 3.4.3. [See Exercise 3.4.3]

Solve the following systems:

1.
$$Y' = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} Y + \begin{pmatrix} e^{2t} \\ e^{t} \end{pmatrix}.$$

The general solution is $Y_H + Y_P$.

Compute Y_H: The eigenvalues are $\lambda_1 = 2,\ \lambda_2 = 1$ with eigenvectors $V_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $V_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Thus

$$Y_H(t) = c_1 V_1 e^{\lambda_1 t} + c_2 V_2 e^{\lambda_2 t} = c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{2t} + c_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{t}, \qquad c_1, c_2 \in \mathbb{R}.$$

Compute Y_P: Use variation of constants. Seek a particular solution in the form

$$Y_P(t) = c_1(t) V_1 e^{\lambda_1 t} + c_2(t) V_2 e^{\lambda_2 t},$$

where c_1, c_2 are differentiable functions satisfying

$$\begin{pmatrix} c_1'(t) \\ c_2'(t) \end{pmatrix} = \left(V_1 e^{\lambda_1 t}\ V_2 e^{\lambda_2 t}\right)^{-1} \begin{pmatrix} e^{2t} \\ e^{t} \end{pmatrix}.$$

Here

$$\left(V_1 e^{\lambda_1 t}\ V_2 e^{\lambda_2 t}\right) = \begin{pmatrix} e^{2t} & 0 \\ 0 & e^{t} \end{pmatrix},$$

so
$$\begin{pmatrix} c_1' \\ c_2' \end{pmatrix} = \begin{pmatrix} e^{-2t} & 0 \\ 0 & e^{-t} \end{pmatrix} \begin{pmatrix} e^{2t} \\ e^{t} \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Thus $c_1'(t) = c_2'(t) = 1$, hence $c_1(t) = t + c_{1,0}$, $c_2(t) = t + c_{2,0}$. Taking one convenient particular choice $c_1(t) = c_2(t) = t$ gives

$$Y_P(t) = t \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{2t} + t \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{t} = \begin{pmatrix} te^{2t} \\ te^{t} \end{pmatrix}.$$

Therefore
$$\boxed{Y(t) = Y_H(t) + Y_P(t) = \begin{pmatrix} (c_1 + t)e^{2t} \\ (c_2 + t)e^{t} \end{pmatrix}, \quad c_1, c_2 \in \mathbb{R}.}$$

2.
$$\begin{cases} Y' = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} Y + \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \\ Y(t_0) = Y_0. \end{cases}$$

The variation-of-constants formula gives

$$Y(t) = e^{(t-t_0)A} Y_0 + \int_{t_0}^{t} e^{(t-u)A} B(u)\, du,$$

with $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $B(u) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

Since $\begin{pmatrix} 0 & t - t_0 \\ 0 & 0 \end{pmatrix}$ is nilpotent, its exponential is

$$e^{(t-t_0)A} = e^{\begin{pmatrix} 0 & t - t_0 \\ 0 & 0 \end{pmatrix}} = I_2 + \begin{pmatrix} 0 & t - t_0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & t - t_0 \\ 0 & 1 \end{pmatrix}.$$

Therefore

$$Y(t) = \begin{pmatrix} 1 & t-t_0 \\ 0 & 1 \end{pmatrix} Y_0 + \int_{t_0}^{t} \begin{pmatrix} 1 & t-u \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} du$$

$$= \begin{pmatrix} 1 & t-t_0 \\ 0 & 1 \end{pmatrix} Y_0 + \begin{pmatrix} \displaystyle\int_{t_0}^{t} 1\, du \\ \displaystyle\int_{t_0}^{t} 0\, du \end{pmatrix} = \begin{pmatrix} 1 & t-t_0 \\ 0 & 1 \end{pmatrix} Y_0 + \begin{pmatrix} t-t_0 \\ 0 \end{pmatrix}.$$

3.

$$Y' = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} Y + \begin{pmatrix} e^t \\ te^{2t} \end{pmatrix}.$$

The general solution is $Y(t) = Y_H(t) + Y_P(t)$, where $Y_H(t)$ is the homogeneous solution

$$Y_H(t) = c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^t + c_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{2t},$$

and $Y_P(t)$ may be found by variation of constants or by solving two scalar non-homogeneous equations separately for each component.

Chapter 4

Introduction to notions of stability

4.0 Introduction

4.0 Autonomous Systems

In this chapter, we first recall the different definitions of stability, then we present the two Lyapunov methods, which are fundamental tools for determining the stability of dynamical systems described by ordinary differential equations.[19]

The nonlinear system given by relation (2.1) is said to be autonomous (or time-invariant) if f does not explicitly depend on time, that is:

$$\dot{x} = f(x). \tag{4.1}$$

Otherwise, the system is called non-autonomous (or time-varying).

In this section, we briefly review the results of Lyapunov theory for autonomous systems.

Theorem 2. Every non-autonomous system is equivalent to an autonomous system.

Proof. Consider the non-autonomous system:

$$\dot{x} = f(x, t).$$

We define the following system:

$$\begin{cases} \dot{x} = f(x, t), \\ \dot{t} = 1. \end{cases}$$

Let

$$y = \begin{pmatrix} x \\ t \end{pmatrix}, \quad \dot{y} = \begin{pmatrix} \dot{x} \\ \dot{t} \end{pmatrix}.$$

Then,

$$\dot{y} = \begin{pmatrix} f(x,t) \\ 1 \end{pmatrix} = \begin{pmatrix} F_1(y) \\ F_2(y) \end{pmatrix} = F(y),$$

which is autonomous. □

4.0.1 Definitions

Consider a finite-dimensional continuous system described by a nonlinear first-order differential equation:

$$\dot{x} = f(x), \qquad x \in \mathbb{R}^n.$$

[19]

Definition 1 (Equilibrium)**.** A state x_e is called an equilibrium point of the autonomous system (4.1) if

$$f(x_e) = 0.$$

Remark 1. Every equilibrium point can be shifted to the origin by a simple change of variables $x \mapsto x - x_e$. Therefore, without loss of generality, the following definitions and theorems will be established for the case $x_e = 0$.

Proof. Consider the two systems:

$$\dot{x} = f(x), \tag{4.2}$$

$$\dot{y} = f(y + a). \tag{4.3}$$

If $x_e = a$ is an equilibrium point of system (4.2), then $y_e = 0$ is an equilibrium point of system (4.3).

Indeed,

$$y(t) = y(t_0) + \int_{t_0}^{t} f(y(s) + a)\, ds,$$
$$y(t) = x(t_0) - a + \int_{t_0}^{t} f(x(s))\, ds,$$
$$y(t) + a = x(t_0) + \int_{t_0}^{t} f(x(s))\, ds.$$

Thus, $x(t)$ is a solution of (4.2) if and only if $x(t) - a$ is a solution of (4.3). In other words, $x_e = a$ is a stable equilibrium of (4.2) if and only if $y(t) \equiv 0$ is a stable equilibrium of (4.3). □

Definition 2 (Stability)**.** [19] The equilibrium point $x_e = 0$ is said to be stable if, for every $\varepsilon > 0$, there exists $\eta(\varepsilon, t_0) > 0$ such that if $\|x(t_0)\| < \eta$, then

$$\|x(t)\| < \varepsilon, \qquad \forall t > t_0.$$

Otherwise, the equilibrium point is unstable [**?**].

Graphically, the stability of x_e means that the trajectory $x(t)$ in the state space remains inside the ball $B(x_e, \varepsilon)$ if its initial point belongs to a ball $B(x_e, \eta)$.

Equivalently, we can reformulate this definition as:

$$\forall \varepsilon > 0,\ \exists\, \eta(\varepsilon, t_0) > 0 \quad \text{such that} \quad x(t_0) \in B_\eta \implies x(t) \in B_\varepsilon, \quad \forall t > t_0.$$

The domain B_η is called the domain of attraction of the equilibrium state. Stability can be represented as shown in Figure (2.1).

Definition 3 (Asymptotic Stability)**.** [19] The equilibrium point $x_e = 0$ is asymptotically stable if:

1. It is stable.
2. And if η can be chosen such that:

$$\|x(t_0)\| < \eta \ \Rightarrow \lim_{t \to +\infty} x(t) = 0$$

It can be represented schematically as shown in Figure (2.1).

It should be noted that the second previous condition does not imply the stability of the equilibrium point.
Asymptotic stability includes the property of stability, but specifies in addition that any trajectory initialized in the ball $B(x_e(t_0), \eta)$ converges towards x_e [3].

Definition 4. (*Marginal stability*)
An equilibrium point which is stable but not asymptotically stable is said to be marginally stable.

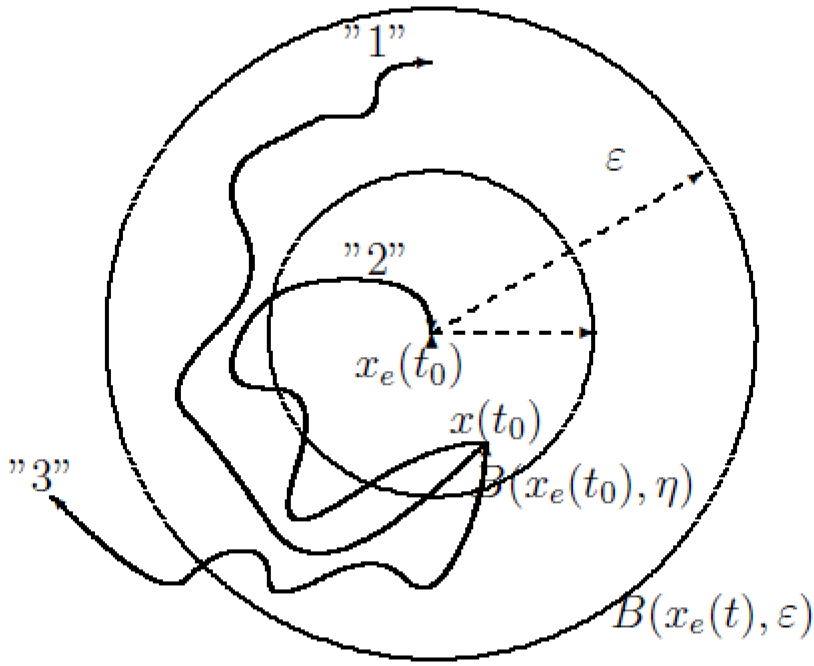


Figure 4.1: Illustration of Stability.

Definition 5. (*Exponential stability*)

An equilibrium point is said to be exponentially stable if there exist two strictly positive numbers α and λ, independent of time and initial conditions, such that:

$$\|x(t)\| \leq \alpha \|x(t_0)\| e^{-\lambda t}, \quad \forall t > t_0 \tag{4.4}$$

for $x(t_0) \in B_\eta$. The scalar λ represents the rate of convergence of the solution $x(t)$.

Remark 2. Exponential stability implies asymptotic stability, the converse is not true; but for time-invariant systems given in the form $\dot{x} = Ax$, asymptotic stability implies exponential stability [7].

Definition 6. (*Uniform stability*)

The equilibrium point $x_e = 0$ is uniformly stable if it is stable with $\eta = \eta(\varepsilon)$ chosen independently of time t_0.

Definition 7. (*Uniform asymptotic stability*)

The equilibrium point $x_e = 0$ is uniformly asymptotically stable if it is uniformly stable and there exists an attraction ball B, independent of t_0, such that $x(t_0) \in B$ implies $x(t) \to 0$ when $t \to \infty$.

Remark 3. Uniform stability implies stability, the converse is not true; but for invariant systems given in the form $\dot{x} = f(x)$, the stability of constant solutions implies uniform stability.

Proof:

Let $y(t) = a$ be a constant stable solution of the autonomous system given by relation (2.2), then

$$\forall \varepsilon > 0, \exists \eta(\varepsilon, t_0) > 0 \text{ such that for all } x(t) \text{ solution of } \dot{x} = f(x) :$$
$$\|x(t_0) - a\| < \eta \Rightarrow \|x(t) - a\| < \varepsilon, \forall t \geq t_0$$

We show that

$$\forall \varepsilon > 0, \exists \eta(\varepsilon) > 0 \text{ such that for all } x(t) \text{ solution of } \dot{x} = f(x) :$$
$$\|x(t_1) - a\| < \eta \Rightarrow \|x(t) - a\| < \varepsilon, \forall t \geq t_1$$

We know that if $x(t)$ is a solution of an autonomous system (2.2), then $x(t+T)$ is also a solution of (2.2). Hence, from the stability of the solution $y(t) = a$, we have:

$$\|x(t_0 + T) - a\| < \eta \Rightarrow \|x(t+T) - a\| < \varepsilon, \forall t \geq t_0$$

which is equivalent to:

$$\|x(t_1) - a\| < \eta \Rightarrow \|x(t) - a\| < \varepsilon, \forall t \geq t_1 \text{ with } t_1 = t_0 + T$$

This completes the proof.

Disadvantages of these definitions

The definitions of stability present some important disadvantages:

• It is necessary to be able to explicitly compute each solution corresponding to each initial condition.

• The manipulation of these definitions is tedious.

Consequently, results that allow determining stability without having to integrate the system equations would be welcome.

4.0.2 Results on Homogeneous Linear Systems

Fundamental Theorem:

We consider the following homogeneous linear system:

$$\dot{x} = A(t)x, \qquad x \in \mathbb{R}^n$$

The fundamental phenomenon is that all the solutions are of the same nature for homogeneous linear systems.

One can therefore speak of stable or unstable systems [8].

Theorem 3. Let $\quad \dot{x} = A(t)x.$

From the stability point of view, all the solutions of $\dot{x} = A(t)x$ are of the same nature (stable, uniformly stable, uniformly asymptotically stable, $\cdots$).

It is therefore enough to know the nature of the zero solution $x(t) \equiv 0$, $\forall t > 0$.

Proof:

Let $x(t)$, $y(t)$ be two solutions of $\dot{x} = A(t)x$.

Suppose $x(t)$ is stable, we show that $y(t)$ is also stable.

If $x(t)$ is stable, then

$$\forall \varepsilon > 0, \exists \eta(\varepsilon, t_0) > 0 \text{ such that for all } \bar{x}(t) \text{ solution of } \dot{x} = A(t)x :$$
$$\|x(t_0) - \bar{x}(t_0)\| < \eta \Rightarrow \|x(t) - \bar{x}(t)\| < \varepsilon, \forall t \geq t_0$$

But

$$\begin{aligned}\|x(t) - \bar{x}(t)\| &= \|R(t, t_0)x(t_0) - R(t, t_0)\bar{x}(t_0)\| \\ &= \|R(t, t_0)(x(t_0) - \bar{x}(t_0))\|\end{aligned}$$

where $R(t, t_0)$ is the resolvent of the equation $\dot{x} = A(t)x$.

$y(t)$ is stable if:

$$\forall \varepsilon > 0, \exists \eta(\varepsilon, t_0) > 0 \text{ such that for all } \bar{y}(t) \text{ solution of } \dot{x} = A(t)x :$$
$$\|y(t_0) - \bar{y}(t_0)\| < \eta \Rightarrow \|y(t) - \bar{y}(t)\| < \varepsilon, \forall t \geq t_0$$

We have:

$$\begin{aligned}\|y(t)-\bar{y}(t)\| &= \|R(t,t_0)y(t_0)-R(t,t_0)\bar{y}(t_0)\|\\ &= \|R(t,t_0)(y(t_0)-\bar{y}(t_0))\|\end{aligned}$$

and thanks to the linearity and continuity of the resolvent, $y(t)$ is stable.

Theorem 4. The system $\dot{x}=A(t)x$ is stable for $t\geq t_0$ if and only if:

$$\exists K(t_0) \text{ such that } \quad \|R(t,t_0)\|\leq K, \forall t\geq t_0$$

Proof:

Suppose that $\|R(t,t_0)\|\leq K$, $\forall t\geq t_0$, (which implies that all solutions are bounded), and we show that $x\equiv 0$ is stable, i.e., we show:

$$\begin{aligned}&\forall\varepsilon>0, \exists\eta(\varepsilon,t_0)>0 \text{ such that for all } \bar{x}(t) \text{ solution of } \dot{x}=A(t)x:\\ &\qquad\qquad \|\bar{x}(t_0)\|<\eta \Rightarrow \|R(t,t_0)\bar{x}(t_0)\|<\varepsilon, \forall t\geq t_0\end{aligned}$$

Let ε be arbitrary. From the previous hypothesis, we have:

$$\|R(t,t_0)\bar{x}(t_0)\|<K\,\|\bar{x}(t_0)\|$$

thus, if $\|\bar{x}(t_0)\|\leq\dfrac{\varepsilon}{K}$, we obtain $\|R(t,t_0)\bar{x}(t_0)\|\leq\varepsilon$. Therefore, it suffices to take $\eta=\dfrac{\varepsilon}{K}$.

Conversely: suppose $x\equiv 0$ is stable, and we show that $\|R(t,t_0)\|$ is bounded. Indeed:

$$\begin{aligned}\|R(t,t_0)\| &= \sup_{\|x_0\|\leq 1}\|R(t,t_0)x_0\|\\ &= \sup_{\|z\|\leq\eta}\left\|R(t,t_0)\frac{1}{\eta}z\right\|\\ &= \frac{1}{\eta}\sup_{\|z\|\leq\eta}\|R(t,t_0)z\|\end{aligned}$$

And from the stability of the zero solution:

$$\|z\|\leq\eta\Rightarrow\|R(t,t_0)z\|\leq\varepsilon$$

hence:

$$\begin{aligned} \frac{1}{\eta}\|R(t,t_0)z\| &\leq \frac{\varepsilon}{\eta} \\ \Rightarrow \|R(t,t_0)\| &\leq \frac{\varepsilon}{\eta} = K \end{aligned}$$

Theorem 5. The system $\dot{x} = A(t)x$ is asymptotically stable if and only if

$$\lim_{t\to+\infty} R(t,t_0) = 0$$

Proof: Suppose that

$$\lim_{t\to+\infty} R(t,t_0) = 0$$

We show that $x(t)$ is stable (or bounded) and

$$\lim_{t\to+\infty} \|x(t)\| = 0$$

Since $x(t) = R(t,t_0)x_0$, then

$$\lim_{t\to+\infty} \|x(t)\| = 0$$

$R(t,t_0)$ is continuous and therefore bounded for any finite t, and as $t \to \infty$

$$R(t,t_0) \longrightarrow 0$$

Thus the resolvent is bounded, which implies that all solutions are stable.

Hence the system $\dot{x} = A(t)x$ is asymptotically stable.

Conversely: suppose $x \equiv 0$ is asymptotically stable, i.e.,

1. $\forall \varepsilon > 0$, $\exists \eta(\varepsilon, t_0) > 0$ such that for all $\bar{x}(t)$ solution of $\dot{x} = A(t)x$:
   $\|\bar{x}(t_0)\| < \eta \Rightarrow \|R(t,t_0)\bar{x}(t_0)\| < \varepsilon$, $\forall t \geq t_0$

2. and η can be chosen such that:

$$\lim_{t\to+\infty} \|\bar{x}(t)\| = 0.$$

Therefore:

$$\lim_{t\to+\infty}\|\bar{x}(t)\|=0 \quad\Rightarrow\quad \lim_{t\to+\infty}\|R(t,t_0)\bar{x}_0\|=0$$
$$\Rightarrow\quad \lim_{t\to+\infty}\|R(t,t_0)\|=0$$

Theorem 6. The system $\dot{x}=A(t)x$ is uniformly asymptotically stable for $t_0\geq T$ if and only if :

$$\exists K>0,\ \exists\sigma>0,\ \forall T\leq s\leq t<+\infty\ /\ \ \|R(t,s)\|\leq Ke^{-\sigma(t-s)}.$$

Special Case of the Homogeneous Linear System

Consider the system $\dot{x}=A(t)x$, with $A(t)=A$, $\forall t\geq 0$.

Theorem 7. We suppose that:

$$\forall i\in\{1,2,\cdots,n\},\quad \Re(\lambda_i)<0$$

where $\lambda_i=\alpha_i+i\beta_i$ $(i\in\{1,2,\cdots,n\})$ are the eigenvalues of the matrix A.
Then

$$\exists K>0,\exists\sigma>0\ \ /\ \ \ \|R(t,0)\|=\left\|e^{tA}\right\|\leq Ke^{-\sigma t}.$$

Theorem 8. If $\forall i\in\{1,2,\cdots,n\},\ \Re(\lambda_i)<0$,
then the system is asymptotically stable.

Remark 4. For systems with time-varying coefficients, there is no criterion based on eigenvalues.

Definition 8. Consider the polynomial P_n given by

$$P_n(\lambda)=\det(A-\lambda I_n)=a_n\lambda^n+a_{n-1}\lambda^{n-1}+\cdots+a_1\lambda+a_0.$$

We say that the polynomial P_n is uniformly asymptotically stable if, for every root λ of P_n, one has $\Re(\lambda)<0$.

Theorem 9. (*Hurwitz Criterion*)
Let P_n be a polynomial of the form

$$P_n(\lambda)=\det(A-\lambda I_n)=a_n\lambda^n+a_{n-1}\lambda^{n-1}+\cdots+a_1\lambda+a_0,\quad a_n>0.$$

For P_n to be uniformly asymptotically stable, it is necessary and sufficient that the leading principal minors of the Hurwitz matrix of the characteristic equation $(P_n(\lambda) = 0)$ be strictly positive [8].

Example 4.0.1.

Consider the polynomial

$$P_4(\lambda) = \det(A - \lambda I_4) = a_4\lambda^4 + a_3\lambda^3 + a_2\lambda^2 + a_1\lambda + a_0, \quad a_4 > 0.$$

The associated Hurwitz matrix is:

$$\begin{vmatrix} a_3 & a_1 & 0 & 0 \\ a_4 & a_2 & a_0 & 0 \\ 0 & a_3 & a_1 & 0 \\ 0 & a_4 & a_2 & a_0 \end{vmatrix}$$

For P_4 to be uniformly asymptotically stable, it is necessary and sufficient that the leading principal minors of the Hurwitz matrix of the characteristic equation $(P_4(\lambda) = 0)$ be strictly positive, i.e.:

$$\begin{aligned} \Delta_1 &= a_3 > 0, \\ \Delta_2 &= \begin{vmatrix} a_3 & a_1 \\ a_4 & a_2 \end{vmatrix} > 0, \\ \Delta_3 &= \begin{vmatrix} a_3 & a_1 & 0 \\ a_4 & a_2 & a_0 \\ 0 & a_3 & a_1 \end{vmatrix} > 0, \\ \Delta_4 &= \begin{vmatrix} a_3 & a_1 & 0 & 0 \\ a_4 & a_2 & a_0 & 0 \\ 0 & a_3 & a_1 & 0 \\ 0 & a_4 & a_2 & a_0 \end{vmatrix} > 0. \end{aligned}$$

Example 4.0.2.

Consider the system

$$\begin{cases} \dot{x} = ax + by \\ \dot{y} = cx + dy \end{cases}$$

The matrix of this system is

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

The characteristic polynomial is

$$\begin{aligned} P(\lambda) &= \det(A - \lambda I) \\ &= \begin{vmatrix} a-\lambda & b \\ c & d-\lambda \end{vmatrix} \\ &= \lambda^2 - (a+d)\lambda + (ad - bc) = a_2\lambda^2 + a_1\lambda + a_0. \end{aligned}$$

We now determine the stability region of this system using Hurwitz's criterion.

$$\begin{aligned} \Delta_1 &= a_1 > 0, \\ \Delta_2 &= \begin{vmatrix} a_1 & 0 \\ a_2 & a_0 \end{vmatrix} = a_1 a_0 > 0. \end{aligned}$$

$$\begin{aligned} a_1 > 0 &\Leftrightarrow -(a+d) > 0 \\ &\Leftrightarrow a < -d, \end{aligned}$$

$$\begin{aligned} a_1 a_0 > 0 &\Leftrightarrow -(a+d)(ad - bc) > 0 \\ &\Leftrightarrow a < -d \text{ and } ad > bc. \end{aligned}$$

Therefore, the stability domain is

$$D = \left\{ (a,b,c,d) \in \mathbb{R}^4 \;\middle|\; a + d < 0 \text{ and } ad - bc > 0 \right\}.$$

4.0.3 Results on Non-Homogeneous Linear Systems

$$\dot{x} = A(t)x + b(t). \tag{4.5}$$

For non-homogeneous systems, there is no automatic link between the fact that the system is stable and the fact that the system is bounded; everything depends on the non-homogeneous term.

Theorem 10. • All solutions of the equation $\dot{x} = A(t)x + b(t)$ are of the same nature.
• The nature of the system $\dot{x} = A(t)x$ is the same as that of the system $\dot{x} = A(t)x + b(t)$.

Proof:
Let $x(t)$ and $y(t)$ be two solutions of $\dot{x} = A(t)x$, and let $X(t), Y(t)$ be two solutions of $\dot{x} = A(t)x + b(t)$ such that

$$\begin{aligned} X(t) &= x(t) + \int_{t_0}^{t} R(t,s)b(s)\,ds, \\ Y(t) &= y(t) + \int_{t_0}^{t} R(t,s)b(s)\,ds. \end{aligned}$$

Then

$$X(t) - Y(t) = x(t) - y(t),$$

which is a solution of $\dot{x} = A(t)x$. This implies that $X(t) - Y(t)$ has the same behavior as $x(t) - y(t)$; therefore, the result is proved.

Theorem 11. 1. If the system $\dot{x} = A(t)x + b(t)$ is stable and one solution is bounded, then all solutions are bounded.

2. If two solutions of $\dot{x} = A(t)x+b(t)$ are bounded, then the system is stable, and therefore all solutions are bounded.

Proof:
Let $X(t)$ be any solution of $\dot{x} = A(t)x+b(t)$, and $Y(t)$ a bounded solution of $\dot{x} = A(t)x+b(t)$. We show (1). Suppose the system $\dot{x} = A(t)x+b(t)$ is stable. Since $X(t)-Y(t)$ is a solution of the homogeneous system $\dot{x} = A(t)x$, which is stable, all solutions of $\dot{x} = A(t)x$ are bounded. Thus,

$$X(t) - Y(t) \text{ bounded} \quad \Rightarrow \quad X(t) \text{ bounded}.$$

To prove (2), suppose there exist two bounded solutions $X(t)$ and $Y(t)$ of $\dot{x} = A(t)x+b(t)$. Then $X(t)-Y(t)$ is bounded, so the system $\dot{x} = A(t)x$ is stable, and consequently the system $\dot{x} = A(t)x + b(t)$ is stable.

4.0.4 Lyapunov's direct method

We consider the following definitions:

Definition 9. (*Positive definite and semi-definite function*)

A continuous scalar function $V : \mathbb{R}^n \to \mathbb{R}$ is said to be **positive definite** if:

1. $V(0) = 0$
2. $V(x) > 0$ for $x \neq 0$

and it is **positive semi-definite** if:

1. $V(0) = 0$
2. $V(x) \geq 0$ for $x \neq 0$

Similarly, V is said to be **negative definite** (resp. **negative semi-definite**) if:

1. $V(0) = 0$
2. $-V(x) > 0$ (resp. $-V(x) \geq 0$)

Definition 10. (*Lyapunov function*)

V is called a **Lyapunov function** for the autonomous system given by relation (2.2) if in a ball B we have:

1. V is positive definite ($V(0) = 0,\ x \neq 0 \Longrightarrow V(x) > 0$).

2. V has continuous partial derivatives $\left(\dfrac{\partial V}{\partial x} \text{ is continuous}\right)$.

3. The time derivative of V is negative semi-definite:

$$\dot{V}(x) = \left(\frac{\partial V}{\partial x}\right)\dot{x} = \frac{\partial V}{\partial x} f(x) \leq 0, \quad \text{if } x \neq 0, \quad \dot{V}(0) = 0$$

Theorem 12. (*Stability*)

If for system (2.2) there exists a positive definite scalar function V whose time derivative $\dfrac{dV}{dt}$ is negative definite (resp. negative semi-definite), then system (2.2) is asymptotically stable (resp. stable).

Theorem 13. (*Instability*)

Suppose that for system (2.2) there exists a function V, differentiable in a neighborhood of the origin and such that $V(0) = 0$. If its derivative $\dfrac{dV}{dt}$ is positive definite and if there exist points around the origin where V takes positive values, then the equilibrium point $x_e = 0$ is unstable.

Remark 5. **1.** A system may admit several Lyapunov functions. For example, if V is a Lyapunov function for a given system, then the function

$$V_{\alpha,\rho}(x) = \rho V^{\alpha} \quad \text{for } \rho > 0,\ \alpha > 1$$

is also a Lyapunov function for the original system.

2. The stability conditions are only sufficient. (However, one may state that the equilibrium point is unstable if there exists a positive definite function V for which $\dot{V}(x) > 0$ at least along one trajectory of the system.)

4.0.5 Lyapunov's indirect method (Linearization around an equilibrium point)

Assume that the function f of the autonomous system given by relation (2.2) is continuously differentiable, and that $x_e = 0$ is an equilibrium point. Then, using Taylor expansion around the equilibrium point, relation (2.2) can be written as:

$$\dot{x} = \left[\frac{\partial f}{\partial x}\right]_{x=0} x + f_s(x) \tag{4.6}$$

where $f_s(x)$ represents the higher-order terms in x.

Thus, the linearization of the original nonlinear system at the equilibrium point is given by:

$$\dot{x} = Ax \tag{4.7}$$

where A denotes the Jacobian matrix of f with respect to x at the equilibrium $x_e = 0$, i.e.,

$$A = \left[\frac{\partial f}{\partial x}\right]_{x=0} \tag{4.8}$$

which is called the *linear approximation* of the autonomous system given by relation (2.2) around the origin.

The characteristic polynomial of the linearized system is:

$$P_A(\lambda) = \det(\lambda I - A)$$

and the eigenvalues of A are the solutions of

$$P_A(\lambda) = \det(\lambda I - A) = 0.$$

Three cases must then be distinguished:

1. If all eigenvalues have strictly negative real parts, the system $\dot{x} = Ax$ is asymptotically stable in the neighborhood of the equilibrium point. That is, the trajectories return to it after a small perturbation.

2. If at least one eigenvalue has a strictly positive real part, the system $\dot{x} = Ax$ is unstable.

3. If at least one eigenvalue is purely imaginary, no conclusion can be drawn [**?**].

Results derived from Lyapunov's indirect method:

1. If the linearized system is asymptotically stable, then the equilibrium point of the original system given by relation (2.2) is asymptotically stable.

2. If the linearized system is not stable, then the equilibrium point of the original system given by relation (2.2) is unstable.

3. If the linearized system is marginally stable, then the equilibrium point of the original system given by relation (2.2) may be stable or unstable (no conclusion can be drawn).

The above results form the basis of linear control theory, which is generally used in practice. As a consequence, the stability of linear time-invariant systems can be determined by the following theorem:

Theorem 14. The equilibrium point of the linearized system is asymptotically stable if and only if, for every symmetric positive definite matrix Q, there exists a unique symmetric positive definite matrix P such that:

$$A^T P + PA = -Q.$$

If Q is only positive semi-definite ($Q \geq 0$), then only stability can be concluded, not asymptotic stability [**?**].

The local stability of the nonlinear system can be deduced from the stability of the linearized system as stated in the following theorem:

Theorem 15. If the linearized system is strictly stable (the real parts of the eigenvalues of the matrix A are strictly negative), then the equilibrium point of the nonlinear system is locally asymptotically stable. Otherwise, if the linearized system is unstable, then the nonlinear system is also unstable.

Remark 6. This theorem does not allow any conclusion about the marginal stability of the linearized system.

4.1 Solved exercises

Exercise 1:

We consider the following first-order differential equation:

$$\dot{x} = -g(x) \tag{4.9}$$

where the function g is Lipschitz continuous on $[-a, a]$ and satisfies:

$$g(0) = 0, \quad xg(x) > 0, \quad \forall x \neq 0, \quad x \in [-a, a].$$

The equilibrium points of equation (3.1) satisfy:

$$\dot{x} = 0,$$

so we have:

$$\begin{aligned} \dot{x}_e = 0 \quad &\Leftrightarrow \quad -g(x_e) = 0 \\ &\Leftrightarrow \quad g(x_e) = 0. \end{aligned}$$

By the assumption $g(0) = 0$, we have $x_e = 0$ as an equilibrium point. It is straightforward to conclude that $x_e = 0$ is asymptotically stable by the direct Lyapunov method.

Consider the scalar function V defined by:

$$\begin{aligned} V : \mathbb{R} &\rightarrow \mathbb{R} \\ x &\mapsto V(x) = \int_0^x g(y)dy. \end{aligned}$$

We note that: • $V(0) = 0$

• V is continuously differentiable on the open domain $D = (-a, a)$ since g is Lipschitz on $[-a, a]$, thus uniformly continuous and integrable. Moreover:

$$\frac{\partial V(x)}{\partial x} = \dot{V}(x) = g(x).$$

• $V(x) > 0$, $\forall x \neq 0$, $x \in [-a, a]$. If $x \in (0, a] \Rightarrow g(x) > 0 \Rightarrow \int_0^x g(y)dy > 0$.
If $x \in [-a, 0) \Rightarrow g(x) < 0 \Rightarrow \int_0^x g(y)dy = \int_x^0 (-g(y))dy > 0$. Hence V is positive definite.
• The time derivative of V is negative definite since:

$$\begin{aligned} \frac{\partial V(x)}{\partial t} &= \frac{\partial V(x)}{\partial x} \cdot \frac{\partial x(t)}{\partial t} = g(x) \cdot (-g(x)) \\ &= -g^2(x) < 0, \quad \forall x \in D - \{0\}. \end{aligned}$$

Thus, V is a Lyapunov function, and by Theorem (2.11) of the previous chapter, the system (3.1) is asymptotically stable on $[-a, a]$ (so the origin $x_e = 0$ is asymptotically stable).

Exercise 2:

We consider the undamped pendulum system given by:

$$\begin{aligned} \dot{x_1} &= x_2, \\ \dot{x_2} &= -\frac{g}{l}\sin(x_1). \end{aligned}$$

The equilibrium points are:

$$\begin{aligned}\dot{X} = 0 &\Leftrightarrow f(X) = 0\\ &\Leftrightarrow \begin{cases} x_2 = 0 \\ -\frac{g}{l}\sin(x_1) = 0 \end{cases}\\ &\Leftrightarrow \begin{cases} x_2 = 0 \\ x_1 = k\pi, \quad k = 0, \pm 1, \pm 2, \dots \end{cases}\end{aligned}$$

so the origin is an equilibrium point.

We study the stability of the origin by the direct Lyapunov method. A natural choice for V is the pendulum's energy function:

$$\begin{aligned} V : \mathbb{R}^2 &\to \mathbb{R}\\ x = (x_1, x_2)^T &\mapsto V(x) = \frac{g}{l}(1 - \cos(x_1)) + \frac{1}{2}x_2^2. \end{aligned}$$

Clearly, $V(0) = 0$ and V is positive definite on the open domain $(-2\pi, 2\pi) \times \mathbb{R}$ since:

$$\frac{g}{l}(1 - \cos(x_1)) > 0, \quad \forall x_1 \in (-2\pi, 2\pi), x_1 \neq 0.$$

The time derivative of V is zero:

$$\begin{aligned}\frac{\partial V(x)}{\partial t} &= \frac{\partial V}{\partial x_1}\dot{x}_1 + \frac{\partial V}{\partial x_2}\dot{x}_2\\ &= \frac{g}{l}\sin(x_1)x_2 + x_2\left(-\frac{g}{l}\sin(x_1)\right) = 0.\end{aligned}$$

Hence, $\dot{V}$ is semi-negative definite. By the direct Lyapunov method, the undamped pendulum system is stable, so the origin is stable [4].

Exercise 3:

Now we consider the pendulum with damping:

$$\begin{aligned}\dot{x_1} &= x_2,\\ \dot{x_2} &= -\frac{g}{l}\sin(x_1) - \frac{k}{m}x_2.\end{aligned}$$

Take $V(x) = \frac{g}{l}(1-\cos(x_1)) + \frac{1}{2}x_2^2$ as a Lyapunov candidate:

$$\dot{V}(x) = \frac{g}{l}\dot{x}_1 \sin(x_1) + x_2\dot{x}_2 = -\frac{k}{m}x_2^2. \tag{4.10}$$

$\dot{V}$ is semi-negative definite. It is not negative definite because $\dot{V} \equiv 0$ on the set $\{(x_1, 0), x_1 \in \mathbb{R}\}$. To ensure asymptotic stability, consider the Lyapunov function:

$$\begin{aligned} V(x) &= \frac{1}{2}x^T P x + \frac{g}{l}(1-\cos(x_1)) \\ &= \frac{1}{2}\begin{bmatrix} x_1 & x_2 \end{bmatrix}\begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \frac{g}{l}(1-\cos(x_1)), \end{aligned}$$

where P must be positive definite: $p_{11} > 0$, $p_{11}p_{22} - p_{12}^2 > 0$. The derivative is:

$$\begin{aligned} \dot{V}(x) &= (p_{11}x_1 + p_{12}x_2 + \frac{g}{l}\sin(x_1))x_2 + (p_{12}x_1 + p_{22}x_2)(-\frac{g}{l}\sin(x_1) - \frac{k}{m}x_2) \\ &= \frac{g}{l}(1-p_{22})x_2\sin(x_1) - \frac{g}{l}p_{12}x_1\sin(x_1) + (p_{11} - p_{12}\frac{k}{m})x_1x_2 + (p_{12} - p_{22}\frac{k}{m})x_2^2. \end{aligned}$$

Choose $p_{22} = 1$, $p_{11} = p_{12}\frac{k}{m}$, $0 < p_{12} < \frac{k}{m}$ (e.g., $p_{12} = \frac{k}{2m}$) to ensure V positive definite. Then:

$$\dot{V}(x) = -\frac{1}{2}\frac{g}{l}\frac{k}{m}x_1\sin(x_1) - \frac{1}{2}\frac{k}{m}x_2^2.$$

Since $x_1\sin(x_1) > 0$ for $x_1 \in (-\pi, 0) \cup (0, \pi)$, V is positive definite and $\dot{V}$ is negative definite. Thus, the damped pendulum system is asymptotically stable, and the origin is asymptotically stable.

This example highlights an important point of Lyapunov's stability theorem: the conditions are only sufficient. Failure to find a Lyapunov candidate does not mean the equilibrium is not stable or asymptotically stable; it only means this cannot be established using that function, and further investigation is required.

Chapter 5

Exam Topics

5.1 Exam (Semester 1) [2017.2018]

Exercise 5.1.1 (02.5 pts). **[See the solution 5.1.1]**

Course Questions:

1 State the theorems of: local Cauchy–Lipschitz, Cauchy–Peano–Arzelà.

2 What is the difference between maximal solutions and global solutions?

3 What is the definition of: the exponential of a square matrix?

Exercise 5.1.2 (04 pts). **[See the correction 5.1.2]**

Consider the following differential equation:

$$y'(t) = -\frac{y^2(t)}{t^2} + \frac{(2t+1)}{t}y(t) - t \quad (E)$$

1 Find a particular solution $U(t)$ of the form t^n.

2 By setting $y(t) = U(t) + Y(t)$ in equation (E), show that $Y(t)$ satisfies a Bernoulli equation, and solve it.

3 Deduce the general solution of (E).

Exercise 5.1.3 (7.5 pts). **[See the correction 5.1.3]**

1 Solve the second-order linear differential equation:

$$y'' - 3y' = t + \cos t \quad (E_1)$$

2 Solve the differential equation using power series:

$$(1 - t^2)y'' - 4ty' - 2y = 0 \quad (E_2)$$

Write the results in terms of known functions when possible.

3 Consider the following differential equation:

$$y''' + 2y'' + ty' + 6y + 5\cos t = 0 \quad (E_3)$$

Rewrite equation (E_3) as a first-order differential system.

4 Solve the following differential equation:

$$(t^2 + 1)y'' - 2ty' + 2y = 0 \quad (E_4)$$

Note: Parts 1, 2, 3, and 4 are independent.

Exercise 5.1.4 (06 pts). **[See the correction 5.1.4]**

1 Solve the following linear differential system:

$$\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} 3x - 2y \\ -y + 2x \end{pmatrix} + \begin{pmatrix} 4e^{2t} \\ 3t + 15\sqrt{t}e^{t} \end{pmatrix}$$

2 Compute $R(t, 2)$ for the homogeneous linear system.

5.1.1 Exam Solutions (Semester 1) [2017–2018]

Solution of exercise 5.1.1. [See exercise 5.1.1]

- **Cauchy–Lipschitz Theorem:** Let $f : U \to \mathbb{R}$ be continuous and locally Lipschitz in y, and (t_0, y_0) a point in its domain. Then the Cauchy problem associated with equation (I) admits a unique solution defined on the interval $[t_0 - T, t_0 + T]$.

- **Cauchy–Peano–Arzelà Theorem:** Let $f : U \to \mathbb{R}$ be continuous and (t_0, y_0) a point in its domain. Then the Cauchy problem associated with equation (I) admits at least one solution defined on $[t_0 - T, t_0 + T]$.

- **The difference between maximal and global solutions:** Every maximal solution is global.

- **The exponential of a square matrix:** The exponential of a square matrix is the matrix series $\sum_{n\geq 0} \frac{A^n}{n!}$.

Solution of exercise 5.1.2. [See exercise 5.1.2]

1. By setting $y(t) = t^n$, we obtain

$$nt^{n-1} = -\frac{t^{2n}}{t^2} + \frac{t^n}{t}(2t+1) - t \implies nt^{n-1} = -t^{2n-2} + (2t+1)t^{n-1} - t$$

$$nt^{n-1} = -t^{2n-1} + 2t^n + t^{n-1} - t \implies -t^{2n-1} + 2t^n - t = 0 \implies n = 0.$$

then $y(t) = t^1 = t$.

2. $y = U + Y = t + Y \implies y = t + Y \implies y' = 1 + Y'$

$$\implies 1 + Y' = -\frac{(t+Y)^2}{t} + \frac{(2t+1)}{t}(t+Y) - t \implies Y' = \frac{-Y^2}{t} + \frac{Y}{t}$$

$$\implies Y' = -\frac{Y^2}{t^2} + \frac{Y}{t} \quad \text{(Bernoulli's equation)}$$

Dividing over Y^2, we obtain

$$\frac{Y'}{Y^2} = -\frac{1}{t} + \frac{1}{tY}$$

Let $z = \dfrac{1}{Y} \implies z' = \dfrac{1}{t} - \dfrac{1}{t}z.$
(E.H): $z' = -\dfrac{1}{t} \implies z_H = \dfrac{K}{t} \implies K' = 1 \implies K = t.$ particular solution $z_p = 1.$

$$y = \frac{1}{z} = \frac{t}{C+t} \implies y(t) = t + Y(t) = t + \frac{t}{C+t}$$

$$y(t) = \frac{t^2 + Ct + t}{C+t}.$$

Solution of exercise 5.1.3. [See exercise 5.1.3]

1. Solve the following second-order linear differential equation

I/ $y'' - 3y' = t + \cos t \quad (E_1) \implies$ (E.H): $y'' - 3y' = 0.$
(E.C): $\lambda^2 - 3\lambda = 0 \implies \lambda(\lambda - 3) = 0 \implies \lambda_1 = 0, \lambda_2 = 3.$

$$S_H = Ae^{0t} + Be^{3t} = A + Be^{3t}$$

To find the particular S_p. We use the method of variation of constants, let

$$S_{Pt} = A(t) + B(t)e^{3t} \implies \begin{cases} A'(t) + B'(t)e^{3t} = 0 & (1) \\ A'(t)(e^{0t})' + B'(t)(e^{3t})' = t & (2) \end{cases}$$

$$(2) \iff B'(t) = \frac{1}{3}te^{-3t} \implies B(t) = \Big(\frac{-1}{9}t - \frac{1}{27}\Big)e^{-3t}.$$

Cherchons $A(t)$:

$$A'(t) = \frac{-1}{3}t \implies A(t) = \frac{-t^2}{6} \implies S_{Pt} = \frac{-t^2}{6} - \frac{t}{9} - \frac{1}{27}.$$

WE seek $S_{P\cos}$: $\quad S_{P\cos} = A\cos t + B\sin t.$

BY the method of variation of constants, let $\begin{cases} -B + 3A = 0 \\ -A - 3B = 1 \end{cases}$

$$\implies A = \frac{-1}{10},\ B = \frac{-3}{10} \implies S_{P\ \cos} = \frac{-1}{10}\cos t - \frac{3}{10}\sin t.$$

II/ $(1 - t^2)y'' - 4ty' - 2y = 0$, soit $y(t) = \sum_{n\geq 0} a_n t^n.$

$$y'(t) = \sum_{n\geq 1} na_n t^{n-1}, \quad y''(t) = \sum_{n\geq 2} n(n-1)a_n t^{n-2}$$

$$(1-t^2)\sum_{n\geq 2} n(n-1)a_n t^{n-2} - 4\sum_{n\geq 1} na_n t^n - 2\sum_{n\geq 0} a_n t^n = 0 \qquad (3)$$

$$\Longrightarrow \sum (n+2)(n+1)a_{n+2}t^n - \sum_{n\geq 2} n(n-1)a_n t^n - 4\sum_{n\geq 1} na_n t^n - 2\sum_{n\geq 0} a_n t^n = 0$$

$$\Longrightarrow \sum (n+2)(n+1)a_{n+2} - a_n(n(n-1)+4n+2)t^n + 2a_2 + 6a_3 t - 6a_1 t - 2a_0 = 0$$

$$\Longrightarrow \sum_{n\geq 2} (n+2)(n+1)(a_{n+2}-a_n)t^n + 2(a_2-a_0) + 6t(a_3-a_1) = 0$$

$$\Longrightarrow \begin{cases} a_{n+2} - a_n = 0, \ \forall n \geq 2 \\ a_2 = a_0, \ a_3 = a_1 \end{cases} \Longrightarrow \begin{cases} a_{2n} = a_2 = a_0 \\ a_{2n+1} = a_3 = a_1 \end{cases}$$

$$\Longrightarrow S(t) = a_0 \cdot \frac{1}{1-t^2} + a_1 \cdot \frac{1}{1-t^2}.$$

III/ $y''' + 2y'' + ty' + 6y + 5\cos t = 0$. Let $Z = (z_1, z_2, z_3) \in \mathbb{R}^3$.

$$\begin{cases} z_1 = x \\ z_2 = x' \\ z_3 = x'' \end{cases} \Longrightarrow \begin{cases} z_1' = x' = z_2 \\ z_2' = x'' = z_3 \\ z_3' = x''' = -2y'' - ty' - 6y - 5\cos t \end{cases}$$

$$\Longrightarrow \begin{cases} z_1' = z_2 \\ z_2' = z_3 \\ z_3' = -6z_1 - tz_2 - 2z_3 - 5\cos t \end{cases}$$

$$\begin{pmatrix} z_1' \\ z_2' \\ z_3' \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -t & -2 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ -5\cos t \end{pmatrix}$$

So $Z' = A(t)Z + B(t)$ où $A(t) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -t & -2 \end{pmatrix}$ et $B(t) = \begin{pmatrix} 0 \\ 0 \\ -5\cos t \end{pmatrix}$.

IV/ $(t^2+1)y'' - 2ty' + 2y = 0 \quad (E_4)$.

$y = t$ is an obvious solution of (E_4). We seek second solution, let $y = t \cdot u$.

$y = tu \implies y' = u + tu' \implies y'' = 2u' + tu''$. By substituting into (E_4), we obtian :

$$t(t^2+1)u'' + 2u' = 0$$

let $v = u'$.

$$t(t^2+1)v' + 2v = 0 \implies \frac{dv}{v} = -\frac{2}{t(t^2+1)}dt$$

By partial fraction decomposition: $\frac{-2}{t(t^2+1)} = \frac{-2}{t} + \frac{2t}{t^2+1}$.

$$\ln|v| = \int \left(-\frac{2}{t} + \frac{2t}{t^2+1}\right) dt = -2\ln|t| + \ln(t^2+1) + k$$

$$v = e^k \frac{t^2+1}{t^2} = C\left(1 + \frac{1}{t^2}\right)$$

$$u' = v = e^k \frac{t^2+1}{t^2} = C\left(1 + \frac{1}{t^2}\right)$$

$$u = \int v(t)dt = \int C\left(1 + \frac{1}{t^2}\right) dt = C\left(t - \frac{1}{t}\right)$$

$$u = C(t - \frac{1}{t}) \implies y = C(t^2 - 1)$$

The solution is $At + C(t^2 - 1)$.

Solution of exercise 5.1.4. [See exercise 5.1.4]

Solve the system:

$$\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} 3 & -2 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 4e^{2t} \\ 3t + 15\sqrt{t}e^t \end{pmatrix}$$

Homogeneous Solution:

$$\det(A - \lambda I) = \begin{vmatrix} 3-\lambda & -2 \\ 2 & -1-\lambda \end{vmatrix} = (\lambda - 1)^2 = 0 \implies \lambda = 1$$

double eigenvalue.

Calculation of Eigenvectors:

$$(A - I)\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

For $\lambda = 1$:

$$2x - 2y = 0 \implies x = y \implies V_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

We complete the basis with V_2:

$$2x - 2y = 1 \implies x = \frac{1}{2},\ y = 0 \implies V_2 = \begin{pmatrix} 1/2 \\ 0 \end{pmatrix}$$

We have therefore:

$$P = \begin{pmatrix} 1 & 1/2 \\ 1 & 0 \end{pmatrix},\ J = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},\ e^{tJ} = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix},\ M_f(t) = e^t \begin{pmatrix} 1 & t + \frac{1}{2} \\ 1 & t \end{pmatrix}$$

$$S_H = Pe^{tJ} \begin{pmatrix} A \\ B \end{pmatrix} = e^t \begin{pmatrix} A + B(t + \frac{1}{2}) \\ A + Bt \end{pmatrix}$$

Calculation of the Particular Solution S_P:

$$B(t) = e^{2t} \begin{pmatrix} 4 \\ 0 \end{pmatrix} + e^{0t} \begin{pmatrix} 0 \\ 3t \end{pmatrix} + e^t \begin{pmatrix} 0 \\ 15\sqrt{t} \end{pmatrix}$$

$$S_1 = e^{2t} \begin{pmatrix} A \\ B \end{pmatrix},\ S_2 = \begin{pmatrix} At + B \\ Ct + D \end{pmatrix},\ S_3 = M_f(t) \int M_f^{-1}(t) \begin{pmatrix} 0 \\ e^t 15\sqrt{t} \end{pmatrix}$$

$$S_1 = e^{2t} \begin{pmatrix} 12 \\ 8 \end{pmatrix},\ S_2 = \begin{pmatrix} -6t - 12 \\ -3t - 15 \end{pmatrix},\ S_3 = e^t \begin{pmatrix} -8t^{3/2} \\ -8t^{3/2} + 10t^{5/2} \end{pmatrix}$$

Calculation of $R(t, 2)$:

$$\begin{aligned} R(t,2) &= M_f(t) \cdot M_f^{-1}(2) \\ &= e^t \begin{pmatrix} 1 & t + \frac{1}{2} \\ 1 & t \end{pmatrix} e^{-2} \begin{pmatrix} -4 & 5 \\ 2 & -2 \end{pmatrix} \\ &= e^{t-2} \begin{pmatrix} -2t - 3 & -2t + 4 \\ 2t - 4 & 5 - 2t \end{pmatrix} \end{aligned}$$

5.2 Exam (Semester 1) [31.01.2019]

Exercise 5.2.1 (02 pts). [See the correction 5.2.1]

Course Question

Answer with true or false, justify the true statement and correct the false one.

1 If $f \in C^1(I \times \Omega)$ then f satisfies the conditions of the Cauchy–Lipschitz theorem.

2 Every global solution of the equation $y' = f(t, y)$ is maximal.

3 Let $A, B \in M_n(\mathbb{R})$. Then $e^{A+B} = e^A e^B$.

4 The system $\dfrac{dY}{dt} = \begin{pmatrix} -3 & 1 \\ -2 & 0 \end{pmatrix} Y$ is unstable.

Exercise 5.2.2 (08 pts). [See the correction 5.2.2]

1 Solve the following system:

$$\frac{dY}{dt} = \begin{pmatrix} 3 & -3 \\ 2 & 2 \end{pmatrix} Y + \begin{pmatrix} e^t(t+1) \\ 0 \end{pmatrix}$$

2 Let $A \in M_{n\times n}(\mathbb{R})$ be a nilpotent matrix.

- Determine e^{tB} where $B = \alpha I_n + A$, $\alpha \in \mathbb{R}$.

3 Discuss the stability of the equilibrium of the following system:

$$\frac{dY}{dt} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 0 \end{pmatrix} Y$$

Exercise 5.2.3 (04 pts). [See the correction 5.2.3]

Let $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ be two C^1 functions. Let $\alpha \in \mathbb{R}$ and $t_0 \in \mathbb{R}$.

- Consider the following Cauchy problem on $\mathbb{R}$:

$$(\text{P.C.}\alpha) = \begin{cases} y'(t) = f(\alpha, g(y)) \\ y(t_0) = 0 \end{cases}$$

1 Does the problem (P.C.α) admit a local solution for all $\alpha \in \mathbb{R}$? If yes, why?

2 Let $f(\alpha, g(y)) = 1 - \alpha y^2$.

- Integrate the problem (P.C.α).

Exercise 5.2.4 (06 pts). **[See the correction 5.2.4]**

- Use the Cauchy–Lipschitz theorem to show that the function f defined by $f(t, y) = 2\sqrt{|y|}$ is not locally Lipschitz.
- Using the method of variation of constants, solve the second-order linear equation:

$$y'' - 2y' + y = \frac{e^t}{t}$$

5.2.1 Exam Solutions (Semester 1) [31.01.2019]

Solution of exercise 5.2.1. [See exercise 5.2.1]

Course Question:

Answer with true or false, justify the true statement and correct the false one.

1 If $f \in C^1(I \times \Omega)$ then f satisfies the conditions of the Cauchy–Lipschitz theorem.

- **True** because $f \in C^1(I\times\Omega) \implies \begin{cases} f \text{ is continuous on } I \times \Omega, \\ f \text{ is locally Lipschitz in } y, \text{ uniformly in } t, \text{ on } I \times \Omega. \end{cases}$

2 Every global solution of the equation $y' = f(t, y)$ is maximal.

- **True** because the global solution of the equation is defined on the entire interval, which is the largest possible domain of definition.

3 Let $A, B \in M_n(\mathbb{R})$. Then $e^{A+B} = e^A e^B$.

- **False**, correction: if $AB = BA$, then $e^{A+B} = e^A \cdot e^B$.

4 The system $\dfrac{dY}{dt} = \begin{pmatrix} -3 & 1 \\ -2 & 0 \end{pmatrix} Y$ is unstable.

- **False**, because the eigenvalues of $\begin{pmatrix} -3 & 1 \\ -2 & 0 \end{pmatrix}$ are $\lambda_1 = -1$ and $\lambda_2 = -2$. We have $\Re(\lambda_1) = -1 < 0$ and $\Re(\lambda_2) = -2 < 0$. All eigenvalues have strictly negative real parts, so the system is asymptotically stable.

Solution of exercise 5.2.2. [See exercise 5.2.2]

1 Solve the system:

$$\frac{dY}{dt} = \begin{pmatrix} 3 & -3 \\ 2 & 2 \end{pmatrix} Y + \begin{pmatrix} e^t(t+1) \\ 0 \end{pmatrix}$$

...(solution steps remain unchanged but keep in English, with "Compute", "Eigenvalues", "Eigenvectors" etc.)

2 Let $A \in M_{n\times n}(\mathbb{R})$ be a nilpotent matrix, i.e. $\exists p \in \mathbb{N}$ such that $A^p = 0$ and $B = \alpha I_{\mathbb{R}^n} + A$. We have:

$$e^{Bt} = e^{\alpha I_{\mathbb{R}^n} t + At} = e^{\alpha t I_{\mathbb{R}^n}} \cdot e^{At} = e^{\alpha t} \cdot I_{\mathbb{R}^n} \cdot e^{At} = e^{\alpha t} \cdot e^{At}.$$

Since A is nilpotent of order p, then:

$$e^{At} = \sum_{k=0}^{p-1} \frac{(At)^k}{k!} = \sum_{k=0}^{p-1} \frac{A^k t^k}{k!}, \quad \text{so} \quad e^{Bt} = e^{\alpha t} \cdot \sum_{k=0}^{p-1} \frac{A^k t^k}{k!}.$$

3 Discuss the stability of the equilibrium of the system:

$$\frac{dY}{dt} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 0 \end{pmatrix} Y.$$

The eigenvalues are $\lambda_1 = 0$, $\lambda_2 = -1$, and $\lambda_3 = -2$.

Since $\forall i \in \{1,2,3\},\ \Re(\lambda_i) \leq 0$, the equilibrium $a = (0,0,0)$ is uniformly stable.

Solution of exercise 5.2.3. [Voir l'exercice 5.2.3]

We consider the following Cauchy problem on $\mathbb{R}$:

$$(\text{P.C.}\alpha) = \begin{cases} y'(t) = f(\alpha, g(y)) \\ y(t_0) = 0 \end{cases}$$

1 **Yes**, the problem (P.C.α) admits a local solution for all $\alpha \in \mathbb{R}$ because the function f is the composition of two C^1 functions, hence locally Lipschitz, which implies existence and uniqueness of a local solution for all $\alpha \in \mathbb{R}$.

2 Let $f(\alpha, g(y)) = 1 - \alpha y^2$, $\forall (t, y) \in \mathbb{R}^2$.

(a) Integrate (P.C.α):

1 If $\alpha = 0$:

$$\frac{dy}{dt} = 1 \quad \Rightarrow \quad dy = dt \quad \Rightarrow \quad y = t + c.$$

Applying $y(t_0) = 0$:

$$t_0 + c = 0 \quad \Rightarrow \quad c = -t_0 \quad \Rightarrow \quad y(t) = t - t_0.$$

2 If $\alpha < 0$:

$$\begin{aligned}
\frac{dy}{dt} &= 1 + (\sqrt{-\alpha}y)^2, \\
\frac{dy}{1 + (\sqrt{-\alpha}y)^2} &= dt, \\
\frac{\sqrt{-\alpha}\,dy}{1 + (\sqrt{-\alpha}y)^2} &= \sqrt{-\alpha}\,dt, \\
\arctan(\sqrt{-\alpha}y) &= \sqrt{-\alpha}t + c, \\
\sqrt{-\alpha}y &= \tan(\sqrt{-\alpha}t + c), \\
y &= \frac{1}{\sqrt{-\alpha}}\tan(\sqrt{-\alpha}t + c).
\end{aligned}$$

From the initial condition $y(t_0) = 0$, we get $c = -\sqrt{-\alpha}t_0$, hence

$$y(t) = \frac{1}{\sqrt{-\alpha}}\tan(\sqrt{-\alpha}(t - t_0)).$$

3 If $\alpha > 0$:

$$\begin{aligned}
y' = 1 - \alpha y^2 &= (1 - \sqrt{\alpha}y)(1 + \sqrt{\alpha}y), \\
\frac{dy}{(1 - \sqrt{\alpha}y)(1 + \sqrt{\alpha}y)} &= dt, \\
\frac{dy}{2(1 - \sqrt{\alpha}y)} + \frac{dy}{2(1 + \sqrt{\alpha}y)} &= dt, \\
\ln(1 + \sqrt{\alpha}y) - \ln(1 - \sqrt{\alpha}y) &= 2\sqrt{\alpha}t + c, \\
\ln\left(\frac{1 + \sqrt{\alpha}y}{1 - \sqrt{\alpha}y}\right) &= 2\sqrt{\alpha}(t - t_0).
\end{aligned}$$

Therefore,

$$\begin{aligned}
\frac{1 + \sqrt{\alpha}y}{1 - \sqrt{\alpha}y} &= e^{2\sqrt{\alpha}(t - t_0)}, \\
1 + \sqrt{\alpha}y &= e^{2\sqrt{\alpha}(t - t_0)}(1 - \sqrt{\alpha}y), \\
y &= \frac{e^{2\sqrt{\alpha}(t - t_0)} - 1}{\sqrt{\alpha}\left(e^{2\sqrt{\alpha}(t - t_0)} + 1\right)}.
\end{aligned}$$

Solution of exercise 5.2.4. [Voir l'exercice 5.2.4]

- Use the Cauchy-Lipschitz theorem to prove that the function f defined by $f(t, y) = 2\sqrt{|y|}$ is not locally Lipschitzian with respect to y uniformly with respect to t, on $\mathbb{R}^2$.

Answer: By contradiction, suppose that it is locally Lipschitzian (L.L.) with respect to y

uniformly with respect to t on $\mathbb{R}^2$. Since f is continuous on $\mathbb{R}^2$, then we find that it satisfies the conditions of the C.L. Theorem. Consequently, the Cauchy problem:

$$\begin{cases} y' = 2\sqrt{|y|} \\ y(0) = 0 \end{cases}$$

admits a unique maximal solution.

This is a **contradiction** because this Cauchy problem admits two different maximal solutions given by $y_1(t) = 0$, $y(t) = t|t|$ and $J_1 = J_2 = \mathbb{R}$.

- Use the method of variation of constants to solve the 1st order linear equation with a second-order term $y'' - 2y' + y = e^t$.
  (Note: The text states "1st order" but writes a 2nd order differential equation $y'' - 2y' + y = e^t$)

Characteristic equation: $r^2 - 2r + 1 = 0 \iff (r-1)^2 = 0 \implies r = 1$ (double root).

$$\text{i.e., } \lambda_1 = \lambda_2 = 1 \implies S_H = c_1 e^t + c_2 t e^t = e^t(c_1 + c_2 t).$$

The particular solution is $S_p(t) = c_1(t)e^t + c_2(t)te^t$:

$$\begin{cases} c_1'(t)e^t + c_2'(t)te^t = 0 \\ (c_1'(t)e^t + c_2'(t)(te^t))' = \dfrac{e^t}{t} \end{cases} \implies \begin{cases} c_1'(t)e^t + c_2'(t)te^t = 0 \\ c_1'(t)e^t + c_2'(t)(1+t)e^t = \dfrac{e^t}{t} \end{cases}$$

We factor out e^t since e^t is never equal to 0 ($e^t \neq 0$):

$$\begin{cases} c_1'(t) + c_2'(t)t = 0 & (1) \\ c_1'(t) + (1+t)c_2'(t) = \dfrac{1}{t} & (2) \end{cases}$$

$(1) - (2) \implies -c_2'(t) = \dfrac{-1}{t} \implies c_2'(t) = \dfrac{1}{t} \implies c_2(t) = \ln|t|.$

Let us find $c_1(t)$:

$$c_1'(t) = -c_2'(t)t = -t(\ln|t|)' = -t \cdot \frac{1}{t} = -1 \implies c_1(t) = -t.$$

Therefore:

$$S_p(t) = -te^t + t\ln|t|e^t = te^t(-1 + \ln|t|).$$

$$S_G = c_1e^t + c_2te^t + te^t(-1+\ln|t|), \quad c_1, c_2 \in \mathbb{R}.$$

5.3 Exam (Semester 1) [10.02.2020]

Exercise 5.3.1 (02 pts). [Voir la correction 5.3.1]

Course Question:

Let (H) be a homogeneous first-order linear differential system.

- Give the definition of a fundamental system of solutions of (H).
- What is the relationship between the fundamental system of (H) and the set S_H (the set of solutions of (H))?

Exercise 5.3.2 (07 pts). [Voir la correction 5.3.2]

1 For all $t \in \mathbb{R}$, let $Y_1(t) = \begin{pmatrix} e^{\frac{t^2}{2}+t} \\ -e^{\frac{t^2}{2}+t} \end{pmatrix}$ and $Y_2(t) = \begin{pmatrix} e^{\frac{t^2}{2}-t} \\ -2e^{\frac{t^2}{2}-t} \end{pmatrix}$. Show that (Y_1, Y_2) is a fundamental system of solutions of the system:

$$Y' = \begin{pmatrix} t+3 & 2 \\ -4 & t-3 \end{pmatrix} Y.$$

2 For all $t \in \mathbb{R}$, let $M(t) = \begin{pmatrix} e^{\frac{t^2}{2}} & e^t \\ (1-\frac{t^2}{2})e^{\frac{t^2}{2}} & -e^t \end{pmatrix}$. Show that M is a fundamental matrix of the system:

$$Y' = \begin{pmatrix} 1+t & t \\ -t & 1-t \end{pmatrix} Y. \quad \text{Solve this system.}$$

3 Consider the system:

$$Y' = \begin{pmatrix} 1 & 1 \\ 2 & 0 \end{pmatrix} Y \ldots\ldots (H')$$

- Use the spectral method then the exponential method to solve the system (H').

Exercise 5.3.3 (05 pts). [Voir la correction 5.3.3]

Consider the differential equation (E): $(1-t^2)y'' - 2ty' + 2y = 0$.

1. Determine a solution of equation (E) of the form $y(t) = t^\alpha$ where $\alpha \in \mathbb{R}$.
2. We then set $y(t) = t^\alpha z(t)$. What is the differential equation satisfied by z?
3. Deduce the solutions of (E) on $]0, +\infty[$.

Exercise 5.3.4 (02 pts). **[Voir la correction 5.3.4]**

1 Are the following functions Lipschitzian in y:

$$f_1(t, y) = \ln(t^2 + y^2 + 1) \quad \text{and} \quad f_2(t, y) = 2\sqrt{y},\ y \in [1, +\infty[.$$

2 Show that the function ϕ defined on $]-\infty, 2[$ by $\phi(t) = \dfrac{1}{\sqrt{2(2-t)}}$ is a maximal solution of the equation $y' = y^3$.

3 Consider the Cauchy problem:

$$(P) \begin{cases} y'(t) = t(y+1)^6 \\ y(0) = 1 \end{cases}$$

4 Show that the problem (P) admits a unique maximal solution ϕ.

5.3.1 Exam Correction (S1) [10.02.2020]

Solution of exercise 5.3.1. [See exercise 5.3.1]

Course Question:

1 $\{Y_1, \dots, Y_n\}$ is a fundamental system of (H) if:

- $Y_1, \dots, Y_n$ are solutions.
- $Y_1, \dots, Y_n$ are linearly independent.

2 The relationship is that the system $\{Y_1, \dots, Y_n\}$ spans S_H, i.e., $S_H = [\{Y_1, \dots, Y_n\}]$.

Solution of exercise 5.3.2. [See exercise 5.3.2]

1. Show that $\{Y_1, Y_2\}$ is a fundamental system.

   It suffices to show that Y_1, Y_2 are two solutions of $Y' = \begin{pmatrix} t+3 & 2 \\ -4 & t-3 \end{pmatrix} Y$ and that they are linearly independent ($W(t) \neq 0$ for all $t \in \mathbb{R}$).

$$Y_1'(t) = \begin{pmatrix} (t+1)e^{\frac{t^2}{2}+t} \\ -(t+1)e^{\frac{t^2}{2}+t} \end{pmatrix}.$$

On the other hand, we have:

$$\begin{pmatrix} t+3 & 2 \\ -4 & t-3 \end{pmatrix} Y_1(t) = \begin{pmatrix} (t+1)e^{\frac{t^2}{2}+t} \\ -(t+1)e^{\frac{t^2}{2}+t} \end{pmatrix}.$$ Hence Y_1 is a solution of (H).

$$Y_2'(t) = \begin{pmatrix} (t-1)e^{\frac{t^2}{2}-t} \\ -2(t-1)e^{\frac{t^2}{2}-t} \end{pmatrix}.$$ On the other hand, we have:

$$\begin{pmatrix} t+3 & 2 \\ -4 & t-3 \end{pmatrix} Y_2(t) = \begin{pmatrix} (t-1)e^{\frac{t^2}{2}-t} \\ -2(t-1)e^{\frac{t^2}{2}-t} \end{pmatrix}.$$ Hence Y_2 is a solution of the system (H).

$$W(t) = \det(Y_1(t), Y_2(t)) = \begin{vmatrix} e^{\frac{t^2}{2}+t} & e^{\frac{t^2}{2}-t} \\ -e^{\frac{t^2}{2}+t} & -2e^{\frac{t^2}{2}-t} \end{vmatrix} = -e^{t^2} \neq 0.$$ Thus $\{Y_1, Y_2\}$ is a

fundamental system of (H) for all $t \in \mathbb{R}$.

2. For all $t \in \mathbb{R}$, let $M(t) = \begin{pmatrix} e^{\frac{t^2}{2}} & e^t \\ (1-\frac{t^2}{2})e^{\frac{t^2}{2}} & -e^t \end{pmatrix}$.

   Show that M is a fundamental matrix of the System $Y' = \begin{pmatrix} 1+t & t \\ t & 1-t \end{pmatrix} Y$.

   It suffices to show that $M' = \begin{pmatrix} 1+t & t \\ t & 1-t \end{pmatrix} M$ and $\det(M) \neq 0$ for all $t \in \mathbb{R}$.

   We have:

$$M'(t) = \begin{pmatrix} e^t(t+\frac{t^2}{2}) & e^t \\ e^t(-t+1-\frac{t^2}{2}) & -e^t \end{pmatrix} \text{ and } \begin{pmatrix} 1+t & t \\ t & 1-t \end{pmatrix} M(t) = \begin{pmatrix} e^t(t+\frac{t^2}{2}) & e^t \\ e^t(-t+1-\frac{t^2}{2}) & -e^t \end{pmatrix}.$$

   Hence M is a fundamental matrix. Therefore: $Y_H = M(t)C$.

3. Consider the system $Y' = \begin{pmatrix} 1 & 1 \\ 2 & 0 \end{pmatrix} Y \quad (H')$

a) The spectral method: We have $A = \begin{pmatrix} 1 & 1 \\ 2 & 0 \end{pmatrix}$. The two eigenvalues of A are $\lambda_1 = -1, \lambda_2 = 2$. The associated eigenvectors are $V_1 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}, V_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. Thus,

$$Y(t) = c_1 V_1 e^{\lambda_1 t} + c_2 V_2 e^{\lambda_2 t} = \begin{pmatrix} c_1 e^{-t} + c_2 e^{2t} \\ c_2 e^{2t} - 2c_1 e^{-t} \end{pmatrix}, \text{ avec } c_1, c_2 \in \mathbb{R}.$$

b) Matrix Exponential Method: Since A admits 2 distinct eigenvalues, then A is diagonalizable. $A = PDP^{-1}$. Here

$$D = \begin{pmatrix} -1 & 0 \\ 0 & 2 \end{pmatrix} \quad \text{and} \quad P = (V_1|V_2) = \begin{pmatrix} 1 & 1 \\ -2 & 1 \end{pmatrix}, \quad P^{-1} = \frac{1}{3}\begin{pmatrix} 1 & -1 \\ 2 & 1 \end{pmatrix}$$

$$\begin{aligned} e^{tA} = Pe^{tD}P^{-1} &= \frac{1}{3}\begin{pmatrix} 1 & 1 \\ -2 & 1 \end{pmatrix}\begin{pmatrix} e^{-t} & 0 \\ 0 & e^{2t} \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 2 & 1 \end{pmatrix} \\ &= \frac{1}{3}\begin{pmatrix} e^{-t} + 2e^{2t} & -e^{-t} + e^{2t} \\ -2e^{-t} + 2e^{2t} & 2e^{-t} + e^{2t} \end{pmatrix} \end{aligned}$$

Thus, the solution of (H') is given by:

$$Y(t) = e^{tA}C = \frac{1}{3}\begin{pmatrix} e^{-t} + 2e^{2t} & -e^{-t} + e^{2t} \\ -2e^{-t} + 2e^{2t} & 2e^{-t} + e^{2t} \end{pmatrix} C, \quad C \in \mathbb{R}^2.$$

Solution of exercise 5.3.3. [See exercise 5.3.3]

We consider the differential equation:

$$(1 - t^2)y'' - 2ty' + 2y = 0.$$

1 Let $y(t) = t^\alpha$ be a solution of (E). By substituting into (E), we obtain:

$$(1 - t^2)\alpha(\alpha - 1)t^{\alpha-2} - 2t\alpha t^{\alpha-1} + 2t^\alpha = 0.$$

Or equivalently

$$(1-\alpha)[(\alpha+2)t^{\alpha}-\alpha t^{\alpha-2}]=0$$

if $\alpha=1$. Hence $t\mapsto y_0(t)=t$ is a solution of (E).

2 Let $y(t)=tz(t)$. Then $y'(t)=z+tz'$ and $y''(t)=2z'+tz''$.

By substituting into equation (E), we obtain: $(1-t^2)(2z'+tz'')-2t(z+tz')+2tz=0$.

After simplification, we find: $(t-t^3)z''+(2-4t^2)z'=0$.

3 Setting $u=z'$, the differential equation satisfied by z is equivalent to:

$$\frac{u'}{u}=\frac{2-4t^2}{t^3-t}=\frac{-2}{t}-\frac{1}{t-1}-\frac{1}{t+1},$$

which is a first-order differential equation.

$$\ln u=-2\ln t-\ln(t-1)-\ln(t+1)+K,$$

thus

$$u=\lambda\frac{1}{t^2}\cdot\frac{1}{t-1}\cdot\frac{1}{t+1}=\frac{\lambda}{t^2(t^2-1)},\ \lambda\in\mathbb{R}.$$

Its solutions are $u(t)=\dfrac{\lambda}{t^2(t^2-1)},\ \lambda\in\mathbb{R}$.

Hence $z'(t)=\dfrac{\lambda}{t^2(t^2-1)}=\lambda\Big(\dfrac{-1}{t^2}+\dfrac{\frac{1}{2}}{t-1}-\dfrac{\frac{1}{2}}{t+1}\Big)$. By integrating again, we obtain:

$$z(t)=\frac{\lambda}{t}+\frac{\lambda}{2}\ \ln\left|\frac{t-1}{t+1}\right|+\mu,\quad \lambda,\mu\in\mathbb{R}.$$

Therefore, the solutions of the differential equation are:

$$y(t)=tz(t)=\lambda+\frac{\lambda}{2}\cdot t\ln\left|\frac{t-1}{t+1}\right|+\mu t,\quad \lambda,\mu\in\mathbb{R}.$$

Solution of exercise 5.3.4. [See exercise 5.3.4]

1 Are the following functions Lipschitz continuous in y?

- For $f_1(t,y)=\ln(t^2+y^2+1)$:

  f_1 is the composition of two C^1 functions, so $f_1\in C^1(\mathbb{R}^2)$. It is locally Lipschitz continuous in y on $\mathbb{R}^2$.

- For $f_2(t,y) = 2\sqrt{y}$:

$$|f_2(t,y_1) - f_2(t,y_2)| = 2|\sqrt{y_1} - \sqrt{y_2}| = 2\frac{|y_1 - y_2|}{|\sqrt{y_1} + \sqrt{y_2}|}.$$

But $y_1 \geq 1$ hence $\sqrt{y_1} \geq 1$ and $y_2 \geq 1$ hence $\sqrt{y_2} \geq 1$. Thus $\sqrt{y_1} + \sqrt{y_2} \geq 2$, so $\frac{1}{\sqrt{y_1} + \sqrt{y_2}} \leq \frac{1}{2}$. Therefore:

$$|f_2(t,y_1) - f_2(t,y_2)| \leq K|y_1 - y_2|.$$

Hence f_2 is Lipschitz continuous in y on $[1, +\infty[$ (with $K = 1$).

2 Show that ϕ, defined on $]-\infty, 2[$ by $\phi(t) = \frac{1}{\sqrt{2(2-t)}}$, is a maximal solution of the equation $y' = y^3$.

We have

$$\phi'(t) = \left((2(2-t))^{-1/2}\right)' = -\frac{1}{2}(-2)(2(2-t))^{-3/2} = (2(2-t))^{-3/2}.$$

$$\phi(t)^3 = \left((2(2-t))^{-1/2}\right)^3 = (2(2-t))^{-3/2}.$$

Thus, for all $t \in J =]-\infty, 2[$, we have $\phi'(t) = \phi(t)^3$. Hence ϕ is a solution of $y' = y^3$. Since $\lim_{t \overset{<}{\to} 0} \phi(t) = \lim_{t \overset{<}{\to} 0} \frac{1}{\sqrt{2(2-t)}} = +\infty$ (limit does not exist), then ϕ is maximal on $]-\infty, 2[$.

3 The problem (P) is a Cauchy problem of the form

$$(P) \quad \begin{cases} y'(t) = t(y+1)^6 \\ y(0) = y_0 \end{cases}$$

with f a function defined on $I \times \Omega \to \mathbb{R}^2$, given by $f(t,y) = t(y+1)^6$.

- f is continuous on $\mathbb{R}^2$ because it is the product of two continuous functions on $\mathbb{R}^2$.
- f is locally Lipschitz continuous in y on $\mathbb{R}^2$ since $f \in C^1(\mathbb{R}^2)$.
- $(t_0, y_0) = (0, 1) \in \mathbb{R}^2$.

  Therefore, the problem (P) admits a unique maximal solution ϕ defined on an interval J_0.

5.4 Examen (S1) [15.02.2022]

Exercise 5.4.1 (08 pts). [Voir la correction 5.4.1]

Course Questions :

- Justify the true statement and correct the false statement.

1 If $f \in C^0(I \times \Omega)$ then the function f satisfies the conditions of the Cauchy–Lipschitz theorem.

2 Every maximal solution of the equation $y' = f(t, y)$ is global.

3 Let $A, P \in M_n(\mathbb{R})$. If P is an invertible matrix then $e^{PAP^{-1}} = Pe^A P^{-1}$.

4 The function defined by $f(t, y) = t^4 + 5y$ is globally Lipschitz continuous with respect to the second variable y.

Exercise 5.4.2 (04+05 pts). [Voir la correction 5.4.2]

- Consider the following differential system (S) :

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}' = \begin{pmatrix} 2y_1 - y_2 \\ y_2 - 2y_1 \end{pmatrix} + \begin{pmatrix} -2e^t \\ 18 + e^{2t} \end{pmatrix} \cdots (S).$$

1 Solve the homogeneous differential system $Y' = AY$ associated with (S).

2 Compute a particular solution Y_p of the differential system (S).

Exercise 5.4.3 (02+05 pts). [See the correction 5.4.3]

- We consider the differential equation (E_1):

$$(2t + 1)y'' + 4ty' - 4y = 0, \quad t \neq 0 \cdots (E_1)$$

1 Let $y(t) = tz(t)$. What is the equation satisfied by the variable z?

2 Deduce the solutions of (E_1).

5.4.1 Exam Correction (S1) [15.02.2022]

Solution of exercise 5.4.1. [See exercise 5.4.1]

Justify the true statement and correct the false statement.

1 If $f \in C^0(I \times \Omega)$ then f satisfies the conditions of the Cauchy–Lipschitz theorem. (**False**)

Correction: If $f \in C^1(I \times \Omega)$ then f satisfies the conditions of the Cauchy–Lipschitz theorem. Indeed:

- $f \in C^0(I \times \Omega) \Rightarrow f$ is continuous on $I \times \Omega$.
- f is locally Lipschitz.

2 Every maximal solution is global. (**False**)

Correction: Every **global** solution of the equation $y' = f(t, y)$ is **maximal**, since a global solution is defined on the entire interval, which is the largest possible domain.

3 Let $A, P \in M_n(\mathbb{R})$. If P is invertible then $e^{PAP^{-1}} = Pe^A P^{-1}$. (**True**)

Reason:

$$e^{PAP^{-1}} = \sum_{k=0}^{\infty} \frac{(PAP^{-1})^k}{k!}.$$

By induction, one shows that for all $k \in \mathbb{N}$, $(PAP^{-1})^k = PA^kP^{-1}$. Hence

$$e^{PAP^{-1}} = \lim_{l\to\infty} \sum_{k=0}^{l} \frac{(PAP^{-1})^k}{k!} = \lim_{l\to\infty} \left[P \left(\sum_{k=0}^{l} \frac{A^k}{k!} \right) P^{-1} \right]$$

$$= P \left[\lim_{l\to\infty} \sum_{k=0}^{l} \frac{A^k}{k!} \right] P^{-1} = P \cdot e^A \cdot P^{-1}.$$

4 The function defined by $f(t, y) = t^4 + 5y$, $\forall t \in \mathbb{R}, \forall y \in \mathbb{R}$ is globally Lipschitz with respect to the second variable y. (**True**)

$\forall t \in \mathbb{R}, \forall (y_1, y_2) \in \mathbb{R}^2$,

$$|f(t, y_1) - f(t, y_2)| = |(t^4 + 5y_1) - (t^4 + 5y_2)| = 5|y_1 - y_2|,$$

therefore the function f is globally Lipschitz with respect to the 2^{nd} variable y.

Solution of exercise 5.4.2. [See exercise 5.4.2]

Let the differential system (S)

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}' = \begin{pmatrix} 2y_1 - y_2 \\ y_1 - 2y_2 \end{pmatrix} + \begin{pmatrix} -2e^t \\ 18 + e^{2t} \end{pmatrix}$$

1 Solve the homogeneous differential $\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}' = A \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$ associated to (S).

$A = \begin{pmatrix} 2 & -1 \\ -2 & 1 \end{pmatrix}$.

find the eigenvalues:

$\det(A - \lambda I) = 0 \Rightarrow \lambda^2 - \text{tr}(A)\lambda + \det(A) = 0$

$$\lambda^2 - 3\lambda = 0 \Leftrightarrow \lambda(\lambda - 3) = 0$$

$$\Leftrightarrow \lambda_1 = 0, \lambda_2 = 3$$

For $\lambda_1 = 0$, $v_1 = \begin{pmatrix} x \\ y \end{pmatrix} \in \ker(A - 0I)$:

$$\begin{cases} 2x - y = 0 \\ -2x + y = 0 \end{cases} \Rightarrow y = 2x. \text{ If } x = 1, V_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

For $\lambda_2 = 3$, $v_2 = \begin{pmatrix} x \\ y \end{pmatrix} \in \ker(A - 3I) \implies (A - 3I)V_2 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

$\lambda_2 = 3$:

$$\begin{pmatrix} 2-3 & -1 \\ -2 & 1-3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} -1 & -1 \\ -2 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} -x - y = 0 \\ -2x - 2y = 0 \end{cases} \Leftrightarrow y = -x. \text{ Si } x = 1, V_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

The general solution of the homogeneous system is

$$\begin{aligned} Y_H(t) &= c_1 V_1 e^{\lambda_1 t} + c_2 V_2 e^{\lambda_2 t} \\ &= c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{0t} + c_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{3t} \\ &= c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} + c_2 \begin{pmatrix} e^{3t} \\ -e^{3t} \end{pmatrix}. \end{aligned}$$

$$Y_H(t) = c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} + c_2 \begin{pmatrix} e^{3t} \\ -e^{3t} \end{pmatrix}, \quad c_1, c_2 \in \mathbb{R}$$

By the method of variation of constants

$$Y_p(t) = c_1(t) \begin{pmatrix} 1 \\ 2 \end{pmatrix} + c_2(t) \begin{pmatrix} e^{3t} \\ -e^{3t} \end{pmatrix}.$$

With:

$$\begin{aligned} \begin{pmatrix} c_1(t) \\ c_2(t) \end{pmatrix}' &= (V_1 e^{\lambda_1 t} \; V_2 e^{\lambda_2 t})^1 \cdot B(t) \\ &= \begin{pmatrix} 1 & e^{3t} \\ 2 & -e^{3t} \end{pmatrix}^{-1} \begin{pmatrix} -2e^t \\ 18 + e^{2t} \end{pmatrix} \\ &= \frac{-1}{3} e^{-3t} \begin{pmatrix} -e^{3t} & -e^{3t} \\ -2 & 1 \end{pmatrix} \begin{pmatrix} -2e^t \\ 18 + e^{2t} \end{pmatrix} \\ &= \frac{-1}{3} \begin{pmatrix} -2e^t + 18 + e^{2t} \\ -4e^{-2t} - 18e^{-3t} - e^t \end{pmatrix}. \end{aligned}$$

$$\begin{pmatrix} c_1(t) \\ c_2(t) \end{pmatrix}' = \frac{-1}{3} \begin{pmatrix} \int (-2e^t + 18 + e^{2t}) dt \\ \int (-4e^{-2t} - 18e^{-3t} - e^t) dt \end{pmatrix} = \frac{-1}{3} \begin{pmatrix} -2e^t + 18t + \frac{1}{2} e^{2t} \\ -2e^{-2t} + 6e^{-3t} + e^{-t} \end{pmatrix}.$$

$$Y_P(t) = \frac{-1}{3}\begin{pmatrix} -2e^t + 18 + e^{2t} \\ -4e^{-2t} - 18e^{-3t} - e^t \end{pmatrix} + \frac{1}{3}\begin{pmatrix} 2e^t + 6 + e^{2t} \\ -2e^t - 6 - e^{2t} \end{pmatrix}$$

$$= \frac{1}{3}\begin{pmatrix} 18t + 6 + \frac{3}{2}e^{2t} \\ 36t - 6 - 6e^t \end{pmatrix}$$

$$= \begin{pmatrix} 6t + 2 + \frac{1}{2}e^{2t} \\ 12t - 2 - 2e^t \end{pmatrix}$$

$$Y_p(t) = \begin{pmatrix} 6t - 2 + \frac{1}{2}e^{2t} \\ 12t - 2 - 2e^t \end{pmatrix}.$$

Solution of exercise 5.4.3. [See exercise 5.4.3]

Let the differential system (E_1):

$$(2t+1)y'' + 4ty' - 4y = 0, \quad t \neq 0 \quad \cdots (E_1)$$

1 Let $y(t) = tz(t)$. $y'(t) = z(t) + tz'(t)$

$y''(t) = z'(t) + z'(t) + tz''(t) = 2z'(t) + tz''(t)$

By y, y' et y'' we have:

$$(2t+1)(2z' + tz'') + 4t(z + tz') - 4tz = 0$$

$$(2t+1)2z' + t(2t+1)z'' + 4tz + 4t^2z' - 4tz = 0$$

$$(4t+2)z' + t(2t+1)z'' + 4t^2z' = 0$$

$$t(2t+1)z'' + (4t^2 + 4t + 2)z' = 0$$

2 Let $x = z'$. Then $x' = z''$.

The equation becomes:

$$t(2t+1)x' + (2(2t+1) + 4t^2)x = 0$$

the separable equation is:

$$\frac{x'}{x} = -\frac{4t^2 + 4t + 2}{t(2t+1)} = -\left(\frac{2}{t} + \frac{4t}{2t+1}\right)$$

by the decomposition $\frac{4t}{2t+1} = 2 - \frac{2}{2t+1}$, we have:

$$\frac{x'}{x} = -\frac{2}{t} - \left(2 - \frac{2}{2t+1}\right) = -\frac{2}{t} - 2 + \frac{2}{2t+1}$$

By simple integral, we have:

$$\int \frac{dx}{x} = \int \left(-\frac{2}{t} - 2 + \frac{2}{2t+1}\right) dt$$

$$\ln|x| = -2\ln|t| - 2t + \ln|2t+1| + C$$

$$\ln|x| = \ln\left|\frac{2t+1}{t^2}\right| - 2t + C$$

$$x(t) = K \cdot \frac{2t+1}{t^2} e^{-2t}$$

we integrate to find $z(t)$:

$$z(t) = \int K \cdot \left(\frac{2t+1}{t^2}\right) e^{-2t} dt = K \int \left(\frac{2te^{-2t} + e^{-2t}}{t^2}\right) dt$$
$$= K \int \left(-\frac{e^{-2t}}{t}\right)' dt = \frac{-Ke^{-2t}}{t} + u.$$

$$y(t) = -Ke^{-2t} + ut, \; K, u \in \mathbb{R}.$$

5.5 Exam (S1) [15.01.2023]

Exercise 5.5.1 (04 pts). **[See the correction 5.5.1]**

Course Questions:

1 State Gronwall's lemma in the form of a differential inequality.

2 State the local Cauchy–Lipschitz theorem.

3 Show that every maximal solution of

$$\begin{cases} y' = f(t, y) = t\sqrt{t^2 + y^2}, \\ y(t_0) = y_0, \end{cases}$$

is global.

Exercise 5.5.2 (05 pts). **[See the correction 5.5.2]**

We consider the Riccati differential equation

$$(E_1): \quad y' + y^2 = \frac{2}{t^2}.$$

1 Determine $a \in \mathbb{R}_+^*$ such that

$$y_0(t) = \frac{a}{t}$$

is a particular solution of (E_1).

2 Show that the change of the unknown function

$$y(t) = y_0(t) + \frac{1}{z(t)}$$

transforms equation (E_1) into a differential equation (E_2).

3 Determine the general solution of the differential equation (E_2).

4 Hence deduce the solutions of (E_1) on

$$J = (0, +\infty).$$

Exercise 5.5.3 (03 pts). **[See the correction 5.5.3]**

Consider the differential equation

$$(E): \quad (1 - t^2)y'' - 2ty' + 2y = 0.$$

1 Determine a solution of equation (E) of the form

$$y(t) = t^{\alpha},$$

where $\alpha \in \mathbb{R}$.

2 Let

$$y(t) = t^{\alpha} z(t).$$

Determine the differential equation satisfied by z.

3 Hence, deduce the solutions of (E) on the interval

$$(0, +\infty).$$

Exercise 5.5.4 (08 pts). **[See the correction 5.5.4]**

Solve the following differential system $(E1)$:

$$Y = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} Y' + \begin{pmatrix} e^{2t} \\ e^{t} \end{pmatrix} \quad \dots\dots\dots (E1)$$

where $Y = \begin{pmatrix} x \\ y \end{pmatrix}$ and $Y' = \begin{pmatrix} x' \\ y' \end{pmatrix}$.

5.5.1 Exam Correction (S1) [15.01.2023]

Solution of exercise 5.5.1. **[See exercise 5.5.1]**

1 **State Gronwall's Lemma in the form of a differential inequality:** Let $\phi, y,$ and Q be three continuous functions on $[a, b]$ taking positive values and satisfying the inequality

$$\forall t \in [a, b], \quad y(t) \leq Q(t) + \int_a^t \phi(s) y(s)\, ds \cdots [E].$$

Then,

$$\forall t \in [a, b], \quad y(t) \leq Q(t) + \int_a^b Q(s)\phi(s)\, e^{\int_s^t \phi(u)\, du}\, ds.$$

2 **State the local Cauchy–Lipschitz Theorem:** Let $f : I \times \Omega \to \mathbb{R}$, where $I \subset \mathbb{R}$, $\Omega \subset \mathbb{R}^n$. Suppose f is continuous and locally Lipschitz with respect to y. Then, for every $(t_0, y_0) \in I \times \Omega$, there exists a neighborhood J of t_0 in $\mathbb{R}$ and a function $\phi : J \to \mathbb{R}^n$ solving $\dot{y} = f(t, y)$ such that $\phi(t_0) = y_0$. Moreover, the problem admits a unique maximal solution.

3 **Show that every maximal solution of**

$$\begin{cases} y' = t\sqrt{t^2 + y^2}, \\ y(t_0) = y_0, \end{cases}$$

is global.

We show that f is Lipschitz continuous:

$$\begin{aligned}|f(t,y_1)-f(t,y_2)| &\leq |t\sqrt{t^2+y_1^2}-t\sqrt{t^2+y_2^2}| \leq |t|\,|\sqrt{t^2+y_1^2}-\sqrt{t^2+y_2^2}| \\ &< |t|\frac{|y_1^2-y_2^2|}{\sqrt{t^2+y_1^2}+\sqrt{t^2+y_2^2}} \\ &\leq |t|\frac{|y_1+y_2||y_1-y_2|}{\sqrt{t^2+y_1^2}+\sqrt{t^2+y_2^2}} \\ &< \frac{|t|(|y_1|+|y_2|)}{\sqrt{t^2+y_1^2}+\sqrt{t^2+y_2^2}}|y_1-y_2|.\end{aligned}$$

Hence, $|f(t,y_1)-f(t,y_2)| < |t|\,|y_1-y_2|$, because

$$\frac{|y_1|+|y_2|}{\sqrt{t^2+y_1^2}+\sqrt{t^2+y_2^2}} \leq 1.$$

Solution of exercise 5.5.2. **[See exercise 5.5.2]**

$$(E_1): \quad y'+y^2=\frac{2}{t^2}$$

1 Determine $a \in \mathbb{R}_+^*$ such that $y_0(t)=\frac{a}{t}$ is a particular solution.

Compute:

$$y_0'(t)+(y_0(t))^2=\frac{-a+a^2}{t^2}.$$

Thus, y_0 is a solution if and only if

$$a^2-a-2=0, \quad \text{which gives } a=2 \text{ or } a=-1.$$

Since $a \in \mathbb{R}_+$, we take $a=2$.

2 If z is a C^1 function that never vanishes, set

$$y(t)=\frac{2}{t}+\frac{1}{z(t)}.$$

Then y is a solution if and only if

$$-\frac{2}{t^2}-\frac{z'(t)}{(z(t))^2}+\left(\frac{2}{t}+\frac{1}{z(t)}\right)^2=\frac{2}{t^2}.$$

This reduces to

$$-\frac{z'(t)}{(z(t))^2}+\frac{4}{tz(t)}+\frac{1}{(z(t))^2}=0.$$

Multiplying through by $z(t)^2$, we see that z satisfies

$$(E_2): \quad z'(t) - \frac{4}{t}z(t) = 1.$$

The general solution of (E_2) is

$$z(t) = -\frac{1}{3}t + \lambda t^4, \quad \lambda \in \mathbb{R}.$$

Solution of exercise 5.5.3. **[See exercise 5.5.3]**

We consider the differential equation:

$$(1-t^2)y'' - 2ty' + 2y = 0.$$

1 Suppose $y(t) = t^\alpha$ is a solution of (E). Substituting into (E), we get:

$$(1-t^2)\alpha(\alpha-1)t^{\alpha-2} - 2t\alpha t^{\alpha-1} + 2t^\alpha = 0.$$

Equivalently,

$$(1-\alpha)\left[(\alpha+2)t^\alpha - \alpha t^{\alpha-2}\right] = 0.$$

If $\alpha = 1$, then $y_0(t) = t$ is a solution.

2 Let $y(t) = tz(t)$. Then $y'(t) = z + tz'$, $y''(t) = 2z' + tz''$. Substituting into (E), we get:

$$(1-t^2)(2z' + tz'') - 2t(z + tz') + 2tz = 0.$$

Simplifying gives

$$(t - t^3)z'' + (2 - 4t^2)z' = 0.$$

3 Setting $u = z'$, the differential equation becomes

$$\frac{u'}{u} = \frac{2-4t^2}{t^3 - t} = \frac{-2}{t} - \frac{1}{t-1} - \frac{1}{t+1},$$

which is a first-order ODE.

Integrating:

$$\ln u = -2\ln t - \ln(t-1) - \ln(t+1) + K,$$

hence

$$u = \frac{\lambda}{t^2(t^2-1)}, \quad \lambda \in \mathbb{R}.$$

Thus,

$$z'(t) = \frac{\lambda}{t^2(t^2-1)} = \lambda\Big(\frac{-1}{t^2} + \frac{1}{2(t-1)} - \frac{1}{2(t+1)}\Big).$$

Integrating once more:

$$z(t) = \frac{\lambda}{t} + \frac{\lambda}{2}\ln\Big|\frac{t-1}{t+1}\Big| + \mu, \quad \lambda, \mu \in \mathbb{R}.$$

Therefore, the solutions are

$$y(t) = tz(t) = \lambda + \frac{\lambda}{2}t\ln\Big|\frac{t-1}{t+1}\Big| + \mu t, \quad \lambda, \mu \in \mathbb{R}.$$

Solution of exercise 5.5.4. [See exercise 5.5.4]

5.6 Exam (S1) [18.01.2024]

Exercise 5.6.1 (06 pts). [See the correction 5.6.1]

Course Questions: Answer *True or False*. Justify the true statements and correct the false ones.

1 Every global solution of the equation $y' = f(t, y)$ is maximal.

2 If $f \in C^1(I \times \Omega)$ then f satisfies the conditions of the Cauchy–Lipschitz theorem.

3 Let y be the solution of the equation $y' = -y^2$ defined on $J =]-\infty, 0[$ by $y(t) = \frac{1}{t}$. Then y is maximal.

4 The solutions of the equation $y' = e^t y + y^2$ are of class $C^3(J)$.

5 For $A, B \in M_n(\mathbb{R})$, we have $e^{A+B} = e^A e^B$.

Exercise 5.6.2 (04 pts). [See the correction 5.6.2]

1 Study the local Lipschitz continuity of the function f, defined on $\mathbb{R}$ by

$$f(y) = 3\sqrt{|y|},$$

in a neighborhood of 0.

2 Let $a \geq 0$. Verify that the function y, defined on $\mathbb{R}$ by

$$y(t) = \begin{cases} \frac{9}{4}(t-a)^2, & t > a, \\ 0, & t \leq a, \end{cases}$$

is a solution of the Cauchy problem

$$y'(t) = 3\sqrt{|y(t)|}.$$

Exercise 5.6.3 (07 pts). **[See the correction 5.6.3]**

Consider the differential equation

$$(E_2): \quad y'' = -y - \frac{5}{2}y',$$

subject to the initial conditions

$$y(0) = a, \qquad y'(0) = 0.$$

1 Solve this initial value problem on $\mathbb{R}$.

2 Rewrite the differential equation (E_2) as a first-order system of differential equations

$$Y' = AY.$$

3 Show that there exist a diagonal matrix D and an invertible matrix P such that

$$A = PDP^{-1}.$$

4 Compute the matrix exponential e^{tD}, and hence determine the solution of the differential system

$$X' = DX,$$

satisfying the initial condition

$$X(0) = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

Exercise 5.6.4 (04 pts)**.** **[See the correction 5.6.4]**

1 Solve the differential system. $y' = \begin{pmatrix} -2 & 1 \\ 0 & -2 \end{pmatrix} y \cdots\cdots (H)$

2 Determine the resolvent matrix, a fundamental matrix, and a fundamental system of solutions of the system. (H).

5.6.1 Exam Correction (S1) [18.01.2024]

Solution of exercise 5.6.1. **[See exercise 5.6.1]**

1 Every global solution of the equation $y' = f(t, y)$ is maximal.

True, because the global solution is defined on the entire interval, which is the largest possible interval of definition.

2 If $f \in C^1(I \times \Omega)$ then f satisfies the conditions of the Cauchy–Lipschitz theorem.

True, because

$$f \in C^1(I \times \Omega) \Longrightarrow \begin{cases} f \text{ is continuous on } I \times \Omega, \\ \quad \text{and} \\ f \text{ is locally Lipschitz with respect to } y. \end{cases}$$

3 Let y be the solution of the equation $y' = -y^2$ defined on $J =]-\infty, 0[$ by $y(t) = \frac{1}{t}$, then y is maximal.

True, because

$$\lim_{t \overset{<}{\to} 0} \frac{1}{t} \qquad \text{does not exist.} \qquad (0.5 \; point)$$

4 The solutions of the equation $y' = e^t y + y^2$ are of class $C^3(J)$.

True, because the function defined by $f(t, y) = e^t y + y^2$ belongs to $C^3(J)$.

5 Let $A, B \in M_n(\mathbb{R})$. Then $e^{A+B} = e^A \cdot e^B$.

If $AB = BA$, then $e^{A+B} = e^A \cdot e^B$.

Solution of exercise 5.6.2. [See exercise 5.6.2]

1 Study the Lipschitz property in the neighborhood of "0" of the function f defined on $\mathbb{R}$ by :

$$f(y) = 3\sqrt{|y|}.$$

In the neighborhood of $y = 0$, the function f is not locally Lipschitz. Indeed, for $k > 0$, if we take $y_1 = 0$ and $y_2 = \frac{u}{k^2}$, we have

$$|f(y_2) - f(y_1)| = |3\sqrt{|y_2|} - 0| = \frac{6}{k} > k|\frac{u}{k^2} - 0| = k|y_2 - y_1|.$$

2 Let $a \geq 0$, verify that the function y defined on $\mathbb{R}$ by

$$y(t) = \begin{cases} \frac{9}{4}(t-a)^2 & \text{if } t > a, \\ 0 & \text{if } t \leq a. \end{cases}$$

We have for $t = a,\ y(a) = 0$

$$\lim_{t \to a^+} \frac{y(t) - y(a)}{t - a} = \lim_{t \to a^+} \frac{\frac{9}{4}(t-a)^2 - 0}{t - a} = 0,$$

and

$$\lim_{t \to a^-} \frac{y(t) - y(a)}{t - a} = \lim_{t \to a^-} \frac{0 - 0}{t - a} = 0.$$

$$y'(t) = \begin{cases} \frac{9}{2}(t-a) & \text{if } t > a, \\ 0 & \text{if } t = a = 3\sqrt{|y(t)|}, \\ 0 & \text{if } t < a. \end{cases}$$

It follows that

$$y(t) = \begin{cases} \frac{9}{4}(t-a)^2 & \text{if } t > a, \\ 0 & \text{if } t \leq a. \end{cases}$$

is a solution of the problem

$$\begin{cases} y'(t) = 3\sqrt{|y(t)|}, \\ y(0) = 0. \end{cases}$$

Solution of exercise 5.6.3. [See exercise 5.6.3]

Consider the differential equation

$$(E_2): \quad y'' = -y - \frac{5}{2}y',$$

subject to the initial conditions

$$\begin{cases} y(0) = a, \\ y'(0) = 0. \end{cases}$$

1 Solve this initial value problem on $\mathbb{R}$.

Equation (E_2) is a second-order homogeneous linear differential equation.

The associated characteristic equation is $(EC) : r^2 + \frac{5}{2}r + 1 = 0$, $\quad s = \frac{25}{4} - 4 = \frac{25}{4} - \frac{16}{4} = \frac{9}{4}$

$$\begin{cases} r_1 = \dfrac{-\frac{5}{2} - \frac{3}{2}}{2} = \dfrac{-8}{4} = -2 \\ r_2 = \dfrac{-\frac{5}{2} + \frac{3}{2}}{2} = \dfrac{-2}{4} = \dfrac{-1}{2}. \end{cases}$$

Therefore, the general solution of (E_2) is given by

$$y(t) = \lambda e^{-2t} + \mu e^{-t/2}, \qquad \lambda, \mu \in \mathbb{R}.$$

It remains to satisfy the initial conditions.

If

$$y(t) = \lambda e^{-2t} + \mu e^{-t/2}, \qquad \lambda, \mu \in \mathbb{R},$$

then we have: (1 point)

$$\begin{cases} y(0) = a \\ y'(0) = 0 \end{cases} \iff \begin{cases} \lambda + \mu = a \\ -2\lambda - \frac{5}{2}\mu = 0 \end{cases} \iff \begin{cases} \lambda = a - \mu \\ -2(a - \mu) - \frac{1}{2}\mu = 0 \end{cases}$$

$$\iff \begin{cases} \lambda = a - \mu \\ -2a + 2\mu - \frac{1}{2}\mu = 0 \end{cases} \iff \begin{cases} \lambda = a - \mu \\ -2a + \frac{3}{2}\mu = 0 \end{cases} \iff \begin{cases} \lambda = a - \mu \\ -4a + 3\mu = 0 \end{cases}$$

$$\iff \begin{cases} \lambda = a - \mu \\ \mu = \frac{+4}{3}a \end{cases} \iff \begin{cases} \lambda = a - \frac{4}{3}a \\ \mu = \frac{+4}{3}a \end{cases} \iff \begin{cases} \lambda = \frac{-1}{3}a \\ \mu = \frac{+4}{3}a. \end{cases}$$

Finally,

$$\forall t \in \mathbb{R}, \qquad y(t) = -\frac{a}{3}e^{-2t} + \frac{4a}{3}e^{-t/2}.$$

2 Rewrite (E_2) as the first-order system

$$Y' = AY.$$

We have:

$$y'' = -y - \frac{5}{2}y' \Longrightarrow \begin{cases} y_1(t) = y(t) \\ y_2(t) = y'(t) \end{cases} \Longrightarrow \begin{cases} y_1'(t) = y'(t) \\ y_2'(t) = y''(t) \end{cases} \qquad (0.5\ point)$$

Let

$$y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \Longrightarrow y' = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}'$$

$$\Longrightarrow \begin{cases} y_1'(t) = y_2 \\ y_2'(t) = -y_1 - \frac{5}{2}y_2 \end{cases} \Longrightarrow \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ -1 & \frac{5}{2} \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \qquad (1\ point)$$

$$(E_2) \qquad \Longleftrightarrow y' = Ay \Longrightarrow A = \begin{pmatrix} 0 & 1 \\ -1 & \frac{5}{2} \end{pmatrix}$$

3 Show that there exist matrices D and P.

That is, show that A is diagonalizable.

$$P_A(\lambda) = \lambda^2 - \lambda T_r A + \det A = \det(A - \lambda I_2) = \det \begin{pmatrix} -\lambda & 1 \\ -1 & \frac{5}{2} - \lambda \end{pmatrix}$$

$$= \lambda(\lambda + \frac{5}{2}) + 1 = \lambda^2 + \frac{5}{2}\lambda + 1$$

$$P_A(\lambda) = \lambda^2 + \frac{5}{2}\lambda + 1 \Longrightarrow \Delta = \frac{25}{4} - \frac{16}{4} = \frac{9}{4} > 0$$

$$\begin{cases} \lambda_1 = \dfrac{\frac{-5}{2} - \frac{3}{2}}{2} = \dfrac{-8}{4} = -2 \\ \lambda_2 = \dfrac{\frac{-5}{2} + \frac{3}{2}}{2} = \dfrac{-1}{2} \end{cases}$$

Since the matrix A has two distinct eigenvalues, it follows that it is diagonalizable.

$$\exists P \in GL_2 \Longleftrightarrow A = P \cdot \begin{pmatrix} -2 & 0 \\ 0 & \frac{-1}{2} \end{pmatrix} \cdot P^{-1}.$$

4 Compute the matrix exponential e^{tD}, and hence determine the solution of the differential system

$$X' = DX,$$

subject to the initial condition

$$tD = t\begin{pmatrix} -2 & 0 \\ 0 & \frac{-1}{2} \end{pmatrix} = \begin{pmatrix} -2t & 0 \\ 0 & \frac{-t}{2} \end{pmatrix}$$

$$e^{tD} = \sum_{n=0}^{+\infty} \frac{(tD)^n}{n!} = \sum_{n=0}^{+\infty} \frac{t^n D^n}{n!} = \sum_{n=0}^{+\infty} \frac{t^n}{n!} \begin{pmatrix} (-2)^n & 0 \\ 0 & (\frac{-1}{2})^n \end{pmatrix}$$
$$= \begin{pmatrix} \sum_{n=0}^{+\infty} \frac{(-2t)^n}{n!} & 0 \\ 0 & \sum_{n=0}^{+\infty} \frac{(\frac{-t}{2})^n}{n!} \end{pmatrix}$$

D'où

$$e^{tA} = \begin{pmatrix} e^{-2t} & 0 \\ 0 & e^{\frac{-t}{2}} \end{pmatrix}$$

The general solution of the differential system

$$X' = DX$$

is of the form

$X(t) = e^{tD}X_0$ with $X(0) = X_0$. Hence, the required solution is :

$$X(t) = e^{tD}X_0 = \begin{pmatrix} e^{-2t} & 0 \\ 0 & e^{\frac{-t}{2}} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} e^{-2t}x_1 \\ e^{\frac{-t}{2}}x_2 \end{pmatrix}$$

Solution of exercise 5.6.4. [See exercise 5.6.4]

1 Solve the differential system:

$$y' = \begin{pmatrix} -2 & 1 \\ 0 & -2 \end{pmatrix} y$$

The solution of the differential system

$$y' = \begin{pmatrix} -2 & 1 \\ 0 & -2 \end{pmatrix} y$$

is given, for all $t \in \mathbb{R}$, by

$$y(t) = e^{tA}C, \qquad y \in \mathbb{R}^2, \text{ here } A = \begin{pmatrix} -2 & 1 \\ 0 & -2 \end{pmatrix}$$

$$e^{tA} = e^{t\begin{pmatrix} -2 & 1 \\ 0 & -2 \end{pmatrix}} = e^{\begin{pmatrix} -2t & t \\ 0 & -2t \end{pmatrix}} = e^{-2tI_2 + \begin{pmatrix} 0 & t \\ 0 & 0 \end{pmatrix}} = e^{-2t} \cdot e^{\begin{pmatrix} 0 & t \\ 0 & 0 \end{pmatrix}}$$

It is easy to verify that

$$\begin{pmatrix} 0 & t \\ 0 & 0 \end{pmatrix}$$

is a nilpotent matrix of index 2. Hence,

$$e^{tA} = e^{-2t}\Big(I_2 + \frac{\begin{pmatrix} 0 & t \\ 0 & 0 \end{pmatrix}}{1!}\Big) = \begin{pmatrix} e^{-2t} & te^{-2t} \\ 0 & e^{-2t} \end{pmatrix}.$$

Therefore,

$$\forall t \in \mathbb{R}, \quad y(t) = \begin{pmatrix} e^{-2t} & te^{-2t} \\ 0 & e^{-2t} \end{pmatrix} C \quad \text{with } C \in \mathbb{R}^2.$$

2 The resolvent matrix.

Let $t, t_0 \in \mathbb{R}$. Then

$$R(t, t_0) = e^{(t-t_0)A} = \begin{pmatrix} e^{-2(t-t_0)} & (t-t_0)e^{-2(t-t_0)} \\ 0 & e^{-2(t-t_0)} \end{pmatrix}.$$

- A fundamental matrix M:

For every $t \in \mathbb{R}$,

$$M(t) = R(t, 0) = \begin{pmatrix} e^{-2t} & te^{-2t} \\ 0 & e^{-2t} \end{pmatrix}.$$

- A fundamental system of solutions $\{y_1, y_2\}$:

Since y_1 and y_2 are the columns of M, for every $t \in \mathbb{R}$,

$$y_1(t) = \begin{pmatrix} e^{-2t} \\ 0 \end{pmatrix}, \qquad y_2(t) = \begin{pmatrix} te^{-2t} \\ e^{-2t} \end{pmatrix}.$$

5.7 Exam (S1) [2024–2025]

Exercise 5.7.1 (03 pts). **[See the correction 5.7.1]**

Course questions:

Consider the Cauchy problem (P)

$$\begin{cases} y'(t) = \sqrt{y^2 + t^2} \\ y(t_0) = y_0 \end{cases}.$$

Prove the existence of a unique maximal solution to problem (P). (Apply the Cauchy–Lipschitz theorem).

Exercise 5.7.2 (07 pts). **[See the correction 5.7.2]**

Consider the Cauchy problem (1)

$$\begin{cases} y'(t) - \dfrac{y(t)}{t} - y^2(t) = -9t^2, \quad t > 0, \\ y(1) = 4. \end{cases} \tag{1}$$

1 Explain why the Cauchy problem (1) admits a unique maximal solution $\varphi : J \mapsto \mathbb{R}$.

2 Verify that the function $t \mapsto 3t$ is a particular solution of the differential equation

$$y'(t) - \frac{y(t)}{t} - y^2(t) = -9t^2, \quad t > 0.$$

3 For all $t \in J$, define the function Ψ by

$$\Psi(t) = \frac{1}{3t - \varphi(t)}.$$

- Verify that $\forall t \in J,\ \varphi(t) = -\dfrac{1}{\Psi(t)} + 3t.$
- Show that Ψ is a solution of the Cauchy problem

$$\begin{cases} \Psi'(t) + (6t + \dfrac{1}{t})\Psi(t) = 1, \quad t > 0, \\ \Psi(1) = -1. \end{cases} \tag{2}$$

- Determine the solution $\Psi(t)$ of the Cauchy problem (2) for all $t > 0$.

Exercise 5.7.3 (04 pts). **[See the correction 5.7.3]**

Solve the differential equation

$$t^2 y''(t) + t y'(t) + \left(t^2 - \frac{1}{4}\right) y(t) = 0,$$

by setting $y(t) = u(t)z(t)$.

Exercise 5.7.4 (06 pts). **[See the correction 5.7.4]**

We consider the following system of differential equations:

$$\begin{cases} x'(t) = -x + 3y + e^t, \\ y'(t) = -2x + 4y. \end{cases}$$

1 Write in the form $z' = Az + b$ where A is a 2×2 matrix, and $b, z : \mathbb{R} \mapsto \mathbb{R}^2$.

2 Solve the homogeneous system $z' = Az$.

3 Find the unique solution of (3) that satisfies $x(0) = y(0) = 1$.

5.7.1 Exam Correction (S1) [2024–2025]

Solution of exercise 5.7.1. [See exercise 5.7.1]

We have $f(t, y) = \sqrt{y^2 + t^2}$, where f is a continuous function on $\mathbb{R} \times \mathbb{R}$ since it is the composition of a polynomial function and the square root function.

Method 1: Let $r_0 > 0$, $T > 0$ and $C = [t_0 - T, t_0 + T] \times [y_0 - r_0, y_0 + r_0]$. For $(t, y_1), (t, y_2) \in C$, we have:

$$\begin{aligned}
|f(t, y_1) - f(t, y_2)| &= \left| \sqrt{y_1^2 + t^2} - \sqrt{y_2^2 + t^2} \right| \\
&= \left| (\sqrt{y_1^2 + t^2} - \sqrt{y_2^2 + t^2}) \times \frac{\sqrt{y_1^2 + t^2} + \sqrt{y_2^2 + t^2}}{\sqrt{y_1^2 + t^2} + \sqrt{y_2^2 + t^2}} \right| \\
&= \frac{|(y_1^2 + t^2) - (y_2^2 + t^2)|}{\sqrt{y_1^2 + t^2} + \sqrt{y_2^2 + t^2}} \\
&= \frac{|y_1^2 - y_2^2|}{\sqrt{y_1^2 + t^2} + \sqrt{y_2^2 + t^2}} \\
&= \frac{|y_1 - y_2||y_1 + y_2|}{\sqrt{y_1^2 + t^2} + \sqrt{y_2^2 + t^2}}
\end{aligned}$$

$$\begin{cases} y_1^2 \leq y_1^2 + t^2 \\ y_2^2 \leq y_2^2 + t^2 \end{cases} \implies \begin{cases} \sqrt{y_1^2 + t^2} \geq |y_1| \\ \sqrt{y_2^2 + t^2} \geq |y_2| \end{cases} \implies \sqrt{y_1^2 + t^2} + \sqrt{y_2^2 + t^2} \geq |y_1| + |y_2|$$

Thus:

$$\frac{|y_1| + |y_2|}{\sqrt{y_1^2 + t^2} + \sqrt{y_2^2 + t^2}} \leq 1$$

Therefore, f is locally Lipschitz.

Solution of exercise 5.7.2. [See exercise 5.7.2]

Consider the Cauchy problem

$$\begin{cases} y'(t) - \dfrac{y(t)}{t} - y^2(t) = -9t^2, & t > 0, \\ y(1) = 4. \end{cases} \tag{1}$$

By the Cauchy–Lipschitz theorem, this problem admits a unique solution.

1. Explain why the Cauchy problem (1) admits a unique maximal solution

$$\varphi : J \longrightarrow \mathbb{R}.$$

Since the function

$$g(t, y) = \frac{y}{t} + y^2 - 9t^2$$

is continuous on $\mathbb{R}_+^* \times \mathbb{R}$ and locally Lipschitz continuous with respect to the second variable, it follows from the Cauchy–Lipschitz theorem that the Cauchy problem (1) admits a unique maximal solution φ, defined on an open interval $J \subset \mathbb{R}$.

2. Verify that the function $t \mapsto 3t$ is a particular solution of the differential equation

$$y'(t) - \frac{y(t)}{t} - y^2(t) = -9t^2, \qquad t > 0.$$

Indeed,

$$y(t) = 3t, \qquad t > 0,$$

implies

$$y'(t) = 3.$$

Hence,

$$y'(t) - \frac{y(t)}{t} - y^2(t) = 3 - \frac{3t}{t} - (3t)^2 = 3 - 3 - 9t^2 = -9t^2.$$

Therefore, the function $t \mapsto 3t$ is a particular solution of the differential equation.

3. For every $t \in J$, define the function

$$\Psi(t) = \frac{1}{3t - \varphi(t)}.$$

- Verify that
$$\forall t \in J, \qquad \varphi(t) = 3t - \frac{1}{\Psi(t)}.$$
Indeed,
$$\Psi(t) = \frac{1}{3t - \varphi(t)} \iff 3t - \varphi(t) = \frac{1}{\Psi(t)} \iff \varphi(t) = 3t - \frac{1}{\Psi(t)}.$$
Next, show that Ψ is a solution of the Cauchy problem
$$\begin{cases} \Psi'(t) + \left(6t + \frac{1}{t}\right)\Psi(t) = 1, \qquad t > 0, \\ \Psi(1) = -1. \end{cases}$$
Since
$$\varphi(t) = 3t - \frac{1}{\Psi(t)},$$
we obtain
$$\varphi'(t) = 3 + \frac{\Psi'(t)}{\Psi^2(t)}.$$
On the other hand,
$$\begin{aligned} \varphi'(t) &= \frac{\varphi(t)}{t} + \varphi^2(t) - 9t^2 \\ &= -\frac{1}{t\Psi(t)} + 3 + \left(3t - \frac{1}{\Psi(t)}\right)^2 - 9t^2 \\ &= -\frac{1}{t\Psi(t)} + 3 + \frac{1}{\Psi^2(t)} - \frac{6t}{\Psi(t)}. \end{aligned}$$
Comparing the two expressions yields
$$\frac{\Psi'(t)}{\Psi^2(t)} = -\left(\frac{1}{t} + 6t\right)\frac{1}{\Psi(t)} + \frac{1}{\Psi^2(t)},$$
which is equivalent to
$$\Psi'(t) + \left(\frac{1}{t} + 6t\right)\Psi(t) = 1.$$
Moreover,
$$\Psi(1) = \frac{1}{3 - \varphi(1)} = \frac{1}{3 - 4} = -1.$$

Therefore, Ψ satisfies the Cauchy problem

$$\begin{cases} \Psi'(t) + \left(\dfrac{1}{t} + 6t\right)\Psi(t) = 1, & t > 0, \\ \Psi(1) = -1. \end{cases}$$

- Determine the solution $\Psi(t)$ of the Cauchy problem for all $t > 0$. The associated homogeneous equation is

$$\Psi'(t) + \left(\frac{1}{t} + 6t\right)\Psi(t) = 0,$$

which gives

$$\frac{\Psi'(t)}{\Psi(t)} = -\left(\frac{1}{t} + 6t\right).$$

Integrating both sides,

$$\ln|\Psi(t)| = -(\ln t + 3t^2) + C,$$

and therefore

$$\Psi(t) = K\,\frac{e^{-3t^2}}{t}, \qquad K \in \mathbb{R}.$$

Using the method of variation of parameters, let

$$\Psi_p(t) = C(t)\frac{e^{-3t^2}}{t}.$$

Differentiating,

$$\Psi_p'(t) = C'(t)\frac{e^{-3t^2}}{t} - C(t)e^{-3t^2}\left(6 + \frac{1}{t^2}\right).$$

Substituting into the complete equation gives

$$C'(t)\frac{e^{-3t^2}}{t} = 1,$$

hence

$$C'(t) = te^{3t^2}.$$

Integrating,

$$C(t) = \int te^{3t^2}\,dt = \frac{1}{6}e^{3t^2}.$$

Thus,

$$\Psi_p(t) = \frac{1}{6t},$$

and the general solution is

$$\Psi(t) = K\frac{e^{-3t^2}}{t} + \frac{1}{6t}.$$

Using the initial condition $\Psi(1) = -1$, we obtain

$$K = -\frac{7}{6}e^3.$$

Consequently,

$$\boxed{\Psi(t) = -\frac{7}{6t}e^{3(1-t^2)} + \frac{1}{6t}, \qquad t > 0.}$$

Solution of exercise 5.7.3. [See exercise 5.7.3]

Solve the following differential equation.

$$t^2y'' + ty' + (t^2 - \frac{1}{4})y = 0$$

by setting

$$\begin{cases} y' = u'z + uz' \\ y'' = u''z + 2u'z' + uz'' \end{cases}$$

Hence

$$t^2(u''z + 2u'z' + uz'') + t(u'z + uz') + (t^2 - \frac{1}{4})uz = 0$$

$$t^2uz'' + z'\left[2u't^2 + tz\right] + z[t^2u'' + tu' + ut^2 - \frac{u}{4}] = 0 \tag{5.1}$$

Since

$$II = \left(2t^2u' + tu\right)z + \left(t^2u'' + tu' + t^2u - \frac{u}{4}\right)z,$$

we set each term equal to zero separately. Thus,

$$2t^2u' + tu = 0 \quad\Longrightarrow\quad 2tu' = -u \quad\Longrightarrow\quad \frac{u'}{u} = -\frac{1}{2t}.$$

Integrating, we obtain

$$\ln|u| = -\frac{1}{2}\ln t + C,$$

hence

$$u(t) = Ct^{-1/2}.$$

Without loss of generality, we take

$$u(t) = t^{-1/2} = \frac{1}{\sqrt{t}}.$$

Now compute u' and u''.

$$u' = -\frac{1}{2}t^{\frac{-3}{2}} = -\frac{1}{2t\sqrt{t}}.$$

$$u'' = \frac{3}{4}t^{-\frac{5}{2}} = \frac{3}{4t^2\sqrt{t}}.$$

Substituting u, u', and u'' into equation (II), we obtain

$$t^2\left(\frac{3}{4t^2\sqrt{t}}\right) + t\left(-\frac{1}{2t\sqrt{t}}\right) + t^2\left(\frac{1}{\sqrt{t}}\right) - \frac{1}{4\sqrt{t}} = \frac{t^2}{\sqrt{t}}.$$

Consequently, equation (5.1) is equivalent to

$$z'' + z = 0,$$

which is a second-order homogeneous linear differential equation with constant coefficients. The associated characteristic equation is

$$r^2 + 1 = 0,$$

whose roots are

$$r = \pm i.$$

Hence,

$$z(t) = C_1 \cos t + C_2 \sin t,$$

where $C_1, C_2 \in \mathbb{R}$. Therefore, the general solution of the original differential equation is

$$y(t) = \frac{1}{\sqrt{t}}\left(C_1 \cos t + C_2 \sin t\right), \qquad C_1, C_2 \in \mathbb{R}.$$

Solution of exercise 5.7.4. **[See the correction 5.7.4]**

1 Let

$$\begin{cases} x'(t) = -x + 3y + e^t, \\ y'(t) = -2x + 4y, \end{cases}$$

and let

$$z = \begin{pmatrix} x \\ y \end{pmatrix}.$$

This is a linear differential system with constant coefficients, whose matrix form is

$$Z' = AZ + B(t).$$

$$z = \begin{pmatrix} x \\ y \end{pmatrix}, A = \begin{pmatrix} -1 & 3 \\ -2 & 4 \end{pmatrix} \text{ et } B(t) = \begin{pmatrix} e^t \\ 0 \end{pmatrix}.$$

2 Homogeneous equation: $Z' = AZ$

$$S(\lambda) = \det(A - \lambda I) = \det \begin{pmatrix} -1-\lambda & 3 \\ -2 & 4-\lambda \end{pmatrix} = (-1-\lambda)(4-\lambda) + 6 = \lambda^2 - 3\lambda + 2.$$

The eigenvalues are

$$\lambda_1 = 1, \qquad \lambda_2 = 2.$$

The corresponding eigenvectors are

$$v_1 = \begin{pmatrix} 3 \\ 2 \end{pmatrix}, \qquad v_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Hence, $D = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}, P = \begin{pmatrix} 3 & 1 \\ 2 & 1 \end{pmatrix}, P^{-1} = \frac{1}{\det P} \begin{pmatrix} 1 & -1 \\ -2 & 3 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ -2 & 3 \end{pmatrix}.$
The homogeneous solution is :

$$Y(t) = C_1 v_1 e^{\lambda_1 t} + C_2 v_2 e^{\lambda_2 t} = C_1 \begin{pmatrix} 3 \\ 2 \end{pmatrix} e^t + C_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{2t}, \quad C_1, C_2 \in \mathbb{R}$$

$$Y(t) = \begin{pmatrix} 3C_1 e^t + C_2 e^{2t} \\ 2C_1 e^t + C_2 e^{2t} \end{pmatrix}$$

3 Let $y_p(t) = C_1(t)v_1e^{\lambda_1 t} + C_2(t)v_2e^{\lambda_2 t}$. We have

$$\begin{pmatrix} C_1'(t) \\ C_2'(t) \end{pmatrix} = \begin{pmatrix} v_1e^{\lambda_1 t} & v_2e^{\lambda_2 t} \end{pmatrix}^{-1} \cdot B(t) = \begin{pmatrix} 3e^t & e^{2t} \\ 2e^t & e^{2t} \end{pmatrix}^{-1} \cdot B(t)$$

$$\begin{pmatrix} C_1'(t) \\ C_2'(t) \end{pmatrix} = \frac{1}{e^{3t}} \begin{pmatrix} e^{2t} & -e^{2t} \\ -2e^t & 3e^t \end{pmatrix} \begin{pmatrix} e^t \\ 0 \end{pmatrix} = e^{-3t} \begin{pmatrix} e^{3t} \\ -2e^{2t} \end{pmatrix} = \begin{pmatrix} 1 \\ -2e^{-t} \end{pmatrix}.$$

$$\Longrightarrow \begin{pmatrix} C_1(t) \\ C_2(t) \end{pmatrix} = \begin{pmatrix} \int 1 \; dt \\ \int -2e^{-t}dt \end{pmatrix} = \begin{pmatrix} t \\ 2e^{-t} \end{pmatrix}$$

Hence, the particular solution is

$$Y_p(t) = t \begin{pmatrix} 3 \\ 2 \end{pmatrix} e^t + 2e^{-t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{2t} = \begin{pmatrix} 3te^t + 2e^t \\ 2te^t + 2e^t \end{pmatrix}$$

Hence, the general solution is

$$Y_G(t) = Y_H(t) + Y_P(t) = \begin{pmatrix} 3C_1e^t + C_2e^{2t} \\ 2C_1e^t + C_2e^{2t} \end{pmatrix} + \begin{pmatrix} 3te^t + 2e^t \\ 2te^t + 2e^t \end{pmatrix}$$

The unique solution satisfying:

$$\begin{cases} x(0) = 1 \\ y(0) = 1 \end{cases} \Longrightarrow \begin{cases} 3C_1 + C_2 + 2 = 1 & (I) \\ 2C_1 + C_2 + 2 = 1 & (II) \end{cases}$$

Subtracting (II) from (I), we obtain

$$C_1 = 0,$$

and

$$C_2 = -1.$$

$$Y_G = \begin{pmatrix} -e^{2t} + 3te^t + 2e^t \\ -e^{2t} + 2t + 2e^t \end{pmatrix}$$

Chapter 6

Make-up exam topics

6.1 Make-up (S1) [Avril 2018]

Exercise 6.1.1 (04 pts). **[See the correction 6.1.1]**

Solve the following differential equations:

1 $t \cdot y' = y\big(1 + \ln y - \ln t\big)$.

2 $y' + t\,y = t^3 y^3$.

3 $y' + \dfrac{1}{2t}y = \dfrac{1}{2t(t+1)}, \quad t > 0$.

Exercise 6.1.2 (08 pts). **[See the correction 6.1.2]**

Solve the following differential equations:

1 $y'' + 3y' + 2y = (t^2 + 1)e^{-t}$.

2 $(2t + 1)\,y'' - 4t\,y' - 4y = 0$ (with the obvious solution $y = t$).

3 $y''' + 3y'' + 4y' + 2y = \sin(2t)$.

Exercise 6.1.3 (08 pts). **[See the correction 6.1.3]**

Consider the system

$$(S) \quad \begin{cases} \dfrac{dx}{dt} = 5x - 3y, \\ \dfrac{dy}{dt} = x + y. \end{cases}$$

1 Find the general solution of system (S).

2 Deduce the solution satisfying $x(-1) = \dfrac{11}{2}$, $y(-1) = \pi - e$.

3 Solve the nonhomogeneous system

$$\begin{cases} \dfrac{dx}{dt} = 5x - 3y + 2e^{3t}, \\ \dfrac{dy}{dt} = x + y + 5e^{-t}. \end{cases}$$

Solutions (Rattrapage — April 2018)

Solution of exercise 6.1.1. [See exercise 6.1.1]

1 **Equation:** $ty' = y\big(1 + \ln y - \ln t\big)$.

Write it as

$$y' = \frac{y}{t}\Big(1 + \ln\frac{y}{t}\Big).$$

Set $v = \dfrac{y}{t}$, so $y = vt$. Then

$$y' = v + tv' = \frac{y}{t}\big(1 + \ln v\big) = v\big(1 + \ln v\big).$$

Hence

$$tv' = v \ln v \quad \Longrightarrow \quad \frac{dv}{v \ln v} = \frac{dt}{t}.$$

Integrate:

$$\ln|\ln v| = \ln|t| + C \quad \Longrightarrow \quad \ln v = C't.$$

(Interpreting constants appropriately yields) $v = e^{t+C_1}$, and choosing the integration constant so that $v = e^t$ gives

$$y(t) = te^t$$

as a solution (the displayed steps follow the integration idea).

2 **Equation:** $y' + ty = t^3y^3$.

Divide by y^3 (assuming $y \neq 0$) and set $z = y^{-2}$:

$$\frac{y'}{y^3} + t\frac{1}{y^2} = t^3 \quad \Longrightarrow \quad -\frac{1}{2}z' + tz = t^3,$$

so

$$z' - 2tz = -2t^3.$$

This is linear for z. The homogeneous equation $z' - 2tz = 0$ has solution $z_H = Ke^{t^2}$. Use variation of constants: set $z(t) = K(t)e^{t^2}$, then

$$K'(t)e^{t^2} = -2t^3 \quad \Longrightarrow \quad K'(t) = -2t^3e^{-t^2}.$$

Integrating (for instance by recognizing derivative of $t^2 e^{-t^2}$), one finds

$$K(t) = e^{-t^2}(t^2 + 1),$$

hence

$$z_p(t) = t^2 + 1, \qquad z(t) = Ke^{t^2} + (t^2 + 1).$$

Recalling $z = 1/y^2$,

$$\boxed{y(t) = \pm \frac{1}{\sqrt{Ke^{t^2} + (t^2 + 1)}}}.$$

3 **Equation:** $y' + \dfrac{1}{2t} y = \dfrac{1}{2t(1+t)}$, $t > 0$.

Homogeneous part: $y' + \dfrac{1}{2t} y = 0$ gives

$$\ln y = -\tfrac{1}{2} \ln t + C \quad \Longrightarrow \quad y_H = \frac{C}{\sqrt{t}}.$$

For a particular solution use variation of constants: set $y_p(t) = \dfrac{K(t)}{\sqrt{t}}$. Compute $K'(t)$ from the equation and integrate. The computation yields

$$K'(t) = \frac{\sqrt{t}}{2t(1+t)} = \frac{1}{2} \cdot \frac{1}{\sqrt{t}(1+t)},$$

and integrating (with the substitution $x = \sqrt{t}$) leads to an explicit $K(t)$ and thus

$$y(t) = \frac{C}{\sqrt{t}} + \frac{1}{2\sqrt{t}} \left(\ln \sqrt{\frac{t}{t+1}} + C_1 \right).$$

(One may simplify constants; the method is variation of constants.)

Solution of exercise 6.1.2. **[See exercise 6.1.2]**

(a) $y'' + 3y' + 2y = (t^2 + 1)e^{-t}$.

Characteristic equation: $\lambda^2 + 3\lambda + 2 = 0$, roots $\lambda = -1, -2$. So

$$y_H = Ae^{-2t} + Be^{-t}.$$

For the right-hand side $(t^2 + 1)e^{-t}$ note e^{-t} corresponds to the root $\lambda = -1$ (simple), so try a particular solution of the form

$$y_p(t) = (a + bt + ct^2)\, te^{-t}.$$

Insert and determine the coefficients (a, b, c); the algebra yields the particular solution (omitted for brevity).

(b) $(2t + 1)y'' - 4ty' - 4y = 0$, with known solution $y = t$.

Seek second independent solution by reduction of order: set $y = tz(t)$. Substituting and simplifying gives a first-order equation for $x = z'$, which can be solved by separation. The computation leads to a second solution

$$S_2(t) = -e^{-2t},$$

so the general solution is

$$y(t) = At + B(-e^{-2t}) = At - Be^{-2t}.$$

(c) $y''' + 3y'' + 4y' + 2y = \sin(2t)$.

Characteristic polynomial: $\lambda^3 + 3\lambda^2 + 4\lambda + 2 = (\lambda + 1)(\lambda^2 + 2\lambda + 2)$, roots $\lambda = -1$ and $\lambda = -1 \pm i$. Hence

$$y_H = Ae^{-t} + e^{-t}\big(B\cos t + C\sin t\big).$$

For the RHS $\sin(2t)$ try $y_p = \alpha\cos(2t) + \beta\sin(2t)$; substitute and solve for α, β. The particular solution obtained is (after calculation)

$$y_p(t) = -\tfrac{1}{10}\sin(2t).$$

So the general solution is $y = y_H + y_p$.

Solution of exercise 6.1.3. [See exercise 6.1.3]

Matrix form: $Z' = AZ$ with $A = \begin{pmatrix} 5 & -3 \\ 1 & 1 \end{pmatrix}$.

1 **Eigenvalues and eigenvectors.** The characteristic polynomial is

$$\det(A - \lambda I) = \lambda^2 - 6\lambda + 8 = 0,$$

so $\lambda_1 = 2,\ \lambda_2 = 4$. Eigenvectors:

$$\lambda = 2:\ v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \qquad \lambda = 4:\ v_2 = \begin{pmatrix} 3 \\ 1 \end{pmatrix}.$$

Thus the general solution of the homogeneous system is

$$Z_H(t) = C_1 v_1 e^{2t} + C_2 v_2 e^{4t},$$

i.e.

$$x(t) = C_1 e^{2t} + 3C_2 e^{4t}, \qquad y(t) = C_1 e^{2t} + C_2 e^{4t}.$$

2 **Find constants for the initial conditions at $t = -1$.**

Solve

$$\begin{cases} C_1 e^{-2} + 3C_2 e^{-4} = \dfrac{11}{2}, \\ C_1 e^{-2} + C_2 e^{-4} = \pi - e. \end{cases}$$

Subtracting gives $2C_2 e^{-4} = \dfrac{11}{2} + e - \pi$, hence

$$C_2 = \frac{e^4}{2}\Big(\frac{11}{2} + e - \pi\Big), \qquad C_1 = e^2\Big(\pi - e - C_2 e^{-4}\Big).$$

Plug these into the general formula to obtain the requested solution.

3 **Nonhomogeneous system.**

The forcing term is

$$B(t) = \begin{pmatrix} 2e^{3t} \\ 5e^{-t} \end{pmatrix}.$$

Find particular solutions for each exponential term separately.

For $2e^{3t}$ try $(x, y) = (a, b)e^{3t}$. Substitution yields the linear algebraic system

$$3\begin{pmatrix} a \\ b \end{pmatrix} = A\begin{pmatrix} a \\ b \end{pmatrix} + \begin{pmatrix} 2 \\ 0 \end{pmatrix},$$

which solves to $a = -4, b = -2$.

For $5e^{-t}$ try $(x, y) = (a, b)e^{-t}$: substitution yields

$$-\begin{pmatrix} a \\ b \end{pmatrix} = A\begin{pmatrix} a \\ b \end{pmatrix} + \begin{pmatrix} 0 \\ 5 \end{pmatrix},$$

which solves to $a = -1, b = -2$.

Thus one particular solution is

$$Z_p(t) = \begin{pmatrix} -4 \\ -2 \end{pmatrix} e^{3t} + \begin{pmatrix} -1 \\ -2 \end{pmatrix} e^{-t}.$$

The general solution of the nonhomogeneous system is

$$Z(t) = C_1 v_1 e^{2t} + C_2 v_2 e^{4t} + Z_p(t),$$

i.e.

$$x(t) = C_1 e^{2t} + 3C_2 e^{4t} - 4e^{3t} - e^{-t}, \qquad y(t) = C_1 e^{2t} + C_2 e^{4t} - 2e^{3t} - 2e^{-t}.$$

6.2 Rattrapage (S1) [30.05.2019]

Exercise 6.2.1 (06 pts). **[See the solution. 6.2.1]**

Consider the differential equation

$$y'(t) - \frac{y(t)}{t} - y^2(t) = -9t^2. \qquad (E)$$

- Determine $a \in (0, +\infty)$ such that

$$y_0(t) = at$$

  is a particular solution of (E).

- Show that the change of the unknown function

$$y(t) = y_0(t) - \frac{1}{z(t)}$$

transforms equation (E) into the differential equation

$$z'(t) + \left(6t + \frac{1}{t}\right) z(t) = 1. \qquad (E_1)$$

Exercise 6.2.2 (08 pts). [See the solution. 6.2.2]

1 Solve the following linear differential system:

$$\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} 3x - 2y \\ -y + 2x \end{pmatrix} + \begin{pmatrix} 4e^{3t} \\ 3t \end{pmatrix}.$$

2 Compute $R(t, 2)$.

Exercise 6.2.3 (06 pts). [See the solution. 6.2.3]

Consider the differential equation

$$(E): \quad y''(t) + 2y'(t) + 4y(t) = te^t.$$

- Solve the associated homogeneous differential equation.
- Find a particular solution of (E), and hence determine the general solution of (E).
- Determine the unique solution h of (E) satisfying

$$h(0) = 1, \qquad h(1) = 0.$$

6.2.1 Make-up Exam Correction (S1) [30.05.2019]

Solution of exercise 6.2.1. [See exercise 6.2.1]

1 We look for a particular solution of (E) of the form $y(t) = at$ for $t \in]0, +\infty[$. Substituting into (E), we obtain:

$$a - \frac{at}{t} - a^2t^2 = -9t^2 \quad \Longrightarrow \quad a^2 = 9 \quad \Longrightarrow \quad a = \pm 3.$$

Thus, $y_0(t) = 3t$ is a particular solution of (E) defined on $]0, +\infty[$.

2 We make the following change of variable:

$$y(t) = y_0(t) - \frac{1}{z(t)} = 3t - \frac{1}{z(t)},$$

where z is a function defined on $]0, +\infty[$ to be determined.

3 We compute y' and y^2:

$$y'(t) = 3 + \frac{z'(t)}{z^2(t)}, \qquad y^2(t) = 9t^2 - \frac{6t}{z(t)} + \frac{1}{z^2(t)}.$$

Substituting into (E), we obtain:

$$3 + \frac{z'(t)}{z^2(t)} - \left(\frac{3t - 1/z}{t}\right) - \left(9t^2 - \frac{6t}{z(t)} + \frac{1}{z^2(t)}\right) = -9t^2,$$

which simplifies to:

$$z'(t) + z(t)\left(\frac{1}{t} + 6t\right) = 1.$$

Solution of exercise 6.2.2. [See exercise 6.2.2]

1 Solve the following linear differential system:

$$\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} 3x - 2y \\ -y + 2x \end{pmatrix} + \begin{pmatrix} 4e^{3t} \\ 3t \end{pmatrix}.$$

Correction:

$$\det(A - \lambda I) = \begin{vmatrix} 3-\lambda & -2 \\ 2 & -1-\lambda \end{vmatrix} = (\lambda - 1)^2,$$

so we have a double eigenvalue $\lambda = 1$.

Eigenvectors:

$$V_1 = \begin{pmatrix} x \\ y \end{pmatrix} \in \ker(A - \lambda I) \implies (A - \lambda I)V_1 = 0,$$

$$\implies \begin{pmatrix} 2 & -2 \\ 2 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \implies x = y \implies V_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

For V_2: $(A - \lambda I)V_2 = V_1 \implies 2x - 2y = 1$. So $y = 0,\ x = \frac{1}{2} \implies V_2 = \begin{pmatrix} \frac{1}{2} \\ 0 \end{pmatrix}$.

Thus:

$$P = \begin{pmatrix} 1 & \frac{1}{2} \\ 1 & 0 \end{pmatrix}, \quad J = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad e^{tJ} = e^t \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}.$$

$$M_f(t) = Pe^{tJ} = \begin{pmatrix} 1 & 1/2 \\ 1 & 0 \end{pmatrix} e^t \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} = e^t \begin{pmatrix} 1 & t+\frac{1}{2} \\ 1 & t \end{pmatrix}.$$

Hence the homogeneous solution:

$$S_H = Pe^{tJ} \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = e^t \begin{pmatrix} C_1 + C_2(t+\frac{1}{2}) \\ C_1 + C_2 t \end{pmatrix}.$$

For the particular solution S_P:

- First part $S_1 = e^{3t} \begin{pmatrix} A \\ B \end{pmatrix}$ leads to:

$$A = 12, \quad B = 18.$$

- Second part $S_2 = \begin{pmatrix} -6t-12 \\ -9t-15 \end{pmatrix}$.

2 Transition matrix:

$$R(t,2) = M_f(t) \cdot M_f^{-1}(2) = e^{t-2} \begin{pmatrix} 2t-3 & 4-2t \\ 2t-4 & 5-2t \end{pmatrix}.$$

Solution of exercise 6.2.3. [See exercise 6.2.3]

We consider the equation:

$$y'' + 2y' + 4y = te^t \quad (E)$$

1 Homogeneous part: $y'' + 2y' + 4y = 0$.

The characteristic polynomial is $P(\lambda) = \lambda^2 + 2\lambda + 4$.

$$\Delta = -12 \implies \lambda_{1,2} = -1 \pm i\sqrt{3}.$$

Thus:

$$y_H(t) = e^{-t}\left[C_1 \cos(\sqrt{3}t) + C_2 \sin(\sqrt{3}t)\right].$$

2 Particular solution: we try $y_p(t) = e^t(at+b)$. After substitution, we get:

$$y_p(t) = \frac{1}{7}\left(t - \frac{4}{7}\right)e^t.$$

So the general solution is:

$$y_G(t) = e^{-t}[C_1\cos(\sqrt{3}t) + C_2\sin(\sqrt{3}t)] + \frac{1}{7}\left(t - \tfrac{4}{7}\right)e^t.$$

3 With initial conditions $h(0) = 1,\ h'(1) = 0$:

From $h(0) = 1$:

$$C_1 = \tfrac{53}{49}.$$

From $h'(1) = 0$ we compute C_2:

$$C_2 = \frac{3e^2 + 53\cos(\sqrt{3})}{49\sin(\sqrt{3})}.$$

6.3 Make-up Exam (S1) [28.12.2020]

Exercise 6.3.1 (08 pts). [See the correction 6.3.1]

Consider the system

$$Y' = \begin{pmatrix} 1 & 1 \\ 2 & 0 \end{pmatrix} Y \cdots\cdots (H)$$

- Use the spectral method and then the matrix exponential method to solve the system (H').

Exercise 6.3.2 (05 pts). [See the correction 6.3.2]

Solve the following differential equation:

$$ty'' + y' = t^2.$$

Exercise 6.3.3 (07 pts). [See the correction 6.3.3]

Consider the equation:

$$y''(t) + 2y'(t) + 4y(t) = te^t \qquad (E)$$

1 Solve the homogeneous differential equation associated with (E).

2 Find a particular solution of (E), then give the set of all solutions of (E).

6.3.1 Correction Make-up Exam (S1) [28.12.2020]

Solution of exercise 6.3.1. [See exercise 6.3.1]

Consider the system $Y' = \begin{pmatrix} 1 & 1 \\ 2 & 0 \end{pmatrix} Y \cdots (H)$

a) The spectral method: We have $A = \begin{pmatrix} 1 & 1 \\ 2 & 0 \end{pmatrix}$. The two eigenvalues of A are $\lambda_1 = -1, \lambda_2 = 2$. The associated eigenvectors are $V_1 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}, V_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. Thus, the general solution is given, for all $t \in \mathbb{R}$, by

$$Y(t) = c_1 V_1 e^{\lambda_1 t} + c_2 V_2 e^{\lambda_2 t} = \begin{pmatrix} c_1 e^{-t} + c_2 e^{2t} \\ c_2 e^{2t} - 2c_1 e^{-t} \end{pmatrix}, \text{ with } c_1, c_2 \in \mathbb{R}.$$

b) The matrix exponential method: Since A admits 2 distinct eigenvalues, A is diagonalizable. $A = PDP^{-1}$. Here

$$D = \begin{pmatrix} -1 & 0 \\ 0 & 2 \end{pmatrix}, \quad P = (V_1|V_2) = \begin{pmatrix} 1 & 1 \\ -2 & 1 \end{pmatrix}, \quad P^{-1} = \frac{1}{3} \begin{pmatrix} 1 & -1 \\ 2 & 1 \end{pmatrix}.$$

$$\begin{aligned} e^{tA} = Pe^{tD}P^{-1} &= \frac{1}{3} \begin{pmatrix} 1 & 1 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} e^{-t} & 0 \\ 0 & e^{2t} \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 2 & 1 \end{pmatrix} \\ &= \frac{1}{3} \begin{pmatrix} e^{-t} + 2e^{2t} & -e^{-t} + e^{2t} \\ -2e^{-t} + 2e^{2t} & 2e^{-t} + e^{2t} \end{pmatrix}. \end{aligned}$$

Hence, the solution of (H') is given by:

$$Y(t) = e^{tA} C = \frac{1}{3} \begin{pmatrix} e^{-t} + 2e^{2t} & -e^{-t} + e^{2t} \\ -2e^{-t} + 2e^{2t} & 2e^{-t} + e^{2t} \end{pmatrix} C, \quad C \in \mathbb{R}^2.$$

Solution of exercise 6.3.2. [See exercise 6.3.2]

1 Solve the following differential equation:

$$ty'' + y' = t^2$$

Dividing by t (for $t \neq 0$), we get: $y'' + \frac{1}{t}y' = t$. Let $\begin{cases} z = y' \\ z' = y'' \end{cases}$

The equation becomes:

$$z' + \frac{1}{t}z = t.$$

We look for the homogeneous solution of the equation $z' + \frac{1}{t}z = 0$.

$$z' + \frac{1}{t}z = 0 \implies \frac{z'}{z} = -\frac{1}{t} \implies \frac{dz}{z} = -\frac{dt}{t}$$

$$\implies \ln z = -\ln|t| + C \implies z_H = \frac{K}{t},\ K \in \mathbb{R} \implies y' = \frac{K}{t} \implies \frac{dy}{dt} = \frac{K}{t}$$

$$dy = \frac{K}{t}dt \implies \int dy = K\int \frac{1}{t}dt \implies y_H = K\ln(t) + K_1,\ K, K_1 \in \mathbb{R}.$$

2 Look for a particular solution z_p of the equation $z' + \frac{1}{t}z = t$.

Let $z_p(t) = \frac{K(t)}{t}$. Differentiating: $z_p' = \frac{K't - K}{t^2} = \frac{K'}{t} - \frac{K}{t^2}$.

$$\frac{K'}{t} - \frac{K}{t^2} + \frac{K}{t^2} = t \implies \frac{K'}{t} = t \implies K' = t^2 \implies K(t) = \int t^2 dt.$$

This implies that the particular solution for z is $z_p(t) = \frac{1}{3}t^2$. Thus,

$$y_p' = \frac{1}{3}t^2 \implies y_p(t) = \int \frac{1}{3}t^2 dt = \frac{1}{9}t^3 \implies z_p(t) = \frac{1}{9}t^3.$$

$$y_G(t) = K_1\ln(t) + K_2 + \frac{1}{9}t^3.$$

Solution of exercise 6.3.3. [See exercise 6.3.3]

Consider the equation

$$y'' + 2y' + 4y = te^t \cdots (E)$$

1 The homogeneous equation $y'' + 2y' + 4y = 0$.

The characteristic polynomial associated with (E) is

$P(\lambda) = \lambda^2 + 2\lambda + 4$. Its discriminant is $\Delta = -12 \implies \begin{cases} \lambda_1 = \dfrac{-2 - i\sqrt{12}}{2} \\ \lambda_2 = \dfrac{-2 + i\sqrt{12}}{2} \end{cases}$.

$$\begin{cases} \lambda_1 = -1 - i\sqrt{3} \\ \lambda_2 = -1 + i\sqrt{3} \end{cases} \implies \begin{cases} \alpha = -1 \\ \beta = \sqrt{3} \end{cases}.$$

The solution of the homogeneous equation:

$$y_H(t) = e^{-t}[C_1 \cos(\sqrt{3}t) + C_2 \sin(\sqrt{3}t)],\ C_1, C_2 \in \mathbb{R}.$$

2 The right-hand side is of the form $e^{\lambda t} P_n(t)$ with $\lambda = 1$.

$P_n(t) = t$. We look for a solution of the equation in the form $y_p(t) = e^{\lambda t} Q(t)$ with Q a polynomial of the same degree as P_n. So we set $Q(t) = at + b$. We have

$y_p(t) = e^t(at + b) \implies 4y_p(t) = e^t(4at + 4b)$

$y'_p(t) = e^t(a + at + b) \implies 2y'_p(t) = e^t(2at + 2(a + b))$

$y''_p(t) = e^t(a + a + at + b) = e^t(at + 2a + b)$

hence: $y''_p + 2y'_p + 4y_p(t) = e^t(7at + 4a + 7b)$

Thus y_p is a solution iff $7at + 4a + 7b = t$.

By coefficient comparison, we find:

$$\begin{cases} 7a = 1 \\ 4a + 7b = 0 \end{cases} \implies \begin{cases} a = \dfrac{1}{7} \\ b = -\dfrac{4}{49} \end{cases}.$$

The function $y_p(t) = \frac{1}{7}\left(t - \frac{4}{7}\right)e^t$ is therefore a particular solution, and the general solution of (E) is

$$y_G(t) = e^{-t}\Big[C_1 \cos(\sqrt{3}t) + C_2 \sin(\sqrt{3}t)\Big] + \frac{1}{7}\Big(t - \frac{4}{7}\Big)e^t.$$

6.4 Rattrapage (S1) [02.07.2023]

Exercise 6.4.1 (07 pts). [See the correction 6.4.1]

Consider the differential equation

$$(E): \quad y'(t) - \frac{y(t)}{t} - y^2(t) = -9t^2.$$

- Determine $a \in (0, +\infty)$ such that

$$y_0(t) = at$$

  is a particular solution of (E).

- Show that the change of the unknown function

$$y(t) = y_0(t) - \frac{1}{z(t)}$$

  transforms equation (E) into the differential equation

$$(E_1): \quad z'(t) + \left(6t + \frac{1}{t}\right) z(t) = 1.$$

- Solve the differential equation (E_1).

- Hence, determine the solutions of (E) on the interval

$$(0, +\infty).$$

Exercise 6.4.2 (04 pts). **[See the correction 6.4.2]**

1 Study the local Lipschitz continuity of the function f, defined on $\mathbb{R}$ by

$$f(y) = 3\sqrt{|y|},$$

in a neighborhood of 0.

2 Let $a \geq 0$. Verify that the function y, defined on $\mathbb{R}$ by

$$y(t) = \begin{cases} \frac{9}{4}(t-a)^2, & t > a, \\ 0, & t \leq a, \end{cases}$$

is a solution of the Cauchy problem

$$y'(t) = 3\sqrt{|y(t)|}.$$

Exercise 6.4.3 (05 pts). **[See the correction6.4.3]**

Consider the differential equation

$$(E_2): \quad t^2y'' - 3ty' + 4y = 0, \qquad \text{on } (0, +\infty).$$

1 Determine a solution of equation (E_2) of the form

$$y(t) = t^\alpha,$$

where $\alpha \in \mathbb{R}$.

2 Let

$$y(t) = t^\alpha z(t).$$

Determine the differential equation satisfied by z.

3 Hence, determine the solutions of (E_2) on the interval

$$(0, +\infty).$$

Exercise 6.4.4 (04 pts). **[see the correction 6.4.4]**

Consider the following differential system:

$$(S) \qquad \begin{cases} y_1' = y_2 + 2e^t, \\ y_2' = y_1 + t^2. \end{cases}$$

1 Solve the associated homogeneous differential system.

2 Find a particular solution y_p of the system (S).

6.4.1 Make-up Exam Correction (S1) [02.07.2023]

Solution of exercise 6.4.1. [See exercise 6.4.1]

1 We look for a particular solution of (E) of the form $y(t) = at$ for $t \in]0, +\infty[$. Substituting $y(t)$ into (E) we obtain:

$$a - \frac{at}{t} - a^2t^2 = -9t^2 \implies a^2 = 9 \implies a = \pm 3.$$

Hence we take $y_0(t) = 3t$ as a particular solution of (E) defined on $]0, \infty[$.

2 We perform the following change of the unknown function:
$y(t) = y_0(t) - \dfrac{1}{z(t)} = 3t - \dfrac{1}{z(t)}$, where z is a function defined on $]0, \infty[$ to be determined.

3 We compute y' and y^2 to substitute into (E): $y'(t) = 3 + \dfrac{z'(t)}{z^2(t)}$ and $y^2(t) = 9t^2 - \dfrac{6t}{z(t)} + \dfrac{1}{z^2(t)}$.
Thus, substituting into (E) we have:

$$3 + \frac{z'(t)}{z^2(t)} - \left(\frac{3t - 1/z}{t}\right) - \left(9t^2 - \frac{6t}{z(t)} + \frac{1}{z^2(t)}\right) = -9t^2$$

$$z'(t) + z(t)\left(\frac{1}{t} + 6t\right) = 1.$$

4 The homogeneous equation associated to (E_1) is $z' + \left(6t + \dfrac{1}{t}\right)z = 0$. Its general solution is $z = C \cdot \dfrac{e^{-3t^2}}{t}$.

Let us find the general solution of equation (E_1) in the form $z(t) = C \cdot \dfrac{e^{-3t^2}}{t}$. Substituting into (E_1) gives

$$C'(t) \cdot \frac{e^{-3t^2}}{t} + \left(-6t - \frac{1}{t}\right) C(t) \cdot \frac{e^{-3t^2}}{t} + \left(6t + \frac{1}{t}\right) C(t) = 1.$$

After simplification we obtain $C'(t) = te^{-3t^2}$, which yields

$$C(t) = \frac{1}{6} e^{3t^2}.$$

Therefore, the general solution of equation (E_2) is

$$z(t) = \frac{1}{6t} + \lambda \frac{e^{-3t^2}}{t}, \quad \lambda \in \mathbb{R}.$$

5 We now deduce the solutions of (E) defined on $]0, +\infty[$. Let y be a C^1 solution defined on $]0, +\infty[$. From the previous question, necessarily

$$y(t) = 3t - \frac{1}{z(t)} = 3t - \frac{6t}{1 + 6\lambda e^{-3t^2}}.$$

For all $t > 0$ we have $1 + 6\lambda e^{-3t^2} \neq 0$, which is equivalent to $-3t^2 \neq \log\left(\frac{-1}{-6\lambda}\right) = -\log(-6\lambda)$ for all $t > 0$. This is possible if $\log(-6\lambda) < 0$, or equivalently $\lambda > \frac{-1}{6}$. Therefore, if y is a solution of (E), then:

$$y(t) = 3t \quad \text{or} \quad y(t) = 3t - \frac{6t}{1 + 6\lambda e^{-3t^2}}, \quad \text{with } \lambda > -\tfrac{1}{6}.$$

Solution of exercise 6.4.2. [See exercise 6.4.2]

1 Study the local Lipschitz continuity of the function f, defined on $\mathbb{R}$ by

$$f(y) = 3\sqrt{|y|}.$$

In a neighborhood of $y = 0$, the function f is not locally Lipschitz. Indeed, for any $k > 0$, let

$$y_1 = 0, \qquad y_2 = \frac{4}{k^2}.$$

Then,

$$|f(y_2) - f(y_1)| = \left|3\sqrt{|y_2|} - 0\right| = \frac{6}{k} > k\left|\frac{4}{k^2} - 0\right| = k|y_2 - y_1|.$$

Therefore, f is not locally Lipschitz in a neighborhood of 0.

2 Let $a \geq 0$. Verify that the function y, defined on $\mathbb{R}$ by

$$y(t) = \begin{cases} \frac{9}{4}(t-a)^2, & \text{if } t > a, \\ 0, & \text{if } t \leq a, \end{cases}$$

is a solution of the Cauchy problem.

Since $y(a) = 0$, we have

$$\lim_{t \to a^+} \frac{y(t) - y(a)}{t - a} = \lim_{t \to a^+} \frac{\frac{9}{4}(t-a)^2}{t-a} = 0,$$

and

$$\lim_{t \to a^-} \frac{y(t) - y(a)}{t - a} = \lim_{t \to a^-} \frac{0}{t-a} = 0.$$

Hence,

$$y'(t) = \begin{cases} \frac{9}{2}(t-a), & \text{if } t > a, \\ 0, & \text{if } t = a, \\ 0, & \text{if } t < a. \end{cases}$$

Moreover,

$$3\sqrt{|y(t)|} = \begin{cases} \frac{9}{2}(t-a), & \text{if } t > a, \\ 0, & \text{if } t \leq a. \end{cases}$$

Therefore,

$$y'(t) = 3\sqrt{|y(t)|},$$

and consequently,

$$y(t) = \begin{cases} \frac{9}{4}(t-a)^2, & \text{if } t > a, \\ 0, & \text{if } t \leq a, \end{cases}$$

is a solution of the Cauchy problem

$$\begin{cases} y'(t) = 3\sqrt{|y(t)|}, \\ y(0) = 0. \end{cases}$$

Solution of exercise 6.4.3. [See exercise 6.4.3]

Consider the differential equation

$$(E_2): \quad t^2y'' - 3ty' + 4t = 0, \qquad t \in (0, +\infty).$$

1 Determine a solution of equation (E_2) of the form

$$y(t) = t^\alpha, \qquad \alpha \in \mathbb{R}.$$

Let $y_p(t) = t^\alpha$. Substituting into (E_2), we obtain

$$\alpha(\alpha - 1)t^\alpha - 3\alpha t^\alpha + 4t^\alpha = 0,$$

or equivalently,

$$(\alpha - 2)^2 t^\alpha = 0.$$

Hence, if $\alpha = 2$, the function

$$y_0(t) = t^2$$

is a solution of (E_2).

2 Let

$$y(t) = t^\alpha z(t).$$

Determine the differential equation satisfied by z.

Setting

$$y(t) = t^2 z(t),$$

we obtain

$$y'(t) = 2tz + t^2 z',$$

and

$$y''(t) = 2z + 4tz' + t^2 z''.$$

Substituting into (E_2), we get

$$t^2 \left(2z + 4tz' + t^2 z''\right) - 3t \left(2tz + t^2 z'\right) + 4t^2 z = 0.$$

After simplification, we find

$$t^4 z'' + t^3 z' = 0.$$

Since $t > 0$, the differential equation satisfied by z is

$$tz'' + z' = 0.$$

3 Hence, determine the solutions of (E_2) on $(0, +\infty)$.

Let

$$u = z'.$$

Then the equation satisfied by z is equivalent to

$$u' + \frac{1}{t}u = 0.$$

Its solutions are

$$u(t) = \lambda t e^{-t}, \qquad \lambda \in \mathbb{R}.$$

Hence,

$$\begin{aligned} z'(t) = \lambda t e^{-t} \quad \Longrightarrow \quad z(t) &= \lambda \int t e^{-t}\, dt \\ &= \lambda t e^{-t} + \int \lambda e^{-t}\, dt \\ &= \lambda (t+1) e^{-t} + \mu, \qquad \lambda, \mu \in \mathbb{R}. \end{aligned}$$

Therefore,

$$y(t) = z e^t = -\lambda(t+1) + \mu e^t, \qquad \lambda, \mu \in \mathbb{R}.$$

4 Determine the solutions satisfying

$$\begin{cases} y(0) = 1, \\ y'(0) = 1. \end{cases}$$

The first condition $y(0) = 1$ gives

$$\lambda + \mu = 1.$$

The second condition $y'(0) = 1$ gives

$$-\lambda + \mu = 1.$$

Therefore, the solutions are

$$y(t) = -\lambda(t+1) + (\lambda+1)e^t, \qquad \lambda \in \mathbb{R}.$$

Solution of exercise 6.4.4. **[See exercise 6.4.4]**

1 Solve the homogeneous differential system $y' = Ay$.

Let us compute y_H.

Let P_A denote the characteristic polynomial of A,

$$P_A(\lambda) = \det(A - \lambda I).$$

We have

$$P_A(\lambda) = \lambda^2 - 1,$$

hence

$$\lambda_1 = -1, \qquad \lambda_2 = 1.$$

Compute the eigenvectors.

Let

$$V_1 = \begin{pmatrix} x \\ y \end{pmatrix}$$

be the eigenvector associated with $\lambda_1 = -1$. Then

$$(A + I_2)\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

which gives

$$\begin{cases} x + y = 0, \\ x + y = 0, \end{cases}$$

and therefore

$$y = -x.$$

Hence,

$$V_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Similarly, let V_2 be the eigenvector associated with $\lambda_2 = 1$. Then

$$(A - I_2) \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

which yields

$$\begin{cases} -x + y = 0, \\ x - y = 0, \end{cases}$$

and consequently

$$x = y.$$

Thus,

$$V_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

The homogeneous solution is given by

$$y_H(t) = c_1 V_1 e^{\lambda_1 t} + c_2 V_2 e^{\lambda_2 t} = c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{t}.$$

Hence,

$$y_H(t) = \begin{pmatrix} c_1 e^{-t} + c_2 e^{t} \\ -c_1 e^{-t} + c_2 e^{t} \end{pmatrix}, \qquad c_1, c_2 \in \mathbb{R}.$$

2 Find a particular solution y_p of system (S).

Let us compute y_p using the method of variation of parameters.

$$y_p(t) = c_1(t) V_1 e^{\lambda_1 t} + c_2(t) V_2 e^{\lambda_2 t}, \qquad t \in \mathbb{R},$$

where

$$\begin{pmatrix} c_1' \\ c_2' \end{pmatrix} = \begin{pmatrix} V_1 e^{\lambda_1 t} & V_2 e^{\lambda_2 t} \end{pmatrix}^{-1} B(t).$$

Thus,

$$\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}' = \begin{pmatrix} e^{-t} & e^t \\ -e^{-t} & e^t \end{pmatrix}^{-1} \begin{pmatrix} 2e^t \\ t^2 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} e^t & -e^t \\ e^{-t} & e^{-t} \end{pmatrix} \begin{pmatrix} 2e^t \\ t^2 \end{pmatrix}.$$

Therefore,

$$\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}' = \frac{1}{2} \begin{pmatrix} 2e^{2t} - t^2 e^t \\ 2 + t^2 e^{-t} \end{pmatrix},$$

which gives

$$\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} e^{2t} - e^t(t^2 - 2t + 2) \\ 2t - e^{-t}(t^2 + 2t + 2) \end{pmatrix}.$$

Hence,

$$Y_p(t) = \begin{pmatrix} c_1(t)e^{-t} + c_2(t)e^t \\ -c_1(t)e^{-t} + c_2(t)e^t \end{pmatrix} = \frac{1}{2} \begin{pmatrix} e^t(2t+1) - 2t^2 - 4 \\ e^t(2t-1) - 4t \end{pmatrix}.$$

6.5 Make-up Exam (S1) [2024–2025]

Exercise 6.5.1 (05 pts). [See the correction 6.5.1]

We consider the following Cauchy problem:

$$\begin{cases} x'(t) + x(t) - 5e^{-t}x^5(t) = 0, \\ x(t_0) = 0. \end{cases} \tag{1}$$

1 State the existence theorem for a solution of the Cauchy problem.

2 Show that the problem admits a unique maximal solution.

3 Using the change of variable $z(t) = x^{-4}(t)$, determine the maximal solution of (1) with $x(1) = 0$.

Exercise 6.5.2 (04 pts). [See the correction 6.5.2]

We consider the differential equation

$$y'' + a(x)y' + b(x)y = 0, \tag{2}$$

where a and b are two continuous functions on I (an interval of $\mathbb{R}$).

— Let u and v be two linearly independent solutions of equation (2).

1 Show that the Wronskian $w(x)$ of u and v satisfies the relation

$$w(x) = w(x_0)e^{-\int_{x_0}^{x} a(s)\,ds}.$$

Exercise 6.5.3 (07 pts). **[See the correction 6.5.3]**

We consider the following system of differential equations:

$$\begin{cases} y_1'(t) = y_1 + t^2 + e^{4t}, \\ y_2'(t) = 2y_2 + t + e^{2t}. \end{cases} \tag{3}$$

1 Write it in the form $Y' = AY + B$ where A is a (2×2) matrix, and $B, Y : \mathbb{R} \to \mathbb{R}^2$.

2 Solve the homogeneous system $Y' = AY$.

3 Compute a particular solution Y_p of equation (3).

Exercise 6.5.4 (04 pts). **[See the correction 6.5.4]**

We consider the following differential system:

$$X'(t) = A(t)X(t), \tag{4}$$

where for all $t \in \mathbb{R}$,

$$X(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}, \quad A(t) = \begin{pmatrix} -t & 1 \\ 1 - t^2 & t \end{pmatrix}.$$

1 Show that $\phi(t) = \begin{pmatrix} 1 \\ t \end{pmatrix}$ and $\psi(t) = \begin{pmatrix} t \\ 1 + t^2 \end{pmatrix}$ are solutions of system (4).

2 Show that the family $\{\phi(t), \psi(t)\}$ is a fundamental system of solutions of system (4).

3 Give the fundamental matrix of system (4).

4 Deduce the solutions of system (4).

6.5.1 Correction of Make-up Exam (S1) [2024–2025]

Solution of exercise 6.5.1. [See exercise 6.5.1]

We consider the following Cauchy problem:

$$\begin{cases} x'(t) + x(t) - 5e^{-t}x^5(t) = 0, \\ x(t_0) = 0. \end{cases} \tag{1}$$

1 **State the existence theorem of a solution to the Cauchy problem.**

Theorem (Cauchy-Lipschitz Theorem): Suppose that f is a continuous and locally Lipschitz function with respect to y on $I \times \Omega$. Then, for every $(t_0, y_0) \in I \times \Omega$, the Cauchy problem (CP) admits a unique solution defined on $[t_0 - T, t_0 + T]$ with values in $[y_0 - \delta_0, y_0 + \delta_0]$.

2 **Show that the problem admits a unique maximal solution.**

Let $f(t, x) = -x(t) + 5e^{-t}x^5(t)$. This function is defined and of class C^1 on $\mathbb{R}^2$, since its partial derivative with respect to x,

$$\frac{\partial f}{\partial x}(t, x) = -1 + 25e^{-t}x^4,$$

exists and is continuous on $\mathbb{R}^2$. By the Cauchy–Lipschitz Theorem, the problem admits a unique maximal solution defined on an open interval $]a, b[$ containing t_0.

3 **Using the change of variable.**

Let $z(t) = x^{-4}(t) \implies z'(t) = -4\frac{x'}{x^5}$. Differentiating gives:

$$z'(t) = -4x^{-5}(t)x'(t).$$

The initial equation is $x'(t) + x(t) - 5e^{-t}x^5(t) = 0$. Dividing by $x^5(t)$ (for $x \neq 0$):

$$\frac{x'(t)}{x^5(t)} + \frac{x(t)}{x^5(t)} - 5e^{-t} = 0,$$

$$\frac{x'(t)}{x^5(t)} + \frac{1}{x^4(t)} - 5e^{-t} = 0.$$

Using z and z', we get:

$$-\frac{1}{4}z'(t) + z(t) - 5e^{-t} = 0 \quad \implies \quad z'(t) = 4z(t) - 20e^{-t}.$$

Then solving $z' = 4z$ gives $\dfrac{z'}{z} = 4 \implies \ln z = 4t + C \implies Z_H = Ke^{4t},\ K \in \mathbb{R}$.

Setting $z_p(t) = K(t)e^{4t}$, we obtain:

$$z_p'(t) = K'(t)e^{4t} + 4K(t)e^{4t},$$

$$K'(t)e^{4t} = -20e^{-t} \quad \implies \quad K'(t) = -20e^{-5t} \quad \implies \quad K = 4e^{-5t}.$$

Thus,

$$z_p = 4e^{-t}, \quad z(t) = Ce^{4t} + 4e^{-t}.$$

Finally,

$$x(t) = \left(Ce^{4t} + 4e^{-t}\right)^{-\frac{1}{4}}.$$

Solution of exercise 6.5.2. [See exercise 6.5.2]

Consider the differential equation

$$y'' + a(x)y' + b(x)y = 0. \qquad (2)$$

Let u and v be two linearly independent solutions of equation (2).

1. Show that

$$W(x) = W(x_0)e^{-\int_{x_0}^{x} a(s)\,ds}.$$

Since u and v are solutions of (2), we have

$$\begin{cases} u'' + a(x)u' + b(x)u = 0 & \times v, \\ v'' + a(x)v' + b(x)v = 0 & \times (-u), \end{cases}$$

which gives

$$\begin{cases} vu'' + a(x)vu' + b(x)vu = 0, & (3), \\ -uv'' - a(x)uv' - b(x)uv = 0. & (4) \end{cases}$$

Adding equations (3) and (4), we obtain

$$(vu'' - uv'') + a(x)(vu' - uv') = 0. \qquad (*)$$

The Wronskian $W(u,v)$ is defined by

$$W(u,v) = uv' - vu' = -(vu' - uv').$$

Differentiating, we obtain

$$W'(u,v) = u'v' + uv'' - v'u' - vu'' = uv'' - vu'' = -(vu'' - uv'').$$

Therefore, equation $(*)$ can be rewritten as

$$W' + a(x)W = 0.$$

This is a first-order linear differential equation for W. Hence,

$$\frac{W'}{W} = -a(x),$$

and integrating from x_0 to x, we obtain

$$\ln\left|\frac{W(x)}{W(x_0)}\right| = -\int_{x_0}^{x} a(s)\,ds.$$

Therefore,

$$W(x) = W(x_0)e^{-\int_{x_0}^{x} a(s)\,ds}.$$

Solution of exercise 6.5.3. **[See exercise 6.5.3]**

Consider the following system of differential equations:

$$\begin{cases} y_1'(t) = y_1(t) + t^2 + e^{4t}, \\ y_2'(t) = 2y_2(t) + t + e^{2t}. \end{cases} \tag{3}$$

1 Rewrite system (3) in the form $Y' = AY + B$.

System (3) can be written as

$$Y'(t) = \begin{pmatrix} y_1'(t) \\ y_2'(t) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix} + \begin{pmatrix} t^2 + e^{4t} \\ t + e^{2t} \end{pmatrix}.$$

Therefore,

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}, \qquad B(t) = \begin{pmatrix} t^2 + e^{4t} \\ t + e^{2t} \end{pmatrix}.$$

2 Solve the homogeneous system $Y' = AY$.

Since

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix},$$

its eigenvalues are

$$\lambda_1 = 1, \qquad \lambda_2 = 2.$$

The corresponding eigenvectors are

$$v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad v_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Hence, the general solution of the homogeneous system is

$$Y_H(t) = c_1 v_1 e^{\lambda_1 t} + c_2 v_2 e^{\lambda_2 t} = c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^t + c_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{2t},$$

that is,

$$Y_H(t) = \begin{pmatrix} c_1 e^t \\ c_2 e^{2t} \end{pmatrix}.$$

Let us compute Y_p.

Using the method of variation of parameters, we seek a particular solution of the form

$$Y_p(t) = \begin{pmatrix} c_1(t) e^t \\ c_2(t) e^{2t} \end{pmatrix},$$

where c_1 and c_2 are differentiable functions satisfying

$$\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}' = \left(v_1 e^{\lambda_1 t} \; v_2 e^{\lambda_2 t} \right)^{-1} B(t) = \begin{pmatrix} e^t & 0 \\ 0 & e^{2t} \end{pmatrix}^{-1} \begin{pmatrix} t^2 + e^{4t} \\ t + e^{2t} \end{pmatrix}.$$

Therefore,

$$\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}' = \begin{pmatrix} e^{-t} & 0 \\ 0 & e^{-2t} \end{pmatrix} \begin{pmatrix} t^2 + e^{4t} \\ t + e^{2t} \end{pmatrix} = \begin{pmatrix} t^2 e^{-t} + e^{3t} \\ te^{-2t} + 1 \end{pmatrix}.$$

Integrating, we obtain

$$\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} \displaystyle\int \left(t^2 e^{-t} + e^{3t}\right) dt \\ \displaystyle\int \left(te^{-2t} + 1\right) dt \end{pmatrix},$$

that is,

$$\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} -e^{-t}(t^2 + 2t + 2) + \frac{1}{3} e^{3t} \\ t - \frac{1}{2} e^{-2t} \left(t + \frac{1}{2}\right) \end{pmatrix}.$$

Solution of exercise 6.5.4. **[See exercise 6.5.4]**

1 Show that

$$\phi(t) = \begin{pmatrix} 1 \\ t \end{pmatrix}, \qquad \psi(t) = \begin{pmatrix} t \\ 1 + t^2 \end{pmatrix}$$

are solutions of system (4).

For $\phi(t)$:

$$\phi'(t) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Moreover,

$$A(t)\phi(t) = \begin{pmatrix} -t & 1 \\ 1 - t^2 & t \end{pmatrix} \begin{pmatrix} 1 \\ t \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Since

$$\phi'(t) = A(t)\phi(t),$$

it follows that $\phi(t)$ is a solution of

$$X'(t) = A(t)X(t).$$

For $\psi(t)$:

$$\psi'(t) = \begin{pmatrix} 1 \\ 2t \end{pmatrix}.$$

Furthermore,

$$A(t)\psi(t) = \begin{pmatrix} -t & 1 \\ 1-t^2 & t \end{pmatrix} \begin{pmatrix} t \\ 1+t^2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2t \end{pmatrix}.$$

Since

$$\psi'(t) = A(t)\psi(t),$$

it follows that $\psi(t)$ is also a solution of

$$X'(t) = A(t)X(t).$$

2 Show that $\{\phi(t), \psi(t)\}$ is a fundamental system.

The Wronskian matrix is

$$W(t) = \begin{pmatrix} 1 & t \\ t & 1+t^2 \end{pmatrix},$$

whose determinant is

$$\det W(t) = \begin{vmatrix} 1 & t \\ t & 1+t^2 \end{vmatrix} = 1(1+t^2) - t^2 = 1 \neq 0.$$

Since

$$\det W(t) \neq 0 \qquad \text{for all } t \in \mathbb{R},$$

the solutions $\phi(t)$ and $\psi(t)$ are linearly independent. Therefore,

$$\{\phi(t), \psi(t)\}$$

forms a fundamental system of solutions of

$$X' = AX.$$

3 Determine a fundamental matrix $M(t)$ of the system

$$X'(t) = A(t)X(t).$$

A fundamental matrix is

$$M(t) = \begin{pmatrix} 1 & t \\ t & 1+t^2 \end{pmatrix}.$$

4 Deduce the general solution of system (4).

The general solution is given by

$$X(t) = M(t)C,$$

where

$$C = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}, \qquad c_1, c_2 \in \mathbb{R}.$$

Hence,

$$X(t) = \begin{pmatrix} 1 & t \\ t & 1+t^2 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} c_1 + c_2 t \\ c_1 t + c_2(1+t^2) \end{pmatrix}.$$

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