People's Democratic Republic of Algeria
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ORDINARY DIFFERENTIAL EQUATIONS

Theory and Practice with Solutions

Intended for M1 students in Mathematical Analysis and Applications

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Contents

General Introduction

The course **"Ordinary Differential Equations"** constitutes a fundamental module of the Master's program in *Mathematical Analysis and Applications*, Semester 1. It builds on the knowledge acquired at the undergraduate level, deepening the theoretical and qualitative study of differential equations, both in classical contexts and in abstract functional spaces. This discipline occupies a central place in the analysis of dynamic phenomena in applied mathematics, physics, engineering, and numerous scientific fields, as it allows modeling and understanding the evolving behavior of various systems.

The main objective of this course is the study of **Cauchy problems in infinite-dimensional Banach spaces**. This general framework requires the use of sophisticated analytical methods, among which *fixed-point theory* plays a crucial role. The latter serves as a fundamental tool for proving the existence and uniqueness of solutions, and it allows extending classical results obtained in finite-dimensional spaces to more abstract contexts. In parallel, the course addresses issues related to **the stability of solutions**, providing students with a deep understanding of the behavior of differential systems under various initial conditions and constraints.

The program begins with the **fundamentals of differential equations**, introducing the notion of a solution and the different types of solutions, including local, maximal, global, and saturated solutions. This classification allows for understanding the possible behaviors of solutions and precisely defining the contexts in which they are studied. Special attention is given to the **qualitative analysis of ordinary differential equations in finite dimension**, relying on classical theorems such as *Peano* and *Cauchy–Lipschitz*, which respectively guarantee the existence and uniqueness of solutions under appropriate conditions.

A natural extension of this study concerns **differential equations in infinite-dimensional spaces**, where the notions of convergence, continuity, and completeness play a critical role. Qualitative analysis in this framework allows for understanding complex phenomena and prepares students for advanced applications in functional analysis and dynamical systems. The theoretical tools developed here are essential for addressing scientific and technical problems involving continuous models and infinite-dimensional spaces.

A significant part of the course is devoted to **differential equations under constraints**, where the solution must remain within a given set. This study involves geometric concepts such as the *Bouligand–Severi tangent cone* and other notions of tangent cones, which formalize the conditions that an admissible trajectory must satisfy. The *Nagumo theorem* constitutes a key result in this context, providing a necessary and sufficient criterion for the viability of solutions within the constraint set.

Finally, the course addresses the **notion of stability**, an essential element for the study of dynamical systems. Students will analyze general properties of stability, the stability of linear differential systems, as well as stability in the sense of *Lyapunov*, which provides a robust theoretical framework to evaluate the persistence of solutions under small initial perturbations. This part enables understanding the long-term behavior of systems and anticipating the sensitivity of solutions, which is crucial in scientific modeling and technical applications.

In summary, this module offers a rigorous and comprehensive approach to ordinary differential equations, combining **theoretical foundations**, **qualitative analysis**, and **advanced concepts such as constraints and stability**. It provides a solid base for students, allowing them to pursue advanced studies in *functional analysis*, *dynamical systems*, and other applied areas of mathematics. Upon completing this course, students will be able to formulate, analyze, and solve complex differential problems, understand stability conditions, and integrate abstract tools into the study of dynamical systems.

Chapter 1

Generalization on Differential Equations

1.1 Notation of Solutions and Types of Solutions (Local, Maximal, Global, and Saturated Solutions)

Definition 1.1.1.

- **A differential equation** is a relation involving the derivatives of the unknown function.
- **A differential equation** in which the unknown function depends on a **single** variable is called an ordinary differential equation, abbreviated as **ODE**.

Example 1.1.1.

The equations:

$$y'(t) = 3y^3(t) - 2y(t) + 1, \quad y''(t) = \cos(y) + 3$$

are ordinary differential equations, whereas the following are not.

Definition 1.1.2.

The **order of an ODE** is the order of the highest derivative appearing in the equation. More precisely, an ODE is said to be of order n if it can be written as:

$$F(t, y(t), y'(t), \ldots, y^{(n)}(t)) = 0,$$

where F is a function defined on $D(F)$, a subset of $\mathbb{R} \times (\mathbb{R}^n)^{n+1}$, with values in $\mathbb{R}^n$.

Example 1.1.2.

The ODE:

$$3y' + y = t^2 + t \quad \text{is of order 1.}$$

$$\sin(y'') = t \quad \text{is of order 2.}$$

Remark 1.1.1.

Any ODE can be rewritten as a first-order ODE. For instance, if we have

$$F(t, y(t), y'(t), \ldots, y^{(n)}(t)) = 0,$$

we can define an equivalent system in the form:

$$F(t, z_1(t), z_2(t), \ldots, z_{n-1}(t), z'_{n-1}(t)) = 0.$$

Example 1.1.3.

Consider the polynomial ODE

$$y''(t) = y^3(t) - ay^2(t) + 2ty'(t), \quad t \in \mathbb{R}.$$

If we set

$$z(t) = (y(t), y'(t)) = (z_1(t), z_2(t)),$$

then

$$\begin{aligned} z_1'(t) &= y'(t) = z_2(t), \\ z_2'(t) &= y''(t) \\ &= y^3(t) - ay^2(t) + 2ty'(t) \\ &= z_1^3(t) - az_1^2(t) + 2tz_2(t). \end{aligned}$$

Thus, if we define

$$F(t, z_1(t), z_2(t)) = \Big(z_2(t),\ z_1^3(t) - az_1^2(t) + 2tz_2(t)\Big),$$

the ODE can be rewritten in the form

$$z'(t) = F(t, z(t)).$$

Definition 1.1.3.

An ODE is said to be a **linear ODE of order** n if it can be written as

$$a_n(t)y^{(n)}(t) + a_{n-1}(t)y^{(n-1)}(t) + \cdots + a_1(t)y'(t) + a_0(t)y(t) = b(t),$$

where a_i for $i = 0, 1, \ldots, n$ and b are functions defined on a subinterval of $\mathbb{R}$.

Example 1.1.4.

The ODE

$$y' + y = 2t$$

is linear, whereas

$$\sin(y'(t)) = 2$$

is nonlinear.

Definition 1.1.4.

An ODE is said to be **autonomous** if the independent variable does not explicitly appear in the equation. More precisely, an ODE of order n is autonomous if it can be written in the form

$$F\Big(y(t), y'(t), \ldots, y^{(n)}(t)\Big) = 0,$$

where F is a function defined on $D(F)$, a subset of $(\mathbb{R}^n)^{n+1}$, with values in $\mathbb{R}^n$.

Example 1.1.5.

The ODE: $y' + y = 2$ is autonomous, whereas the ODE: $y' + y = 2t$ is not autonomous.

1.1.1 Notions of Solution and Types of Solutions

We consider the ODE:

$$y' = f(t, y) \tag{1.1}$$

where $f : I \times \Omega \to \mathbb{R}^n,\ \Omega \subset \mathbb{R}^n$.

Definition 1.1.5.

(1) The function $y : J \to \Omega$ is a solution of equation (1.1), where J is a non-empty subinterval of I, if $J \subset I$.

- $\forall t \in J,\ y(t) \in \Omega$.
- y is differentiable on J and $\forall t \in J,\ y' = f(t, y)$.

(2) If in addition $y \in C^1(J)$, we say that y is a solution of class C^1.

Example 1.1.6. Determine the C^1 solutions of the ODE:

1. $y' = 3t^2 y$ admits infinitely many C^1 solutions defined on $\mathbb{R}$ by

$$y(t) = ce^{t^3}, \quad y(t) = c,$$

   where c is an arbitrary constant.

2. The ODE $y' = 1 + y^2$ admits infinitely many C^1 solutions defined on any interval of the form

$$]c - \frac{\pi}{2}, c + \frac{\pi}{2}[, \quad y(t) = \tan(t - c),$$

   where c is a constant.

It may happen that the domain of definition of a solution of equation (1.1) is different from I (since f is defined on $I \times \Omega$).

Definition 1.1.6. Let $y : J \mapsto \Omega$ be a solution of problem (1.1).

1. The solution y is called **global** if $J = I$. Otherwise ($J \subset I$), y is called **local**.

2. The solution y is called **maximal** if it admits no extension to another solution of problem (1.1).

• Reminder:

$$\tilde{y} : \tilde{J} \to \Omega \text{ is an extension of } \; y : J \mapsto \Omega \text{ if } J \subset \tilde{J} \text{ and } \forall t \in J, \; \tilde{y}(t) = y(t).$$

Example 1.1.7. Consider the Cauchy problem:

$$\begin{cases} y'(t) = -2t\, y^2(t), \quad t \in \mathbb{R}, \\ y(0) = 1. \end{cases}$$

1. Assume that $y(t) \neq 0,\ \forall t \in \mathbb{R}$. Then one can write

$$\frac{y'(t)}{y^2(t)} = -2t.$$

   Integrating with respect to t, we obtain

$$-\frac{1}{y(t)} - \left(\frac{-1}{y(0)}\right) = -t^2.$$

   Hence a solution is

$$y(t) = \frac{y(0)}{t^2 y(0) + 1}.$$

   Since $y(0) = 1$, we get

$$y(t) = \frac{1}{t^2 + 1}.$$

   Notice that $y(t)$ is well defined on $\mathbb{R}$, hence $J = \mathbb{R}$, i.e. the solution is **global**.

2. Consider the Cauchy problem:

$$\begin{cases} y'(t) = 2t\, y^2(t), \\ y(0) = 1. \end{cases}$$

   It admits a solution of the form

$$y(t) = \frac{1}{1 - t^2}, \quad t \in J =]-1, 1[.$$

   This solution is **maximal** but not global ($J \neq \mathbb{R}$).

Remark 1.1.2. Any solution $y : J \longmapsto \Omega$ of the ODE (1.1) such that

$$\lim_{t \to a} y(t) = \infty \quad \text{where } a = \sup J \text{ or } a = \inf J$$

cannot be extended to another solution of the same problem.

1.2 Qualitative Study of ODEs in Finite Dimension (Peano and Cauchy–Lipschitz Theorems)

Cauchy Problem and Integral Equation

Definition 1.2.1.
Consider the ODE

$$y'(t) = f(t, y(t)), \tag{1.2}$$

where $f : I \times \Omega \to \mathbb{R}^n$, I is a subinterval of $\mathbb{R}$, Ω is a subset of $\mathbb{R}^n$, and let $(t_0, y_0) \in I \times \Omega$.

• A **Cauchy problem** is the task of finding a solution of (1.2) satisfying the initial condition $y(t_0) = y_0$. It is written in the form:

$$\begin{cases} y'(t) = f(t, y(t)), \\ y(t_0) = y_0. \end{cases} \tag{1.3}$$

Definition 1.2.2.
A function $y : J \to \Omega$ is called a solution of the Cauchy problem (1.3), where J is a subinterval of I containing t_0, if $y(t_0) = y_0$ and y is a solution of the ODE (1.2).

Remark 1.2.1.
Let $(t_0, y_0) \in I \times \Omega$ and let y be a solution of the Cauchy problem (1.3).

The solution y is called a **right-hand solution** if

$$J = [t_0, T] \quad \text{or} \quad J = [t_0, T[.$$

Similarly, y is called a **left-hand solution** if

$$J = [T, t_0] \quad \text{or} \quad J =]T, t_0].$$

Otherwise, the solution y is called **bilateral** ($\inf J < t_0 < \sup J$).

Proposition 1.2.1.
Suppose that f is continuous on $I \times \Omega$.
Then a function $y : J \to \Omega$ is a solution of the Cauchy problem (1.3) if and only if y is **continuous** on J and

$$\forall t \in J, \quad y(t) = y_0 + \int_{t_0}^{t} f(s, y(s))\, ds.$$

Proof.

Assume that y is a solution of the Cauchy problem (1.3). Since y is differentiable on J, it is therefore **continuous**. As f is continuous on $I \times \Omega$, the mapping $s \mapsto f(s, y(s))$ is also

continuous.

The solution y satisfies:

$$\forall t \in J, \quad y'(t) = f(t, y(t)).$$

Integrating both sides from t_0 to t, we obtain:

$$\forall t \in J, \quad \int_{t_0}^{t} y'(s)\, ds = \int_{t_0}^{t} f(s, y(s))\, ds.$$

Since $y(t_0) = y_0$, it follows that:

$$\forall t \in J, \quad y(t) = y(t_0) + \int_{t_0}^{t} f(s, y(s))\, ds.$$

Conversely: Since f is continuous on $I \times \Omega$ and y is continuous on J, the mapping $s \mapsto f(s, y(s))$ is continuous on J.

Moreover, from the integral equation we have:

$$\forall t \in J, \quad y(t) = y_0 + \int_{t_0}^{t} f(s, y(s))\, ds.$$

Differentiating with respect to t, we get:

$$\forall t \in J, \quad y'(t) = f(t, y(t)).$$

This completes the proof.

□

Local Existence of Solutions for a Cauchy Problem

Theorem 1.2.2 (Peano's Theorem)**.** *If f is continuous on $I \times \Omega$, where $\Omega \subset \mathbb{R}^2$ is open, then for every $(t_0, y_0) \in I \times \Omega$, the problem (1.3) admits **at least** one **local** solution $y : J \subset I \to \Omega$ defined on a subinterval $J \subset I$ containing t_0.*

Proof. We present a proof method based on delay integral equations. □

Proposition 1.2.2. Consider the delay integral equation for $\lambda \geq 0$:

$$y_\lambda(t) = \begin{cases} x, & t \in [t_0 - \lambda, t_0] \\ x + \int_{t_0}^{t} f(s, y_\lambda(s-\lambda))ds, & t \in [t_0, t_0+\delta]. \end{cases} \tag{1.4}$$

where $f : I \times \Omega \to \mathbb{R}^n$, $\Omega \subset \mathbb{R}^n$ is open and $(t_0, x) \in I \times \Omega$. Choose $\delta > 0$ such that $[t_0, t_0+\delta] \subset I$ and $\lambda \geq 0$. Note that if $\lambda = 0$, then equation (1.4) reduces to the integral equation

$$y(t) = x + \int_{t_0}^{t} f(s, y(s))ds, \quad \forall t \in [t_0, t_0+\delta],$$

which is exactly the integral equation associated with the Cauchy problem (1.3).

• **Proof Method:** The idea of the proof is as follows:
First, we show that the integral equation (1.4)

admits for every $\lambda > 0$, at least one solution $y_\lambda : [t_0-\lambda, t_0+\delta] \to \mathbb{R}^n$.

Next, by letting $\lambda \to 0$, we show that $(y_\lambda)_\lambda$ converges uniformly to a function y. Finally, we show that the limit y is a solution of the considered Cauchy problem.
We first consider the case $\Omega = \mathbb{R}^n$ and f is **continuous** on $I \times \mathbb{R}^n$. The following result will be useful.

Lemma 1.2.3. *If $f : I \times \mathbb{R}^n \to \mathbb{R}^n$ is* ***continuous****, then for every $(t_0, x) \in I \times \mathbb{R}^n$, $\lambda > 0$, and $\delta > 0$ such that $[t_0, t_0+\delta] \subset I$, the equation (1.4) admits a* ***unique solution*** *defined on $[t_0 - \lambda, t_0+\delta]$.*

Proof.

The construction of y_λ is done in two steps.

• Suppose $\delta \leq \lambda$. On the interval $[t_0 - \lambda, t_0]$, we have $y_\lambda \equiv x$. Now, for $t \in [t_0, t_0 + \delta]$, clearly if $s \in [t_0, t]$, then $s - \lambda \in [t_0 - \lambda, t_0]$, giving:

$$y_\lambda(s - \lambda) = x, \quad \forall s \in [t_0, t].$$

Consequently,

$$y_\lambda(t) = x + \int_{t_0}^{t} f(s, x) ds, \quad \forall t \in [t_0, t_0 + \lambda].$$

The construction is thus complete.

• Suppose $\lambda < \delta$. The construction of y_λ is done on $[t_0 - \lambda, t_0 + \lambda]$.

- If $t_0 + \delta \leq t_0 + 2\lambda$, then for $t \in [t_0 + \lambda, t_0 + \delta]$,

$$s - \lambda \in [t_0, t_0 + \lambda], \quad \forall s \in [t_0 + \lambda, t],$$

  so

$$y_\lambda(s - \lambda) = x + \int_{t_0}^{t} f(\theta, x) d\theta.$$

  Hence, y_λ is defined on $[t_0 - \lambda, t_0 + \delta]$.

- If $2\lambda \leq \delta$, repeat the same procedure.

After finitely many steps, there exists a natural number n such that $t_0 + n\lambda \geq t_0 + \delta$. This completes the construction of y_λ. Since y_λ is defined by integrals, it is continuous.

The proof is complete.

□

We have thus constructed a solution of the integral equation defined on $[t_0 - \lambda, t_0 + \delta]$ for an arbitrary choice of delay λ and any $\delta > 0$ such that $[t_0, t_0 + \delta] \subset I$.

The lemma below allows us to remove the delay $\lambda > 0$.

Lemma 1.2.4. *Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be continuous and bounded on $I \times \mathbb{R}^n$. Then for every $(t_0, x) \in I \times \mathbb{R}^n$ and $\delta > 0$ such that $[t_0, t_0 + \delta] \subset I$, the delay equation (1.4) has a* ***unique solution*** *defined on $[t_0, t_0 + \delta]$.*

Proof.

Let $(t_0, x) \in I \times \mathbb{R}^n$ and $\delta > 0$ such that $[t_0, t_0 + \delta] \subset I$. Let $n \in \mathbb{N}$. Consider the delay integral equation with $\delta_n = \frac{\delta}{n+1}$:

$$y_n(t) = \begin{cases} x, & t \in [t_0 - \delta_n, t_0] \\ x + \int_{t_0}^{t} f(s, y_n(s - \delta_n))ds, & t \in (t_0, t_0 + \delta_n] \end{cases} \tag{1.5}$$

By Lemma 1.2.4, this integral equation admits a **unique solution**:

$$y_n : [t_0 - \delta_n, t_0 + \delta] \to \mathbb{R}^n, \quad \forall n \in \mathbb{N}.$$

Consider the family $\mathcal{F}$ of restrictions of the solutions y_n to the interval $[t_0, t_0 + \delta]$, also denoted y_n. To prove that the solutions y_n converge uniformly on $[t_0, t_0 + \delta]$, it suffices to apply the Arzelà-Ascoli theorem, which requires showing that the family $\mathcal{F}$ is equicontinuous and uniformly bounded.

The family $\mathcal{F}$ is uniformly bounded: We show that all functions y_n are bounded by a constant independent of n. Indeed, f is bounded on $I \times \mathbb{R}^n$ (assumption of the lemma). Thus,

$$\exists M > 0, \quad \forall (t, y) \in I \times \mathbb{R}^n, \quad \|f(t, y)\| \leq M.$$

From equation (1.5) we have:

$$\forall n \in \mathbb{N},\ \forall t \in [t_0, t_0 + \delta], \quad \|y_n(t)\| \leq \|x\| + (t - t_0)M \leq \|x\| + \delta M.$$

Hence, the family $\mathcal{F}$ is uniformly bounded on $[t_0, t_0 + \delta]$.

The family $\mathcal{F}$ is equicontinuous: We observe that

$$\forall t, s \in [t_0, t_0 + \delta], \quad \|y_n(t) - y_n(s)\| \leq \left| \int_s^t \|f(\tau, y_n(\tau - \delta_n))\| d\tau \right| \leq M|t - s|.$$

This shows that $\mathcal{F}$ is equicontinuous on $[t_0, t_0 + \delta]$.

By the Arzelà-Ascoli theorem, $(y_n)_n$ converges uniformly to a continuous function

$y : [t_0, t_0 + \delta] \to \mathbb{R}^n$. Moreover,

$$\lim_{n\to\infty} (s - \delta_n) = s,$$

uniformly on $[t_0, t_0 + \delta]$. Since f is continuous on $I \times \mathbb{R}^n$, passing to the limit in (1.5) as $n \to +\infty$, we deduce:

$$\forall t \in [t_0, t_0 + \delta], \quad y(t) = x + \int_{t_0}^{t} f(s, y(s))ds,$$

so $y : [t_0, t_0 + \delta] \to \mathbb{R}^n$ is a solution of the Cauchy problem.

□

• We now move to the proof of Peano's main result.

Proof.

Let $(t_0, x) \in \Omega$. Since I and Ω are open, there exist $d > 0$ and $r > 0$ such that

$$[t_0 - d, t_0 + d] \subset I, \quad B(x, r) = \{\eta \in X; \|\eta - x\| \leq r\} \subset \Omega.$$

Define the function $\rho : \mathbb{R}^n \to \mathbb{R}^n$ by

$$\rho(y) = \begin{cases} y, & y \in B(x, r) \\ \dfrac{r}{\|y - x\|}(y - x) + x, & y \in \mathbb{R}^n \setminus B(x, r). \end{cases}$$

The map ρ is continuous and $\rho(\mathbb{R}^n) \subset B(x, r)$. Define now $g : (t_0 - d, t_0 + d) \times \mathbb{R}^n \to \mathbb{R}^n$ by

$$g(t, y) = f(t, \rho(y)), \quad \forall (t, y) \in (t_0 - d, t_0 + d) \times \mathbb{R}^n.$$

Since f is continuous, by Weierstrass' theorem its restriction to $(t_0 - d, t_0 + d) \times \mathbb{R}^n$ is bounded. By Lemma 1.2.4, for any $d' \in (0, d)$, the Cauchy problem

$$\begin{cases} y' = g(t, y) \\ y(t_0) = x \end{cases}$$

admits at least one solution $y : [t_0, t_0 + d'] \to \mathbb{R}^n$. Since y is continuous at t_0 and $y(t_0) = x$, for any $r > 0$, there exists $\delta \in (0, d')$ such that

$$\forall t \in [t_0, t_0 + \delta], \quad \|y(t) - x\| \leq r.$$

Hence, $g(t, y(t)) = f(t, y(t))$, so y is indeed a solution of the Cauchy problem. The proof is complete.

□

Remark 1.2.5.

(i) Continuity of f is not sufficient to guarantee uniqueness of the solution. The example below (constructed by G. Peano himself) illustrates this phenomenon.

(ii) The conditions of Peano's theorem are sufficient to guarantee the existence of at least one local solution to the Cauchy problem (1.3). It may happen that f is not continuous on $I \times \Omega$, yet the Cauchy problem (1.3) still admits at least one local solution (see exercises).

Example 1.2.1. It is easy to verify that the Cauchy problem

$$\begin{cases} y'(t) = 3y^{2/3} \\ y(0) = 0 \end{cases}$$

admits two solutions: the zero function $y_1 \equiv 0$ and the function $y_2 : \mathbb{R} \to \mathbb{R}$ defined by $y_2(t) = t^3$.

Cauchy–Lipschitz Theorem (Uniqueness of a Local Solution to a Cauchy Problem)

We are interested in the uniqueness of a local solution of problem (1.3) when it exists. As already noted in Example 1.2.1, the continuity of f is not sufficient to guarantee uniqueness.

For this purpose, additional conditions on the function f must be imposed.

Definition 1.2.3.

The Cauchy problem (1.3) is said to have local uniqueness if for every $(t_0, x) \in I \times \Omega$, whenever u and v are two solutions of problem (1.3), there exists $\delta > 0$ such that $[t_0, t_0 + \delta] \subset I$ and

$$\forall t \in [t_0, t_0 + \delta],\ u(t) = v(t).$$

The Cauchy problem (1.3) has the property of global uniqueness if for every $(t_0, x) \in I \times \Omega$, all solutions of problem (1.3) coincide on the common part of their domains of definition.

Proposition 1.2.3.

The properties of local uniqueness and global uniqueness of a Cauchy problem are equivalent.

Proof.

We need to prove that the property of local uniqueness implies global uniqueness (the reverse implication is trivial). Let $(t_0, x) \in I \times \Omega$. Consider two solutions $u : \mathbb{J} \to \Omega$ and $v : \mathbb{K} \to \Omega$ of problem (1.3). Define the set $\mathcal{C}(u, v)$ by

$$\mathcal{C}(u, v) = \{t \in \mathbb{J} \cap \mathbb{K} :\ u(s) = v(s),\ \forall s \in [t_0, t].\}$$

Since (1.3) has the property of local uniqueness, the set $\mathcal{C}(u, v)$ is nonempty. We show that $\mathcal{C}(u, v)$ is closed: let $(t_n)_n \subset \mathcal{C}(u, v)$ converge to i. Then $i \in \mathcal{C}(u, v)$. We must show that

$$\sup \mathcal{C}(u, v) = \sup(\mathbb{J} \cap \mathbb{K}).$$

For contradiction, assume $\sup \mathcal{C}(u, v) < \sup(\mathbb{J} \cap \mathbb{K})$. Let $b = \sup \mathcal{C}(u, v)$. Consider the Cauchy problem

$$\begin{cases} z' = f(t, z(t)), \\ z(b) = y(b). \end{cases}$$

The solution is defined on $[b, \delta] \subset I$. By the patching principle, we can thus define a solution of the Cauchy problem (1.3) on $[t_0, b + \delta]$, which contradicts the maximality of the solution. The proof is therefore complete.

□

Remark 1.2.6.

Since there is equivalence between the two uniqueness properties, we will henceforth simply say that the Cauchy problem has the uniqueness property whenever it applies.

Definition 1.2.4.

A function $f : I \times \Omega \to \mathbb{R}^n$ is said to be locally Lipschitz on Ω if for every compact set $\mathcal{K} \subset I \times \Omega$, there exists a constant $L = L(\mathcal{K}) > 0$ such that

$$\forall (t, x_1), (t, x_2) \in \mathcal{K}, \ \|f(t, x_1) - f(t, x_2)\| \leq L \|x_1 - x_2\|.$$

If the above condition holds on $I \times \Omega$, we say that the function f is Lipschitz on Ω.

Example 1.2.2.

1. Clearly, every Lipschitz function is locally Lipschitz.
2. The function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(t, y) = |ty|$ is locally Lipschitz on $\mathbb{R}$.

   Indeed,

   $$|f(t, y_1) - f(t, y_2)| = |t||y_1 - y_2| \leq L(t)|y_1 - y_2|, \ \forall t \in [a, b] \times \mathbb{R}^n.$$

3. The function $f : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$ defined by $f(t, y) = \sqrt{ty}$ is not locally Lipschitz on $\mathbb{R}_+$.

Remark 1.2.7.

If $f : I \times \Omega \to \mathbb{R}$ satisfies the Cauchy condition on Ω, i.e., for all $i, j = 1, 2, \ldots, n$ the partial derivatives $\dfrac{df_j}{dx_i}$ are continuous on $I \times \Omega$, then f is locally Lipschitz on Ω.

Proposition 1.2.4.

By virtue of Proposition 1.2.3, it suffices to prove that the Cauchy problem (1.3) has the property of local uniqueness. Let $(t_0, x) \in I \times \Omega$ and let $u : \mathbb{J} \to \Omega,\ v : \mathbb{K} \to \Omega$ be two solutions of problem (1.3). Since $I \times \Omega$ is open, there exist $\rho > 0$ and $d > 0$ such that

$$a + d \leq \sup(\mathbb{J} \cap \mathbb{K}), \text{ and } B(x, \rho) \subset \Omega.$$

The functions u and v are continuous at t_0. We may therefore choose δ such that

$$\forall t \in [t_0, t_0 + d[, \|u(t) - v(t)\| \leq \int_{t_0}^{t} \|f(s, u(s)) - f(s, v(s))\| ds \leq \int_{t_0}^{t} \|u(s) - v(s)\| ds.$$

Gronwall's Lemma allows us to conclude that

$$\forall t \in [t_0, t_0 + d[, \ \|y(t) - z(t)\| = 0.$$

Consequently,

$$\forall t \in J',\ y(t) = z(t).$$

Theorem 1.2.8.

If the function $f : I \times \Omega \to \mathbb{R}^n$ is continuous on $I \times \Omega$ and locally Lipschitz on Ω, then for every $(t_0, x) \in I \times \Omega$, there exists $\delta > 0$ such that $[t_0, t_0 + \delta[\subset I$ and the Cauchy problem (1.3) admits a unique solution defined on $[t_0, t_0 + \delta[$.

Proof. We apply Peano's theorem. □

1.2.1 Maximal Solutions of a Cauchy Problem

In this paragraph we are interested in maximal solutions of problem (1.3).

Recall that a solution $y : \mathbb{J} \to \Omega$ is said to be maximal if it cannot be extended to another solution of the same problem.

Without loss of generality, we assume that $y : [t_0, T) \to \Omega$, where $[t_0, T) \subset I$, is a solution of the Cauchy problem (1.3). We will study the possibility of extending y to another solution $\tilde{y}$ defined on the interval $[t_0, T')$ where $T < T'$ and $[t_0, T') \subset I$. The following lemma will be of great importance in what follows.

Lemma 1.2.9.

Let $f : I \times \Omega \to \mathbb{R}^n$. Then the solution $y : [t_0, T) \to \mathbb{R}^n$ of problem (1.3) is extendable if:

(i) $T < \sup I$.

(ii) There exists $y^* \in \Omega$ such that $\lim_{t \to T^-} y(t) = y^*$.

Proof.

If the solution y is extendable, then conditions (i) and (ii) are obviously satisfied.

Now suppose that $y : [t_0, T) \to \Omega$ is a solution of problem (1.3) satisfying conditions (i) and (ii). We define an extension of y to the interval $[t_0, T]$ by:

$$\begin{cases} u(t) = y(t), & t \in [t_0, T[\\ u(T) = y^*. \end{cases}$$

A simple argument shows that u is a solution of problem (1.3) on the interval $[t_0, T]$.

Indeed, it suffices to show that

$$u'(T) = f(T, u(T)) = f(T, y^*).$$

Let $t \in [t_0, T[$ be arbitrary and $h < 0$ sufficiently small (so that $t + h \in [t_0, T[$). Then

$$y(t+h) = y(t) + hy'(t) + o(h) = y(t) + hf(t, y(t)) + o(h).$$

Taking the limit as $t \to T^-$ and using (ii) together with the continuity of f on $I \times \Omega$, we obtain:

$$y(T+h) = y^* + hf(T, y^*) + o(h).$$

It follows that:

$$\frac{u(T+h) - u(T)}{h} = \frac{y^* + hf(T, y^*) + o(h) - y^*}{h} = f(T, y^*) + o(1).$$

Now, letting $h \to 0^-$, we obtain

$$u'(T) = f(T, y^*).$$

By Peano's theorem, the Cauchy problem

$$\begin{cases} y' = f(t, y(t)), \\ y(T) = y^* \end{cases}$$

admits at least one solution v defined on an interval $[T, T') \subset I$. It follows, by the patching principle, that the function $\tilde{y}$ defined by:

$$\begin{cases} \tilde{y}(t) = u(t), & t \in [t_0, T], \\ \tilde{y}(t) = v(t), & t \in [T, T'), \end{cases}$$

is a solution of the Cauchy problem (1.3) defined on $[t_0, T')$. This completes the proof. $\square$

Remark 1.2.10.

The previous lemma remains valid if we consider a solution $y :]T, t_0] \to \Omega$, replacing (i) with $T > \inf I$, and in (ii) replacing $t \to T^-$ by $t \to T^+$. In this case, any extension of y will be defined on an interval of the form $]T', t_0]$ where $\inf I < T' < T$.

Remark 1.2.11.

From the previous lemma we deduce that if $y : \mathbb{J} \to \Omega$ is a maximal solution of the Cauchy problem (1.3), and f is continuous on $I \times \Omega$, where I and Ω are open sets, then necessarily the interval $\mathbb{J}$ is open.

The following lemma provides sufficient conditions for condition (ii) of Lemma 1.2.9 to be satisfied.

Proposition 1.2.5.

Let $y : [t_0, T) \to \Omega$ be a solution of problem (1.3) where $T < +\infty$. Suppose there exists a constant $M > 0$ such that:

$$\forall t \in [t_0, T),\ \|f(t, y(t))\| \leq M.$$

Then, there exists $y^* \in \overline{\Omega}$ such that:

$$\lim_{t \to T^-} y(t) = y^*.$$

Proof.

We deduce that:

$$\forall t, s \in [t_0, T),\ \|y(t) - y(s)\| \leq \left| \int_s^t \|f(s, y(s))\| ds \right| \leq M|t - s|.$$

Thus, the Cauchy convergence criterion guarantees that:

$$\lim_{t \to T^-} y(t) \text{ exists.}$$

□

Theorem 1.2.12.

Every solution $y : \mathbb{J} \to \Omega$ of the Cauchy problem (1.3) can be extended to a maximal solution.

Proof.

Let y be a solution of problem (1.3). Suppose that y is not maximal. Let $\mathcal{S}$ be the set of all solutions extending y. We endow $\mathcal{S}$ with a binary relation, denoted by $\preceq$, defined as follows:

$$\forall x, z \in \mathcal{S}, \; x \preceq z \Longleftrightarrow z \text{ extends } x.$$

The set $\mathcal{S}$ is nonempty (since $y \in \mathcal{S}$). Moreover, the set $\mathcal{S}$ endowed with the binary relation $\preceq$ is ordered. Hence, Zorn's lemma allows us to deduce that there exists a maximal element $y^* \in \mathcal{S}$ such that $y \preceq y^*$. Therefore, y^* is maximal. This completes the proof.

□

Remark 1.2.13.

(i) The previous theorem remains valid even if the function f is not continuous or if one of the sets I or Ω is not open.

(ii) A Cauchy problem may admit several maximal solutions.

We conclude this section with a fundamental result concerning the existence of maximal solutions of a Cauchy problem.

Corollary 1.2.14.
Suppose that I *and* Ω *are open. If* $f : I \times \Omega \to \mathbb{R}^n$ *is continuous on* $I \times \Omega$*, then for every* $(t_0, x) \in I \times \Omega$*, the Cauchy problem (1.3) admits at least one maximal solution defined on an open interval* $\mathbb{J}$ *containing* t_0*.*

Proof.

The proof follows immediately from Peano's theorem and Theorem 1.2.12.
If the Cauchy problem has the uniqueness property of the solution, for example if f is Lipschitz continuous with respect to the second variable, then the maximal solution, if it exists, is unique. More precisely, we obtain the following result: □

Corollary 1.2.15.
Suppose that I *and* Ω *are open. If* $f : I \times \Omega \to \mathbb{R}^n$ *is continuous on* $I \times \Omega$ *and locally Lipschitz on* Ω*, then for every* $(t_0, x) \in I \times \Omega$ *the Cauchy problem (1.3) admits a unique maximal solution defined on an open interval* $\mathbb{J}$ *containing* t_0*.*

Proof. The proof follows immediately from the Cauchy–Lipschitz theorem (i.e. Theorem 1.2.8) and Theorem 1.2.12.

□

Example 1.2.3.

We seek all the maximal solutions of the ODE

$$\begin{cases} y' = y^2, \\ y(0) = 0. \end{cases} \tag{1.6}$$

In order to discuss the nature of the maximal solutions, we must declare the domain of definition of the ODE dynamics. Let $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and

$$\forall (t, y) \in \mathbb{R} \times \mathbb{R},\ f(t, y) = y^2.$$

Thus, the (trivial) zero solution is global. On the other hand, we have

$$\frac{df}{dy}(t, y) = 2y,$$

which is continuous on $\mathbb{R}$. We deduce that f is locally Lipschitz with respect to the second variable. Consequently, the zero solution is unique.

Example 1.2.4.

We consider the same previous problem, but with a different initial condition. Let

$$\begin{cases} y' = y^2, \\ y(0) = 1. \end{cases} \tag{1.7}$$

The dynamics are defined as before. Clearly, the zero solution does not satisfy the ODE (1.7). Let us solve the problem explicitly. To this end, we use the method of separation of variables. We obtain:

$$\frac{dy}{y^2} = 1 \Longrightarrow \int \frac{dy}{y^2} = \int 1 \Longrightarrow \frac{-1}{y} = t + C.$$

Hence, every solution of the ODE (1.7) is of the form:

$$y(t) = \frac{-1}{t+C}.$$

Since $y(0) = 1$, we deduce that $C = -1$. Consequently,

$$y(t) = \frac{-1}{t-1}.$$

The solution y is defined on $]-\infty, 1[$. The solution y is maximal. Indeed,

$$\lim_{t \to 1^-} y(t) = -\infty.$$

1.2.2 Global solutions of a Cauchy problem

This subsection is devoted to the study of the existence of solutions of the Cauchy problem (1.3).

Theorem 1.2.16. *Let I be an open interval of $\mathbb{R}$ and let $f : I \times \mathbb{R}^n \to \mathbb{R}^n$ be continuous on $I \times \mathbb{R}^n$. Suppose there exist two functions $h, k : I \to \mathbb{R}$ such that*

$$\forall (t, y) \in I \times \mathbb{R}^n, \ \|f(t, y)\| \leq k(t)\|y\| + h(t).$$

Then, for every $(t_0, x) \in I \times \mathbb{R}^n$, the Cauchy problem (1.3) admits at least one global solution.

Proof.

By virtue of Corollary 1.2.15, the Cauchy problem (1.3) admits at least one maximal solution $y : [t_0, b) \to \Omega$ with $[t_0, b) \subset I$. We deduce that the solution y satisfies

$$\forall t \in [t_0, b), \ \|y(t)\| \leq \|x\| + \int_{t_0}^{t} h(s)\, ds + \int_{t_0}^{t} k(s)\|y(s)\|\, ds.$$

We now show that $b = \sup I$. For contradiction, assume $b \neq \sup I$. Since h and k are continuous on $\mathbb{J}$ and $\mathbb{J}$ is compact, we deduce that there exists a constant $M > 0$ such that

$$\forall t \in [a, b], \ h(t) \leq M \text{ and } k(t) \leq M.$$

Applying Gronwall's Lemma, we therefore obtain

$$\forall t \in [a, b], \ \|y(t)\| \leq \Big(\|x\| + M(b - a)\Big)e^{M(b-a)}.$$

□

1.2.3 Study of differentiable and integrable functions in normed vector spaces

Normed vector spaces, Banach spaces

An $\mathbb{R}$-vector space X is said to be normed if it can be endowed with a norm $N : X \to \mathbb{R}_+$. We define the following.

Definition 1.2.5. A norm on a vector space X is a mapping $N : X \to \mathbb{R}_+$ satisfying the following three axioms:

- For all $x \in X$, we have $N(x) = 0$ if and only if $x = 0$.
- For all $x \in X$ and $\lambda \in \mathbb{R}$, we have $N(\lambda x) = |\lambda| N(x)$.
- For all $x, y \in X$, we have $N(x + y) \leq N(x) + N(y)$.

Example 1.2.5.

1. The vector space $X = \mathbb{R}^n$ can be equipped with three norms, namely:
   - $N_1(x) = N_1(x_1, x_2, \dots, x_n) = \sum_{i=1}^{n} |x_i|$.
   - $N_2(x) = N_2(x_1, x_2, \dots, x_n) = \left(\sum_{i=1}^{n} x_i^2\right)^{\frac{1}{2}}$.
   - $N_\infty(x) = \max\{|x_i| \mid i = 1, 2, \dots, n\}$.
2. The vector space of bounded real sequences, denoted $l^\infty(\mathbb{R})$, is endowed with the following norm: If $x = (x_n)_n$ is bounded, then

$$N_\infty(x) = \|x\|_\infty = \sup_{n \in \mathbb{N}} |x_n|.$$

Definition 1.2.6. A normed vector space is called a **Banach space** if it is complete.

A topological space is complete if every Cauchy sequence in the space converges.

Example 1.2.6.

1. Let $(E, \|\cdot\|)$ be a Banach space. Let I be an arbitrary set. We denote by $l^\infty(I, E)$ the vector space of bounded functions defined on I with values in E. We endow $l^\infty(I, E)$ with a norm, denoted by $\|\cdot\|_\infty$, defined by:

$$\forall f \in l^\infty(I, E), \quad \|f\|_\infty = \sup\{f(x),\ x \in I\}.$$

   It can be shown (as an exercise) that $(l^\infty(I, E), \|\cdot\|_\infty)$ is a Banach space.

2. In the particular case where $I = \mathbb{N}$ and $E = \mathbb{R}$, $l^\infty(I, E)$ becomes the vector space of bounded real sequences, and we write:

$$l^\infty(I, E) = l^\infty(\mathbb{R}),$$

   which is a Banach space.

1.2.4 Differential of a mapping

Let X and Y be two normed vector spaces over a field $\mathbb{K}$, and let $\Omega \subset X$ be an open set. We consider the function $F : \Omega \to Y$. We denote by $L(X, Y)$ the $\mathbb{K}$-vector space of linear mappings defined on X with values in Y.

Definition 1.2.7. The function f is said to be Fréchet differentiable at $x_0 \in \Omega$ if there exists a linear map denoted $f'(x_0) \in L(X, Y)$ such that:

$$f(x_0 + h) = f(x_0) + f'(x_0)h + \omega(x_0, h), \quad \text{where } \omega(x_0, h) = o(|h|),\ h \to 0.$$

If f is differentiable in the sense of Fréchet at every point $x_0 \in \Omega$, we say that it is Fréchet differentiable on Ω.

Remark 1.2.17.

(i) If the function $f : \Omega \to Y$ is differentiable on X, then we define the derivative function $f' : \Omega \to L(X, Y)$.
If $X = \mathbb{K}$, we can identify $L(\mathbb{K}, Y)$ with Y, and thus $f'(x_0)$ will be identified with a vector.

(ii) If the function f' is continuous, then we say that f is continuously differentiable on Ω and we write $f \in C^1(\Omega)$.

Example 1.2.7. Let the integral operator $F : X \to X$, where $X = C(\mathbb{J})$ and $\mathbb{J} = [a, b]$, be defined by:

$$F(x)(t) = \int_a^b k(t,s) f(s, x(s))\, ds, \quad t \in \mathbb{J}.$$

If $k : \mathbb{J} \times \mathbb{J} \to \mathbb{R}$, f and $\dfrac{\partial f}{\partial x}$ are continuous, then F is differentiable on X and we have:

$$\forall h \in X, \quad (F'(x)(h))(t) = \int_a^b k(t,s) \frac{\partial f}{\partial x}(s, x(s)) h(s)\, ds.$$

Example 1.2.8. Consider the functional $\psi : C(\mathbb{J}) \to \mathbb{R}$ defined by:

$$\psi(x) = \int_a^b \int_0^{x(t)} f(\tau, s)\, ds\, d\tau,$$

where $f : \mathbb{J} \times \mathbb{R} \to \mathbb{R}$ is continuous. Then ψ is continuously differentiable and

$$\psi'(x)(h) = \int_a^b f(\tau, x(\tau)) h(\tau)\, d\tau.$$

Proposition 1.2.6. Let X be a Banach space. If $f : \Omega \to Y$, then:

$$\int_a^b f'(t) = y(b) - y(a) \quad \text{if and only if} \quad y : [a, b] \to X$$

is continuously differentiable.

Proposition 1.2.7. If $f : \Omega \to Y$ is differentiable at $a \in \Omega$, then for all $h \in X$, we have:

$$\lim_{t \to 0} \frac{f(a + th) - f(a)}{t} = df(a) \cdot h.$$

Proposition 1.2.8. If f is differentiable at a, then it is continuous at a.

Remark 1.2.18. We show that f is differentiable at x_0 if and only if there exists a continuous linear mapping, denoted $f'(x_0) \in L(X, Y)$, such that:

$$\lim_{x \to x_0} \frac{f(x) - f(x_0) - f'(x_0)(x - x_0)}{\|x - x_0\|} = 0.$$

Example 1.2.9. Let $X = C^0$ and $I = [0, 1]$. We consider the function $y : I \to X$ defined by:

$$\forall t \in I, \quad y(t) = y_n(t) = \frac{1}{n}\sin(nt).$$

(i) Show that y is Lipschitz. Indeed,

(ii) Show that y is nowhere differentiable.

1.2.5 Solution in the sense of Carathéodory

The existence theorems studied so far require the continuity of the right-hand side defining the ODE. If the continuity condition is relaxed, the existence of a classical local solution (of class C^1) is no longer guaranteed. The following example illustrates this phenomenon.

Example 1.2.10. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by:

$$f(x) = \begin{cases} 1 & \text{if } x < 0, \\ -1 & \text{if } x > 0, \\ -1 & \text{if } x = 0. \end{cases}$$

We consider the Cauchy problem:

$$\begin{cases} y' = f(y(t)), \\ y(0) = 0. \end{cases} \tag{1.8}$$

Suppose that the problem (1.8) admits a C^1 solution y defined to the right of 0, i.e., y is defined at least on an interval $[0, T[$. Then:

$$y'(0) = f(y(0)) = f(0) = -1.$$

Since y' is continuous at 0, we can choose T small enough such that:

$$\forall t \in [0, T[, \qquad y'(t) < 0.$$

It follows that y is strictly decreasing on $[0, T[$. Hence,

$$\forall t \in]0, T[, \qquad y(t) < y(0) = 0.$$

It then follows that:

$$\forall t \in]0, T[, \quad y'(t) = f(y(t)) = 1 \implies y'(0) = 1,$$

which contradicts the fact that we already have $y'(0) = -1$. Finally, we deduce that the Cauchy problem (1.8) does not admit any C^1 solution.

Remark 1.2.19. This phenomenon of non-existence of a C^1 solution for problem (1.8) is, in fact, due to the discontinuity of the function f with respect to the space variable y.

Note here that the problem persists if the function f is discontinuous with respect to the time variable t.

Thus, following the same reasoning as before, the Cauchy problem:

$$\begin{cases} y' = f(t), \\ y(0) = 0. \end{cases} \tag{1.9}$$

admits no classical C^1 solution.

We are therefore led to broaden the class of functions that can be solutions of problem (1.9). In other words, we slightly modify the notion of solution of problem (1.9).

More precisely, we say that $y : \mathbb{J} \to \Omega$ is a solution of problem (1.9) if $y(t) \in \Omega,\ \forall t \in \mathbb{J},\ y(0) = 0,\ y$ is differentiable almost everywhere on $\mathbb{J}$, and

$$y'(t) = f(y(t)) \quad \text{a.e. on } \mathbb{J}.$$

Thus, adopting the previous definition, the function y defined on $\mathbb{R}$ by:

$$y(t) = -|t|$$

becomes a solution of problem (1.9).

It is easy to check that y is defined and continuous on $\mathbb{R}$, and satisfies the relation $y' = f(t)$ almost everywhere on $\mathbb{R}$ (excluding the value 0).

The aim of this section is therefore to define another concept of solution and establish its existence. To this end, a particular class of functions f defining the right-hand side of the ODE will be introduced.

Definition 1.2.8.
A function $f : I \times \Omega \to \mathbb{R}^n$ is said to be of **Carathéodory type** if the following conditions are satisfied:

(i) For almost every $t \in I$, the mapping $x \mapsto f(t,x)$ is continuous on Ω.

(ii) For every $x \in \Omega$, the mapping $t \mapsto f(t,x)$ is Lebesgue-measurable on I.

(iii) For every $(t_0, x) \in I \times \Omega$, there exist $\rho > 0$ and $\delta > 0$ such that

$$[t_0 - \delta,\, t_0 + \delta] \subset I \quad \text{and} \quad B(x,\rho) \subset \Omega.$$

Moreover, there exists a function $h : [t_0 - \delta, t_0 + \delta] \to \mathbb{R}^n$, Lebesgue-integrable, such that

$$\forall (t,y) \in [t_0 - \delta,\, t_0 + \delta] \times B(x,\rho), \quad \|f(t,y)\| \leq h(t).$$

Consider now the Cauchy problem:

$$\begin{cases} y'(t) = f(t, y(t)), \\ y(t_0) = x, \end{cases} \tag{1.10}$$

where $f : I \times \Omega \to \mathbb{R}^n$ is of Carathéodory type.

Definition 1.2.9.

A function $y : \mathbb{J} \to \mathbb{R}^n$ is said to be a **solution** of the Cauchy problem (1.10) if:

(i) For all $t \in \mathbb{J}$, $y(t) \in \Omega$.

(ii) $y(t_0) = x$, y is absolutely continuous on $\mathbb{J}$, and

$$y'(t) = f(t, y(t)) \quad \text{a.e. on } \mathbb{J}.$$

Theorem 1.2.20.

Suppose $f : I \times \Omega \to \mathbb{R}^n$ is of Carathéodory type. Then, for every $(t_0, x) \in I \times \Omega$, the problem (1.10) admits at least one local solution (in the sense of Definition 1.2.9).

To prove the theorem above, we proceed as follows. Let $(t_0, x) \in I \times \Omega$, and consider the delayed integral equation with parameter $\lambda > 0$, defined by:

$$y_\lambda(t) = \begin{cases} x, & \text{if } t \in [t_0 - \lambda, t_0], \\ x + \displaystyle\int_{t_0}^{t} f(s, y_\lambda(s - \lambda))\, ds, & \text{if } t \in (t_0,\, t_0 + \delta]. \end{cases} \tag{1.11}$$

Method of proof: The idea of the proof is as follows (analogous to Peano's Theorem). First, we show that the integral equation (1.11) admits, for every $\lambda > 0$, at least one solution $y_\lambda : [t_0 - \lambda,\, t_0 + \lambda]$. In contrast to the proof of Peano's theorem, here we prove that the solution of the integral equation is absolutely continuous (the variation of the function equals the primitive of its derivative). Next, by letting $\lambda \to 0$, we show that the sequence $(y_\lambda)_\lambda$ converges uniformly to a function y, where the Arzelà–Ascoli theorem plays a fundamental role. Finally, we prove that the limit y is a solution of the considered Cauchy problem.

To avoid technical complications, we assume $f : I \times \mathbb{R}^n \to \mathbb{R}^n$. We now establish the following lemma:

Lemma 1.2.21.
Let $f : I \times \mathbb{R}^n \to \mathbb{R}^n$ be a Carathéodory function. Suppose there exists $h : I \to \mathbb{R}_+$, locally integrable, such that for almost every $t \in I$:

$$\forall y \in \mathbb{R}^n, \quad \|f(t, y)\| \leq h(t). \tag{1.12}$$

Then, for every $(t_0, x) \in I \times \mathbb{R}^n$ and every $\delta > 0$ with $[t_0, t_0 + \delta] \subset I$, the integral equation (1.11) admits a unique absolutely continuous solution defined on $[t_0 - \lambda,\, t_0 + \delta]$.

Proof. If $y : I \to \mathbb{R}^n$ is continuous on I, then the function $t \mapsto f(t, y(t))$ is measurable on I. From inequality (1.12) we obtain, for almost every $t \in I$,

$$\|f(t, y(t))\| \leq h(t).$$

Thus, $t \mapsto f(t, y(t))$ is locally integrable on I. It follows that the function y_λ is defined and continuous on every interval of the form $[t_0 - \lambda,\, t_0 + i\lambda]$ with $t_0 + i\lambda \leq t_0 + \delta$. Hence $s \mapsto f(s, y_\lambda(s - \lambda))$ is Lebesgue integrable on $[t_0 - \lambda,\, t_0 + (i+1)\lambda]$, for $i = 1, 2, \ldots$. By induction, y_λ is defined on $[t_0 - \lambda,\, t_0 + (i+1)\lambda]$.

On the other hand, y_λ is clearly defined on $[t_0 - \lambda,\, t_0]$. For $t \in [t_0,\, t_0 + \lambda]$, we observe that $s - \lambda \in [t_0 - \lambda, t_0]$, which implies $y_\lambda(s - \lambda) = x$. Hence,

$$y_\lambda(t) = x + \int_{t_0}^{t} f(s, x)\, ds.$$

Therefore y_λ is well-defined on $[t_0, t_0 + \lambda]$. Repeating this argument, y_λ is defined on $[t_0 + \lambda, t_0 + 2\lambda]$, and so forth.

By induction, y_λ is defined on the whole interval $[t_0,\, t_0 + \delta]$. Moreover, y_λ is absolutely continuous (being the primitive of its derivative). The proof is complete. □

We now proceed to another auxiliary result.

Lemma 1.2.22.
Let $f : I \times \mathbb{R}^n \to \mathbb{R}^n$ be of Carathéodory type, and suppose there exists a function $h : I \to \mathbb{R}_+$ satisfying property (1.12). Then, for every $(t_0, x) \in I \times \mathbb{R}^n$ and every $\delta > 0$ such that $[t_0, t_0 + \delta] \subset I$, the Cauchy problem (1.10) admits at least one Carathéodory solution defined on $[t_0, t_0 + \delta]$.

Proof. Let $(t_0, x) \in I \times \mathbb{R}^n$ and $\delta > 0$ with $[t_0, t_0 + \delta] \subset I$. Let $n \in \mathbb{N}$ and consider the delayed integral equation with step $\delta_n = \frac{\delta}{n+1}$:

$$y_n(t) = \begin{cases} x, & t \in [t_0 - \delta_n, t_0], \\ x + \int_{t_0}^{t} f(s, y_n(s - \delta_n))\, ds, & t \in (t_0,\, t_0 + \delta_n]. \end{cases} \tag{1.13}$$

By Lemma 1.2.21, for every $n \in \mathbb{N}$, equation (1.13) admits a unique absolutely continuous solution $y_n : [t_0 - \delta_n,\, t_0 + \delta] \to \mathbb{R}^n$.

Consider the family $\mathcal{F}$ of restrictions of the y_n to $[t_0, t_0 + \delta]$. To prove that $(y_n)_n$ converges uniformly, it suffices to apply the Arzelà–Ascoli theorem. Thus, we need to show that $\mathcal{F}$ is uniformly bounded and equicontinuous.

Uniform boundedness. From (1.12), we have:

$$\forall n \in \mathbb{N},\ \forall t \in [t_0, t_0 + \delta], \quad \|y_n(t)\| \le \|x\| + \int_{t_0}^{t} h(s)\, ds \le \|x\| + \int_{t_0}^{t_0+\delta} h(s)\, ds.$$

Thus $\mathcal{F}$ is uniformly bounded.

Equicontinuity. Again from (1.12):

$$\forall n \in \mathbb{N},\ \forall t, s \in [t_0, t_0 + \delta], \quad \|y_n(t) - y_n(s)\| \le \left| \int_{s}^{t} h(u)\, du \right|.$$

Since h is Lebesgue integrable on $[t_0, t_0 + \delta]$, the map $t \mapsto \int_{t_0}^{t} h(s)\, ds$ is absolutely continuous. Hence $\mathcal{F}$ is equicontinuous.

By Arzelà–Ascoli, there exists a subsequence (y_n) converging uniformly to some function y.

Moreover,

$$\lim_{n\to+\infty} y_n(s-\delta_n) = y(s),$$

uniformly on $[t_0, t_0+\delta]$. Since f is Carathéodory, it follows that for almost every $t \in [t_0, t_0+\delta]$,

$$\lim_{n\to+\infty} f(s, y_n(s-\delta_n)) = f(s, y(s)).$$

By the dominated convergence theorem, passing to the limit yields

$$\forall t \in [t_0, t_0+\delta], \quad y(t) = x + \int_{t_0}^{t} f(s, y(s))\, ds.$$

Thus y is a solution of the Cauchy problem. $\square$

We now proceed to the proof of the main result.

Proof. Let $(t_0, x) \in \Omega$. Since I and Ω are open, there exist $d > 0$ and $r > 0$ such that

$$[t_0 - d,\, t_0 + d] \subset I, \quad B(x, r) = \{\eta \in \mathbb{R}^n : \|\eta - x\| \leq r\} \subset \Omega.$$

Define $\rho : \mathbb{R}^n \to \mathbb{R}^n$ by

$$\rho(y) = \begin{cases} y, & y \in B(x, r), \\ \dfrac{r}{\|y - x\|}(y - x) + x, & y \notin B(x, r). \end{cases}$$

The map ρ is continuous and $\rho(\mathbb{R}^n) \subset B(x, r)$. Now define $g : (t_0 - d, t_0 + d) \times \mathbb{R}^n \to \mathbb{R}^n$ by

$$g(t, y) = f(t, \rho(y)).$$

Since f is continuous, the Weierstrass theorem ensures that f is bounded on compact subsets. Thus, for every $d' \in (0, d)$, the Cauchy problem

$$\begin{cases} y' = g(t, y), \\ y(t_0) = x, \end{cases}$$

admits at least one solution $y : [t_0, t_0 + d'] \to \mathbb{R}^n$.

Since $y(t_0) = x$ and y is continuous, for every $r > 0$ there exists $\delta \in (0, d')$ such that

$$\forall t \in [t_0, t_0 + \delta], \quad \|y(t) - x\| \leq r.$$

Thus, $g(t, y(t)) = f(t, y(t))$, and hence y is a solution of the original Cauchy problem. The proof is complete. $\square$

Example 1.2.11.

Consider the Cauchy problem:

$$\begin{cases} y'(t) = y \text{ if } g(t-1), \\ y(0) = 1. \end{cases} \tag{1.14}$$

The Cauchy problem (1.14) has a unique solution $y : \mathbb{R}_+ \to \mathbb{R}$ in the sense of Carathéodory, defined by:

$$\forall t \in \mathbb{R}_+, \; y(t) = e^{|t-1|-1}.$$

In this chapter, we are interested in Cauchy problems of the form:

$$\begin{cases} y' = f(t, y(t)), \\ y(t_0) = x, \end{cases} \tag{1.15}$$

where $f : I \times \Omega \to X$, X is a Banach space, $\Omega \subset X$, and $I \subset \mathbb{R}$ is an open interval.

The concepts of integrability and differentiability in Banach spaces are detailed in Appendix A.

We begin by defining a solution of problem (1.15).

Definition 1.2.10.
A function $y : \mathbb{J} \to \Omega$ is called a solution of problem (1.15), where $\mathbb{J}$ is a non-empty subinterval of I containing t_0, if:

- $\forall t \in \mathbb{J},\ y(t) \in \Omega$ and $y(t_0) = x$.
- y is differentiable on $\mathbb{J}$ and $\forall t \in \mathbb{J},\ y'(t) = f(t, y(t))$.

Remark 1.2.23. The concept of solutions of ODEs in infinite dimensions is a delicate subject. In infinite dimensions, ODEs rarely admit differentiable solutions.

As in finite dimensions, the following result is a characterization of (classical) solutions of problem (1.15).

Proposition 1.2.9. Suppose that f is continuous on $I \times \Omega$. The function $y : \mathbb{J} \to \Omega$ is a solution of the Cauchy problem (1.15) if and only if y is continuous on $\mathbb{J}$ and

$$\forall t \in \mathbb{J},\ y(t) = x + \int_{t_0}^{t} f(s, y(s))ds.$$

Proof 1. We note that

$$\int_{t_0}^{t} f(s, y(s))ds \text{ is in the sense of Bochner.}$$

Remark 1.2.24. The notions of global solution, maximal solution, right solution, left solution, etc. are the same as in finite dimensions.

1.3 Qualitative study of ODEs in infinite dimension (n) (Peano and Cauchy–Lipschitz theorem)

Suppose that $X = \mathbb{R}^n$ and f is continuous on $I \times \Omega$. Then Peano's theorem guarantees that for every $(t_0, x) \in I \times \Omega$, the Cauchy problem (1.15) admits at least one local solution defined on an interval $\mathbb{J}$ containing t_0.

The "naïve" generalization of Peano's theorem to infinite dimension is drastically false.

In 1951, Dieudonné constructed two examples showing that continuity of the dynamics (in our case, the function f) is not sufficient to guarantee the existence of at least one local solution to the Cauchy problem (1.15).

In particular, for every Banach space X of infinite dimension, there exists a Cauchy problem (associated with a continuous function $f : I \times X \to X$) that possesses no local solution.

Thus, to guarantee the existence of at least one local solution, an additional assumption on the function f (redundant in finite dimension) must be imposed.

Definition 1.3.1.

The function f is said to be **compact** if it maps every bounded set in $I \times \Omega$ to a relatively compact set in X.

The theorem below is a variation of Peano's theorem in infinite dimension.

Theorem 1.3.1.

If f is compact and continuous on $I \times \Omega$ where $\Omega \subset \mathbb{R}^n$ is open, then for every $(t_0, x) \in I \times \Omega$, the problem (1.15) admits at least one local solution $y : \mathbb{J} \subset I \to \Omega$ defined on a subinterval $\mathbb{J} \subset I$ containing t_0.

Remark 1.3.2. In finite dimension, the assumption of compactness of f is redundant. Indeed, if f is continuous then it maps every bounded set in $I \times \Omega$ to a bounded set in $\mathbb{R}^n$, which is relatively compact.

Before starting the proof of Theorem 1.3.1, let us first consider the special case where f is continuous on $I \times X$ and $f(I \times X)$ is relatively compact in X. We choose $\lambda > 0$ and $\delta > 0$ such that $[t_0, t_0 + \delta] \subset I$, and consider the delay integral equation:

$$y_\lambda(t) = \begin{cases} x & \text{if } t \in [t_0 - \lambda, t_0], \\ x + \int_{t_0}^{t} f(s, y_\lambda(s - \lambda))ds & \text{if } t \in (t_0, t_0 + \delta]. \end{cases} \tag{1.16}$$

• **Proof method:** First, we show that for every $\lambda > 0$, the integral equation (1.16) admits at least one solution $y_\lambda : [t_0 - \lambda, t_0 + \delta] \to X$.
Next, by letting $\lambda \to 0$, we show that the sequence $(y_\lambda)_\lambda$ converges uniformly to a function y. Finally, we show that the limit y is a solution to the considered Cauchy problem. We note here that the compactness of the function f is essential for the convergence of the approximate solutions to a solution of the considered problem.

Lemma 1.3.3.
If $f : I \times X \to X$ is continuous, then for every $(t_0, x) \in I \times X$, $\lambda > 0$ and $\delta > 0$ such that $[t_0, t_0 + \delta] \subset I$, the delay integral equation (1.16) admits a unique solution defined on $[t_0 - \lambda, t_0 + \delta]$.

Proof.

First, we observe that the function y_λ is uniquely defined on the interval $[t_0 - \lambda, t_0]$ $(y_\lambda \equiv x)$. Now let $t \in [t_0 - \lambda, t_0)$. It is clear that if $s \in [t_0, t]$ then $s - \lambda \in [t_0 - \lambda, t_0]$, which gives $y_\lambda(s - \lambda) = x$. Consequently,

$$y_\lambda(t) = x + \int_{t_0}^{t} f(s, x)ds,$$

which shows that the function y_λ is also determined on the interval $[t_0 - \lambda, t_0]$.
Thus, in a similar manner, we can define y_λ by induction on the intervals
$[t_0 + \lambda, t_0 + 2\lambda]$, $[t_0 + 2\lambda, t_0 + 3\lambda], \cdots, [t_0 + (n-2)\lambda, t_0 + (n-1)\lambda]$ such that $n\lambda \geq t_0 + \delta$.
Therefore, the function y_λ is defined on the interval $[t_0, t_0 + \delta]$ and is continuous. The proof is thus complete.

□

We now move on to another auxiliary existence result.

Lemma 1.3.4.
Let $f : I \times X \to X$ be continuous and suppose $f(I \times X)$ is relatively compact. Then, for every $(t_0, x) \in I \times X$ and $\delta > 0$ such that $[t_0, t_0 + \delta] \subset I$, the delay equation (1.16) has a unique solution defined on $[t_0, t_0 + \delta]$.

Proof.

Let $(t_0, x) \in I \times X$ and $\delta > 0$ such that $[t_0, t_0 + \delta] \subset I$. Let $n \in \mathbb{N}$. Consider the delayed integral equation with $\delta_n = \dfrac{1}{n+1}$:

$$y_n(t) = \begin{cases} x & \text{if } t \in [t_0 - \delta_n, t_0] \\ x + \int_{t_0}^t f(s, y_n(s - \delta_n))ds & \text{if } t \in (t_0, t_0 + \delta_n]. \end{cases} \tag{1.17}$$

By virtue of Lemma 1.3.3, for every $n \in \mathbb{N}$, the delayed integral equation (1.17) admits a unique solution $y_n : [t_0 - \delta_n, t_0 + \delta] \to X$. In the sequel, we consider the family
$\mathcal{F} = \{y_n,\ n \in \mathbb{N}\}$ consisting of the restrictions of y_n to the interval $[t_0, t_0 + \delta]$. We now aim to prove that the family $\mathcal{F}$ is relatively compact in the space $C([t_0, t_0 + \delta], X)$. This amounts to showing that $\mathcal{F}$ satisfies the assumptions of the Arzelà–Ascoli theorem, i.e., $\mathcal{F}$ is uniformly bounded and equicontinuous.
The family $\mathcal{F}(t) = \{y_n(t),\ n \in \mathbb{N}\}$ is relatively compact: We need to prove that for every $t \in [t_0, t_0 + \delta]$, the set

$$\mathcal{F}(t) = \{y_n(t),\ n \in \mathbb{N}\}$$

is relatively compact in X. This follows directly from the assumption that $f(I \times X)$ is relatively compact in X. Therefore, we deduce that the set

$$\mathcal{F}(t) = \{y_n(t),\ n \in \mathbb{N}\} = \left\{x + \int_{t_0}^{t} f(s, y_n(s - \delta_n))ds,\ n \in \mathbb{N}\right\}$$

is relatively compact in X for all $t \in [t_0, t_0 + \delta)$.

The family $\mathcal{F}$ is equicontinuous: The set $\mathcal{F}$ is equicontinuous. Indeed,

$$\forall (t, s) \in [t_0, t_0 + \delta),\ \|y_n(t) - y_n(s)\| \leq \left| \int_{s}^{t} \|f(\sigma, y_n(\sigma - \delta_n))\| d\sigma \right| \leq M|t - s|,$$

for some constant M and for all $n \in \mathbb{N}$. Consequently, $\mathcal{F}$ is equicontinuous on $[t_0, t_0 + \delta]$. By applying the Arzelà–Ascoli theorem, we deduce that $(y_n)_n$ converges uniformly to a function $y : [t_0, t_0 + \delta] \to X$.

On the other hand, we have:

$$\lim_{n} y_n(s - \delta_n) = y(s)$$

uniformly on $[t_0, t_0 + \delta]$. Since f is continuous on $I \times X$, by passing to the limit in (1.17), we deduce that

$$\forall t \in [t_0, t_0 + \delta],\ y(t) = x + \int_{t_0}^{t} f(s, y(s))ds.$$

It follows that $y : [t_0, t_0 + \delta] \to X$ is a solution to the Cauchy problem (1.15).

□

We now proceed to the proof of the main theorem of this section (Theorem 1.3.1).

Proof.

Let $(t_0, x) \in \Omega$. Since Ω is open, there exist $d > 0$ and $r > 0$ such that

$$[t_0 - d, t_0 + d] \times B(x, r) \subset D \text{ and } B(x, r) = \{\eta \in X,\ \|\eta - x\| \leq r\}.$$

Define the function $\rho : X \to X$ by:

$$\rho(y) = \begin{cases} y & \text{if } y \in B(x, r) \\ \dfrac{r}{\|y - x\|}(y - x) + x & \text{if } y \in X - B(x, r). \end{cases}$$

Hence, ρ is continuous and $\rho(X) \subset B(x,r)$. Now define $g : (t_0 - d, t_0 + d) \times X \to X$ by:

$$\forall (t,y) \in (t_0 - d, t_0 + d) \times X,\ g(t,y) = f(t, \rho(y)).$$

The set $f((t_0 - d, t_0 + d) \times B(x,r))$ is relatively compact since f is b-compact. Therefore, g is continuous and $g((t_0 - d, t_0 + d) \times X)$ is relatively compact. By virtue of Lemma 1.3.3, we deduce that for every $d' \in (0,d)$, the Cauchy problem

$$\begin{cases} y' = g(t,y) \\ y(t_0) = x, \end{cases}$$

admits at least one solution $y : [t_0, t_0 + d'] \to X$.

Since y is continuous at t_0 and $y(t_0) = x$, we deduce that for any $r > 0$, there exists $\delta \in (0, d')$ such that for all $t \in [t_0, t_0 + \delta]$, we have $\|y(t) - x\| \leq r$. Thus,

$$g(t, y(t)) = f(t, y(t)).$$

Consequently, the function y is a solution to the Cauchy problem (1.15). The proof is complete.

$\square$

Remark 1.3.5.

We note here that the missing argument in Peano's result is the impossibility of making the local approximate solutions converge by means of Ascoli's theorem. The latter itself fails because the closed balls of X are not compact: by Riesz's theorem, the closed unit ball of an infinite-dimensional normed space is never compact. More precisely, one can prove that in any separable infinite-dimensional Banach space X there exists a continuous function $f : X \to X$ such that the ODE $y' = f(y)$ admits no local solution.

Remark 1.3.6.
The results established in finite dimensions concerning the existence of maximal solutions remain valid in infinite dimensions. More precisely, we have the above variation of Peano's theorem concerning the existence of a maximal solution of a Cauchy problem in infinite dimension.

Theorem 1.3.7.
If f is compact and continuous on $I \times \Omega$, where $\Omega \subset \mathbb{R}^n$ is open, then for every $(t_0, x) \in I \times \Omega$, problem (1.9) admits at least one maximal solution $y : \mathbb{J} \subset I \to \Omega$ defined on an open interval $\mathbb{J} \subset I$ containing t_0.

1.3.1 Cauchy Problem and Integral Equation

We are interested in providing a result of existence and uniqueness of the solution of a Cauchy problem in infinite dimension.

Remark 1.3.8.

The proof of the Cauchy–Lipschitz theorem is analogous to that established in finite dimensions.

In what follows, we will present and prove it in a different way, based on Picard's approximations. Broadly speaking, the method of proving the existence of a solution in Banach spaces (including in finite dimension) proceeds in three steps:

- Construct a sequence of approximate solutions of the considered Cauchy problem.
- Show that the sequence built in the first step converges uniformly.
- Finally, prove that the limit obtained in the second step is indeed a solution of the Cauchy problem.

We now collect the following assumptions on the function f:

(H1) The function $f : R_0 = [t_0, t_0 + \delta] \times B(x, r) \to X$ is continuous and

$$\exists M_0 > 0,\ \forall (t, y) \in R_0,\ \|f(t, y)\| \leq M_0.$$

(H2) There exists a function $g : [t_0, t_0 + a] \times [0, 2b] \to \mathbb{R}_+$ such that

$$\forall (t, u) \in [t_0, t_0 + a] \times [0, 2b],\ 0 \leq g(t, u) \leq M_1.$$

Moreover, $g(t, 0) \equiv 0$, for all $t \in [t_0, t_0 + \delta]$ the function $g(t, \cdot)$ is strictly increasing, and the zero function is the unique solution of the Cauchy problem

$$u' = g(t, u),\ u(t_0) = 0. \tag{1.18}$$

(H3) f satisfies the following relation (an extension of the Lipschitz condition):

$$\forall (t, y), (t, z) \in R_0,\ \|f(t, y) - f(t, z)\| \leq g(t, \|y - z\|).$$

The above result (Cauchy–Lipschitz theorem in infinite dimension) extends in a natural way the one already established in finite dimension. More precisely, we have the following theorem:

Theorem 1.3.9.
Suppose that assumptions $(H1) \sim (H2)$ are satisfied. Then the successive Picard approximations (in infinite dimension) defined by:

$$y_{n+1}(t) = x + \int_{t_0}^{t} f(s, y_n(s))\, ds, \quad n = 0, 1, 2, \cdots \tag{1.19}$$

exist on $[t_0, t_0+\alpha]$, where $\alpha = \min\{a, \frac{b}{M}\}$ and $M = \max\{M_0, M_1\}$, and converge uniformly to a function y, which is the unique solution of the Cauchy problem (1.9).

Proof.

•**Existence of the solution:** We prove by induction that the successive approximations (1.19) are defined and continuous on $[t_0, t_0 + \alpha]$. Moreover, we have:

$$\forall t \in [t_0, t_0 + \alpha], \ \|y_{n+1}(t) - x\| \leq b, \quad n = 0, 1, 2, \cdots .$$

We now consider the following approximations:

$$\begin{cases} u_0(t) = M(t - t_0), \\ u_{n+1}(t) = \int_{t_0}^{t} g(s, u_n(s))\, ds, \quad t \in [t_0, t_0 + \alpha]. \end{cases}$$

Again, by induction, we show that the successive approximations above are well defined, and we have:

$$0 \leq u_{n+1}(t) \leq u_n(t), \quad t \in [t_0, t_0 + \alpha].$$

By the Ascoli–Arzelà theorem, we deduce that the sequence (u_n) converges uniformly to a function u on $[t_0, t_0 + \alpha]$. From assumption (ii), we deduce that $u \equiv 0$ on $[t_0, t_0 + \alpha]$.
On the other hand, we show by induction that

$$\forall t \in [t_0, t_0 + \alpha], \ \|y_{n+1}(t) - y_n(t)\| \leq u_n(t), \quad n = 0, 1, 2, \cdots .$$

As a consequence,

$$\forall t \in [t_0, t_0 + \alpha],\ \|y'_{n+1}(t) - y'_n(t)\| \leq g(t, u_{n-1}(t)).$$

Now, let $n \leq m$. We then have:

$$\forall t \in [t_0, t_0 + \alpha],\ \|y'_{n+1}(t) - y'_n(t)\| \leq g(t, u_{n-1}(t)) + g(t, u_{m-1}(t)) + g(t, \|y_n(t) - y_m(t)\|).$$

Since $u_{n+1}(t) \leq u_n(t)$ on $[t_0, t_0 + \alpha]$, it follows that:

$$\frac{d}{dt}\Big(\|y_n(t) - y_m(t)\|\Big) \leq g(t, \|y_n(t) - y_m(t)\|) + 2g(t, u_{n-1}(t)).$$

Consequently,

$$\forall t \in [t_0, t_0 + \alpha],\ \|y_n(t) - y_m(t)\| \leq r_n(t),$$

where $r_n(t)$ is the maximal solution of the Cauchy problem:

$$v' = g(t, v) + 2g(t, u_{n-1}(t)), \quad v_n(t_0) = 0, \quad n = 0, 1, \cdots.$$

But $2g(t, u_{n-1}(t))$ tends uniformly to 0 on $[t_0, t_0 + \alpha]$ as $n \to +\infty$. This implies that y_n converges uniformly to y as $n \to +\infty$. Finally, we show that y is a solution of the Cauchy problem.

• **Uniqueness of the solution:** To show that the solution y is unique, suppose that z is another solution of the Cauchy problem. Define

$$\forall t \in [t_0, t_0 + \alpha],\ m(t) = \|y(t) - z(t)\|.$$

Thus, $m(t_0) = 0$. Moreover, we have:

$$\forall t \in [t_0, t_0 + \alpha],\ m'(t) \leq \|y(t) - z(t)\| \leq g(t, m(t)).$$

From assumption (iii), we deduce that:

$$\forall t \in [t_0, t_0 + \alpha],\ m(t) \leq r(t),$$

where r is the unique maximal solution of (1.18). From assumption **(H2)** we obtain $r \equiv 0$, which implies that $y \equiv z$. The proof is thus complete.

$\square$

Remark 1.3.10.

We note here that if assumption (H3) is weakened, the result may fail.

1.3.2 Solved Exercises

Exercise 1:

Let F be a function of class C^2 on $\mathbb{R}^n$ with values in $\mathbb{R}$ such that

$$\lim_{\|x\| \to +\infty} F(x) = +\infty,$$

and let $(t_0, x_0) \in \mathbb{R} \times \mathbb{R}^n$. Consider the Cauchy problem

$$\left\{ \begin{array}{rcl} \dfrac{d}{dt}x(t) & = & -\nabla F(x(t)), \quad t \in \mathbb{R}, \\ x(t_0) & = & x_0, \quad x_0 \in \mathbb{R}^n. \end{array} \right. \tag{1.20}$$

1. Show that (1.20) admits a unique maximal solution defined on an interval $]T_-, T_+[\subset \mathbb{R}$.

Solution:

1. Since F is of class C^2, we have

   $$\nabla F(x) = \left(\frac{\partial F}{\partial x_j}(x) \right)_{j=1,\dots,n},$$

   which is of class C^1. Moreover, $\Omega = \mathbb{R}^n$ is convex, hence ∇F is locally Lipschitz continuous in x, uniformly with respect to t. By the **Cauchy–Lipschitz theorem**, the problem (1.20) admits a unique maximal local solution defined on an interval $J =]T^-, T^+[\subset \mathbb{R}$.

2. For all $t \in J$ and for any solution $x(t) = (x_1(t), \dots, x_n(t))$ of (1.20), we have

$$\frac{d}{dt}F(x(t)) = \sum_{j=1}^{n} \frac{\partial F(x(t))}{\partial x_j} \frac{dx_j(t)}{dt}.$$

Since $x'(t) = -\nabla F(x(t))$, it follows that

$$\frac{d}{dt}F(x(t)) = -\sum_{j=1}^{n} \left| \frac{\partial F(x(t))}{\partial x_j} \right|^2 \leq 0.$$

Thus, $F(x(t))$ is decreasing, and so for all $t \geq t_0$, $F(x(t)) \leq F(x_0)$.

To show that $T^+ = +\infty$, suppose on the contrary that $T^+ < +\infty$. By the **finite-time blow-up theorem**, we would have $\lim_{t \to T^+} \|x(t)\| = +\infty$. But since $\lim_{\|x\| \to +\infty} F(x) = +\infty$ and $F(x(t))$ is decreasing, this leads to a contradiction. Therefore, necessarily $T^+ = +\infty$.

3. Special case $n = 1$: Take $F(x) = \frac{x^4}{4}$. Then $F'(x) = x^3$ and the ODE becomes

$$x'(t) = -x^3(t), \quad x(0) = x_0 \neq 0.$$

This problem admits a unique maximal solution defined on $]T^-, T^+[$ with $T^+ = +\infty$.

If there exists $t_1 \in]T^-, +\infty[$ such that $x(t_1) = 0$, then by uniqueness we would have $x(t) = 0$ for all t, contradicting $x_0 \neq 0$. Hence $x(t) \neq 0$.

We write:

$$\frac{x'(t)}{x^3(t)} = -1.$$

Integrating with respect to t:

$$-\frac{1}{2x^2(t)} + \frac{1}{2x_0^2} = -(t - t_0).$$

Thus,

$$x^2(t) = \frac{x_0^2}{1 + 2x_0^2(t - t_0)}, \qquad t > t_0 - \frac{1}{2x_0^2}.$$

Therefore, the solution is defined on $]T^{-}, +\infty[$ with

$$T^{-} = t_0 - \frac{1}{2x_0^2}.$$

Exercise 2:

Consider the **Riccati equation**

$$\begin{cases} \dfrac{d}{dt}x(t) = x^2(t), & t \in \mathbb{R}, \\ x(0) = x_0, & x_0 \in \mathbb{R}. \end{cases} \tag{1.21}$$

1. Show that (1.21) admits a maximal solution defined on an interval $J \subset \mathbb{R}$ containing $t_0 = 0$.

2. Let $y(\cdot)$ be a continuous function defined on J satisfying

$$y(t) \leq e^{\int_0^t y(s)\,ds}, \quad \forall t \in J.$$

   Using (1), show that

$$y(t) \leq \frac{1}{1-t}, \quad \forall t \in J.$$

3. Now consider the **non-homogeneous Riccati equation**

$$\begin{cases} \dfrac{d}{dt}x(t) = x^2(t) + t^2, & t \in \mathbb{R}, \\ x(0) = 0. \end{cases} \tag{1.22}$$

   (a) Let $x(\cdot)$ be the maximal solution of (1.22). Define

$$z(t) = e^{-\int_0^t x(s)\,ds}.$$

   Write a second-order differential equation for $z(\cdot)$ and show that:

$$z(0) = z''(0) = z^3(0) = 0.$$

(b) Solve the differential equation in $z(\cdot)$ by seeking a solution in the form of a power series.

(c) Deduce that $x(\cdot)$ is defined on an interval $]-a, a[$ with $a \in]2, +\infty[$.

Correction: The existence of a unique maximal solution $x(t)$ of (3.5.2) on an interval J is guaranteed by the Cauchy–Lipschitz theorem. Let us determine the interval of existence J.

To this end, let us first observe that the Riccati equation in (3.5.2) is an autonomous ODE (here $f(x(t)) = x^2(t)$, $t \in \mathbb{R}$ and $\Omega = \mathbb{R}$). Hence, only the zero solution is global, and consequently $x(t) \neq 0, \ \forall t \in \mathbb{R}$. We may then write

$$\frac{x'(t)}{x^2(t)} = 1.$$

By integrating with respect to t, we obtain

$$-\frac{1}{x(t)} + \frac{1}{x(t_0)} = t - t_0,$$

or equivalently

$$\frac{1}{x(t)} = \frac{1}{x_0} - (t - t_0).$$

Thus,

$$t - t_0 < \frac{1}{x_0}, \quad \text{i.e.} \quad t < t_0 + \frac{1}{x_0}.$$

Therefore,

$$\lim_{t \to t_0 + \frac{1}{x_0}} |x(t)| = +\infty.$$

In this exercise, $t_0 = 0$. Hence:

$$\begin{cases} \text{if } x_0 < 0, \quad \text{then the maximal solution } x(t) \text{ is defined on } \left[\frac{1}{x_0}, +\infty\right[, \\ \text{if } x_0 > 0, \quad \text{then the maximal solution is defined on } \left]-\infty, \frac{1}{x_0}\right[. \end{cases}$$

If we set

$$h(t) = e^{\int_0^t y(s)\, ds},$$

we see that h is well-defined since y is continuous on J. Moreover, h is differentiable and

$$h'(t) = y(t)h(t) \leq h^2(t).$$

Since h never vanishes, we have

$$\frac{h'(t)}{h^2(t)} \leq 1.$$

Integrating with respect to t, we obtain (noting that $h(0) = 1$):

$$-\frac{1}{h(t)} + 1 \leq t,$$

which yields

$$h(t) \leq \frac{1}{1-t}, \qquad t < 1,$$

since h is positive by definition. Finally, by assumption $y(t) \leq h(t)$, therefore

$$y(t) \leq \frac{1}{1-t}.$$

Since

$$z(t) = e^{-\int_0^t x(s)\,ds},$$

where $x(\cdot)$ is a maximal solution of (3.5.3) (i.e. $x'(t) = x^2(t) + t^2,\ x(0) = 0$), we have

$$z'(t) = -x(t)z(t), \qquad z''(t) - x'(t)z(t) + x^2(t)z(t) = -t^2 z(t), \qquad z^{(3)}(t) = -2tz(t) - t^2 z(t).$$

The second-order differential equation in $z(\cdot)$ is

$$\frac{d^2}{dt^2} z(t) = -t^2 z(t), \qquad t \in \mathbb{R},$$

with initial conditions

$$z(0) = 1, \quad z'(0) = 0, \quad z''(0) = 0, \quad z^{(3)}(0) = 0.$$

We seek a solution of the ODE

$$z''(t) = -t^2 z(t),$$

under the initial conditions $z(0) = 1$, $z'(0) = 0$, in the form of a power series

$$z(t) = \sum_{n=0}^{+\infty} a_n t^n, \quad t \in]-R, R[,$$

with R the radius of convergence and

$$a_k = \frac{z^{(k)}(0)}{k!}.$$

We find that

$$a_0 = z(0) = 1, \quad a_1 = z'(0) = 0, \quad a_2 = \frac{z''(0)}{2} = 0, \quad a_3 = \frac{z^{(3)}(0)}{6} = 0.$$

Moreover, for all $t \in]-R, R[$:

$$z''(t) = \sum_{n=0}^{+\infty} (n+2)(n+1) a_{n+2} t^n,$$

and

$$-t^2 z(t) = -t^2 \left(\sum_{n=0}^{+\infty} a_n t^n \right) = -\sum_{n=0}^{+\infty} a_n t^{n+2} = -\sum_{n=2}^{+\infty} a_{n-2} t^n.$$

By identification:

$$(n+2)(n+1) a_{n+2} = -a_{n-2}, \quad \text{or equivalently} \quad a_n = \frac{a_{n-4}}{n(n-1)}, \quad n \geq 4.$$

This last relation shows that the coefficients a_n vanish if $n \neq 4k,\ k \in \mathbb{N}^*$.

Therefore, we can write

$$a_{4k} = \frac{(-1)^k}{4k(4k-1)(4k-4)(4k-5)\cdots 4 \cdot 3}.$$

In particular,

$$|a_{4k}| \leq \frac{1}{k!\, 4^k},$$

hence

$$\sum_k |a_{4k} t^{4k}| \leq e^{t^4/4}.$$

This implies that $R = +\infty$.

Exercise 3:

Consider the Cauchy problem:

$$\begin{cases} \dfrac{d}{dt} x(t) = 2te^{-t^2} + \dfrac{x^5(t)}{1 + x^4(t)} \cos(te^{x(t)}), \quad t \in \mathbb{R}, \\ x(0) = 1. \end{cases} \tag{1.23}$$

1. Show that (1.23) admits a unique maximal solution $x(\cdot)$ defined on an interval $]T_-, T_+[$ of $\mathbb{R}$ containing $t_0 = 0$.

2. Let $\varphi(\cdot)$ denote the maximal solution of (1.23). Using the integral formulation of (1.23), show that:
$$|\varphi(t)| \leq 2 + \int_0^t |\varphi(s)| \, ds, \quad \forall t \in J.$$

3. Deduce that:
$$|\varphi(t)| \leq 2e^{|t|}, \quad \forall t \in J.$$

Solution: Let

$$f(t, x) = 2te^{-t^2} + \frac{x^5}{1 + x^4} \cos(te^x), \qquad (t, x) \in \mathbb{R}^2.$$

We see that f is well-defined and of class C^1 on the convex set $\mathbb{R}^2$, hence it is locally Lipschitz with respect to x, uniformly in t. Therefore, by the Cauchy–Lipschitz theorem, problem (3.5.4) admits a unique maximal solution defined on $J =]T^-, T^+[$ taking values in $\mathbb{R}$, with $t_0 = 0 \in J$. Moreover, this solution is of class C^2 since f is of class C^1, hence the solution is C^1.
For all $t \in [0, T^+[$, we have

$$\varphi(t) = \varphi(0) + \int_0^t \varphi'(s) \, ds = 1 + \int_0^t f(s, \varphi(s)) \, ds,$$

(since φ is a solution of (3.5.4), so $\varphi'(t) = f(t, \varphi(t))$ and $\varphi(0) = 1$).

Thus,

$$|\varphi(t)| \leq 1 + \int_0^t 2se^{-s^2}\,ds + \int_0^t \frac{|\varphi(s)|^5}{1+\varphi^4(s)}\,ds.$$

Using the fact that

$$\frac{|a|^5}{1+a^4} \leq |a|,$$

we obtain

$$|\varphi(t)| \leq 1 + \Big[-e^{-s^2}\Big]_0^t + \int_0^t |\varphi(s)|\,ds \leq 2 + \int_0^t |\varphi(s)|\,ds.$$

Similarly, for $t \in]T^-, 0]$, we obtain:

$$\left|\int_t^0 f(s,\varphi(s))\,ds\right| \leq 1 + \int_t^0 |\varphi(s)|\,ds.$$

Hence, for all $t \in]T^-, T^+[$ we have

$$|\varphi(t)| \leq 2 + \left|\int_0^t |\varphi(s)|\,ds\right|.$$

Applying Gronwall's lemma to the inequality

$$|\varphi(t)| \leq 2 + \int_0^t |\varphi(s)|\,ds,$$

we find

$$|\varphi(t)| \leq 2e^{\int_0^t ds} = 2e^{|t|}.$$

By contradiction, suppose $T^+ < +\infty$. Then, by the blow-up criterion in finite time, we would necessarily have

$$\lim_{t\to T^+} |\varphi(t)| = +\infty.$$

But from part (3), we have

$$|\varphi(t)| \leq 2e^{|t|},$$

which implies

$$\lim_{t\to T^+} |\varphi(t)| \leq \lim_{t\to T^+} 2e^{|t|} = 2e^{T^+} < +\infty.$$

This is a contradiction. Therefore, we conclude that $T^+ = +\infty$. Similarly, one obtains $T^- = -\infty$.

Hence $J = \mathbb{R}$, i.e. the maximal solution of (3.5.4) is global.

Exercise 4:

1. Give the maximal solutions of the following Cauchy problems:

$$x'(t) = x^3(t),\ x(0) = 0 \quad \text{and} \quad x'(t) = \frac{1}{x(t)},\ x(0) = 1.$$

2. Show that every maximal solution of the Cauchy problem

$$x'(t) = t\sqrt{t^2 + x^2(t)}, \quad x(t_0) = x_0$$

is global.

Solution: Let us give the maximal solutions of the following Cauchy problems:

$$\begin{cases} x'(t) = x^3(t), \quad x(0) = 0, \\ x'(t) = \dfrac{1}{x(t)}, \quad x(0) = 1. \end{cases}$$

— The function $f(x) = x^3$ is locally Lipschitz; hence the associated Cauchy problem admits a unique maximal solution $x(t)$, $t \in J \subset \mathbb{R}$. Suppose this solution never vanishes for all t, then

$$\frac{x'(t)}{x^3(t)} = 1.$$

Integrating over $[t_0, t]$, we find

$$-\frac{1}{2x^2(t)} + \frac{1}{2x^2(t_0)} = t - t_0.$$

Thus

$$x^2(t) = \frac{x^2(t_0)}{1 - 2x^2(t_0)(t - t_0)}.$$

From the initial condition $x(0) = 0$, it follows that $x^2(t) = 0$. Hence, the zero function $x(t) = 0,\ \forall t$ is the unique maximal solution satisfying $x(0) = 0$.

— The function $f(x) = \dfrac{1}{x}$ is locally Lipschitz; hence the associated Cauchy problem admits a unique maximal solution $x(t)$, $t \in J \subset \mathbb{R}$. Suppose this solution never vanishes for all t, then

$$x'(t)x(t) = 1.$$

Integrating over $[t_0, t]$, we find

$$x^2(t) - x^2(t_0) = 2(t - t_0).$$

Therefore

$$x^2(t) = x^2(t_0) + 2(t - t_0).$$

But $x(0) = 1$, so

$$x(t) = \sqrt{1 + 2t}$$

is the unique maximal solution of this Cauchy problem with $x(0) = 1$.

(2) Show that every maximal solution of the following Cauchy problem is global:

$$\begin{cases} x'(t) = t\sqrt{t^2 + x^2(t)}, \\ x(t_0) = x_0. \end{cases}$$

— Let $f(t, x) = t\sqrt{t^2 + x^2}$ and show that this function is globally Lipschitz. Indeed, for all $t \in \mathbb{R}$ and $(x, y) \in \mathbb{R}^2$, we have:

$$\begin{aligned} |f(t,x) - f(t,y)| &= |t(\sqrt{t^2 + x^2} - \sqrt{t^2 + y^2})| \\ &\leq |t| \frac{|x^2 - y^2|}{\sqrt{t^2 + x^2} + \sqrt{t^2 + y^2}} \\ &\leq |t| \frac{|x| + |y|}{\sqrt{t^2 + x^2} + \sqrt{t^2 + y^2}} |x - y| \\ &\leq |t|\,|x - y|. \end{aligned}$$

This means that f is globally Lipschitz with Lipschitz constant

$$k(t) = |t|, \quad \forall t \in \mathbb{R}.$$

By the Cauchy–Lipschitz theorem (for global solutions), the above problem admits a unique global solution.

1.3.3 Unsolved Exercises

Exercise 5:

We consider the differential equation:

$$x'(t) = f(t, x(t)),$$

where $f : \mathbb{R}^2 \to \mathbb{R}$ is a C^1 function. Suppose that ϕ and ψ are two solutions of this equation such that there exists $t_0 \in \mathbb{R}$ with

$$\phi(t_0) < \psi(t_0).$$

Show that:

$$\phi(t) < \psi(t), \quad \forall t \in \mathbb{R}.$$

Hint: Use proof by contradiction.

Exercise 6:

Consider the differential equation

$$x'(t) = (a - x(t))(b - x(t)),$$

where a and b are two real constants with $a \leq b$.

1. Show that for any fixed initial data $x_0 \in \mathbb{R}$, this equation admits a unique maximal solution $x(\cdot)$ such that $x(0) = x_0$.

2. What is this solution if $x_0 = a$ or $x_0 = b$?

3. Suppose that $a = b$.
   Find all solutions of the differential equation. Sketch the shape of these solutions depending on a and x_0.

4. Suppose that $a = 1,\ b = 2,\ x_0 \neq a$ and $x_0 \neq b$.

Determine in this case the solution in terms of $x(0) = x_0$ and then sketch the shape of the solution depending on x_0.

Exercise 7:

Consider the Cauchy problem

$$\begin{cases} \dfrac{d}{dt}x(t) = (1+\cos t)x(t) - x^3(t), & t \in \mathbb{R}, \\ x(0) = x_0, \quad x_0 \in \mathbb{R}_+^*. \end{cases} \tag{1.24}$$

1. Show that (1.24) admits a unique local maximal solution ϕ defined on an interval J of $\mathbb{R}$ and that ϕ is of class C^2 on J.

2. Show that if there exists $t_1 \in J$ such that $\phi(t_1) = 0$, then $\phi(t) = 0$ for all $t \in J$.

3. Show that there exists $C > 0$ such that for $t \in J$, we have:

$$0 < \phi(t) \leq \phi(0)e^{Ct}.$$

4. Show that the maximal solution ϕ is global.

5. Let $\psi(t, x)$ be the flow associated with problem (1.24) at $t = 0$. We set

$$P(x) = \psi(2\pi, x), \text{ where } x \in \mathbb{R}_+.$$

   Verify that $P(0) = 0$ and $P'(0) = e^{2\pi}$, then deduce that P is a solution of the differential equation

$$P'(x) = e^{-4\pi}\left(\frac{P(x)}{x}\right)^3.$$

6. Solve this differential equation, then deduce that ϕ is 2π-periodic.

Chapter 2

Ordinary Differential Equations under Constraint

We consider the ODE:

$$y'(t) = f(t, y(t)) \tag{2.1}$$

where $f : I \times \Omega \to \mathbb{R}^n,\ I \subset \mathbb{R},\ \Omega \subset \mathbb{R}^n$.

For various reasons (physical, chemical, $\cdots$), the solutions of the ODE (2.1) must satisfy certain requirements called "viability constraints."

The theory of viability aims to establish a link between the dynamics (in our case the ODE (2.1)) and a given constraint, which in this context is considered as a nonempty subset $\mathcal{K}$ of $I \times \Omega$. More precisely, we have the following definitions:

Definition 2.0.1.

(i) A solution y of the Cauchy problem

$$\begin{cases} y'(t) = f(t, y(t)), \\ y(t_0) = x \end{cases} \tag{2.2}$$

with $(t_0, x) \in \mathcal{K}$ is said to be viable with respect to the constraint "$\mathcal{K}$" if there exists $T > t_0$ such that $[t_0, T] \subset I$ and

$$\forall t \in [t_0, T], \ (t, y(t)) \in \mathcal{K}.$$

(ii) The constraint $\mathcal{K}$ is said to be viable with respect to (2.1) if for all $(t_0, x) \in \mathcal{K}$, the Cauchy problem (2.2) admits a viable solution $y : [t_0, T] \to \Omega$ with $[t_0, T] \subset I$.

Remark 2.0.1. In this chapter, we only consider C^1 solutions. Nevertheless, the viability of a solution can also be defined if it is of Carathéodory type or weak...

We note here that in the case where $\mathcal{K} = I \times \Omega$ and Ω is an open subset of $\mathbb{R}^n$, Peano's theorem guarantees that if f is continuous on $\mathcal{K}$, then for all $(t_0, x) \in \mathcal{K}$, there exists $T > t_0$ such that $[t_0, T] \subset I$ and a solution $y : [t_0, T] \to \Omega$ to the Cauchy problem $y' = f(t, y), \ y(t_0) = x$.
Thus, if Ω is an open set and f is continuous on $\mathcal{K}$, then $\mathcal{K}$ is viable with respect to the ODE (2.1).
In the case where Ω is not open, the previous result is no longer valid.
The following example will illustrate this possibility.

Example 2.0.1. We consider the Cauchy problem

$$y'(t) = f(t, y(t)) = e^{y(t)} \tag{2.3}$$

where $f : [0,1] \times [0,1] \to \mathbb{R}$, and f is continuous on $[0,1] \times [0,1]$. The unique solution y of the problem (2.3) satisfying $y(0) = 1$ is given by

$$\forall t \in [0,1],\;\; y(t) = e^t.$$

We immediately observe that $\forall t \in (0,1),\;\; y(t) \notin \Omega$. That is, the solution graph leaves (immediately after its start) the set $\mathcal{K} = [0,1] \times [0,1]$. Hence, $\mathcal{K}$ is not viable with respect to (2.3).

The first viability theorem was announced by Nagumo in 1942. In fact, Nagumo discovered the missing condition required to ensure the viability of a set that is not necessarily open. In order to understand it properly, we will focus in the next section on the notion of the tangent cone.

2.1 Bouligand-Severi Tangent Cone

The notion of a tangent vector to a set at a given point was introduced simultaneously and independently by Bouligand and Severi in 1932. In what follows, let Ω be a nonempty subset of $\mathbb{R}^n$. Recall that the distance between a point $v \in \mathbb{R}^n$ and the set Ω is defined by:

$$\operatorname{dist}(\eta;\Omega) = \inf_{\omega\in\Omega} \|\eta - \omega\|.$$

Definition 2.1.1. We say that the vector η is tangent to Ω at x if:

$$\lim_{h\to 0^+} \inf \frac{1}{h}\operatorname{dist}(x + hv, \Omega) = 0.$$

The set of all tangent vectors to Ω at x is denoted by $\mathcal{T}_\Omega(x)$.

Example 2.1.1. Let $\Omega = [a, b] \subset \mathbb{R}$. We compute the set $\mathcal{T}_\Omega(a)$. Two cases are distinguished:

- If $v \in \mathbb{R}_+$, then
$$\lim_{h \to 0^+} \inf \frac{1}{h} \text{dist}(a + hv, [a, b]) = 0.$$
It follows that
$$\mathbb{R}_+ \subset \mathcal{T}_\Omega(a).$$

- If $v \in \mathbb{R}_-^*$, then
$$\lim_{h \to 0^+} \inf \frac{1}{h} \text{dist}(a + hv, [a, b]) = \lim_{h \to 0^+} \frac{|a + hv - a|}{h} = |v| \neq 0.$$
Hence,
$$\mathcal{T}_\Omega(a) = \mathbb{R}_+.$$

Example 2.1.2. Let $\Omega = \{(x, y) \in \mathbb{R}^2,\ x^2 + y^2 = 1\} \subset \mathbb{R}^2$. We want to determine all tangent vectors η to Ω at $x = (1, 0)$. Let $\eta = (\eta_1, \eta_2)$. Then:

$$\begin{aligned} \text{dist}(x + h\eta, \Omega) &= \sqrt{(1 + h\eta_1)^2 + (h\eta_2)^2} - 1 \\ &= \frac{2h\eta_1 + h^2\eta_1^2 + h^2\eta_2^2}{\sqrt{(1 + h\eta_1)^2 + (h\eta_2)^2} + 1}. \end{aligned} \tag{2.4}$$

Consequently,
$$\lim_{h \to 0} \inf \frac{1}{h} \text{dist}(x + h\eta, \Omega) = \eta_1.$$

Thus, the vector η is tangent to Ω at $x = (1, 0)$ if and only if $\eta_1 = 0$. Hence we can write:

$$\mathcal{T}_\Omega((1, 0)) = \{(\eta_1, \eta_2) \in \mathbb{R}^2,\ \eta_1 = 0\}.$$

Proposition 2.1.1. The set $\mathcal{T}_\Omega(x)$ is a closed cone.

Proof.

Recall that $\mathcal{T}_\Omega(x)$ is a cone if:

- $0 \in \mathcal{T}_\Omega(x)$. Indeed,

$$\lim_{h\to 0^+}\inf \frac{1}{h}\mathrm{dist}(x+hv,\Omega) = \lim_{h\to 0^+} \frac{1}{h}\mathrm{dist}(x,\Omega) = 0.$$

- $\forall s \in \mathbb{R},\ sv \in \mathcal{T}_\Omega(x)$. If $v \in \mathcal{T}_\Omega(x)$ and $\forall s \in \mathbb{R}$, then:

$$\lim_{h\to 0^+}\inf \frac{1}{h}\mathrm{dist}(x+hsv,\Omega) = \lim_{\theta\to 0^+}\inf s\frac{1}{\theta}\mathrm{dist}(x+\theta v,\Omega) = 0.$$

  This proves that $sv \in \mathcal{T}_\Omega(x)$.

  It remains to show that $\mathcal{T}_\Omega(x)$ is closed. Let $(v_n)_n$ be a sequence in $\mathcal{T}_\Omega(x)$ converging to $\overline{v}$. For all $n \in \mathbb{N}$ and $h > 0$, we have:

$$\frac{1}{h}\mathrm{dist}(x+h\overline{v},\Omega) \leq \|v_n - \overline{v}\| + \frac{1}{h}\mathrm{dist}(x+hv_n,\Omega). \tag{2.5}$$

  Since $v_n \to \overline{v}$, passing to the limit as $n \to +\infty$ gives:

$$\lim_{h\to 0^+}\inf \frac{1}{h}\mathrm{dist}(x+h\overline{v},\Omega) = 0.$$

Thus the proof is complete.

□

Proposition 2.1.2. If x belongs to the interior of Ω, then $\mathcal{T}_\Omega(x) = X$.

Proof.

If x belongs to the interior of Ω, then there exists an open ball centered at x with radius r such that $B(x,r) \subset \Omega$. For any $v \in X$, we can choose h sufficiently close to 0 such that $x + hv \in B(x,r)$. Indeed,

$$\|x + hv - x\| = h\|v\|.$$

To ensure $x + hv \in B(x,r)$, i.e., $h\|v\| < r$, it suffices to choose $h \in \left(0, \frac{r}{\|v\|}\right)$. Thus,

$$\lim_{h \to 0^+} \inf \frac{1}{h} \mathrm{dist}(x + hv, \Omega) = 0.$$

□

Proposition 2.1.3. Let $\Omega \subset \mathbb{R}^n$ and $x \in \Omega$. The following assertions are equivalent:

(i) v is tangent to Ω at x, $(v \in \mathcal{T}_\Omega(x))$.

(ii) For all $\delta, \epsilon > 0$, there exist $h \in (0, \delta)$ and $p_h \in B(0, \epsilon)$ such that

$$x + hv + hp_h \in \Omega.$$

(iii) There exist three sequences:

$$(h_n)_n \subset \mathbb{R}_+, \quad \lim_n h_n = 0,$$

$$(v_n)_n \subset \mathbb{R}^n, \quad \lim_n v_n = 0,$$

$$(p_n)_n \subset \mathbb{R}^n, \quad \lim_n p_n = 0,$$

such that $\forall n \in \mathbb{N}, \ x + h_n v_n + h_n p_n \in \Omega$.

Proof.

(i)$\Longrightarrow$(ii): Suppose v is tangent to Ω at x. From Definition 2.1.1, we have:

$$\lim_{h\to 0^+} \inf \frac{1}{h}\mathrm{dist}(x+hv,\Omega)=0.$$

This is equivalent to:

$$\forall \delta>0,\ \forall \epsilon>0,\ \exists h\in(0,\delta),\ \exists \omega_h\in\Omega,\ \frac{1}{h}\|x+hv-\omega_h\|\leq\epsilon.$$

Setting $p_h=\dfrac{-x-hv+\omega_h}{h}$, we obtain:

$$x+hv+hp_h=\omega_h\in\Omega,$$

with $p_h\in B(0,\epsilon)$.

(ii)$\Longrightarrow$(iii): it suffices to choose $\delta=\epsilon=\frac{1}{n},\ n=1,2,\ldots.$

(iii)$\Longrightarrow$(i): follows immediately from the definition.

□

2.2 Other Types of Tangent Cones

2.2.1 Bouligand-Severi Tangent Cone

Definition 2.2.1. Let $\Omega\subset\mathbb{R}^n$ and $x\in\Omega$. The **Bouligand-Severi tangent cone** (or contingent cone) at x is defined by:

$$T_\Omega(x)=\left\{v\in\mathbb{R}^n:\lim_{h\to 0^+}\frac{1}{h}\mathrm{dist}(x+hv,\Omega)=0\right\}.$$

Example 2.2.1. Let $\Omega = \mathbb{R}^2_+ = \{(x, y) : x \geq 0,\, y \geq 0\}$ and $x = (0, 0)$. Then:

$$T_\Omega(0, 0) = \mathbb{R}^2_+.$$

2.2.2 Interior Tangent Cone

Definition 2.2.2. The **interior tangent cone** at $x \in \Omega$ is defined by:

$$T^i_\Omega(x) = \left\{ v \in \mathbb{R}^n : \exists\, h_k \downarrow 0^+,\, \exists\, v_k \to v \text{ with } x + h_k v_k \in \Omega \ \forall k \right\}.$$

Example 2.2.2. For $\Omega = \mathbb{R}^2_+$ and $x = (1, 1)$ (interior point), we have:

$$T^i_\Omega(1, 1) = \mathbb{R}^2.$$

For $x = (0, 1)$ (boundary point):

$$T^i_\Omega(0, 1) = \{(v_1, v_2) \in \mathbb{R}^2 : v_1 \geq 0\}.$$

2.2.3 Exterior Tangent Cone

Definition 2.2.3. The **exterior tangent cone** at $x \in \Omega$ is defined by:

$$T^e_\Omega(x) = \{v \in \mathbb{R}^n : -v \in T^i_\Omega(x)\}.$$

Example 2.2.3. For $\Omega = \mathbb{R}^2_+$ and $x = (0, 0)$:

$$T^e_\Omega(0, 0) = \mathbb{R}^2_- = \{(v_1, v_2) : v_1 \leq 0, v_2 \leq 0\}.$$

2.2.4 Clarke Tangent Cone

Definition 2.2.4. The **Clarke tangent cone** at $x \in \Omega$ is defined as the convex closure of the Bouligand cone:

$$T^C_\Omega(x) = \overline{\text{co}}\, T_\Omega(x).$$

Example 2.2.4. Let $\Omega = \{(x, y) \in \mathbb{R}^2 : y \geq |x|\}$ and $x = (0, 0)$. Then:

$$T^C_\Omega(0, 0) = \{(v_1, v_2) : v_2 \geq |v_1|\}.$$

2.2.5 Proximal Tangent Cone

Definition 2.2.5. The **proximal tangent cone** at $x \in \Omega$ is defined by:

$$T^P_\Omega(x) = \Big\{ v \in \mathbb{R}^n : \exists\, \alpha > 0 \text{ such that } \langle v, y - x \rangle \leq \alpha \|y - x\|^2,\ \forall y \in \Omega \Big\}.$$

Example 2.2.5. Let $\Omega = \{(x, y) : x^2 + y^2 \leq 1\}$ (unit disk) and $x = (1, 0)$. Then:

$$T^P_\Omega(1, 0) = \{(v_1, v_2) : v_1 \leq 0\}.$$

2.3 Comparison Between Types of Tangent Cones

Main Differences

- **Contingent cone (Bouligand-Severi)**: directions accessible by small perturbations. May be non-convex.
- **Interior cone**: directions where one immediately stays inside Ω. Always included in the contingent cone.
- **Exterior cone**: outward directions, defined as the opposite of the interior cone.
- **Clarke cone**: convex closure of the contingent cone, always convex.
- **Proximal cone**: defined by a quadratic condition, used in optimization.

Relations Between Cones

$$T^P_\Omega(x) \subseteq T^i_\Omega(x) \subseteq T_\Omega(x), \quad T^C_\Omega(x) = \overline{\text{co}}(T_\Omega(x)), \quad T^e_\Omega(x) = -T^i_\Omega(x).$$

Summary Table

Type of cone	Definition	Relation
Contingent $T_\Omega(x)$	$\liminf_{h\to 0^+} \frac{1}{h}\text{dist}(x+hv,\Omega)=0$	General, not always convex
Interior $T^i_\Omega(x)$	Directions remaining in Ω right from the start	$T^i_\Omega(x) \subseteq T_\Omega(x)$
Exterior $T^e_\Omega(x)$	$T^e_\Omega(x) = -T^i_\Omega(x)$	Cone of outgoing directions
Clarke $T^C_\Omega(x)$	$\overline{\text{co}}(T_\Omega(x))$	Convex, larger than $T_\Omega(x)$
Proximal $T^P_\Omega(x)$	$\langle v, y-x\rangle \le \alpha\|y-x\|^2$	$T^P_\Omega(x) \subseteq T_\Omega(x)$

2.4 Nagumo's Theorem

As previously mentioned, Nagumo discovered in 1942 the missing condition to ensure the viability of a set with respect to an ordinary differential equation (ODE). More precisely, we have the following result:

Theorem 2.4.1 (Nagumo)**.** *Let $\Omega \subset \mathbb{R}^n$ be a closed set and*

$$f : I \times \Omega \longrightarrow \mathbb{R}^n$$

a continuous mapping.

A necessary and sufficient condition for

$$K = I \times \Omega$$

to be viable with respect to the ODE (1.22) is the ***tangential condition****:*

$$\forall (t_0, x) \in K, \quad f(t_0, x) \in T_\Omega(x),$$

where $T_\Omega(x)$ denotes the tangent cone (or contingent) to Ω at x.

Remark 2.4.2.

- Nagumo's theorem ensures that if $f : I \times \Omega \to \mathbb{R}^n$ is continuous and Ω is closed, then $K = I \times \Omega$ is viable with respect to (1.21) if and only if the tangential condition (TC) is satisfied.
- Nagumo's theorem generalizes the one established by **Peano** in 1890. Indeed, if Ω is open, the tangential condition becomes redundant.

2.4.1 Proof of Nagumo's Theorem

This paragraph is devoted to the proof of Nagumo's theorem.

Proof of the necessary condition

Let $f : I \times \Omega \to \mathbb{R}^n$. We need to show that if $\mathcal{K} = I \times \Omega$ is viable with respect to (2.2) then the tangential condition (TC) is satisfied.

Proof.

Let $(t_0, x) \in I \times \Omega$. Since $\mathcal{K} = I \times \Omega$ is viable with respect to (2.2), there exists $T > t_0$ such that $[t_0, T] \subset I$ and a solution $y : [t_0, T] \to \Omega$ of the Cauchy problem:

$$\begin{cases} y' = f(t, y) \\ y(t_0) = x. \end{cases}$$

A first-order Taylor expansion of y near t_0 gives:

$$y(t_0 + h) = y(t_0) + y'(t_0)(h) + o(h) = x + f(t_0, x)h + o(h), \;\; h \simeq 0^+.$$

Consequently,

$$\begin{aligned} \mathrm{dist}(x + f(t_0, x)h, \Omega) &= \mathrm{dist}(y(t_0 + h) - o(h), \Omega) \\ &\leq \mathrm{dist}(y(t_0 + h) - o(h), y(t_0 + h)) + \mathrm{dist}(y(t_0 + h), \Omega) \end{aligned} \tag{2.6}$$

Knowing that h is sufficiently close to 0, we have $y(t_0 + h) \in \Omega$ and thus:

$$\frac{1}{h}\mathrm{dist}(x + f(t_0, x)h, \Omega) \leq \frac{1}{h}(|o(h)| + 0) \leq |o(1)|.$$

Taking the limit as $h \to 0^+$:

$$\lim_{h \to 0^+} \inf \frac{1}{h}\mathrm{dist}(x + f(t_0, x)h, \Omega) = 0.$$

Conclusion: $f(t_0, x) \in \mathcal{T}_\Omega(x)$. Therefore, the necessary condition is satisfied.

□

Proof of the sufficient condition

We present a sketch of the proof of Nagumo's theorem.

Let $(t_0, x) \in I \times \Omega$. Consider the Cauchy problem:

$$\begin{cases} y' = f(t, y(t)) \\ y(t_0) = x. \end{cases} \tag{2.7}$$

We aim to construct, from the tangential condition (TC), a viable solution of the problem (2.7). In this part, the continuity of f is crucial, and the proof proceeds in three steps:

(i) The first step is to construct, from the tangential condition (TC), a sequence of approximate solutions to the Cauchy problem (2.7).

(ii) Show the convergence of the constructed sequence to a continuous function y.

(iii) Show that the limit function y is viable.

• **Step 1:** Construction of a sequence of approximate solutions to the Cauchy problem (2.7). Let $(t_0, x) \in \mathcal{K}$ and consider the Cauchy problem (2.7). We fix in advance the domain of definition of the approximate solutions. In fact, the domain must be uniform for all solutions (it does not depend on the chosen approximation). This will help in the limit passage step.
Since Ω is closed and $x \in \Omega$, there exists $\rho > 0$ such that $B(x, \rho) \cap \Omega$ is closed.
The function f being continuous is thus bounded on $B(x, \rho) \cap \Omega$, so let:

$$M = \sup_{y \in B(x,\rho) \cap \Omega} \|f(t_0, y)\|. \tag{2.8}$$

We choose T such that:

$$(T - t_0)(M + 1) \leq \rho. \tag{2.9}$$

The following lemma is constructive and allows us to build a sequence of approximate solutions for (2.7).

Lemma 2.4.3.

Let T, ρ and $M > 0$ be fixed as above. For any $\epsilon > 0$, there exist three functions $\sigma : [t_0, T] \to [t_0, T]$ (Lebesgue integrable) and $y : [t_0, T] \to \mathbb{R}^n$ satisfying:

(i) $\sigma(t) \leq t$ *and* $t - \sigma(t) \leq \epsilon$, *for all* $t \in [t_0, T]$.

(ii) $\|g(t)\| \leq \epsilon$, *for all* $t \in [t_0, T]$.

(iii) $y(\sigma(t)) \in B(x, \rho) \cap \Omega$ *for all* $t \in [t_0, T]$ *and* $y(T) \in B(x, \rho) \cap \Omega$.

(iv)

$$y(t) = x + \int_{t_0}^{t} f(\sigma(s), y(\sigma(s)))ds + \int_{t_0}^{t} g(s)ds, \text{ for all } t \in [t_0, T].$$

Proof. **Initialization of the problem:** This step initializes the construction of the functions in the previous lemma.

This means constructing the functions σ, g, and y on the interval $[t_0, t_0 + h]$ while satisfying the lemma's statements.

Let $(t_0, x) \in \mathcal{K}$ and $\epsilon \in (0, 1)$. Since $f(t_0, x)$ is tangent to $\mathcal{K}$, there exist $\delta > 0, h \in (0, \delta)$, and $p \in X$ such that $\|p\| \leq \epsilon$ and

$$x + f(t_0, x)h + hp \in \Omega,$$

we define the functions

$$\sigma : [t_0, t_0 + h] \to [t_0, t_0 + h],\ g : [t_0, t_0 + h] \to \mathbb{R}^n, \text{ and } y : [t_0, t_0 + h] \to \mathbb{R}^n$$

as follows:

$$\begin{cases} \forall t \in [t_0, t_0 + h],\ \sigma(t) = t_0, \\ \forall t \in [t_0, t_0 + h],\ g(t) = p, \\ \forall t \in [t_0, t_0 + h],\ y(t) = x + (t - t_0)f(t_0, x) + (t - t_0)p. \end{cases}$$

We check that the functions σ, g, and y satisfy statements (i)–(iv) of the above lemma. Indeed, conditions (i), (ii), and (iv) are satisfied. We show the truth of condition (iii).

Using relations (2.8) and (2.9) we immediately deduce:

• **Step 1:** Now we show the existence of the functions σ, g, and y on the interval $[t_0, T]$. We define the set $\mathcal{S}$ of all triplets (σ, g, y) satisfying the lemma (2.4.3) on the interval $[t_0, t_0 + h] \subset [t_0, T]$. We equip $\mathcal{S}$ with a binary relation $\preceq$ defined as follows:

Let $(\sigma_1, g_1, y_1), (\sigma_2, g_2, y_2) \in \mathcal{S}$ defined respectively on $[t_0, t_0 + h_1]$ and $[t_0, t_0 + h_2]$. Then:

$$(\sigma_1, g_1, y_1) \preceq (\sigma_2, g_2, y_2) \Leftrightarrow \begin{cases} h_1 \leq h_2 \\ \forall t \in [t_0, t_0 + h_1],\ (\sigma_1(t), g_1(t), y_1(t)) = (\sigma_2(t), g_2(t), y_2(t)). \end{cases}$$

Thus, we show that the set $\mathcal{S}$ admits a maximal element $\overline{y} : [t_0, \overline{T}] \to \mathbb{R}^n$. The next step is to prove that $\overline{T} = T$. We reason by contradiction and assume that $\overline{T} < T$. The tangential condition is satisfied for all $(\tau, \xi) \in \mathcal{K}$, which allows us to construct a solution defined on $[\overline{T}, \overline{T} + \delta] \subset I$. The patching principle then allows us to define a solution of problem (2.7) on $[t_0, \overline{T} + \delta]$, which contradicts the maximality of $\overline{y}$.

• **Step 2:** By setting $\epsilon = \frac{1}{n},\ n = 1, 2, \ldots,$ we construct a sequence of approximate functions (y_n). The Ascoli-Arzelà theorem plays a crucial role in showing that the sequence (y_n) converges uniformly to a function y solving problem (2.7).

• **Step 3:** We show that the solution $y : [t_0, T] \to \mathbb{R}^n$ is viable, i.e.

$$\forall t \in [t_0, T], y(t) \in \Omega.$$

□

2.4.2 Application

In this paragraph, we present some applications of viability theory.

Lipschitz dependence of solutions on initial conditions

Consider the ODE:

$$y' = f(y(t)) \tag{2.10}$$

where $f : \Omega \to \mathbb{R}^n$ and $\Omega \subset \mathbb{R}^n$ is open. We show that the solutions of ODE (2.10) depend in a Lipschitz manner on the initial conditions.

Theorem 2.4.4.

Assume that f is Lipschitz on Ω, i.e., there exists a constant $k > 0$ such that:

$$\forall (y, z) \in \Omega,\ \|f(y) - f(z)\| \leq k\|y - z\|.$$

Let $y_0, z_0 \in \Omega$. If $y : [t_0, T] \to \mathbb{R}^n$ is a solution of ODE (2.10) with $y(t_0) = y_0$, then there exists $z : [t_0, T] \to \mathbb{R}^n$ solution of ODE (2.10) with $z(t_0) = z_0$ such that:

$$\forall t \in [t_0, T],\ \|y(t) - z(t)\| \leq e^{k(T-t_0)}\|y_0 - z_0\|.$$

Proof.

We present a sketch of the proof. Consider the Banach space $X = \mathbb{R}^n \times \mathbb{R}$.
Let

$$\mathcal{K} = \{(y, z) \in X \mid \|y\| \leq z\}.$$

Consider the ODE:

$$u' = F(u), \tag{2.11}$$

where $F : X \to X$ is defined by:

$$\forall u \in X,\ F(u) = (f(y), z),\ u = (y, z).$$

The idea is to show that the set $\mathcal{K}$ is viable with respect to ODE (2.11).

□

Banach fixed-point theorem

We provide an extension of Banach's fixed-point theorem.

Theorem 2.4.5.

Let Ω be a closed subset of $\mathbb{R}^n$ and $g : \Omega \to \mathbb{R}^n$ a Lipschitz function with constant $L < 1$. If:

$$\forall x \in K, \lim_{h \to 0^+} \inf \frac{1}{h} \mathrm{dist}(x + h(f(x) - x), K) = 0,$$

then g has a unique fixed point.

Proof.

Consider $F : K \to \mathbb{R}^n$ defined by:

$$\forall x \in K, \; F(x) = g(x) - x.$$

Consider the ODE:

$$u' = F(u). \tag{2.12}$$

We show that K is viable with respect to ODE (2.12). Let $T > 0$ and consider the operator $\mathbb{Q} : K \to K$ defined by:

$$\forall x \in K, \; \mathbb{Q}(x) = u(T).$$

We show that the operator $\mathbb{Q}$ is contractive, hence it admits a fixed point. A simple calculation gives $g(x) - x = 0$. The proof is thus complete.

□

2.5 Solved Exercises

• **Exercise 01:** [Contingent — square at the corner] Let $\Omega = \{(x, y) \in \mathbb{R}^2 : x \geq 0, \; y \geq 0\}$ (first quadrant). Compute $T_\Omega(0, 0)$ (Bouligand tangent cone at the origin).

• **Solution:**

Intuition: from the origin, one can only approach points of Ω along directions with nonnegative components.

Quick proof: Let $v = (v_1, v_2)$. If $v_1 < 0$, then for small $h > 0$, $x + hv_1 < 0$, so $\mathrm{dist}((0,0) + hv, \Omega) \geq -hv_1 > 0$ proportional to h, hence the fraction is nonzero and $v \notin T_\Omega(0,0)$. Same reasoning if $v_2 < 0$.

If $v_1 \geq 0$ and $v_2 \geq 0$, then for all $h > 0$, $(0,0) + hv = (hv_1, hv_2) \in \Omega$, so dist $= 0$ and the limit is 0. Thus,

$$T_\Omega(0,0) = \{(v_1, v_2) : v_1 \geq 0,\ v_2 \geq 0\} = \mathbb{R}^2_+.$$

• **Exercise 02:** [Interior cone — half-space] Let $\Omega = \{(x_1, x_2) : x_1 \leq 0\}$ (closed half-space). Find $T^i_\Omega(x)$ for (i) x interior ($x_1 < 0$) and (ii) x on the boundary ($x_1 = 0$).

• **Solution:**

(i) If $x_1 < 0$, then x is interior: there exists a neighborhood around x included in Ω, so for all small vectors v we have $x + hv \in \Omega$ for h small enough. Thus

$$T^i_\Omega(x) = \mathbb{R}^2.$$

(ii) If $x_1 = 0$ (boundary), for $x + hv \in \Omega$ for all small $h \geq 0$, the first component must be nonpositive: $v_1 \leq 0$. Hence,

$$T^i_\Omega(x) = \{v = (v_1, v_2) : v_1 \leq 0,\ v_2 \in \mathbb{R}\}.$$

Remark: $T^i_\Omega(x) \subset T_\Omega(x)$ (general inclusion).

• **Exercise 03:** [Exterior cone — interpretation] For the same Ω as in Exercise 2.5, give $T^e_\Omega(x)$ at x on the boundary ($x_1 = 0$) and comment.

• **Solution:**

By definition, $T^e_\Omega(x) = -T^i_\Omega(x)$. So, for $x_1 = 0$,

$$T^e_\Omega(x) = \{v : -v_1 \leq 0\} = \{v : v_1 \geq 0\},$$

which is the cone of directions that "exit" the half-space (toward $x_1 > 0$). Intuition: if we move backward along v, we remain in Ω (since $-v$ is an interior direction).

• **Exercise 04:** [Clarke cone — V-shaped region] Let $\Omega = \{(x, y) : y \geq |x|\}$ (region above the absolute-value parabola, V-shaped) and $x = (0, 0)$. Compute $T_\Omega(0, 0)$ (contingent) and then $T_\Omega^C(0, 0)$ (Clarke cone).

• **Solution:**

Contingent: a direction $v = (v_1, v_2)$ is admissible if, for small $h > 0$, hv can be approximated by points of Ω. The exact condition is:

$$T_\Omega(0, 0) = \{v : v_2 \geq |v_1|\}.$$

Indeed, for small $h > 0$, $hv = (hv_1, hv_2)$ must satisfy $hv_2 \gtrsim |hv_1|$, giving $v_2 \geq |v_1|$ in the limit.
Clarke cone: it is the closed convex hull of the contingent. Here the contingent is already convex and closed (verifiable), so

$$T_\Omega^C(0, 0) = T_\Omega(0, 0) = \{(v_1, v_2) : v_2 \geq |v_1|\}.$$

Remark: in other examples (more complex corners), T^C may be the strict convex closure of T.

• **Exercise 05:** [Proximal cone — unit disk] Let $\Omega = \{(x, y) : x^2 + y^2 \leq 1\}$ (unit disk) and $x = (1, 0)$ (boundary point). Show that

$$T_\Omega^P(1, 0) = \{(v_1, v_2) : v_1 \leq 0\}$$

using the proximal definition (quadratic inequality).

• **Solution:**

At $(1, 0)$, the outward normal to the disk is $(1, 0)$. The admissible directions via the proximal definition are those without a positive outward component at x.
Formally, take v such that the proximal condition exists with $\alpha > 0$:

$$\langle v, y - x\rangle \leq \alpha\|y - x\|^2 \quad \forall y \in \Omega.$$

Choose $y = (1 - \varepsilon, 0) \in \Omega$ (small $\varepsilon > 0$). Then $y - x = (-\varepsilon, 0)$ and the inequality gives

$$\langle v, (-\varepsilon, 0)\rangle = -\varepsilon v_1 \leq \alpha\varepsilon^2.$$

Dividing by $\varepsilon > 0$ and taking $\varepsilon \downarrow 0$ yields $-v_1 \leq 0$, hence $v_1 \geq 0$. But this describes outward-pointing vectors, contradictory to intuition.
Conclusion: the proximal formulation must be applied with the correct convention. A common equivalent formulation (for smooth convex sets) is:

$$T_\Omega^P(x) = \{v : \exists\alpha > 0,\ x \in \mathrm{proj}_\Omega(x + \alpha v)\},$$

i.e., x is a projection of $x + \alpha v$ onto Ω; for boundary points of the disk, this imposes v pointing inward: $v_1 \leq 0$.
Intuition and final result (for the disk): admissible directions point inward radially. Thus,

$$T_\Omega^P(1, 0) = \{(v_1, v_2) : v_1 \leq 0\},$$

consistent with the usual tangent cone at smooth boundaries ($T_\Omega(1, 0) = \{v_1 \leq 0\}$).

2.5.1 Solved Exercises on Nagumo's Theorem

• **Exercise 06:** Consider the simple ODE:

$$x'(t) = -x(t), \qquad x(0) = x_0 \geq 0,$$

and the set $\Omega = \mathbb{R}_+ = \{x \in \mathbb{R} : x \geq 0\}$.

1. Show that Ω is closed in $\mathbb{R}$.
2. Check Nagumo's tangency condition at the boundary $\partial\Omega = \{0\}$.
3. Conclude on the viability of Ω for this ODE.

Solution:

1. $\mathbb{R}_+$ is closed because its complement $(-\infty, 0)$ is open.
2. At the boundary $x = 0$, we have $f(t, 0) = -0 = 0$. The tangent cone at $x = 0$ is $T_\Omega(0) = \mathbb{R}_+$. Since $0 \in \mathbb{R}_+$, the tangency condition is satisfied.

3. By Nagumo's theorem, Ω is viable for this ODE. Hence, any trajectory starting from $x_0 \geq 0$ remains in $\mathbb{R}_+$.

• **Exercise 07:** Consider:

$$x'(t) = x(t), \qquad x(0) = x_0 \geq 0,$$

and $\Omega = \mathbb{R}_+$.

1. Check the tangency condition at the boundary $x = 0$.
2. Conclude on the viability of Ω.

Solution: At the boundary $x = 0$, $f(t, 0) = 0 \in T_\Omega(0) = \mathbb{R}_+$. Thus, the tangency condition is satisfied. Hence, Ω is viable: if $x_0 \geq 0$, the solution $x(t) = x_0 e^t$ remains nonnegative for all t.

• **Exercise 08:** Consider the ODE:

$$x'(t) = -1, \qquad x(0) = x_0,$$

and $\Omega = \mathbb{R}_+$.

1. Check the tangency condition at the boundary $x = 0$.
2. What can be concluded about the viability of Ω?

Solution: At the boundary $x = 0$, $f(t, 0) = -1 \notin T_\Omega(0) = \mathbb{R}_+$. The tangency condition is not satisfied. Therefore, Ω is not viable for this ODE: solutions starting from $x_0 > 0$ leave $\mathbb{R}_+$ in finite time.

• **Exercise 09:** In dimension 2, consider:

$$\dot{x}(t) = \begin{pmatrix} -x_1(t) \\ x_2(t) \end{pmatrix}, \quad x(0) = x_0,$$

and the set $\Omega = \mathbb{R}_+^2 = \{(x_1, x_2) : x_1 \geq 0, x_2 \geq 0\}$.

1. Check Nagumo's tangency condition on the boundary $\partial\Omega$.

2. Conclude on the viability of Ω.

Solution: - If $x_1 = 0$, then $f(t,(0,x_2)) = (-0,x_2) = (0,x_2)$, which is in the tangent cone $T_\Omega(0,x_2) = \{(v_1,v_2) : v_1 \geq 0\}$. - If $x_2 = 0$, then $f(t,(x_1,0)) = (-x_1,0)$ belongs to $T_\Omega(x_1,0) = \{(v_1,v_2) : v_2 \geq 0\}$ only if $x_1 = 0$.

Thus, Ω is not globally viable. The tangency condition fails on the boundary portion where $x_2 = 0$ and $x_1 > 0$.

Chapter 3

Notion of Stability

In this chapter, we study the behavior of solutions of an ODE. Generally speaking, a solution is said to be *stable* if solutions starting from nearby initial values remain close to the considered solution for all future times. For linear differential systems, the stability of solutions is governed by the sign of the real parts of the eigenvalues of the matrix associated with the linear part of the ODE.

We conclude this chapter with a brief introduction to Lyapunov functions, which, under certain conditions, are very useful tools for studying the stability of ODEs.

3.1 Generalities: Stability of Linear Differential Systems

Consider the ODE:

$$y'(t) = f(t, y(t)), \tag{3.1}$$

where $f : \mathbb{R}_+ \times \Omega \to \mathbb{R}^n$ is continuous on $\mathbb{R}_+ \times \Omega$ and $\Omega \subset \mathbb{R}^n$ is an open set. We further assume that f is locally Lipschitz on Ω. Suppose that the ODE (3.1) has a global solution:

$$\varphi : \mathbb{R}^+ \to \Omega.$$

Definition 3.1.1.

The solution φ of (3.1) is said to be *stable* if the following two conditions are satisfied:

(i) For every $a \geq 0$, there exists $\mu(a) > 0$ such that for every $\xi \in B(\varphi(a), \mu(a))$, the maximal solution y of the Cauchy problem

$$\begin{cases} y' = f(t, y), \\ y(a) = \xi \end{cases} \tag{3.2}$$

is defined on $[a, +\infty)$.

(ii) For every $a \geq 0$ and every $\epsilon > 0$, there exists $\delta(a, \epsilon) \in [0, \mu(a))$ such that for every $\xi \in B(\varphi(a), \delta(a, \epsilon))$, the maximal solution of the Cauchy problem (3.2) satisfies:

$$\forall t \in [a, \infty), \quad \|y(t) - \varphi(t)\| \leq \epsilon.$$

Remark 3.1.1.

Condition (i) of the above definition ensures that for any $a \geq 0$, there exists a unique solution of the Cauchy problem (3.2) defined on $[a, \infty)$, provided that ξ is sufficiently close to $\varphi(a)$. Moreover, condition (ii) guarantees that for any $\epsilon > 0$ and for ξ sufficiently close to $\varphi(a)$, the graph of the solution y of the Cauchy problem (3.2) (whose existence is ensured by condition (i)) remains sufficiently close to the graph of the solution φ.

Example 3.1.1. Consider the ODE $y' = 0$. We study the stability of the zero solution, i.e.,

$$\varphi : \mathbb{R}_+ \to \mathbb{R}, \quad \varphi \equiv 0.$$

• $a \geq 0$: for any $\xi \in \mathbb{R}$, the unique maximal solution of the Cauchy problem

$$\begin{cases} y' = 0, \\ y(a) = \xi \end{cases}$$

is defined on $[a, +\infty)$ and satisfies $y \equiv \xi$. Therefore, $\mu(a)$ can be chosen arbitrarily. Condition (i) of Definition 3.1.1 is satisfied.

• $\epsilon > 0$ fixed but arbitrary. We have:

$$\forall t \geq a, \ |y(t) - \varphi(t)| = |y(t)| \leq \epsilon \Longrightarrow |\xi| \leq \epsilon.$$

It follows that we can choose $\delta(a, \epsilon) = \epsilon$. Hence, condition (ii) of Definition 3.1.1 is satisfied.

Conclusion: the zero solution is stable.

Example 3.1.2. Consider the ODE $y' = y$. We study the stability of the zero solution.

• Let $a \geq 0$. The unique maximal solution of the Cauchy problem

$$y' = y, \quad y(a) = \xi$$

is defined on $[a, +\infty)$ and is given by

$$y(t) = \xi e^{t-a}.$$

Thus, $\mu(a)$ can be chosen arbitrarily, so condition (i) of Definition 3.1.1 is satisfied.

• However, condition (ii) is not satisfied since

$$\lim_{t \to +\infty} |y(t)| = +\infty.$$

Conclusion: the zero solution is unstable.

Other concepts of stability can also be defined.

Definition 3.1.2. The solution φ of (3.1) is said to be *uniformly stable* if it is stable and the constants $\mu(a)$ and $\delta(a, \epsilon)$ in Definition 3.1.1 do not depend on a and ϵ.

Definition 3.1.3. The solution φ of (3.1) is said to be *asymptotically stable* if it is stable and, for every $\xi \in B(\varphi(a), \mu(a))$, the maximal solution y of the Cauchy problem (3.2) satisfies

$$\lim_{t \to +\infty} \|y(t) - \varphi(t)\| = 0.$$

Remark 3.1.2. The notions of stability defined above are relative to a particular solution of the ODE (3.1). Therefore, two different solutions of the same ODE may differ in their stability properties.

Example 3.1.3. Consider the ODE, called the population dynamics model:

$$y' = by(p - y),$$

where $b > 0$. We study the stability of the solutions $\varphi_1 \equiv 0$ and $\varphi_2 \equiv p$.

• Let $a \geq 0$ and $\xi \in \mathbb{R}$. The Cauchy problem

$$\begin{cases} y' = by(p - y), \\ y(a) = \xi \end{cases}$$

admits a maximal solution defined on $[a, +\infty)$ given by

$$y(t) = \frac{p\xi e^{bp(t-a)}}{p + \xi(e^{bp(t-a)} - 1)}.$$

Condition (i) of Definition 3.1.1 is satisfied.

• We notice that

$$\lim_{t \to +\infty} |y(t)| = p^2 \neq 0.$$

Thus, condition (ii) of Definition 3.1.1 is not satisfied. Conclusion: the zero solution φ_1 is unstable. As for the solution φ_2, we have

$$\lim_{t \to +\infty} |y(t) - \varphi_2(t)| = \lim_{t \to +\infty} \left| \frac{p\xi e^{bp(t-a)}}{p + \xi(e^{bp(t-a)} - 1)} - p \right| = 0.$$

Conclusion: φ_2 is asymptotically stable.

Remark 3.1.3. Consider the ODE (3.1) and let $\varphi : \mathbb{R}^+ \to \Omega$ be a global solution of (3.1). The change of unknown function

$$y = x - \varphi$$

leads to the following ODE:

$$x'(t) = f(t, x(t) + \varphi(t)) - \varphi'(t). \tag{3.3}$$

In this chapter, we are only interested in the stability of the zero solution of the ODE (3.1). For this purpose, we assume that $0 \in \Omega$ and $\forall t \in \mathbb{R}_+, f(t, 0) = 0$.

Considering the zero solution, we rewrite Definitions 3.1.1, 3.1.2 and 3.1.3 as follows:

Definition 3.1.4. The zero function of the ODE (3.1) is said to be stable if:

(i) For all $a \geq 0$, there exists $\mu(a) > 0$ such that for every $\xi \in B(0, \mu(a))$, the maximal solution y of (3.2) is defined on $[a, +\infty)$.

(ii) For all $\epsilon > 0$, there exists $\delta(a, \epsilon) \in [a, \mu(a))$ such that for every $\xi \in B(0, \delta(a))$, the maximal solution of the Cauchy problem (3.2) satisfies:

$$\forall t \in [a, +\infty), \quad \|y(t)\| \leq \epsilon.$$

Definition 3.1.5. The zero solution of (3.1) is said to be *uniformly stable* if it is stable and the constants $\mu(a)$ and $\delta(a, \epsilon)$ in Definition 3.1.4 do not depend on a and ϵ.

Definition 3.1.6. The zero solution of (3.1) is said to be *asymptotically stable* if it is stable and for every $\xi \in B(0, \mu(a))$, the maximal solution y of the Cauchy problem (3.2) satisfies:

$$\lim_{t \to +\infty} \|y(t)\| = 0.$$

Next, we establish a connection between equilibrium points of an ODE and the stability of the zero solution.

Definition 3.1.7. A point $x^* \in \Omega$ is called a *stationary point* or *equilibrium point* of the ODE (3.1) if:

$$\forall t \in \mathbb{R}_+, \quad f(t, x^*) = 0.$$

Remark 3.1.4. It is clear that if x^* is a stationary point of the ODE (3.1), then the constant function $y \equiv x^*$ is a constant solution of the ODE (3.1), called a *stationary solution*. Moreover, it is clear that the identically zero solution is also a stationary solution of the ODE (3.1).

We assume that the ODE (3.1) is autonomous, i.e., the function f does not depend on t. We have the following result.

Theorem 3.1.5. *Let $f : \Omega \to \mathbb{R}^n$ be continuous, and let $y : [a, +\infty) \to \Omega$ be a solution of the ODE*

$$y'(t) = f(y(t)). \tag{3.4}$$

If

$$\lim_{t \to +\infty} y(t) = y^* \quad \text{and} \quad y^* \in \Omega,$$

then y^* is an equilibrium point of the ODE (3.4).

Proof.

Let $f = (f_1, f_2, \ldots, f_n)$. Suppose $y = (y_1, y_2, \ldots, y_n)$ is a solution of the ODE (3.7). Consider a component y_i where $i \in \{1, 2, \ldots, n\}$ defined on an interval $[m, m+1]$ with $m \in \mathbb{N} \cap [a, +\infty)$. Apply the mean value theorem to y_i on $[m, m+1]$.
Then, there exists a real number $\theta_m \in (m, m+1)$ such that:

$$y_i(m+1) - y_i(m) = y_i'(\theta_m) = f_i(y(\theta_m)).$$

Since

$$\lim_{m \to +\infty} \Big(y_i(m+1) - y_i(m) \Big) = 0,$$

and

$$\lim_{m \to +\infty} f_i(y(\theta_m)) = f_i(y^*),$$

we deduce that

$$\lim_{t \to +\infty} f_i(y(t)) = f_i(y^*)$$

for all $i \in \{1, 2, \ldots, n\}$. Thus, $f(y^*) = 0$.

□

3.1.1 Stability of Linear Systems

We first study the simplest case, namely a linear system without forcing term:

$$y'(t) = \mathcal{A}y(t), \tag{3.5}$$

where $\mathcal{A} \in M_{n \times n}(\mathbb{C})$. The previous theorem will be useful.

Theorem 3.1.6. *The zero solution of the ODE (3.1) is stable (asymptotically stable, uniformly stable) if and only if every maximal solution of the ODE (3.5) is stable (asymptotically stable, uniformly stable).*

Proof.

Suppose $y = \varphi$ is a maximal solution of the ODE (3.5). Using the change of variable $x = y - \varphi$, we deduce that φ corresponds to the maximal solution $x = 0$.
Thus, it suffices to notice that φ satisfies the conditions of Definitions 3.1.1, 3.1.2, and 3.1.3 if and only if the zero solution $x = 0$ satisfies the conditions of Definitions 3.1.4, 3.1.5, and 3.1.6.

$\square$

Remark 3.1.7. By the previous theorem, all maximal solutions of the ODE (3.5) have the same type of stability. Hence, we will speak of the stability of the entire differential system.

To study the stability of the ODE (3.5), recall that for any $t_0 \geq 0$, the solution of the Cauchy problem

$$\begin{cases} y' = \mathcal{A}y, \\ y(t_0) = \xi \end{cases}$$

is defined on $[t_0, +\infty)$ and given by

$$y(t) = e^{(t-t_0)\mathcal{A}}\xi.$$

Thus, condition (i) in Definition 3.1.4 is satisfied. To understand what happens, consider the particular case: $n = 1$ and $\mathcal{A} = (a)$ where $a \in \mathbb{C}$.

$$\forall t \in [t_0, +\infty), \quad \|y(t)\| = e^{(t-t_0)\mathrm{Re}(a)}.$$

Hence, condition (ii) in Definition 3.1.5 is satisfied if and only if $\mathrm{Re}(a) \leq 0$.
Conclusion: the zero solution of the ODE $y' = ay$ is stable if and only if $\mathrm{Re}(a) \leq 0$. Moreover,

$$\lim_{t \to +\infty} |y(t)| = 0$$

if and only if $\mathrm{Re}(a) < 0$.

Conclusion: the zero solution of the ODE $y' = ay$ is asymptotically stable if and only if $\mathrm{Re}(a) < 0$. The following theorem characterizes the stability of solutions of the ODE (3.5).

Theorem 3.1.8. *Let λ_i, $i \in \{1, 2, \ldots, n\}$, be the eigenvalues of the matrix $\mathcal{A}$. Then the ODE (3.5) is:*

(i) Stable if and only if for every $i \in \{1, 2, \ldots, n\}$ either $\mathrm{Re}(\lambda_i) < 0$ or $\mathrm{Re}(\lambda_i) = 0$ and the Jordan block corresponding to λ_i is diagonalizable.

(ii) Asymptotically stable if and only if $\mathrm{Re}(\lambda_i) < 0$ for all $i \in \{1, 2, \ldots, n\}$.

Proof.

We distinguish the following cases:

• **The matrix $\mathcal{A}$ is diagonalizable:**

In this case, after a linear change of coordinates, $\mathcal{A}$ reduces to the matrix

$$\overline{\mathcal{A}} = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix},$$

where $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of $\mathcal{A}$. Thus, the ODE (3.5) reduces to the linear ODEs:

$$y_j' = \lambda_j y_j, \quad y_j(t_0) = \xi_j, \quad j \in \{1, 2, \ldots, n\}.$$

The solutions are

$$y_j = \xi_j e^{\lambda_j (t - t_0)}, \quad j \in \{1, 2, \ldots, n\}.$$

The solutions are stable if and only if $\mathrm{Re}(\lambda_j) \leq 0$ for all j, and asymptotically stable if and only if $\mathrm{Re}(\lambda_j) < 0$ for all j.

• **The matrix $\mathcal{A}$ is non-diagonalizable:**

In this case, we treat each block in a triangularization of $\mathcal{A}$. Specifically, suppose

$$\begin{pmatrix} \lambda & \cdots & * \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda \end{pmatrix} = \lambda I + N,$$

where N is a nonzero upper-triangular nilpotent matrix. Then

$$e^{(t-t_0)\mathcal{A}} = e^{(t-t_0)\lambda I} e^{(t-t_0)N} = e^{\lambda(t-t_0)} \sum_{k=0}^{n-1} \frac{(t-t_0)^k}{k!} N^k.$$

The result follows from the properties of the Jordan blocks and the growth of $e^{(t-t_0)\mathcal{A}}$.

$\square$

3.1.2 Stability of Perturbed Linear Systems

Consider the system:

$$y' = \mathcal{A}y + f(t, y(t)), \qquad (3.6)$$

where $\mathcal{A} \in M_{n\times n}(\mathbb{C})$ and $f : \mathbb{R}_+ \times \Omega \to \mathbb{R}^n$ with Ω an open subset of $\mathbb{R}^n$ containing 0. Consider the following conditions:

On the matrix $\mathcal{A}$:

(A) There exist constants $M \geq 1$, $\omega > 0$ and $L > 0$ such that:

$$\forall t \geq 0, \quad \|e^{t\mathcal{A}}\| \leq M e^{-\omega t}.$$

On the function f:

(H_1) f is continuous on $\mathbb{R}_+ \times \Omega$ and locally Lipschitz on Ω.

(H_2) $\forall t \in \mathbb{R}_+,\ f(t, 0) = 0.$

(H_3) $\forall (t, x) \in \mathbb{R}_+ \times \Omega,\ \|f(t, x)\| \leq L\|x\|.$

We start with the following fundamental result.

Theorem 3.1.9. *Suppose that conditions (A), $(H_1) \sim (H_3)$ and (D) are satisfied. If*

$$LM - \omega < 0,$$

then the zero solution of the ODE (3.6) is asymptotically stable.

Proof.

Let $\xi \in \Omega$, $t_0 \in \mathbb{R}_+$. Consider the Cauchy problem:

$$\begin{cases} y' = \mathcal{A}y + f(t, y(t)), \\ y(t_0) = \xi. \end{cases}$$

Recall that this Cauchy problem admits a unique maximal solution defined on $[t_0, T_n)$ given by:

$$y(t) = e^{(t-t_0)\mathcal{A}}\xi + \int_{t_0}^{t} e^{(t-s)\mathcal{A}} f(s, y(s))\, ds.$$

We first show that if ξ is sufficiently close to 0, then the solution y is defined on $[t_0, +\infty)$, and we have:

$$\forall t \in [t_0, T_n), \quad \|y(t)\| \leq \|e^{(t-t_0)\mathcal{A}}\|\|\xi\| + \int_{t_0}^{t} \|e^{(t-s)\mathcal{A}}\|\|f(s, y(s))\|ds.$$

Using hypotheses (H_3) and (A), we get:

$$\forall t \in [t_0, T_n), \quad \|y(t)\| \leq Me^{-\omega(t-t_0)}\|\xi\| + \int_{t_0}^{t} LMe^{-\omega(t-s)}\|y(s)\|ds.$$

Multiplying both sides by $e^{\omega t}$, we obtain:

$$\forall t \in [t_0, T_n), \quad e^{\omega t}\|y(t)\| \leq Me^{\omega t_0}\|\xi\| + \int_{t_0}^{t} LMe^{\omega s}\|y(s)\|ds.$$

Define the function $x : [t_0, T_n) \to \mathbb{R}_+$ by:

$$x(t) = e^{\omega t}\|y(t)\|.$$

Then

$$x(t) \le Me^{\omega t_0}\|\xi\| + \int_{t_0}^{t} LMx(s)ds.$$

By Gronwall's inequality, we get:

$$x(t) \le Me^{\omega t_0}\|\xi\|e^{LM(t-t_0)},$$

hence

$$\|y(t)\| \le M\|\xi\|e^{(LM-\omega)(t-t_0)}.$$

Choose $\rho > 0$ such that $B(0,\rho) \subset \Omega$ and define $\mu(\rho) = \rho/(2M)$. Then

$$\|y(t)\| \le \rho/2, \quad \forall t \in [t_0, T_n).$$

Suppose $T_n < \infty$. Then

$$\lim_{t\to T_n} y(t) = y^* \in B(0,\rho/2) \subset \Omega,$$

which contradicts maximality of y.

Conclusion: $T_n = \infty$, satisfying the condition in Definition 3.1.6. Condition (ii) is also satisfied: if $\|\xi\| \le \mu(\rho)$, then

$$\lim_{t\to+\infty} y(t) = 0.$$

The proof is complete.

□

The following result is a consequence of the previous theorem.

Theorem 3.1.10. *Let $\mathcal{A} \in M_{n\times n}(\mathbb{R})$. Suppose all eigenvalues of $\mathcal{A}$ have negative real parts. Suppose conditions (D) and $(H_1) \sim (H_3)$ are satisfied. If there exists a function $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ such that*

$$\forall (t, x) \in \mathbb{R}_+ \times \Omega, \quad \|f(t, x)\| \leq \alpha(\|x\|),$$

and

$$\lim_{r\to 0^+} \frac{\alpha(r)}{r} = 0,$$

then the zero solution of the ODE (3.6) is asymptotically stable.

Proof.

Since all eigenvalues of $\mathcal{A}$ have negative real parts, there exist constants $M \geq 1$ and $\omega > 0$ such that condition (A) is satisfied. Choose $L > 0$ such that

$$ML - \omega < 0.$$

Let $\delta > 0$ satisfy

$$\forall r \in (0, \delta), \quad \alpha(r) \leq Lr.$$

Thus, the matrix $\mathcal{A}$ and function f satisfy the assumptions of the previous theorem on $\mathbb{R}_+ \times \{x \in \Omega \mid \|x\| \leq \delta\}$.
The proof is complete.

□

We now consider the stability of the zero solution of the autonomous ODE:

$$y' = f(y), \qquad (3.7)$$

where $f : \Omega \to \mathbb{R}^n$. We have the following result:

Theorem 3.1.11. *Let Ω be an open subset of $\mathbb{R}^n$ containing 0. If f is C^1 on Ω, $f(0) = 0$, and all eigenvalues of the Jacobian matrix $\mathcal{A} = f_x(0)$ have negative real parts, then the zero solution of the ODE (3.7) is asymptotically stable.*

Proof.

Since f is C^1 on Ω, we can write

$$f(x) = f(0) + f_x(0)x + g(x) = \mathcal{A}x + g(x),$$

with

$$\lim_{x \to 0} \frac{\|g(x)\|}{\|x\|} = 0.$$

Define $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ by

$$\alpha(r) = \sup\{\|f_x(\theta) - f_x(0)\| : \theta \in \Omega,\ \|\theta\| \leq r\}.$$

Then the hypotheses of Theorem 3.1.10 are satisfied. The proof is complete.

□

3.1.3 Stability in the Sense of Lyapunov

Consider the ODE

$$y' = f(t, y), \tag{3.8}$$

where $f : \mathbb{R}_+ \times \Omega \to \mathbb{R}^n$ with Ω an open subset of $\mathbb{R}^n$ containing 0. We assume that f satisfies the following two conditions:

(i) f is continuous on $\mathbb{R}_+ \times \Omega$ and locally Lipschitz on Ω.

(ii) $f(t, 0) = 0$, $\forall t \in \mathbb{R}_+$.

Remark 3.1.12. The first condition guarantees the existence and uniqueness of a solution to the Cauchy problem associated with the ODE (3.8). The second condition shows that the function $\varphi \equiv 0$ is a solution of the ODE (3.8).

Definition 3.1.8. A function $V : \mathbb{R}_+ \times \Omega \to \mathbb{R}_+$ is said to be positive definite on $\mathbb{R}_+ \times \Omega$ if there exists a continuous, increasing function $\omega : \mathbb{R}_+ \to \mathbb{R}_+$ such that $\omega(r) = 0$ if and only if $r = 0$ and

$$\forall (t, x) \in \mathbb{R}_+ \times \Omega, \quad V(t, x) \geq \omega(\|x\|). \tag{3.9}$$

A function $V : \mathbb{R}_+ \times \Omega \to \mathbb{R}_-$ is negative definite on $\mathbb{R}_+ \times \Omega$ if $-V$ is positive definite on $\mathbb{R}_+ \times \Omega$.

Definition 3.1.9. A function $V : \mathbb{R}_+ \times \Omega \to \mathbb{R}_+$ is called a Lyapunov function of the system (3.8) if it satisfies the following conditions:

(i) V is of class C^1 on $\mathbb{R}_+ \times \Omega$ and $\forall t \in \mathbb{R}_+,\ V(t, 0) = 0$.

(ii) V is positive definite on $\mathbb{R}_+ \times \Omega$.

(iii) For all $(t, x) \in \mathbb{R}_+ \times \Omega$,

$$\frac{\partial V}{\partial t}(t, x) + \sum_{i=1}^{n} f_i(t, x) \frac{\partial V}{\partial x_i}(t, x) \leq 0. \tag{3.10}$$

Theorem 3.1.13. *If the ODE (3.8) possesses a Lyapunov function, then the zero solution is stable.*

Proof.

Let $a \in \mathbb{R}_+$, $\xi \in \Omega$, and $y : [a, T_m) \to \Omega$ be the unique maximal solution of the ODE (3.8) satisfying $y(a) = \xi$. We first show that if ξ is sufficiently small, then $T_m = +\infty$.
Define the function $g : [a, T_m) \to \mathbb{R}_+$ by

$$g(t) = V(t, y(t)),$$

where V is the Lyapunov function corresponding to the ODE (3.8). The function g is of class C^1 on $[a, T_m)$ and by (3.10) we have

$$\begin{aligned} g'(t) &= \frac{\partial V}{\partial t}(t, y(t)) + \sum_{i=1}^{n} f_i(t, y(t)) \frac{\partial V}{\partial x_i}(t, y(t)) \\ &\leq 0. \end{aligned} \tag{3.11}$$

Hence, g is decreasing on $[a, T_m)$, so that

$$\forall t \in [a, T_m), \quad g(t) \leq g(a) \implies V(t, y(t)) \leq V(a, \xi).$$

From (3.9), it follows that

$$\omega(\|y(t)\|) \leq V(a, \xi), \quad \forall t \in [a, T_m).$$

Let $\rho > 0$ such that $B(0, \rho) \subset \Omega$. Since $V(a, \cdot)$ is continuous at 0 and $V(a, 0) = 0$, there exists $r = r(a) \in (0, \rho)$ such that

$$\forall \xi \in B(0, r), \quad V(a, \xi) < \omega(\rho).$$

Thus,

$$\forall t \in [a, T_m), \quad \omega(\|y(t)\|) \leq \omega(\rho) \implies \|y(t)\| \leq \rho.$$

Since $B(0, \rho) \subset \Omega$ and y is maximal, we conclude that

$$\forall \xi \in \Omega, \ \|\xi\| \leq r(a) \implies T_m = +\infty.$$

Similarly, for any $a \geq 0$ and $\epsilon > 0$, there exists $\delta(a, \epsilon) > 0$ such that

$$\forall \xi \in \Omega, \ \|\xi\| \leq \delta \implies \|y(t)\| \leq \epsilon.$$

Hence, the zero solution is stable. The proof is complete.

□

Theorem 3.1.14. *If the ODE (3.8) possesses a Lyapunov function V and there exist two continuous, strictly increasing functions $\lambda, \eta : \mathbb{R}_+ \to \mathbb{R}_+$ such that $\lambda(r) = 0$ if and only if $r = 0$, $\eta(s) = 0$ if and only if $s = 0$, and*

$$V(t,x) \leq \lambda(\|x\|), \quad \forall (t,x) \in \mathbb{R}_+ \times \Omega, \tag{3.12}$$

$$\frac{\partial V}{\partial t}(t,x) + \sum_{i=1}^{n} f_i(t,x) \frac{\partial V}{\partial x_i}(t,x) \leq -\eta(\|x\|), \quad \forall (t,x) \in \mathbb{R}_+ \times \Omega, \tag{3.13}$$

then the zero solution is asymptotically stable.

Proof.

From Theorem 3.1.13, it follows that the zero solution is stable. The inequalities (3.12) and (3.13) further imply that the zero solution converges to zero as $t \to +\infty$, hence it is asymptotically stable.

□

3.1.4 Solved Exercises

• Exercise 01:

Find the equilibrium points of the differential equation

$$\frac{dx}{dt} = \sin x$$

Solution:

Equilibria satisfy $\sin(x) = 0$.

Thus, there is an infinite number of equilibria:

$$x_k^* = k\pi, \quad k \in \mathbb{Z}.$$

• **Exercise 02:**

Consider the function

$$V(x,y) = x^2 + y^2$$

Use 2 methods to study the stability of the system

$$\begin{cases} \dfrac{dx}{dt} = -x + y \\ \dfrac{dy}{dt} = -x - y \end{cases} \qquad (1)$$

Solution: Lyapunov method

- $V(0,0) = 0^2 + 0^2 = 0.$
- $V(x,y) = x^2 + y^2 > 0$ for all $(x,y) \neq (0,0)$.
- $\dfrac{dV}{dt} = 2x\dfrac{dx}{dt} + 2y\dfrac{dy}{dt} = 2x(-x+y) + 2y(-x-y)$

$$= -2x^2 + 2xy - 2xy - 2y^2 = -2(x^2+y^2) < 0$$

Hence, the system is asymptotically stable.

• **Exercise 03:**

Consider the system

$$\begin{cases} \dfrac{dx}{dt} = -y^3 \\ \dfrac{dy}{dt} = x \end{cases} \qquad (2)$$

1. Can we study the stability of system (2) using linearization?
2. Consider the function $V(x,y) = 2x^2 + y^4$. Study the stability of system (2).

Solution:

1. The Jacobian matrix is

$$Df(x,y) = \begin{pmatrix} 0 & -3y^2 \\ 1 & 0 \end{pmatrix}, \quad Df(0,0) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Eigenvalues: $\lambda_1 = \lambda_2 = 0$, so linearization gives no information. Hence, stability cannot be studied by linearization.

2. Lyapunov method: $V(0,0) = 0$, and $V(x,y) = 2x^2 + y^4 > 0$ for all $(x,y) \neq (0,0)$.
   Compute derivative along solutions:

$$\dot{V}(x,y) = 4x(-y^3) + 4y^3(x) = -4xy^3 + 4xy^3 = 0 \leq 0$$

   Hence, the system is stable.

- **Exercise 04:**

1. Cite 3 methods for studying stability of nonlinear autonomous systems.

2. Let $a \in \mathbb{R}_+^*$. Study the stability of the system

$$Y' = \begin{pmatrix} 0 & a \\ a & 0 \end{pmatrix} Y$$

3. Use the Lyapunov function $V(x,y) = x^2 + y^2$ to study the stability of

$$\begin{cases} x' = y - x(x^2 + y^2) \\ y' = -x - y(x^2 + y^2) \end{cases}$$

Solution:

1. Using the definition of stability, linearization method, and Lyapunov function method.

2. Eigenvalues of the matrix $\begin{pmatrix} 0 & a \\ a & 0 \end{pmatrix}$: $\lambda_1 = a$, $\lambda_2 = -a$.
   Since one eigenvalue has strictly positive real part, the system is unstable.

3. Lyapunov function:

   (a) $V(0,0) = 0$, $V(x,y) > 0$ for $(x,y) \neq (0,0)$.

(b) Along solutions:

$$\dot{V}(x,y) = 2x(y - x(x^2+y^2)) + 2y(-x - y(x^2+y^2)) = -2(x^2+y^2)^2 < 0$$

Hence, the system is asymptotically stable.

• **Exercise 05:**

Consider the system

$$\begin{cases} x' = x - x^2 + 2y \\ \\ y' = -2xy + 2x \end{cases} \qquad (E)$$

Study the nature of the origin of system (E).

Solution:

The Jacobian matrix at $(0,0)$ is

$$A = Df(0,0) = \begin{pmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{pmatrix}_{(0,0)} = \begin{pmatrix} 1 & 2 \\ \\ 2 & 0 \end{pmatrix}.$$

3.1.5 Unsolved Exercises

• **Exercise 01:**

Study the stability of the zero solution of the ODE

$$y' = -2y + \sin y. \tag{3.14}$$

• **Exercise 02:**

Consider the ODE

$$y' = a(t)y, \tag{3.15}$$

where $t \geq 0$ and $a :]0,+\infty[\to \mathbb{R}$ is continuous. Show that:

(i) The ODE (3.15) is stable if and only if there exists a function $K : [0,+\infty[\to \mathbb{R}$ such that

$$\int_{t_0}^{t} a(s)\,ds \leq K(t_0),$$

for all $t_0 \geq 0$ and $t \geq t_0$.

Bibliography

[1] V. Arnold, *Équations différentielles ordinaires*, Éditions Mir, Moscou, 1974.

[2] C. Chicone, *Ordinary Differential Equations with Applications*, Texts in Applied Mathematics, vol. 34, Springer-Verlag, New York, 1999.

[3] E. A. Coddington, N. Levinson, *Theory of Ordinary Differential Equations*, McGraw-Hill, New York–Toronto–London, 1955.

[4] S. B. Gavage, *Calcul différentiel et équations différentielles.*

[5] M. Al. Gwai, *Sturm–Liouville Theory and its Applications*, Springer Undergraduate Mathematics Series, 2008.

[6] P. Hartman, *Ordinary Differential Equations*, Classics in Applied Mathematics, vol. 38, SIAM, Philadelphia, 2002.

[7] S. W. Hirsch, S. Smale, *Differential Equations, Dynamical Systems and Linear Algebra*, Academic Press, University of California, Berkeley, 1970.

[8] S. Benzoni-Gavage, *Calcul différentiel et équations différentielles*, Dunod, 2010.

[9] F. Boyer, *Agrégation externe de Mathématiques, Équations différentielles ordinaires*, poly disponible sur

[10] R. Danchin, *Cours de calcul différentiel*, Cours de L3 mathématiques, poly disponible sur

[11] A. C. King, J. Billingham, S. R. Otto, *Differential Equations: Linear, Nonlinear, Ordinary, Partial*, Cambridge University Press, 2003.

[12] J. P. La Salle, “Some extensions of Liapunov’s second method”, *IRE Transactions on Circuit Theory*, CT-7, pp. 520–527, 1960.

[13] Y. Pinchover, J. Rubinstein, *An Introduction to Partial Differential Equations*, Cambridge University Press, 2005.

[14] W. Rudin, *Analyse réelle et complexe*, Masson, Paris, 1980.

[15] S. Villani, *Équations d’évolution*, Université de Lyon, Institut Henri Poincaré.