Algerian Democratic and Popular Republic

Ministry of Higher Education and Scientific Research

Khemis Miliana University

Faculty of Matter Sciences and Computer science

Department of Mathematics

Introduction to Topology

Course Notes and Solved Exercises

Designed for Students of
Second year licence in mathematics

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Academic Year 2025–2026

Contents

Preface

This document is designed as a rigorous yet accessible introduction to general topology for second-year Licence in mathematics at the Faculty of Matter Sciences and Computer Science, University of Djilali Bounaama, Khemis Miliana. It serves as the primary course notes for the module *Introduction to Topology*, aiming to bridge the gap between elementary analysis and the abstract framework of modern mathematics.

General topology constitutes a cornerstone of advanced mathematical thought, providing the language and tools essential for analysis, geometry, and beyond. Recognizing the conceptual challenges students often face when encountering topological ideas for the first time, this text adopts a deliberately structured and intuitive approach. Abstract notions are introduced gradually, always anchored in concrete examples and motivated by geometric insight.

The content is organized into four coherent chapters. Each chapter presents core definitions, fundamental theorems, and illustrative examples, followed by a curated set of exercises. Detailed solutions are provided for the majority of these exercises to encourage active engagement and support independent study.

This manuscript reflects my ongoing commitment to clarity, precision, and pedagogical effectiveness. While it represents a complete and self-contained first version, it is inevitably subject to improvement. I therefore warmly invite feedback, corrections, and suggestions from students, colleagues, and readers. Such contributions will be instrumental in refining future editions and enhancing the learning experience for subsequent cohorts.

Finally, I extend my deepest appreciation to my students for their intellectual curiosity, thoughtful questions, and unwavering dedication—qualities that continually enrich the teaching and learning process. I also The Author

Introduction

Based on my experience teaching Topology to second-year undergraduate mathematics students, I have observed recurring challenges in grasping the abstract nature of the subject. This document is born from a desire to address these difficulties by providing a clear, structured, and accessible exposition of the fundamental concepts of general topology.

The primary aim of these notes is to serve as a reliable companion for students, helping them to build a solid conceptual foundation, navigate the common pitfalls of the subject, and ultimately strengthen their problem-solving abilities.

The content is organized into five chapters, each designed to progressively develop the theory while maintaining a balance between abstraction and intuition.

Chapter 1 lays the groundwork with the axiomatic definition of a topological space. It introduces essential notions such as interior, closure, boundary, and neighborhoods, all of which rely on a fluent understanding of elementary set theory. The chapter then establishes the concepts of limits and continuity, highlighting the crucial role of the Hausdorff (separated) property in ensuring the uniqueness of limits. A key result, that every open interval in $\mathbb{R}$ is homeomorphic to $\mathbb{R}$ itself, provides an early illustration of the power of topological equivalence. The chapter concludes with a treatment of the subspace topology and the product topology.

Chapter 2 focuses on metric spaces, demonstrating how a distance function naturally induces a topology. This setting provides a more concrete and intuitive framework for understanding limits, continuity, and convergence. The chapter explores metric-specific concepts such as uniform continuity, Lipschitz mappings, and isometries, and extends the notion of distance to sets via the diameter and distance between subsets. It also clarifies the relationship between metric and topological equivalence of metrics and establishes that every metric space is Hausdorff, a property that does not hold for all topological spaces.

Chapter 3 is devoted to compactness, a central and powerful topological property. While the general definition via open coverings (the Borel–Lebesgue property) is foundational, the chapter emphasizes more tangible characterizations in the context of metric spaces, notably the Bolzano–Weierstrass property (sequential compactness). The intrinsic nature of compactness is highlighted, along with its fundamental connection to continuity, as encapsulated in theorems on the compactness of products and the behavior of continuous functions on compact sets.

Chapter 4 explores the concept of connectedness, which captures the idea of a space being "in one piece." The chapter presents several equivalent definitions and examines related notions such as local connectedness and path-connectedness, supported by illustrative examples. A major result is the generalization of the Intermediate Value Theorem to continuous functions on connected spaces, demonstrating its topological underpinning.

Finally, Chapter 5 introduces normed vector spaces, which blend the geometric structure of a metric with the algebraic structure of a vector space. This leads naturally to the study of Banach spaces, a cornerstone of functional analysis where the "vectors" are often functions themselves.

These notes are designed to guide students from the foundational axioms of topology towards its profound applications, fostering both a technical mastery and a deeper conceptual understanding of the subject.

Chapter 1

Topological Spaces

In this chapter, we introduce the fundamental definition of a topological space and establish the essential vocabulary of general topology.

1.1 The Order Topology

Definition 1.1. Let $(E, \leq)$ be a totally ordered set. The *open intervals* in E are subsets of one of the following four types:

1. E itself,
2. $\emptyset$,
3. $]a, b[:= \{x \in E \mid a < x < b\}$ for $a, b \in E$,
4. $\{x \in E \mid x < b\}$ for some $b \in E$,
5. $\{x \in E \mid a < x\}$ for some $a \in E$.

The **order topology** on E is defined as the collection of all arbitrary unions of these open intervals.

Remark 1.2. If $E = \mathbb{R}$ with its standard order, the intervals of types (4) and (5) are conventionally denoted by $]-\infty, b[$ and $]a, +\infty[$, respectively.

If E is the extended real line $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$ (with the natural order $-\infty < x < +\infty$ for all $x \in \mathbb{R}$), then these intervals become $[-\infty, b[$ and $]a, +\infty]$, respectively.

1.2 Topological Spaces

Definition 1.3. A **topological space** is a pair $(E, \mathcal{T})$ where E is a set and $\mathcal{T} \subseteq \mathcal{P}(E)$ is a family of subsets of E, called *open sets*, satisfying the following axioms:

1. $\emptyset \in \mathcal{T}$ and $E \in \mathcal{T}$.
2. The intersection of any two open sets is open: if $U, V \in \mathcal{T}$, then $U \cap V \in \mathcal{T}$.
3. The union of any collection of open sets is open: if $\{U_i\}_{i \in I} \subseteq \mathcal{T}$, then $\bigcup_{i \in I} U_i \in \mathcal{T}$.

The family $\mathcal{T}$ is called a *topology* on E.

1. The elements of $\mathcal{T}$ are called the *open sets* of the topology. The pair $(X, \mathcal{T})$ is called a *topological space*.

2. The complements of open sets in E are called ***closed sets***. Thus, a subset $F \subseteq E$ is closed if and only if $E \setminus F \in \mathcal{T}$.

3. Axiom (ii) implies by induction that any *finite* intersection of open sets is open. In other words, $\mathcal{T}$ is closed under finite intersections.

4. Dually, the collection of closed sets is closed under arbitrary intersections and finite unions. However, an arbitrary intersection of open sets need not be open, and an arbitrary union of closed sets need not be closed. Classic counterexamples in $\mathbb{R}$ with its standard topology $\mathcal{T}_u$ are:
$$\bigcap_{n\in\mathbb{N}^*} \left]-\frac{1}{n}, \frac{1}{n}\right[= \{0\} \quad \text{(a closed set, not open)},$$
$$\bigcup_{n\in\mathbb{N}^*} \left[\frac{1}{n}, 1\right] =]0,1] \quad \text{(neither open nor closed)}.$$

1. **The Standard Topology on $\mathbb{R}$**: A subset $U \subseteq \mathbb{R}$ is declared open if for every $x \in U$, there exists $\varepsilon > 0$ such that the open interval $]x-\varepsilon, x+\varepsilon[$ is contained in U. This defines a topology $\mathcal{T}_u$ on $\mathbb{R}$, known as the *standard topology*.

2. The collection $\{\emptyset, \mathbb{R}\} \cup \{]a, b[\mid a, b \in \mathbb{R}, a < b\}$ is *not* a topology on $\mathbb{R}$ because it is not closed under arbitrary unions (e.g., the union of all $]a, b[$ with $a < 0 < b$ is $\mathbb{R}$, which is included, but the union of $]1/n, 1[$ for $n \in \mathbb{N}^*$ is $]0, 1[$, which *is* included; a better example is that the union of all $]x-1, x+1[$ for $x \in \mathbb{R}\setminus\mathbb{Q}$ is $\mathbb{R}\setminus\mathbb{Q}$, which is not an open interval). The main point is that the set of all open intervals alone is not a topology; one must include all possible unions of them.

3. On any set E, there are two canonical topologies:
   - The **discrete topology** $\mathcal{T}_d = \mathcal{P}(E)$, where every subset is open.
   - The **trivial topology** $\mathcal{T}_t = \{\emptyset, E\}$, where only the empty set and the whole space are open.

4. On a two-point set $E = \{a, b\}$, there are exactly four distinct topologies:
$$\begin{aligned}\mathcal{T}_t &= \{\emptyset, E\},\\ \mathcal{T}_1 &= \{\emptyset, \{a\}, E\},\\ \mathcal{T}_2 &= \{\emptyset, \{b\}, E\},\\ \mathcal{T}_d &= \{\emptyset, \{a\}, \{b\}, E\}.\end{aligned}$$

5. Let $(X, \mathcal{T})$ be a topological space such that $\{x\} \in \mathcal{T}$ for every $x \in X$ (i.e., every singleton is open). Then $(X, \mathcal{T})$ is the discrete topology.

1.2.1 Exercises

Exercise 1. Let E be an arbitrary set. Show that the family consisting of the empty set and the complements of finite subsets of E forms a topology on E.

Solution 1. Denote this family by $\mathcal{T} = \{E \setminus F \mid F \subseteq E \text{ is finite}\} \cup \{\emptyset\}$.

- Since $\emptyset$ is finite, $E \setminus \emptyset = E \in \mathcal{T}$. Also, $\emptyset \in \mathcal{T}$ by definition.

- Let $\{U_i\}_{i\in I} \subseteq \mathcal{T}$ be an arbitrary family of open sets. For each i, either $U_i = \emptyset$ or $U_i = E \setminus F_i$ for some finite $F_i \subseteq E$. Their union is

$$\bigcup_{i\in I} U_i = E \setminus \left(\bigcap_{i\in I} F_i\right).$$

  The intersection $\bigcap_{i\in I} F_i$ is a subset of any F_i, hence finite. Thus, the union is in $\mathcal{T}$.

- Let $U_1, \ldots, U_n \in \mathcal{T}$ be a finite collection. For each k, write $U_k = E \setminus F_k$ with F_k finite (or $U_k = \emptyset = E \setminus E$, and we can take $F_k = E$ which is fine for the complement). Their intersection is

$$\bigcap_{k=1}^{n} U_k = E \setminus \left(\bigcup_{k=1}^{n} F_k\right).$$

  The union $\bigcup_{k=1}^{n} F_k$ is a finite union of finite sets, hence finite. Thus, the intersection is in $\mathcal{T}$.

Therefore, $\mathcal{T}$ is a topology on E, called the *cofinite topology.*

Exercise 2. Show that the set $\mathcal{T}$, consisting of $\emptyset$, $\mathbb{R}$, and arbitrary unions of intervals of the form $]a, b[$, is indeed a topology on $\mathbb{R}$.

Solution 2. This collection $\mathcal{T}$ is precisely the standard topology on $\mathbb{R}$.

- $\emptyset \in \mathcal{T}$ (as the empty union) and $\mathbb{R} \in \mathcal{T}$ since $\mathbb{R} = \bigcup_{n=1}^{\infty}]-n, n[$.

- Let $U, V \in \mathcal{T}$. Then $U = \bigcup_{i\in I}]a_i, b_i[$ and $V = \bigcup_{j\in J}]c_j, d_j[$ for some index sets I, J. Using the distributive law,

$$U \cap V = \bigcup_{i\in I, j\in J} (]a_i, b_i[\cap]c_j, d_j[).$$

  The intersection of two open intervals is either empty or another open interval $]e, f[$. Thus, $U \cap V$ is a union of open intervals, so $U \cap V \in \mathcal{T}$.

- Let $\{U_i\}_{i\in I} \subseteq \mathcal{T}$. Each U_i is a union of open intervals, so their total union $\bigcup_{i\in I} U_i$ is also a union of open intervals, hence in $\mathcal{T}$.

Therefore, $\mathcal{T}$ is a topology on $\mathbb{R}$.

Exercise 3. What conditions must two distinct subsets A and B of a set E satisfy for $\mathcal{T} = \{\emptyset, A, B, E\}$ to be a topology on E?

Solution 3. For $\mathcal{T}$ to be a topology, it must be closed under finite intersections and arbitrary unions. The non-trivial checks are for $A \cap B$ and $A \cup B$.

Since $\mathcal{T}$ has only four elements, $A \cap B$ and $A \cup B$ must each be one of $\emptyset, A, B, E$.

- $A \cap B \in \mathcal{T}$ implies $A \cap B = \emptyset$, A, or B.

- $A \cup B \in \mathcal{T}$ implies $A \cup B = A$, B, or E.

Since A and B are distinct, we cannot have both $A \subseteq B$ and $B \subseteq A$. The only consistent possibilities are:

$$A \subseteq B \quad \text{or} \quad B \subseteq A.$$

To see this, suppose $A \nsubseteq B$ and $B \nsubseteq A$. Then $A \cap B$ is a proper subset of both A and B, so $A \cap B \notin \mathcal{T}$. Also, $A \cup B$ is a proper superset of both A and B, so $A \cup B \notin \mathcal{T}$.

Conversely, if $A \subseteq B$, then $A \cap B = A \in \mathcal{T}$ and $A \cup B = B \in \mathcal{T}$. All other intersections and unions are trivially in $\mathcal{T}$.

Therefore, the necessary and sufficient condition is that one of the sets is contained in the other: $A \subseteq B$ or $B \subseteq A$.

Exercise 4. Let $E :=]0, +\infty[$, and for every $\alpha > 0$, let $O_\alpha :=]0, \alpha[$. Consider the family $\mathcal{T} = \{\emptyset\} \cup \{O_\alpha \mid \alpha > 0\} \cup \{E\}$.

1. Show that $\mathcal{T}$ constitutes a topology on E.

2. Determine the closed sets of the topological space $(E, \mathcal{T})$.

Solution 4.

1. We verify the axioms of a topology.

   - $\emptyset \in \mathcal{T}$ and $E \in \mathcal{T}$ by definition.
   - Let $O_\alpha, O_\beta \in \mathcal{T}$. Then $O_\alpha \cap O_\beta =]0, \min(\alpha, \beta)[= O_{\min(\alpha,\beta)} \in \mathcal{T}$.
   - Let $\{O_{\alpha_i}\}_{i\in I} \subseteq \mathcal{T}$. Their union is

   $$\bigcup_{i\in I} O_{\alpha_i} = \left]0, \sup_{i\in I} \alpha_i\right[.$$

   If $\sup_{i\in I} \alpha_i = +\infty$, the union is $E \in \mathcal{T}$. Otherwise, it is $O_{\sup \alpha_i} \in \mathcal{T}$.

   Thus, $\mathcal{T}$ is a topology on E.

2. A set is closed if and only if its complement in E is open. The open sets are $\emptyset$, E, and $]0, \alpha[$ for $\alpha > 0$. Their complements are:

   $$E \setminus \emptyset = E,$$
   $$E \setminus E = \emptyset,$$
   $$E\setminus]0, \alpha[= [\alpha, +\infty).$$

   Therefore, the closed sets are $\emptyset$, E, and all intervals of the form $[\alpha, +\infty)$ for $\alpha > 0$.

1.3 Neighborhood of a Point or a Set

Definition 1.4. Let $(E, \mathcal{T})$ be a topological space.

- A subset $V \subseteq E$ is a *neighborhood* of a point $a \in E$ if there exists an open set $O \in \mathcal{T}$ such that $a \in O \subseteq V$. The set of all neighborhoods of a is denoted by $\mathcal{V}(a)$.

- More generally, a *neighborhood* of a subset $A \subseteq E$ is any subset $V \subseteq E$ for which there exists an open set $O \in \mathcal{T}$ such that $A \subseteq O \subseteq V$. The set of all neighborhoods of A is denoted by $\mathcal{V}(A)$.

In mathematical notation, for any $a \in E$ and $A \subseteq E$,

$$V \in \mathcal{V}(a) \iff \exists O \in \mathcal{T} \text{ such that } a \in O \subseteq V,$$

$$V \in \mathcal{V}(A) \iff \exists O \in \mathcal{T} \text{ such that } A \subseteq O \subseteq V.$$

Example 1.5. In the topological space $\mathbb{R}$ with its standard topology, a neighborhood of a point $a \in \mathbb{R}$ is any subset of $\mathbb{R}$ that contains an open interval of the form $]a-\varepsilon, a+\varepsilon[$ for some $\varepsilon > 0$.

Example 1.6. Consider the set $E = \{a, b, c\}$ equipped with the topology $\mathcal{T} = \{\emptyset, \{a\}, E\}$. The neighborhoods of the point a are all subsets $V \subseteq E$ that contain the open set $\{a\}$. Thus, $\mathcal{V}(a) = \{\{a\}, \{a, b\}, \{a, c\}, \{a, b, c\}\}$.

Remark 1.7. Let $a \in E$. The family of neighborhoods $\mathcal{V}(a)$ has the following properties:

1. Every neighborhood of a contains a.
2. Every superset of a neighborhood of a is also a neighborhood of a.
3. Any finite intersection of neighborhoods of a is a neighborhood of a.
4. Every open set containing a is a neighborhood of a.

The next theorem provides a fundamental characterization of open sets in terms of neighborhoods.

Theorem 1.8. A subset U of a topological space $(E, \mathcal{T})$ is open if and only if U is a neighborhood of each of its points.

Proof. ($\Rightarrow$) If U is open, then for every $a \in U$, we have $a \in U \subseteq U$, so U is a neighborhood of a.

($\Leftarrow$) Conversely, suppose U is a neighborhood of each of its points. Then for every $a \in U$, there exists an open set $O_a \in \mathcal{T}$ such that $a \in O_a \subseteq U$. Consider the union $O = \bigcup_{a \in U} O_a$. This union O is open as it is a union of open sets. Moreover, for every $a \in U$, $a \in O_a \subseteq O$, so $U \subseteq O$. Conversely, since each $O_a \subseteq U$, we have $O \subseteq U$. Therefore, $U = O$, which is open. ■

1.3.1 Exercises

Exercise 5. Show that in $\mathbb{N} \cup \{+\infty\}$ equipped with the order topology, the neighborhoods of $+\infty$ are precisely the subsets that contain $+\infty$ and all integers from some fixed rank onward.

Solution 5. In the order topology on $\mathbb{N} \cup \{+\infty\}$, a base of open neighborhoods for the point $+\infty$ is given by the intervals of the form $(a, +\infty] = \{x \in \mathbb{N} \cup \{+\infty\} \mid x > a\}$ for $a \in \mathbb{N}$.

Let $V \subseteq \mathbb{N} \cup \{+\infty\}$. By definition, V is a neighborhood of $+\infty$ if and only if there exists an open set O such that $+\infty \in O \subseteq V$. Since the intervals $(a, +\infty]$ form a neighborhood base, this is equivalent to the existence of some $a \in \mathbb{N}$ such that $(a, +\infty] \subseteq V$.

The set $(a, +\infty]$ contains $+\infty$ and all integers $n \in \mathbb{N}$ with $n > a$. Therefore, V is a neighborhood of $+\infty$ if and only if there exists $N \in \mathbb{N}$ (take $N = a + 1$) such that V contains $+\infty$ and all integers $n \geq N$.

Conversely, if a set V contains $+\infty$ and all integers $n \geq N$ for some N, then it contains the open interval $(N - 1, +\infty]$, so V is a neighborhood of $+\infty$.

Exercise 6. (Neighborhood Axioms) Let X be a set. Suppose that for every element $x \in X$, there is a family $\mathcal{V}(x)$ of subsets of X satisfying the following properties:

1. $\mathcal{V}(x) \neq \emptyset$ and for every $V \in \mathcal{V}(x)$, we have $x \in V$.

2. If a subset $A \subseteq X$ contains some $V \in \mathcal{V}(x)$, then $A \in \mathcal{V}(x)$.

3. The intersection of any two elements of $\mathcal{V}(x)$ is also in $\mathcal{V}(x)$.

4. For every $V \in \mathcal{V}(x)$, there exists $W \in \mathcal{V}(x)$ such that for all $y \in W$, we have $W \in \mathcal{V}(y)$.

Show that there exists a unique topology $\mathcal{T}$ on X such that for every $x \in X$, $\mathcal{V}(x)$ is precisely the family of all neighborhoods of x in the topological space $(X, \mathcal{T})$.

Solution 6. We will construct the topology $\mathcal{T}$ from the given neighborhood systems and prove it is the unique one with the desired property.

Step 1: Define the candidate topology. Define a subset $U \subseteq X$ to be *open* if and only if it is a neighborhood of each of its points, i.e.,

$$U \in \mathcal{T} \quad \text{if and only if} \quad \forall x \in U, U \in \mathcal{V}(x).$$

Step 2: Show that $\mathcal{T}$ is a topology. We verify the three axioms of a topology.

- *$\emptyset, X \in \mathcal{T}$:* The statement "$\forall x \in \emptyset$, $\emptyset \in \mathcal{V}(x)$" is vacuously true, so $\emptyset \in \mathcal{T}$.

  For X, let $x \in X$ be arbitrary. By (V1), there exists some $V \in \mathcal{V}(x)$. By (V2), since $V \subseteq X$, we have $X \in \mathcal{V}(x)$. As this holds for all $x \in X$, $X \in \mathcal{T}$.

- *$\mathcal{T}$ is closed under arbitrary unions:* Let $\{U_i\}_{i \in I} \subseteq \mathcal{T}$ and let $U = \bigcup_{i \in I} U_i$. Let $x \in U$. Then $x \in U_j$ for some $j \in I$. Since $U_j \in \mathcal{T}$, we have $U_j \in \mathcal{V}(x)$. But $U_j \subseteq U$, so by (V2), $U \in \mathcal{V}(x)$. Since this holds for all $x \in U$, $U \in \mathcal{T}$.

- *$\mathcal{T}$ is closed under finite intersections:* Let $U_1, U_2 \in \mathcal{T}$ and let $U = U_1 \cap U_2$. Let $x \in U$. Then $x \in U_1$ and $x \in U_2$, so $U_1 \in \mathcal{V}(x)$ and $U_2 \in \mathcal{V}(x)$. By (V3), $U = U_1 \cap U_2 \in \mathcal{V}(x)$. Since this holds for all $x \in U$, $U \in \mathcal{T}$. The result extends to any finite intersection by induction.

Thus, $\mathcal{T}$ is a topology on X.

Step 3: Show that $\mathcal{V}(x)$ is the family of $\mathcal{T}$-neighborhoods of x. Let $x \in X$. We must show that a set V is a $\mathcal{T}$-neighborhood of x if and only if $V \in \mathcal{V}(x)$.

($\Leftarrow$) Suppose $V \in \mathcal{V}(x)$. By (V4), there exists $W \in \mathcal{V}(x)$ such that for all $y \in W$, $W \in \mathcal{V}(y)$. This means, by our definition of $\mathcal{T}$, that W is an open set (i.e., $W \in \mathcal{T}$). Since $x \in W \subseteq V$, V contains an open set containing x, so V is a $\mathcal{T}$-neighborhood of x.

($\Rightarrow$) Conversely, suppose V is a $\mathcal{T}$-neighborhood of x. Then there exists an open set $U \in \mathcal{T}$ such that $x \in U \subseteq V$. Since U is open and $x \in U$, we have $U \in \mathcal{V}(x)$ by the definition of $\mathcal{T}$. Then, by (V2), since $U \subseteq V$, we get $V \in \mathcal{V}(x)$.

Therefore, the two notions of neighborhood coincide.

Step 4: Uniqueness. Suppose $\mathcal{T}'$ is another topology on X for which $\mathcal{V}(x)$ is the family of neighborhoods of x. A set U is open in $\mathcal{T}'$ if and only if it is a neighborhood of each of its points (by Theorem 1.8), which is if and only if $U \in \mathcal{V}(x)$ for all $x \in U$. This is exactly the definition of our topology $\mathcal{T}$. Hence $\mathcal{T}' = \mathcal{T}$, proving uniqueness.

1.4 Bases of Open Sets and Neighborhood Bases

Definition 1.9. Let $(E, \mathcal{T})$ be a topological space, and let $\mathcal{B}$ be a collection of subsets of E. We say that $\mathcal{B}$ is a *basis* for the topology $\mathcal{T}$ (or a *basis of open sets*) if one (and hence all) of the following equivalent conditions is satisfied:

1. The open sets in $\mathcal{T}$ are precisely the unions of elements of $\mathcal{B}$.
2. $\mathcal{B} \subseteq \mathcal{T}$ and every open set in $\mathcal{T}$ is a union of elements of $\mathcal{B}$.
3. $\mathcal{B} \subseteq \mathcal{T}$ and for every $U \in \mathcal{T}$ and every point $x \in U$, there exists a set $B \in \mathcal{B}$ such that $x \in B \subseteq U$.

Remark 1.10. The topology $\mathcal{T}$ itself is always a basis for $\mathcal{T}$. However, it is usually more useful to work with a smaller, more economical basis. For example, the collection of all open intervals $]a, b[$ forms a basis for the standard topology on $\mathbb{R}$.

Example 1.11. The family

$$\left\{ \left] r - \frac{1}{n}, r + \frac{1}{n} \right[\;\middle|\; r \in \mathbb{Q},\ n \in \mathbb{N}^* \right\}$$

forms a countable basis for the standard topology on $\mathbb{R}$. This follows from the density of $\mathbb{Q}$ in $\mathbb{R}$ and the Archimedean property. A topological space that admits a countable basis is said to be *second-countable*.

Definition 1.12 (Fundamental System of Neighborhoods)**.** A *fundamental system of neighborhoods* (or a *neighborhood basis*) of a point $a \in E$ is a family $\mathcal{B}(a)$ of neighborhoods of a such that:

$$\forall V \in \mathcal{V}(a),\ \exists W \in \mathcal{B}(a) \text{ such that } W \subseteq V.$$

Example 1.13. For every $a \in \mathbb{R}$, the family of intervals $\left\{ \left[a - \frac{1}{n}, a + \frac{1}{n}\right] \mid n \in \mathbb{N}^* \right\}$ is a countable neighborhood basis for a in the standard topology. This example shows that the elements of a neighborhood basis need not be open sets themselves.

It is often convenient to work with bases consisting of open sets. The following proposition establishes the precise link between a basis of open sets and neighborhood bases.

Proposition 1.14. Let $(E, \mathcal{T})$ be a topological space and let $\mathcal{B} \subseteq \mathcal{P}(E)$. Then $\mathcal{B}$ is a basis of open sets for $\mathcal{T}$ if and only if, for every point $a \in E$, the family

$$\mathcal{B}(a) := \{B \in \mathcal{B} \mid a \in B\}$$

is a neighborhood basis for a.

Proof. ($\Rightarrow$) Suppose $\mathcal{B}$ is a basis of open sets. Let $a \in E$ and let V be a neighborhood of a. Then there exists an open set $U \in \mathcal{T}$ such that $a \in U \subseteq V$. Since $\mathcal{B}$ is a basis, U is a union of elements of $\mathcal{B}$, so there exists $B \in \mathcal{B}$ with $a \in B \subseteq U \subseteq V$. Thus, $\mathcal{B}(a)$ is a neighborhood basis for a.

($\Leftarrow$) Conversely, suppose that for every $a \in E$, $\mathcal{B}(a)$ is a neighborhood basis for a. Let $U \subseteq E$. By Theorem 1.8, U is open if and only if it is a neighborhood of each of its points. This means that for every $a \in U$, there exists $B_a \in \mathcal{B}(a) \subseteq \mathcal{B}$ such that $B_a \subseteq U$. Therefore, $U = \bigcup_{a \in U} B_a$, which is a union of elements of $\mathcal{B}$. Hence, $\mathcal{B}$ is a basis of open sets. ∎

Exercise 7. Show that $\mathcal{B} = \{[a, b[\mid (a, b) \in \mathbb{R}^2,\ a < b\} \cup \{\emptyset\}$ is a basis for a topology on $\mathbb{R}$.

Solution 7. To verify that $\mathcal{B}$ is a basis, we must check two conditions.

(1) Coverage of $\mathbb{R}$: For any $x \in \mathbb{R}$, the interval $[x, x + 1[$ is in $\mathcal{B}$ and contains x. Thus, $\bigcup_{B \in \mathcal{B}} B = \mathbb{R}$.

(2) Local intersection property: Let $B_1 = [a_1, b_1[$ and $B_2 = [a_2, b_2[$ be two elements of $\mathcal{B}$, and let $x \in B_1 \cap B_2$. Then $x \geq \max(a_1, a_2)$ and $x < \min(b_1, b_2)$. Define $B = [x, \min(b_1, b_2)[$. Then $B \in \mathcal{B}$, and we have $x \in B \subseteq B_1 \cap B_2$.

Since both conditions are satisfied, $\mathcal{B}$ is a basis for a topology on $\mathbb{R}$. This topology is known as the *lower limit topology* or the *Sorgenfrey line*.

1.5 Interior and Closure

1.5.1 Closure

Definition 1.15. In a topological space E, a point x is said to be *adherent* to a subset $A \subseteq E$ if every neighborhood of x intersects A, i.e.,

$$\forall V \in \mathcal{V}(x), \quad V \cap A \neq \emptyset.$$

The set of all points adherent to A is called the *closure* of A and is denoted by $\overline{A}$.

Remark 1.16. Every element $x \in A$ is adherent to A (since $x \in V$ for any neighborhood V of x), so $A \subseteq \overline{A}$.

We can distinguish two types of adherent points.

Definition 1.17. An adherent point x of A is:

- an *isolated point* of A if there exists a neighborhood V of x such that $A \cap V = \{x\}$ (equivalently, $x \in A$ and $\{x\}$ is open in the subspace topology on A).
- a *limit point* (or *accumulation point*) of A if every neighborhood of x intersects $A \setminus \{x\}$ (note that x need not belong to A).

1. A point cannot be both isolated and a limit point of A. If an adherent point x is not isolated, then for every neighborhood V of x, $A \cap V \neq \{x\}$, so x is a limit point. Thus, these are the only two possibilities for adherent points.
2. Every limit point of A is adherent to A, but the converse is generally false (e.g., isolated points are adherent but not limit points).
3. The set of all limit points of A is called the *derived set* of A, denoted by A'. This concept was introduced by the German mathematician Georg Cantor (1845–1918).

Example 1.18. .

1. In $\mathbb{R}$ with the standard topology, the closure of $]0, \sqrt{2}[\cup \{3\}$ is $[0, \sqrt{2}] \cup \{3\}$. The point 3 is an isolated point, while $[0, \sqrt{2}]$ is the set of limit points.
2. The closure of $\{1/n \mid n \in \mathbb{N}^*\}$ is $\{1/n \mid n \in \mathbb{N}^*\} \cup \{0\}$. The points $1/n$ are isolated, and 0 is a limit point.

Example 1.19. If $(E, \mathcal{T}) = (\mathbb{R}, \text{standard topology})$ and $A =]a, b[$, then $\overline{A} = [a, b]$. Indeed, $[a, b]$ is a closed set containing A, so $\overline{A} \subseteq [a, b]$. The only closed set among the possibilities $]a, b[$, $]a, b]$, $[a, b[$, and $[a, b]$ is $[a, b]$, hence $\overline{A} = [a, b]$.

Proposition 1.20. The closure of a subset A is the smallest closed set containing A. In particular, A is closed if and only if $A = \overline{A}$.

Proof. The family of closed sets containing A is nonempty (since E is such a set), so their intersection exists and is closed. We must show that $x \in \overline{A}$ if and only if x belongs to every closed set containing A.

A point x is adherent to A if every open neighborhood of x intersects A. This is equivalent to saying that there is no open set containing x that is disjoint from A, or, by taking complements, that there is no closed set containing A that excludes x. The contrapositive of this is that every closed set containing A must contain x. ■

Exercise 8. Let O be an open subset of a topological space $(E, \mathcal{T})$. Show that for any subset $A \subseteq E$,

$$A \cap O = \emptyset \quad \Longleftrightarrow \quad \overline{A} \cap O = \emptyset.$$

Solution 8. We prove both implications.

($\Rightarrow$) Assume $A \cap O = \emptyset$. Since O is open, its complement $E \setminus O$ is closed. The hypothesis $A \cap O = \emptyset$ implies $A \subseteq E \setminus O$. Because the closure $\overline{A}$ is the smallest closed set containing A, we have $\overline{A} \subseteq E \setminus O$. This is equivalent to $\overline{A} \cap O = \emptyset$.

($\Leftarrow$) Assume $\overline{A} \cap O = \emptyset$. Since $A \subseteq \overline{A}$, it follows immediately that $A \cap O \subseteq \overline{A} \cap O = \emptyset$. Hence, $A \cap O = \emptyset$.

This completes the proof.

1.5.2 Interior

Definition 1.21. For a subset B of a topological space E, the *interior* of B, denoted by $\mathring{B}$, is the largest open set contained in B. Dually to the inclusion $A \subseteq \overline{A}$, we have the immediate inclusion $\mathring{B} \subseteq B$.

Corollary 1.22. .

1. The interior of B is the set of points for which B is a neighborhood.
2. The interior of B is the union of all open sets contained in B (this union exists since $\emptyset$ is open, though it may be empty). That is,

$$\mathring{B} = \bigcup_{\substack{O \in \mathcal{T} \\ O \subseteq B}} O.$$

3. A subset B of E is open if and only if $\mathring{B} = B$.

Theorem 1.23. A point $b \in E$ is in the interior of B (i.e., $b \in \mathring{B}$) if and only if B is a neighborhood of b.

Proof. ($\Rightarrow$) If $b \in \mathring{B}$, then $\mathring{B}$ is an open set with $b \in \mathring{B} \subseteq B$, so B is a neighborhood of b.

($\Leftarrow$) If B is a neighborhood of b, there exists an open set O such that $b \in O \subseteq B$. By definition of the interior, $O \subseteq \mathring{B}$, so $b \in \mathring{B}$. ■

Proposition 1.24. Let E be a topological space and let A and B be subsets of E. Then:

1. If A and B are complements, then $\mathring{B}$ and $\overline{A}$ are complements.
2. If $A \subseteq B$, then $\overline{A} \subseteq \overline{B}$ and $\mathring{A} \subseteq \mathring{B}$.
3. $\overline{A \cup B} = \overline{A} \cup \overline{B}$ and $\overset{\circ}{\overline{A \cap B}}$ $A \overset{\circ}{\cap} B = \mathring{A} \cap \mathring{B}$.

Proof. 1. By Theorem 1.23, $x \notin \mathring{B} \iff B$ is not a neighborhood of $x \iff$ every neighborhood of x meets $A \iff x \in \overline{A}$.

2. Follows directly from the definitions of closure and interior.
3. The inclusion $\overline{A \cup B} \subseteq \overline{A} \cup \overline{B}$ follows from (ii). The reverse inclusion holds because $A \subseteq A \cup B$ and $B \subseteq A \cup B$ imply $\overline{A} \subseteq \overline{A \cup B}$ and $\overline{B} \subseteq \overline{A \cup B}$. The proof for the interior of the intersection is analogous.

■

Remark 1.25. In general, we only have the inclusions $\overline{A \cap B} \subseteq \overline{A} \cap \overline{B}$ and $\overset{\circ}{A \cup B} \supseteq \mathring{A} \cup \mathring{B}$, and equality does not always hold.

Example 1.26. If $A = [0,1]$ and $B = [1,2]$, then $\overset{\circ}{A \cup B} = \overset{\circ}{[0,2]} =]0,2[$. However, $\mathring{A} \cup \mathring{B} =]0,1[\cup]1,2[=]0,2[\setminus\{1\}$. Conversely, if $A =]0,1[$ and $B =]1,2[$, then $A \cap B = \emptyset$, so $\overline{A \cap B} = \emptyset$. But $\overline{A} \cap \overline{B} = [0,1] \cap [1,2] = \{1\}$.

Definition 1.27. The *boundary* of a subset A of a topological space E is defined by

$$\mathrm{Fr}(A) := \overline{A} \setminus \mathring{A} = \overline{A} \cap \overline{E \setminus A}.$$

It is equal to the boundary of the complement of A.

Example 1.28. If $E = \mathbb{R}$ with the standard topology and $A = [0,1[$, then $\overline{A} = [0,1]$, $\mathring{A} =]0,1[$, and $\mathrm{Fr}(A) = \{0,1\}$.

1.5.3 Exercises

Exercise 9. Let E be a topological space and let $A, B \subseteq E$.

1. Show that $\overline{A^c} = (\mathring{A})^c$ and that $\mathring{A^c} = (\overline{A})^c$.
2. Show that
$$\overline{A \cap B} \subseteq \overline{A} \cap \overline{B}, \quad \overline{A \cup B} = \overline{A} \cup \overline{B}, \quad \overset{\circ}{A \cap B} = \mathring{A} \cap \mathring{B}.$$
3. What can be said about $\mathring{A} \cup \mathring{B}$?
4. Define $u(A) = \overset{\circ}{\overline{A}}$ and $v(A) = \overline{\mathring{A}}$.
   (a) Compute $u(A)$ and $v(A)$ for $E = \mathbb{R}$ (with the standard topology) and for $A =]0,2]$ and $A = \mathbb{Q}$.
   (b) Compare the sets A, $\mathring{A}$, $u(A)$, and $v(A)$.

Solution 9. .

1. The interior $\mathring{A}$ is the union of all open sets contained in A. Taking complements, $(\mathring{A})^c$ is the intersection of all closed sets containing A^c. By Proposition 1.20, this intersection is precisely $\overline{A^c}$. The second identity follows by replacing A with A^c in the first.
2. The inclusion $\overline{A \cap B} \subseteq \overline{A} \cap \overline{B}$ is immediate from the monotonicity of closure (since $A \cap B \subseteq A$ and $A \cap B \subseteq B$).

   For the union, we have $A, B \subseteq A \cup B$, so $\overline{A}, \overline{B} \subseteq \overline{A \cup B}$, hence $\overline{A} \cup \overline{B} \subseteq \overline{A \cup B}$. Conversely, $A \cup B \subseteq \overline{A} \cup \overline{B}$, and since the right-hand side is closed (as a finite union of closed sets), it must contain $\overline{A \cup B}$.

   The identity for the interior of an intersection follows by applying the first part to the complements of A and B and using the duality between interior and closure.
3. The set $\mathring{A} \cup \mathring{B}$ is an open set contained in $A \cup B$, so $\mathring{A} \cup \mathring{B} \subseteq \overset{\circ}{A \cup B}$. The reverse inclusion is generally false. A standard counterexample is $A = [0,1]$, $B = [1,2]$ in $\mathbb{R}$. Then $\overset{\circ}{A \cup B} =]0,2[$, but $\mathring{A} \cup \mathring{B} =]0,1[\cup]1,2[$.

4. (a) For $A =]0,2]$:

$$\overline{A} = [0,2], \quad u(A) = \overline{\mathring{A}} =]0,2[.$$
$$\mathring{A} =]0,2[, \quad v(A) = \mathring{\overline{A}} = [0,2].$$

For $A = \mathbb{Q}$:

$$\overline{A} = \mathbb{R}, \quad u(A) = \mathring{\overline{A}} = \mathring{\mathbb{R}} = \mathbb{R}.$$
$$\mathring{A} = \emptyset, \quad v(A) = \overline{\mathring{A}} = \overline{\emptyset} = \emptyset.$$

(b) In general, we have $\mathring{A} \subseteq v(A) \subseteq \overline{A}$ and $\mathring{A} \subseteq u(A) \subseteq \overline{A}$. However, $u(A)$ and $v(A)$ are not comparable in general. For $A = \mathbb{Q}$, $v(A) = \emptyset \subsetneq \mathbb{R} = u(A)$. For a closed ball in $\mathbb{R}^n$, $u(A) \subsetneq v(A)$.

Exercise 10. Consider $\mathbb{R}^2$ with its standard topology. Determine the interior and the closure of the following subsets:

$$\begin{aligned} A &= \{(x,y) \in \mathbb{R}^2 \mid 2x > y+1\}, \\ B &= \{(x,y) \in \mathbb{R}^2 \mid 0 < x < 2 \text{ and } 0 \leq y \leq 1\}, \\ C &= \{(x,y) \in \mathbb{R}^2 \mid 0 \leq x^2 + y^2 \leq 1\}, \\ D &= \{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 \geq 4\} \setminus \mathbb{Q}^2. \end{aligned}$$

Solution 10. .

- The set A is open because it is the preimage of the open set $]1,\infty[$ under the continuous map $(x,y) \mapsto 2x - y - 1$. Hence, $\mathring{A} = A$. Its closure is $\overline{A} = \{(x,y) \mid 2x \geq y+1\}$, as every point on the boundary line can be approached by a sequence from A.
- The set B is a rectangle missing its left and right edges but including its top and bottom edges. Its closure is the closed rectangle $\overline{B} = [0,2] \times [0,1]$. Its interior is the open rectangle $\mathring{B} =]0,2[\times]0,1[$.
- The set C is the closed unit disk, hence it is closed: $\overline{C} = C$. Its interior is the open unit disk $\mathring{C} = \{(x,y) \mid x^2 + y^2 < 1\}$. Note that the origin is an interior point.
- The set D is the exterior of the closed disk of radius 2, with all points having rational coordinates removed. Since $\mathbb{Q}^2$ is dense in $\mathbb{R}^2$, removing it does not change the closure, so $\overline{D} = \{(x,y) \mid x^2 + y^2 \geq 4\}$. However, any nonempty open set in $\mathbb{R}^2$ contains points of $\mathbb{Q}^2$, so no nonempty open set is contained in D. Thus, $\mathring{D} = \emptyset$.

Exercise 11. Let $(E, \mathcal{T})$ be a topological space and let $A, B \subseteq E$ such that $E = A \cup B$. Show that

$$E = \mathring{A} \cup \overline{B}.$$

Solution 11. We will show that every point $x \in E$ belongs to either $\mathring{A}$ or $\overline{B}$.

Let $x \in E$ be arbitrary. We consider two cases.

Case 1: $x \notin \overline{B}$.

If x is not in the closure of B, then by definition there exists an open neighborhood U of x such that $U \cap B = \emptyset$. This implies $U \subseteq B^c$. Since $E = A \cup B$, we have $B^c \subseteq A$, and therefore $U \subseteq A$. This means that x is an interior point of A, so $x \in \mathring{A}$.

Case 2: $x \in \overline{B}$.

In this case, the conclusion is immediate.

Since every point $x \in E$ is in either $\mathring{A}$ or $\overline{B}$, we conclude that $E = \mathring{A} \cup \overline{B}$.

Exercise 12. Determine the boundary of the following subsets of $\mathbb{R}^2$:

$$\begin{aligned} A_1 &= \{(x,y) \in \mathbb{R}^2 \mid 0 < x^2 + y^2 < 2\}, \\ A_2 &= \mathbb{Q} \times \mathbb{Q}, \\ A_3 &=]-2,1[\times[0,1]. \end{aligned}$$

Solution 12. Recall that $\mathrm{Fr}(S) = \overline{S} \setminus \mathring{S}$.

- $\overline{A_1} = \{(x,y) \mid x^2+y^2 \leq 2\}$, $\mathring{A}_1 = A_1$. Thus, $\mathrm{Fr}(A_1) = \{(0,0)\} \cup \{(x,y) \mid x^2+y^2 = 2\}$.
- $\overline{A_2} = \mathbb{R}^2$, $\mathring{A}_2 = \emptyset$. Thus, $\mathrm{Fr}(A_2) = \mathbb{R}^2$.
- $\overline{A_3} = [-2,1] \times [0,1]$, $\mathring{A}_3 =]-2,1[\times]0,1[$. Thus, $\mathrm{Fr}(A_3)$ is the boundary of the rectangle, i.e.,
$$\mathrm{Fr}(A_3) = \{-2,1\} \times [0,1] \cup [-2,1] \times \{0,1\}.$$

Exercise 13. Let $\mathcal{T} = \{]-\alpha,\alpha[\mid \alpha \in [0,+\infty]\}$, where we adopt the conventions $]-\infty,\infty[= \mathbb{R}$ and $]-0,0[= \emptyset$.

1. Show that $\mathcal{T}$ is a topology on $\mathbb{R}$.
2. Determine the closure, interior, and boundary of a singleton $\{a\}$ and of a closed interval $[a,b]$ (with $a \leq b$).

Solution 13. 1. We verify the three axioms of a topology.

- $\emptyset =]-0,0[\in \mathcal{T}$ and $\mathbb{R} =]-\infty,\infty[\in \mathcal{T}$.
- Let $U =]-\alpha,\alpha[$ and $V =]-\beta,\beta[$ be two elements of $\mathcal{T}$. Their intersection is
$$U \cap V =]-\min(\alpha,\beta),\min(\alpha,\beta)[\in \mathcal{T}.$$
By induction, $\mathcal{T}$ is closed under finite intersections.
- Let $\{]-\alpha_i,\alpha_i[\}_{i\in I} \subseteq \mathcal{T}$ be an arbitrary family. Their union is
$$\bigcup_{i\in I}]-\alpha_i,\alpha_i[=]-\sup_{i\in I}\alpha_i, \sup_{i\in I}\alpha_i[\in \mathcal{T}.$$

Therefore, $\mathcal{T}$ is a topology on $\mathbb{R}$. This is known as the *symmetric interval topology.*

2. First, we describe the open and closed sets in this topology.

- **Open sets:** By definition, the non-empty proper open sets are symmetric intervals $]-\alpha,\alpha[$ with $\alpha \in (0,\infty)$.
- **Closed sets:** The complements of open sets are the closed sets. They are $\mathbb{R}$, $\emptyset$, and the sets of the form $]-\infty,-\alpha] \cup [\alpha,\infty[$ for $\alpha \in [0,\infty)$.

Now we analyze the requested sets.

The singleton $\{a\}$ (with $a \neq 0$):

- *Interior:* The only open set contained in $\{a\}$ is $\emptyset$, so $\mathring{\{a\}} = \emptyset$.

- *Closure:* The closure $\overline{\{a\}}$ is the smallest closed set containing a. A closed set $]-\infty,-\alpha]\cup[\alpha,\infty[$ contains a if and only if $|a|\geq\alpha$. The smallest such closed set (for the inclusion order) is obtained when $\alpha=|a|$, so

$$\overline{\{a\}}=]-\infty,-|a|]\cup[|a|,\infty[.$$

- *Boundary:* The boundary is $\mathrm{Fr}(\{a\})=\overline{\{a\}}\setminus\overset{\circ}{\{a\}}=\overline{\{a\}}$.

The singleton $\{0\}$:

- *Interior:* $\overset{\circ}{\{0\}}=\emptyset$, for the same reason as above.
- *Closure:* Every non-empty open set contains 0, so 0 is in the closure of every non-empty set. The smallest closed set containing 0 is the whole space $\mathbb{R}$ (since all other closed sets are of the form $]-\infty,-\alpha]\cup[\alpha,\infty[$ which do not contain 0). Thus, $\overline{\{0\}}=\mathbb{R}$.
- *Boundary:* $\mathrm{Fr}(\{0\})=\mathbb{R}$.

The closed interval $[a,b]$ (with $a\leq b$): We consider cases based on whether the interval contains 0.

Case 1: $0\in[a,b]$ (i.e., $a\leq 0\leq b$).

- *Interior:* The interior is the largest open set contained in $[a,b]$. The open sets are symmetric intervals around 0, so the largest one contained in $[a,b]$ is $]-\alpha,\alpha[$ where $\alpha=\min(|a|,b)$. Thus,

$$\overset{\circ}{[a,b]}=]-\min(|a|,b),\min(|a|,b)[.$$

- *Closure:* Since $[a,b]$ contains 0, its closure must be the entire space $\mathbb{R}$. This is because every non-empty open set (which all contain 0) intersects $[a,b]$, so every point of $\mathbb{R}$ is adherent to $[a,b]$. Thus, $\overline{[a,b]}=\mathbb{R}$.
- *Boundary:* $\mathrm{Fr}([a,b])=\mathbb{R}\setminus]-\min(|a|,b),\min(|a|,b)[$.

Case 2: $0\notin[a,b]$ (so either $a>0$ or $b<0$). By symmetry, assume $a>0$.

- *Interior:* No non-empty symmetric open set $]-\alpha,\alpha[$ can be contained in $[a,b]\subset]0,\infty[$ because $]-\alpha,\alpha[$ always contains negative numbers. Thus, $\overset{\circ}{[a,b]}=\emptyset$.
- *Closure:* The closure is the smallest closed set containing $[a,b]$. The closed sets are $]-\infty,-\alpha]\cup[\alpha,\infty[$. To contain $[a,b]\subset]0,\infty[$, we need $\alpha\leq a$. The smallest (for inclusion) such closed set is obtained for the largest possible α, which is $\alpha=a$. Therefore,

$$\overline{[a,b]}=]-\infty,-a]\cup[a,\infty[.$$

(If $b<0$, the result is $\overline{[a,b]}=]-\infty,b]\cup[-b,\infty[$).
- *Boundary:* Since the interior is empty, the boundary is the closure itself.

Exercise 14. Let $(E,\mathcal{T})$ be a topological space and let $A,B\subseteq E$.

1. Show that $\mathrm{Fr}(A\cup B)\subseteq\mathrm{Fr}(A)\cup\mathrm{Fr}(B)$, and that equality holds if A and B are disjoint. Give an example in $\mathbb{R}$ (with its standard topology) where the inclusion is strict.

2. Show that $\mathrm{Fr}(\overline{A})\subseteq\mathrm{Fr}(A)$ and $\mathrm{Fr}(\mathring{A})\subseteq\mathrm{Fr}(A)$.

Solution 14. We recall that the boundary of a set S is $\mathrm{Fr}(S) = \overline{S} \setminus \mathring{S}$.

1. **Proof of inclusion.** Let $x \in \mathrm{Fr}(A \cup B)$. Then $x \in \overline{A \cup B} = \overline{A} \cup \overline{B}$, so $x \in \overline{A}$ or $x \in \overline{B}$.

   Suppose $x \notin \mathrm{Fr}(A) \cup \mathrm{Fr}(B)$. Then x must be an interior point of both A and B (since it is in their closures but not on their boundaries). This implies $x \in \mathring{A} \cap \mathring{B} \subseteq \overset{\circ}{\overline{A \cup B}}$. But then $x \notin \mathrm{Fr}(A \cup B)$, a contradiction. Therefore, $x \in \mathrm{Fr}(A) \cup \mathrm{Fr}(B)$.

   Equality for disjoint sets. Now assume $A \cap B = \emptyset$. We already have $\mathrm{Fr}(A \cup B) \subseteq \mathrm{Fr}(A) \cup \mathrm{Fr}(B)$. For the reverse inclusion, let $x \in \mathrm{Fr}(A)$ (the argument for $\mathrm{Fr}(B)$ is identical).

   Since $x \in \overline{A} \subseteq \overline{A \cup B}$, we need to show $x \notin A \overset{\circ}{\cup} B$. Suppose for contradiction that $x \in A \overset{\circ}{\cup} B$. Then there is an open neighborhood U of x such that $U \subseteq A \cup B$. Since $x \in \mathrm{Fr}(A)$, every neighborhood of x intersects A^c, so $U \cap A^c \neq \emptyset$, which implies $U \cap B \neq \emptyset$ (because $U \subseteq A \cup B$). Thus, $x \in \overline{B}$. But since A and B are disjoint and $x \in \overline{A}$, if x were also in B, it would be a limit point of A inside B, which is allowed. However, because $x \in \mathrm{Fr}(A)$, it is not in $\mathring{A}$. The key is that U cannot be contained in A, but our assumption that $U \subseteq A \cup B$ with $U \cap B \neq \emptyset$ doesn't immediately give a contradiction.

   A cleaner argument: For disjoint A, B, we have $A \overset{\circ}{\cup} B = \mathring{A} \cup \mathring{B}$ (this is a standard result). Then

   $$\mathrm{Fr}(A \cup B) = \overline{A \cup B} \setminus A \overset{\circ}{\cup} B = (\overline{A} \cup \overline{B}) \setminus (\mathring{A} \cup \mathring{B}) = (\overline{A} \setminus \mathring{A}) \cup (\overline{B} \setminus \mathring{B}) = \mathrm{Fr}(A) \cup \mathrm{Fr}(B).$$

   Counterexample for strict inclusion. Let $A = [0, 1]$ and $B = [1, 2]$ in $\mathbb{R}$. Then $A \cup B = [0, 2]$.

   $$\mathrm{Fr}(A \cup B) = \{0, 2\}, \quad \mathrm{Fr}(A) = \{0, 1\}, \quad \mathrm{Fr}(B) = \{1, 2\}.$$

   Thus, $\mathrm{Fr}(A) \cup \mathrm{Fr}(B) = \{0, 1, 2\} \supsetneq \{0, 2\} = \mathrm{Fr}(A \cup B)$.

2. The original statement "$\mathrm{Fr}(A) \subseteq \mathrm{Fr}(A)$" is a tautology and likely contains a typo. The intended and correct statements are the ones given in the exercise above.

   Proof that $\mathrm{Fr}(\overline{A}) \subseteq \mathrm{Fr}(A)$**.** We have $\overline{\overline{A}} = \overline{A}$. Also, $\mathring{\overline{A}} \supseteq \mathring{A}$. Therefore,

   $$\mathrm{Fr}(\overline{A}) = \overline{A} \setminus \mathring{\overline{A}} \subseteq \overline{A} \setminus \mathring{A} = \mathrm{Fr}(A).$$

   Proof that $\mathrm{Fr}(\mathring{A}) \subseteq \mathrm{Fr}(A)$**.** We have $\overline{\mathring{A}} \subseteq \overline{A}$. Also, $\mathring{\mathring{A}} = \mathring{A}$. Therefore,

   $$\mathrm{Fr}(\mathring{A}) = \overline{\mathring{A}} \setminus \mathring{A} \subseteq \overline{A} \setminus \mathring{A} = \mathrm{Fr}(A).$$

Exercise 15. Let $(E, \mathcal{T})$ be a topological space.

1. Show that $A \subseteq B \implies \overline{A} \subseteq \overline{B}$ for any subsets $A, B \subseteq E$.

2. Deduce that $\bigcup_{i \in I} \overline{A_i} \subseteq \overline{\bigcup_{i \in I} A_i}$, and show that equality holds when the index set I is finite.

3. Give an example in $\mathbb{R}$ (with its standard topology) where this inclusion is strict.

4. Show that $\overline{\bigcap_{i \in I} A_i} \subseteq \bigcap_{i \in I} \overline{A_i}$ and give an example in $\mathbb{R}$ where the inclusion $\overline{A \cap B} \subseteq \overline{A} \cap \overline{B}$ is strict.

5. State the corresponding results for interiors.

Solution 15. 1. Suppose $A \subseteq B$. The closure $\overline{B}$ is a closed set containing B, and hence contains A. Since $\overline{A}$ is the smallest closed set containing A, it must be that $\overline{A} \subseteq \overline{B}$.

2. For each $j \in I$, $A_j \subseteq \bigcup_{i\in I} A_i$, so by part 1, $\overline{A_j} \subseteq \overline{\bigcup_{i\in I} A_i}$. Taking the union over all $j \in I$ gives the inclusion $\bigcup_{i\in I} \overline{A_i} \subseteq \overline{\bigcup_{i\in I} A_i}$.

   Now assume $I = \{1, \ldots, n\}$ is finite. We will show the reverse inclusion $\overline{\bigcup_{k=1}^n A_k} \subseteq \bigcup_{k=1}^n \overline{A_k}$. The right-hand side is a finite union of closed sets, hence closed. It also contains $\bigcup_{k=1}^n A_k$. Since the closure $\overline{\bigcup_{k=1}^n A_k}$ is the smallest closed set with this property, the inclusion follows.

3. Let $A_n = \left\{\frac{1}{n}\right\}$ for $n \in \mathbb{N}^*$. Then $\overline{A_n} = A_n$, so

$$\bigcup_{n=1}^{\infty} \overline{A_n} = \left\{1, \frac{1}{2}, \frac{1}{3}, \ldots\right\}.$$

   However, $\bigcup_{n=1}^{\infty} A_n = \left\{1, \frac{1}{2}, \frac{1}{3}, \ldots\right\}$, and its closure is

$$\overline{\bigcup_{n=1}^{\infty} A_n} = \left\{1, \frac{1}{2}, \frac{1}{3}, \ldots\right\} \cup \{0\}.$$

   Thus, the inclusion is strict.

4. Let $x \in \overline{\bigcap_{i\in I} A_i}$. Then every neighborhood of x intersects $\bigcap_{i\in I} A_i$, and therefore intersects every A_i. This means $x \in \overline{A_i}$ for all $i \in I$, so $x \in \bigcap_{i\in I} \overline{A_i}$.

   For a strict example, let $A =]0,1[$ and $B =]1,2[$ in $\mathbb{R}$. Then $A \cap B = \emptyset$, so $\overline{A \cap B} = \emptyset$. However, $\overline{A} = [0,1]$ and $\overline{B} = [1,2]$, so $\overline{A} \cap \overline{B} = \{1\} \neq \emptyset$.

5. The corresponding results for interiors (denoted by $\mathring{\cdot}$) are:

   - $A \subseteq B \implies \mathring{A} \subseteq \mathring{B}$.
   - $\overset{\circ}{\overline{\bigcap_{i\in I} A_i}}$ — i.e. $\overset{\circ}{\bigcap_{i\in I} A_i} = \bigcap_{i\in I} \mathring{A_i}$.
   - $\overset{\circ}{\bigcup_{i\in I} A_i} \supseteq \bigcup_{i\in I} \mathring{A_i}$, with equality if I is finite.

   These follow from the duality between interior and closure (via complements) or can be proven directly using the definition of interior as the largest open set contained in a given set.

1.6 Dense Subsets

Definition 1.29. Let $(E, \mathcal{T})$ be a topological space and let $D \subseteq E$. The set D is said to be *dense* in E if its closure is the entire space, i.e., $\overline{D} = E$.

The following proposition provides a very useful characterization of dense sets.

Proposition 1.30. A subset $D \subseteq E$ is dense in E if and only if every nonempty open set in E intersects D.

Proof. ($\Rightarrow$) Suppose D is dense, so $\overline{D} = E$. Let O be a nonempty open set. If $O \cap D = \emptyset$, then $O \subseteq E \setminus D$. Since O is open, its complement $E \setminus O$ is closed and contains D. Therefore, $\overline{D} \subseteq E \setminus O$, which implies $E \subseteq E \setminus O$, a contradiction since O is nonempty.

($\Leftarrow$) Conversely, assume every nonempty open set meets D. Suppose $\overline{D} \neq E$. Then the set $O = E \setminus \overline{D}$ is a nonempty open set (as the complement of a proper closed set) that is disjoint from D, contradicting our hypothesis. Hence, $\overline{D} = E$. ∎

Theorem 1.31.

1. The set of rational numbers $\mathbb{Q}$ is dense in $\mathbb{R}$ (i.e., $\overline{\mathbb{Q}} = \mathbb{R}$). Every nonempty open interval in $\mathbb{R}$ contains infinitely many rational numbers.
2. The set of irrational numbers $\mathbb{R} \setminus \mathbb{Q}$ is dense in $\mathbb{R}$ (i.e., $\overline{\mathbb{R} \setminus \mathbb{Q}} = \mathbb{R}$). Every nonempty open interval in $\mathbb{R}$ contains infinitely many irrational numbers.

Proof. Let $I \subseteq \mathbb{R}$ be a nonempty open interval. Then I contains an open interval of the form $]x, y[$ with $x, y \in \mathbb{R}$. We may therefore assume $I =]x, y[$.

1. I contains a rational number. We prove that for all $x, y \in \mathbb{R}$ with $x < y$, there exists a rational number $r \in \mathbb{Q}$ such that $x < r < y$.

By the Archimedean property, there exists $q \in \mathbb{N}^*$ such that $q > 1/(y-x)$, so that $q(y-x) > 1$. The interval $]qx, qy[$ has length greater than 1, so it must contain an integer p. Therefore, $qx < p < qy$, which implies $x < p/q < y$. The number $r = p/q$ is the desired rational.

2. I contains an irrational number. Consider the real numbers $x - \sqrt{2}$ and $y - \sqrt{2}$. Since $x < y$, we have $x - \sqrt{2} < y - \sqrt{2}$. By part 1, there exists a rational number $r \in \mathbb{Q}$ such that

$$x - \sqrt{2} < r < y - \sqrt{2}.$$

Adding $\sqrt{2}$ to all parts of the inequality yields $x < r + \sqrt{2} < y$. The number $\alpha = r + \sqrt{2}$ is irrational, for if it were rational, then $\sqrt{2} = \alpha - r$ would be rational as the difference of two rationals, which is a contradiction. Thus, I contains the irrational number α. ∎

1.6.1 Exercises

Exercise 16. Consider $\mathbb{R}$ equipped with its standard topology. Determine the interior and the closure of $\mathbb{Q}$ and $\mathbb{R} \setminus \mathbb{Q}$.

Solution 16. We will use the fundamental fact that both $\mathbb{Q}$ and $\mathbb{R} \setminus \mathbb{Q}$ are dense in $\mathbb{R}$.

For $\mathbb{Q}$:

- *Closure:* Since $\mathbb{Q}$ is dense in $\mathbb{R}$, its closure is the entire space:

$$\overline{\mathbb{Q}} = \mathbb{R}.$$

- *Interior:* We claim $\mathring{\mathbb{Q}} = \emptyset$. To see this, suppose for contradiction that there exists a nonempty open set $U \subseteq \mathbb{Q}$. In the standard topology on $\mathbb{R}$, every nonempty open set contains an open interval $]a, b[$. However, any such interval is uncountable, while $\mathbb{Q}$ is countable, so $]a, b[\not\subseteq \mathbb{Q}$. Therefore, no nonempty open set is contained in $\mathbb{Q}$, and its interior is empty.

For $\mathbb{R} \setminus \mathbb{Q}$:

- *Closure:* The set of irrationals is also dense in $\mathbb{R}$. Hence,

$$\overline{\mathbb{R} \setminus \mathbb{Q}} = \mathbb{R}.$$

- *Interior:* The argument is analogous to that for $\mathbb{Q}$. Any nonempty open interval $]a, b[$ in $\mathbb{R}$ contains rational numbers (since $\mathbb{Q}$ is dense). Therefore, no nonempty open interval can be contained in $\mathbb{R} \setminus \mathbb{Q}$, which implies that its interior is also empty:

$$\mathbb{R} \overset{\circ}{\setminus} \mathbb{Q} = \emptyset.$$

In summary:

$$\overline{\mathbb{Q}} = \mathbb{R}, \quad \mathring{\mathbb{Q}} = \emptyset, \qquad \overline{\mathbb{R} \setminus \mathbb{Q}} = \mathbb{R}, \quad \mathbb{R} \overset{\circ}{\setminus} \mathbb{Q} = \emptyset.$$

Exercise 17. Show that a subset A of a topological space $(E, \mathcal{T})$ has nonempty interior if and only if it intersects every dense subset of E.

Solution 17. We will prove both implications.

($\Rightarrow$) Assume that $\mathring{A} \neq \emptyset$. Let $D \subseteq E$ be an arbitrary dense subset, so that $\overline{D} = E$.

Since $\mathring{A}$ is a nonempty open set and D is dense, Proposition 1.30 tells us that D must intersect every nonempty open set. In particular, $D \cap \mathring{A} \neq \emptyset$. Since $\mathring{A} \subseteq A$, it follows that $D \cap A \neq \emptyset$.

($\Leftarrow$) We prove the contrapositive. Suppose that $\mathring{A} = \emptyset$. We will construct a dense subset D of E that is disjoint from A.

Consider the complement of A, $D := E \setminus A$. We claim that D is dense in E. To see this, note that the closure of D is

$$\overline{D} = \overline{E \setminus A} = E \setminus \mathring{A}.$$

Since $\mathring{A} = \emptyset$, we have $\overline{D} = E \setminus \emptyset = E$. Therefore, D is dense. By construction, $D \cap A = (E \setminus A) \cap A = \emptyset$.

Thus, if A has empty interior, there exists a dense set (namely, its complement) that it does not meet. This completes the proof of the contrapositive and hence the entire statement.

Exercise 18. Let $(E, \mathcal{T})$ be a topological space, and let A and B be two dense subsets of E. Suppose further that A is open in E. Show that $A \cap B$ is dense in E. (You may use Proposition 1.30: a set is dense if and only if it intersects every nonempty open set).

Solution 18. We will use the characterization of dense sets from Proposition 1.30: a subset $S \subseteq E$ is dense if and only if $S \cap U \neq \emptyset$ for every nonempty open set $U \subseteq E$.

Let U be an arbitrary nonempty open subset of E. We must show that $(A \cap B) \cap U \neq \emptyset$.

Since A is open and U is open, their intersection $A \cap U$ is also open. Moreover, because A is dense in E, it intersects every nonempty open set. In particular, $A \cap U \neq \emptyset$.

Now, $A \cap U$ is a nonempty open set, and B is a dense subset of E. By the same characterization, B must intersect every nonempty open set, so

$$B \cap (A \cap U) \neq \emptyset.$$

This is equivalent to $(A \cap B) \cap U \neq \emptyset$.

Since U was an arbitrary nonempty open set, we conclude that $A \cap B$ intersects every nonempty open set in E. Therefore, by Proposition 1.30, $A \cap B$ is dense in E.

1.7 Hausdorff Spaces

Definition 1.32. A topological space $(E, \mathcal{T})$ is said to be *Hausdorff* (or *separated*) if for every pair of distinct points $x, y \in E$, there exist neighborhoods $V \in \mathcal{V}(x)$ and $W \in \mathcal{V}(y)$ such that $V \cap W = \emptyset$.

Example 1.33.
1. If E is equipped with the discrete topology, then for any distinct $x, y \in E$, the singletons $\{x\}$ and $\{y\}$ are disjoint open sets. Hence, E is Hausdorff.

2. If E is equipped with the trivial topology and contains at least two distinct points, then E is not Hausdorff. Indeed, for any $x \in E$, the only neighborhood of x is E itself, so any two neighborhoods of distinct points must intersect.

3. The set of real numbers $\mathbb{R}$ with its standard topology is a Hausdorff space. Indeed, for any distinct $x, y \in \mathbb{R}$, let $r = |x - y|/3 > 0$. The open intervals $]x - r, x + r[$ and $]y - r, y + r[$ are disjoint neighborhoods of x and y, respectively.

Proposition 1.34. If $(E, \mathcal{T})$ is a Hausdorff space, then for every point $\ell \in E$, we have

$$\bigcap_{V \in \mathcal{V}(\ell)} V = \{\ell\}.$$

Proof. It is clear that $\ell \in V$ for every $V \in \mathcal{V}(\ell)$, so $\ell \in \bigcap_{V \in \mathcal{V}(\ell)} V$.

Conversely, let $y \in E \setminus \{\ell\}$. Since E is Hausdorff, there exists a neighborhood $V \in \mathcal{V}(\ell)$ such that $y \notin V$. Therefore, $y \notin \bigcap_{V \in \mathcal{V}(\ell)} V$.

This shows that $\bigcap_{V \in \mathcal{V}(\ell)} V \subseteq \{\ell\}$, completing the proof. ■

1.7.1 Exercises

Exercise 19. Let $E :=]0, +\infty[$. For every $\alpha \in \mathbb{R}$, define $O_\alpha :=]\alpha, +\infty[$. Consider the family $\mathcal{T}$ of subsets of E given by

$$\mathcal{T} = \{\emptyset, E\} \cup \{O_\alpha \cap E \mid \alpha \in \mathbb{R}\}.$$

1. Show that $\mathcal{T}$ constitutes a topology on E.

2. Determine the closed sets of the topological space $(E, \mathcal{T})$.

3. Give (without proof) the interior $\mathring{A}$ and the closure $\overline{A}$ for the following sets:

$$A = [4, 9[, \quad A = [-2, +\infty[, \quad A = \{2, 3, 4\}, \quad A = \mathbb{N}.$$

   (Note: All sets are considered as subsets of $E =]0, +\infty[$).

4. Show that $(E, \mathcal{T})$ is not a Hausdorff space.

Solution 19. First, we note that the open sets in $\mathcal{T}$ are precisely $\emptyset$, E, and all intervals of the form $]\alpha, +\infty[$ where $\alpha \geq 0$. (If $\alpha < 0$, then $O_\alpha \cap E =]0, +\infty[= E$).

1. We verify the axioms of a topology.

   - $\emptyset, E \in \mathcal{T}$ by definition.
   - Let $\{U_i\}_{i \in I} \subseteq \mathcal{T}$ be an arbitrary collection of open sets. Each U_i is either $\emptyset$, E, or $]\alpha_i, +\infty[$ for some $\alpha_i \geq 0$. Their union is $]\inf_{i \in I} \alpha_i, +\infty[$ (with the convention that $\inf\{\text{set containing } E\} = 0$), which is in $\mathcal{T}$.
   - Let $U_1 =]\alpha, +\infty[$ and $U_2 =]\beta, +\infty[$ be two non-trivial open sets. Their intersection is $]\max(\alpha, \beta), +\infty[\in \mathcal{T}$. The intersection with $\emptyset$ or E is trivial.

   Thus, $\mathcal{T}$ is a topology on E, known as the *right-ray topology*.

2. The closed sets are the complements of the open sets. They are:

$$E, \quad \emptyset, \quad \text{and all sets of the form }]0, \alpha] \text{ for } \alpha \in [0, +\infty[.$$

3. We analyze each set as a subset of $E =]0, +\infty[$.

   - $A = [4, 9[$: The largest open set contained in A is empty, so $\mathring{A} = \emptyset$. The smallest closed set containing A is $]0, 9]$, so $\overline{A} =]0, 9]$.
   - $A = [-2, +\infty[$: Since we are in E, this is just E itself. So $\mathring{A} = E$ and $\overline{A} = E$.
   - $A = \{2, 3, 4\}$: No non-empty open set is contained in a finite set, so $\mathring{A} = \emptyset$. The smallest closed set containing A is $]0, 4]$, so $\overline{A} =]0, 4]$.
   - $A = \mathbb{N}$: Again, $\mathring{A} = \emptyset$. Since $\mathbb{N}$ is unbounded above in E, the only closed set containing it is E itself. Thus, $\overline{A} = E$.

4. To show $(E, \mathcal{T})$ is not Hausdorff, take any two distinct points $x, y \in E$ with $x < y$. Every non-empty open neighborhood of x is of the form $]a, +\infty[$ with $a < x$, and every non-empty open neighborhood of y is of the form $]b, +\infty[$ with $b < y$. Since $x < y$, we can choose $a = b = x/2$, but more importantly, *any* neighborhood of x is $]a, +\infty[$ with $a < x < y$, which necessarily contains y. Similarly, any neighborhood of y is $]b, +\infty[$ with $b < y$, which contains x if $b < x$.

   In fact, for any two points $x < y$, every neighborhood of x contains y. Therefore, there do not exist disjoint neighborhoods of x and y, so the space is not Hausdorff.

1.8 Induced and Product Topologies

1.8.1 Induced Topology

Definition 1.35. Let $(E, \mathcal{T})$ be a topological space and let $A \subseteq E$. The family $\mathcal{T}_A$ of subsets of A defined by

$$\mathcal{T}_A = \{U \subseteq A \mid \exists O \in \mathcal{T} \text{ such that } U = O \cap A\}$$

is a topology on A, called the *subspace topology* (or *induced topology*) on A. The topological space $(A, \mathcal{T}_A)$ is called a *subspace* of E.

Proof. We verify the axioms of a topology for $(A, \mathcal{T}_A)$.

1. Since $\emptyset, E \in \mathcal{T}$, we have $\emptyset = \emptyset \cap A \in \mathcal{T}_A$ and $A = E \cap A \in \mathcal{T}_A$.

2. Let $U_1, U_2 \in \mathcal{T}_A$. Then there exist $O_1, O_2 \in \mathcal{T}$ such that $U_i = O_i \cap A$ for $i = 1, 2$. Their intersection is

$$U_1 \cap U_2 = (O_1 \cap O_2) \cap A.$$

   Since $\mathcal{T}$ is a topology, $O_1 \cap O_2 \in \mathcal{T}$, so $U_1 \cap U_2 \in \mathcal{T}_A$.

3. Let $\{U_i\}_{i \in I} \subseteq \mathcal{T}_A$. For each i, there is an $O_i \in \mathcal{T}$ with $U_i = O_i \cap A$. Their union is

$$\bigcup_{i \in I} U_i = \left(\bigcup_{i \in I} O_i \right) \cap A.$$

   Since $\bigcup_{i \in I} O_i \in \mathcal{T}$, the union is in $\mathcal{T}_A$.

■

Proposition 1.36. Let $(E, \mathcal{T})$ be a topological space and $A \subseteq E$.

1. The closed sets of the subspace $(A, \mathcal{T}_A)$ are precisely the sets of the form $F \cap A$, where F is a closed set in E.
2. The neighborhoods of a point $a \in A$ in the subspace topology are precisely the sets of the form $V \cap A$, where V is a neighborhood of a in E.

Example 1.37. Consider the interval $[0, 2]$ as a subspace of $\mathbb{R}$ with its standard topology. The set $[0, 1[$ is open in $[0, 2]$ because $[0, 1[=]-1, 1[\cap [0, 2]$ and $]-1, 1[$ is open in $\mathbb{R}$. Similarly, $[0, 1[$ is closed in $[-1, 1]$ because $[0, 1[= [0, 3] \cap [-1, 1]$ and $[0, 3]$ is closed in $\mathbb{R}$. However, $[0, 1[$ is neither open nor closed in $\mathbb{R}$ itself.

Proposition 1.38 (Transitivity of the Induced Topology)**.** Let $(E, \mathcal{T})$ be a topological space, and let $B \subseteq A \subseteq E$. The subspace topology induced on B directly from E is the same as the topology obtained by first inducing the topology on A from E and then inducing the topology on B from A.

1.8.2 Product Topology

Let $(E_1, \mathcal{T}_1)$ and $(E_2, \mathcal{T}_2)$ be two topological spaces.

Definition 1.39. An *elementary open set* in the Cartesian product $E_1 \times E_2$ is a set of the form $U_1 \times U_2$, where $U_1 \in \mathcal{T}_1$ and $U_2 \in \mathcal{T}_2$. The *product topology* on $E_1 \times E_2$ is the topology whose open sets are arbitrary unions of elementary open sets.

Remark 1.40. This defines a topology because:

- $E_1 \times E_2$ and $\emptyset = \emptyset \times \emptyset$ are elementary open sets.
- The intersection of two elementary open sets is elementary: $(U_1 \times U_2) \cap (V_1 \times V_2) = (U_1 \cap V_1) \times (U_2 \cap V_2)$.
- Arbitrary unions of unions of elementary sets are again unions of elementary sets.

Definition 1.41. The *standard topology* on $\mathbb{R}^n$ is the product topology obtained by taking the product of n copies of $\mathbb{R}$ with its standard topology.

Example 1.42. The standard topology on $\mathbb{R}^2$ has a basis consisting of open rectangles, i.e., sets of the form $]a, b[\times]c, d[$.

Proposition 1.43 (Transitivity of the Induced Topology)**.** Let $(E, \mathcal{T})$ be a topological space, and let $B \subseteq A \subseteq E$ be subsets of E. The topology induced on B directly by $\mathcal{T}$ is identical to the topology obtained by first inducing the topology on A from $\mathcal{T}$ and then inducing the topology on B from A.

Proof. Let $\mathcal{T}_B^E$ denote the topology on B induced directly from E, and let $\mathcal{T}_B^A$ denote the topology on B induced from the subspace A (which itself has the topology $\mathcal{T}_A$ induced from E).

By definition,

$$\mathcal{T}_B^E = \{U \subseteq B \mid \exists O \in \mathcal{T} \text{ such that } U = O \cap B\}.$$

For $\mathcal{T}_B^A$, a set $U \subseteq B$ is open if and only if there exists an open set V in the subspace A such that $U = V \cap B$. But V is open in A if and only if there exists $O \in \mathcal{T}$ such that $V = O \cap A$. Therefore,

$$U = V \cap B = (O \cap A) \cap B = O \cap (A \cap B) = O \cap B,$$

since $B \subseteq A$. This shows that $\mathcal{T}_B^A = \mathcal{T}_B^E$. ■

1.8.3 Product Topology

Let $(E_1, \mathcal{T}_1)$ and $(E_2, \mathcal{T}_2)$ be two topological spaces.

Definition 1.44. An *elementary open set* in the Cartesian product $E_1 \times E_2$ is a subset $W \subseteq E_1 \times E_2$ of the form $W = O_1 \times O_2$, where $O_1 \in \mathcal{T}_1$ and $O_2 \in \mathcal{T}_2$. The *product topology* on $E_1 \times E_2$ is the topology whose open sets are arbitrary unions of elementary open sets.

Remark 1.45. This indeed defines a topology because:

- $E_1 \times E_2$ and $\emptyset = \emptyset \times \emptyset$ are elementary open sets.
- The intersection of two elementary open sets is elementary:

$$(O_1 \times O_2) \cap (O_1' \times O_2') = (O_1 \cap O_1') \times (O_2 \cap O_2').$$

- Arbitrary unions of unions of elementary sets are unions of elementary sets.

Definition 1.46. The *standard topology* on $\mathbb{R}^n$ is the topology obtained by taking the n-fold product of $\mathbb{R}$ with its standard topology.

Example 1.47. The standard topology on $\mathbb{R}^2$ has a basis consisting of open rectangles, i.e., sets of the form $]a, b[\times]c, d[$.

Proposition 1.48. Let $x = (x_1, x_2) \in E_1 \times E_2$.

1. The collection $\{V_1 \times V_2 \mid V_1 \in \mathcal{V}(x_1),\ V_2 \in \mathcal{V}(x_2)\}$ is a neighborhood basis for x in $E_1 \times E_2$.
2. More generally, if $\mathcal{B}(x_1)$ (resp. $\mathcal{B}(x_2)$) is a neighborhood basis for x_1 in E_1 (resp. for x_2 in E_2), then the collection $\{B_1 \times B_2 \mid B_1 \in \mathcal{B}(x_1),\ B_2 \in \mathcal{B}(x_2)\}$ is a neighborhood basis for x in $E_1 \times E_2$.

Proof. We prove the second, more general statement; the first follows as a special case.

Let W be an arbitrary neighborhood of $x = (x_1, x_2)$ in the product space $E_1 \times E_2$. By definition of the product topology, there exists an elementary open set $O_1 \times O_2$ such that

$$x \in O_1 \times O_2 \subseteq W,$$

where O_1 is open in E_1 and O_2 is open in E_2. Since O_1 is an open neighborhood of x_1, there exists $B_1 \in \mathcal{B}(x_1)$ such that $B_1 \subseteq O_1$. Similarly, there exists $B_2 \in \mathcal{B}(x_2)$ such that $B_2 \subseteq O_2$.

It follows that $B_1 \times B_2 \subseteq O_1 \times O_2 \subseteq W$, and $B_1 \times B_2$ is an element of the proposed neighborhood basis. This shows that every neighborhood of x contains a set from the collection $\{B_1 \times B_2\}$, which is precisely the definition of a neighborhood basis. ∎

Example 1.49. Let $\mathbb{R}^2$ be equipped with its standard topology, and let $x = (x_1, x_2) \in \mathbb{R}^2$. The family

$$\left\{]x_1 - \varepsilon, x_1 + \varepsilon[\times]x_2 - \varepsilon, x_2 + \varepsilon[\;\middle|\; \varepsilon \in \mathbb{R}_+^* \right\}$$

is a fundamental system of neighborhoods (neighborhood basis) for x.

Proposition 1.50. If E_1 and E_2 are Hausdorff spaces, then their product $E_1 \times E_2$ (with the product topology) is also Hausdorff.

Proof. Let (x_1, x_2) and (y_1, y_2) be two distinct points in $E_1 \times E_2$. Then either $x_1 \neq y_1$ or $x_2 \neq y_2$ (or both).

Suppose $x_1 \neq y_1$. Since E_1 is Hausdorff, there exist disjoint open sets $U_1, V_1 \subseteq E_1$ such that $x_1 \in U_1$ and $y_1 \in V_1$. The sets $U_1 \times E_2$ and $V_1 \times E_2$ are open in the product topology, contain (x_1, x_2) and (y_1, y_2) respectively, and are disjoint because $U_1 \cap V_1 = \emptyset$.

The case $x_2 \neq y_2$ is analogous. Therefore, $E_1 \times E_2$ is Hausdorff. ∎

Example 1.51. By induction and Proposition 1.50, the standard topology on $\mathbb{R}^n$ for any $n \in \mathbb{N}^*$ is Hausdorff.

1.8.4 Exercises

Exercise 20. Let $(E_1, \mathcal{T}_1)$ and $(E_2, \mathcal{T}_2)$ be topological spaces, and let $E = E_1 \times E_2$ be their product space. Show that E is Hausdorff if and only if both E_1 and E_2 are Hausdorff.

Solution 20. $(\Leftarrow)$ This is precisely the statement of Proposition 1.50.

$(\Rightarrow)$ Suppose $E_1 \times E_2$ is Hausdorff. We will show E_1 is Hausdorff; the proof for E_2 is identical.

Let $x_1, y_1 \in E_1$ with $x_1 \neq y_1$. Fix an arbitrary point $z_2 \in E_2$. Then (x_1, z_2) and (y_1, z_2) are distinct points in $E_1 \times E_2$. Since the product is Hausdorff, there exist disjoint open sets $W, W' \subseteq E_1 \times E_2$ such that $(x_1, z_2) \in W$ and $(y_1, z_2) \in W'$.

By the definition of the product topology, there exist elementary open sets $U_1 \times U_2 \subseteq W$ and $V_1 \times V_2 \subseteq W'$ containing (x_1, z_2) and (y_1, z_2) respectively. Therefore, $x_1 \in U_1$, $y_1 \in V_1$, and $(U_1 \times U_2) \cap (V_1 \times V_2) = \emptyset$, which implies $U_1 \cap V_1 = \emptyset$. Thus, U_1 and V_1 are disjoint open neighborhoods of x_1 and y_1 in E_1, so E_1 is Hausdorff.

Exercise 21. Closure, Interior, and Boundary of a Product Let E_1 and E_2 be topological spaces, let $A \subseteq E_1$, $B \subseteq E_2$, and let $C = A \times B$ be a subset of the product space $E = E_1 \times E_2$. Show that:

1. $\overline{C} = \overline{A} \times \overline{B}$,
2. $\mathring{C} = \mathring{A} \times \mathring{B}$,
3. $\mathrm{Fr}(C) = \big(\mathrm{Fr}(A) \times \overline{B}\big) \cup \big(\overline{A} \times \mathrm{Fr}(B)\big)$.

Solution 21.

1. **Closure.** $(\supseteq)$ Let $(x, y) \in \overline{A} \times \overline{B}$. Let W be an arbitrary open neighborhood of (x, y) in $E_1 \times E_2$. Then W contains an elementary open set $U \times V$ with $x \in U$ and $y \in V$. Since $x \in \overline{A}$, $U \cap A \neq \emptyset$; since $y \in \overline{B}$, $V \cap B \neq \emptyset$. Therefore, $(U \times V) \cap (A \times B) \neq \emptyset$, so $W \cap C \neq \emptyset$. Hence, $(x, y) \in \overline{C}$.

   $(\subseteq)$ Let $(x, y) \in \overline{C}$. Suppose $x \notin \overline{A}$. Then there exists an open set $U \subseteq E_1$ with $x \in U$ and $U \cap A = \emptyset$. Then $U \times E_2$ is an open neighborhood of (x, y) disjoint from C, contradicting $(x, y) \in \overline{C}$. Thus, $x \in \overline{A}$. Similarly, $y \in \overline{B}$. Therefore, $\overline{C} \subseteq \overline{A} \times \overline{B}$.

2. **Interior.** $(\subseteq)$ $\mathring{A} \times \mathring{B}$ is an open set (as a product of open sets) contained in $A \times B = C$. Therefore, $\mathring{A} \times \mathring{B} \subseteq \mathring{C}$.

   $(\supseteq)$ Let $(x, y) \in \mathring{C}$. Then there exists an elementary open set $U \times V \subseteq C = A \times B$ with $x \in U$ and $y \in V$. This implies $U \subseteq A$ and $V \subseteq B$, so $x \in \mathring{A}$ and $y \in \mathring{B}$. Hence, $(x, y) \in \mathring{A} \times \mathring{B}$.

3. **Boundary.** By definition, $\mathrm{Fr}(C) = \overline{C} \setminus \mathring{C}$. Using parts 1 and 2, this is

   $$(\overline{A} \times \overline{B}) \setminus (\mathring{A} \times \mathring{B}).$$

   A point (x, y) is in this set if and only if ($x \in \overline{A}$ and $y \in \overline{B}$) and *not* ($x \in \mathring{A}$ and $y \in \mathring{B}$). By De Morgan's law, this is equivalent to:

   $$(x \in \overline{A} \text{ and } y \in \overline{B} \text{ and } x \notin \mathring{A}) \quad \text{or} \quad (x \in \overline{A} \text{ and } y \in \overline{B} \text{ and } y \notin \mathring{B}).$$

   This is precisely

   $$(\mathrm{Fr}(A) \times \overline{B}) \cup (\overline{A} \times \mathrm{Fr}(B)),$$

   since $\mathrm{Fr}(A) = \overline{A} \setminus \mathring{A}$ and $\mathrm{Fr}(B) = \overline{B} \setminus \mathring{B}$.

1.9 Convergent Sequences

Throughout this section, $(E, \mathcal{T})$ denotes a topological space.

Definition 1.52. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence of elements of E and let $\ell \in E$. We say that the sequence (x_n) *converges* to ℓ (or *tends* to ℓ) as $n \to +\infty$ if

$$\forall V \in \mathcal{V}(\ell),\ \exists n_0 = n_0(V) \in \mathbb{N} \text{ such that } \forall n \in \mathbb{N},\ n \geq n_0 \implies x_n \in V.$$

If (x_n) converges to ℓ, we say that ℓ is the *limit* of the sequence (x_n).

Remark 1.53. 1. A sequence in a general topological space may have multiple limits. For example, if $\mathcal{T}$ is the trivial topology on E, then every point of E is a limit of every sequence in E.

2. The convergent sequences in a discrete space are precisely the eventually constant (or *stationary*) sequences.

Theorem 1.54. In a Hausdorff space, the limit of any convergent sequence is unique.

Proof. Suppose, for contradiction, that a sequence $(x_n)_{n\in\mathbb{N}}$ in a Hausdorff space $(E, \mathcal{T})$ has two distinct limits ℓ_1 and ℓ_2. Since E is Hausdorff, there exist disjoint neighborhoods $V \in \mathcal{V}(\ell_1)$ and $W \in \mathcal{V}(\ell_2)$ such that $V \cap W = \emptyset$.

Because ℓ_1 is a limit, there exists $n_1 \in \mathbb{N}$ such that $x_n \in V$ for all $n \geq n_1$. Similarly, because ℓ_2 is a limit, there exists $n_2 \in \mathbb{N}$ such that $x_n \in W$ for all $n \geq n_2$. Let $n = \max(n_1, n_2)$. Then $x_n \in V \cap W$, which contradicts $V \cap W = \emptyset$. Therefore, the limit must be unique. ■

Definition 1.55. A sequence $(x_n)_{n\in\mathbb{N}}$ in a topological space E is said to be *convergent* (in E) if there exists $\ell \in E$ such that $x_n \to \ell$ as $n \to +\infty$. Such a point ℓ is called a *limit point* of the sequence (x_n) in E.

Remark 1.56. If $(y_n)_{n\in\mathbb{N}}$ is a sequence in a subset $A \subseteq E$ and (y_n) converges to ℓ in E, then necessarily $\ell \in \overline{A}$.

Definition 1.57. Let $u\colon \mathbb{N} \to E$ be a sequence in a topological space E. A *subsequence* (or *extracted sequence*) of u is a sequence of the form $u \circ \varphi$, where $\varphi\colon \mathbb{N} \to \mathbb{N}$ is a strictly increasing function.

Proposition 1.58. If a sequence $(x_n)_{n\in\mathbb{N}}$ in a topological space E converges to a limit ℓ, then every subsequence of (x_n) also converges to ℓ.

Proof. Let $(x_{n_k})_{k\in\mathbb{N}}$ be a subsequence of (x_n), so that the index map $k \mapsto n_k$ is strictly increasing. Let $V \in \mathcal{V}(\ell)$ be an arbitrary neighborhood of ℓ. Since $x_n \to \ell$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$, $x_n \in V$.

Because the sequence (n_k) is strictly increasing and unbounded, there exists $K \in \mathbb{N}$ such that for all $k \geq K$, $n_k \geq N$. It follows that for all $k \geq K$, $x_{n_k} \in V$. This proves that the subsequence $x_{n_k} \to \ell$. ■

1.9.1 Adherence Values of a Sequence

Definition 1.59. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in a topological space E. A point $\ell \in E$ is an *adherence value* (or *cluster point*) of the sequence if

$$\forall V \in \mathcal{V}(\ell),\ \forall N \in \mathbb{N},\ \exists n \geq N \text{ such that } x_n \in V.$$

In other words, every neighborhood of ℓ contains infinitely many terms of the sequence.

Every limit of a convergent sequence is an adherence value, but the converse is false in general.

Example 1.60. In $\mathbb{R}$, the sequence $x_n = (-1)^n$ has two adherence values: 1 and -1.

Example 1.61. Consider the sequence defined by $u_0 = 0$, $u_1 = 1$, and $u_n = 2(-1)^n$ for $n \geq 2$. In the space $E = \{0, 1, 2, -2\}$, the adherence values of (u_n) are 2 and -2.

Proposition 1.62. The set of adherence values of a sequence $u\colon \mathbb{N} \to E$ is equal to

$$\bigcap_{N \in \mathbb{N}} \overline{\{u_n \mid n \geq N\}}.$$

Proof. Let $\ell \in E$.

($\subseteq$) Suppose ℓ is an adherence value of u. Let $N \in \mathbb{N}$ be arbitrary. We must show $\ell \in \overline{\{u_n \mid n \geq N\}}$. Let V be a neighborhood of ℓ. By definition of an adherence value, there exists some $n \geq N$ such that $u_n \in V$. This shows that every neighborhood of ℓ intersects the set $\{u_n \mid n \geq N\}$, so ℓ is in its closure. As this holds for all N, ℓ is in the intersection.

($\supseteq$) Conversely, suppose $\ell \in \bigcap_{N \in \mathbb{N}} \overline{\{u_n \mid n \geq N\}}$. Let V be a neighborhood of ℓ and let $N \in \mathbb{N}$ be given. Since ℓ is in the closure of $\{u_n \mid n \geq N\}$, the neighborhood V must intersect this set. Therefore, there exists some $n \geq N$ with $u_n \in V$. This is precisely the condition for ℓ to be an adherence value of the sequence u. ■

Proposition 1.63. If a subsequence $(x_{\varphi(n)})_{n \in \mathbb{N}}$ of $(x_n)_{n \in \mathbb{N}}$ converges to ℓ, then ℓ is an adherence value of (x_n).

Proof. Let V be a neighborhood of ℓ. Since the subsequence converges to ℓ, there exists N such that for all $n \geq N$, $x_{\varphi(n)} \in V$. As φ is strictly increasing, the indices $\varphi(n)$ are all distinct and arbitrarily large. Thus, V contains infinitely many terms of the original sequence, so ℓ is an adherence value. ■

1.9.2 Exercises

Exercise 22. Let $(u_n)_{n \in \mathbb{N}}$ be a bounded sequence of real numbers. Prove that (u_n) converges if and only if it has a unique adherence value.

Solution 22. ($\Rightarrow$) If (u_n) converges, then by the previous proposition, its limit is its only adherence value.

($\Leftarrow$) Suppose (u_n) has a unique adherence value ℓ but does not converge to ℓ. Then there exists $\varepsilon > 0$ such that for infinitely many n, $|u_n - \ell| \geq \varepsilon$. From these terms, we can extract a subsequence (u_{n_k}). This subsequence is bounded, so by the Bolzano–Weierstrass theorem, it has a convergent subsequence, whose limit is an adherence value of (u_n) distinct from ℓ, a contradiction. Hence, (u_n) converges to ℓ.

Exercise 23. Determine the adherence values of the sequence $(u_n)_{n \in \mathbb{N}}$ defined by

$$u_n = (-1)^n + \frac{1}{n+1}.$$

Solution 23. The subsequence of even indices is $u_{2k} = 1 + \frac{1}{2k+1} \to 1$. The subsequence of odd indices is $u_{2k+1} = -1 + \frac{1}{2k+2} \to -1$.

Thus, 1 and -1 are adherence values. Any convergent subsequence must contain infinitely many even or infinitely many odd terms (or both), so its limit must be 1 or -1. Therefore, the set of adherence values is $\{-1, 1\}$.

Exercise 24. Let $(u_n)_{n\in\mathbb{N}}$ be a bounded real sequence such that

$$u_n + \frac{1}{2}u_{2n} \xrightarrow[n\to\infty]{} 0.$$

1. Show that if a is an adherence value of (u_n), then $-2a$ is also an adherence value.
2. Deduce that (u_n) converges.

Solution 24.
1. Let a be an adherence value. Then there is a subsequence $u_{n_k} \to a$. The corresponding subsequence u_{2n_k} is bounded, so it has a convergent subsequence $u_{2n_{k_j}} \to b$. From the given relation,
$$u_{n_{k_j}} + \frac{1}{2}u_{2n_{k_j}} \to 0,$$
we get $a + \frac{1}{2}b = 0$, so $b = -2a$. Thus, $-2a$ is an adherence value.
2. Suppose $a \neq 0$ is an adherence value. Then by (1), so are $-2a, 4a, -8a, \ldots$, i.e., $(-2)^k a$ for all $k \in \mathbb{N}$. But this sequence is unbounded, contradicting the boundedness of (u_n). Therefore, the only possible adherence value is 0. Since the sequence is bounded and has a unique adherence value, it converges to 0.

Exercise 25.

1. What are the adherence values of the sequences $(-1)^n$ and $\cos(n\pi/3)$?
2. Give an example of a non-convergent sequence that has a unique adherence value.

Solution 25.
1. The sequence $(-1)^n$ takes only the values 1 and -1, and both are limits of subsequences (the even and odd terms), so its adherence values are $\{-1, 1\}$.

   The sequence $\cos(n\pi/3)$ is periodic with period 6, taking the values $\cos(0) = 1$, $\cos(\pi/3) = 1/2$, $\cos(2\pi/3) = -1/2$, $\cos(\pi) = -1$, $\cos(4\pi/3) = -1/2$, $\cos(5\pi/3) = 1/2$. All these values are adherence values, so the set is $\{-1, -1/2, 1/2, 1\}$.
2. Define u_n by $u_{2n} = 1$ and $u_{2n+1} = n$. The sequence is divergent (since the odd terms go to infinity), but the only adherence value is 1, because any convergent subsequence must eventually consist only of even-indexed terms (since the odd terms are unbounded).

Exercise 26. Let $u = (u_n)_{n\in\mathbb{N}}$ be a real sequence such that $\lim_{n\to\infty}(u_{n+1} - u_n) = 0$. Prove that the set $\mathrm{Adh}(u)$ of adherence values of u is an interval.

Solution 26. Let $a, b \in \mathrm{Adh}(u)$ with $a < b$, and let $c \in]a, b[$. We will show $c \in \mathrm{Adh}(u)$.

Let $\varepsilon > 0$. Since $u_{n+1} - u_n \to 0$, there exists N_0 such that $|u_{n+1} - u_n| < \varepsilon$ for all $n \geq N_0$.

Because a is an adherence value, there exists $n_1 \geq N_0$ with $u_{n_1} < c$. Because b is an adherence value, there exists $n_2 > n_1$ with $u_{n_2} > c$.

Consider the first index $p > n_1$ such that $u_p \geq c$. Then $u_{p-1} < c$ and $p - 1 \geq N_0$, so

$$c \leq u_p < u_{p-1} + \varepsilon < c + \varepsilon.$$

Also, $u_p \geq c > c - \varepsilon$. Thus, $u_p \in]c - \varepsilon, c + \varepsilon[$ and $p \geq N_0$.

Since ε and N_0 were arbitrary, c is an adherence value. Hence, $\mathrm{Adh}(u)$ is an interval.

Exercise 27. Determine the set of adherence values, in $\mathbb{R}$, of the sequences u, v, and w defined by:

$$\forall n \in \mathbb{N},\ u_{2n} = (-2)^n,\ u_{2n+1} = \sqrt{2};\quad v_n = e^{-n};\quad w_{2n} = 1,\ w_{2n+1} = n.$$

Solution 27.

- For u_n: The odd subsequence is constant $\sqrt{2}$, so $\sqrt{2}$ is an adherence value. The even subsequence $(-2)^n$ is unbounded and divergent, and has no convergent subsequence. Thus, the only adherence value is $\{\sqrt{2}\}$.
- For $v_n = e^{-n}$: This sequence converges to 0, so its only adherence value is $\{0\}$.
- For w_n: The even subsequence is constant 1, so 1 is an adherence value. The odd subsequence n is unbounded, so it has no adherence values. Any convergent subsequence must be eventually even, so the only adherence value is $\{1\}$.

Exercise 28. Let $f\colon \mathbb{R} \to \mathbb{R}$ be a continuous function, and let $u = (u_n)_{n\in\mathbb{N}}$ be a sequence defined by

$$u_0 \in \mathbb{R}, \quad u_{n+1} = f(u_n) \text{ for all } n \in \mathbb{N}.$$

Suppose that u has a unique adherence value a. Show that u converges to a.

Solution 28. Since f is continuous, if a subsequence $u_{n_k} \to a$, then $u_{n_k+1} = f(u_{n_k}) \to f(a)$. But u_{n_k+1} is also a subsequence of u, so its limit $f(a)$ must be an adherence value of u. By uniqueness, $f(a) = a$, so a is a fixed point.

Now, suppose for contradiction that u does not converge to a. Then there exists $\varepsilon > 0$ and a subsequence u_{m_k} such that $|u_{m_k} - a| \geq \varepsilon$ for all k. This subsequence is not necessarily bounded, but if it is, it has a convergent subsequence whose limit is an adherence value $\neq a$, a contradiction.

If it is unbounded, we can consider the set $K = \{x \in \mathbb{R} \mid |x - a| \geq \varepsilon\}$. The sequence visits K infinitely often. However, because a is the *only* adherence value, for any $M > 0$, the set $\{n \mid |u_n| > M\}$ must be finite (otherwise, $+\infty$ or $-\infty$ would be an adherence value in the extended reals, and we could argue the sequence must accumulate at a finite point or diverge, but the uniqueness of a precludes other finite accumulation points). A more direct argument: the set $\{u_n \mid n \in \mathbb{N}\}$ is relatively compact in $\mathbb{R}$ because its only limit point is a. Thus, the sequence is bounded. By the Bolzano–Weierstrass theorem, the subsequence u_{m_k} has a convergent subsequence, whose limit is an adherence value different from a, a contradiction. Therefore, $u_n \to a$.

1.10 Continuous Functions

Throughout this section, $(E, \mathcal{T})$ and $(E', \mathcal{T}')$ denote topological spaces.

1.10.1 Limits

Definition 1.64. Let $A \subseteq E$ be a nonempty subset, $f\colon A \to E'$ a function, $a \in \overline{A}$, and $b \in E'$. We say that $f(x)$ *tends to* b *as* x *tends to* a (and write $\lim_{x\to a} f(x) = b$) if

$$\forall W \in \mathcal{V}(b),\ \exists V \in \mathcal{V}(a) \text{ such that } f(V \cap A) \subseteq W.$$

Remark 1.65. In the above definition, the families $\mathcal{V}(b)$ and $\mathcal{V}(a)$ can be replaced by any neighborhood bases at b and a, respectively.

Definition 1.66 (Equivalent formulation)**.** Let $f\colon E \to E'$ be a function, and let $a \in E$, $b \in E'$. Then $\lim_{x\to a} f(x) = b$ if and only if for every neighborhood W of b, the preimage $f^{-1}(W)$ is a neighborhood of a.

Theorem 1.67. If E' is a Hausdorff space, then the limit of a function at a point (if it exists) is unique.

Proof. Suppose f has two distinct limits $b \neq b'$ at a. Since E' is Hausdorff, there exist disjoint neighborhoods $W \in \mathcal{V}(b)$ and $W' \in \mathcal{V}(b')$. By the definition of limit, there exist $V \in \mathcal{V}(a)$ and $V' \in \mathcal{V}(a)$ such that $f(V \cap A) \subseteq W$ and $f(V' \cap A) \subseteq W'$. Then $f((V \cap V') \cap A) \subseteq W \cap W' = \emptyset$, so $(V \cap V') \cap A = \emptyset$. But this contradicts $a \in \overline{A}$, since $V \cap V'$ is a neighborhood of a. ∎

Example 1.68. The function $f\colon \mathbb{R}_+^* \to \mathbb{R}$ defined by $f(x) = 1/x$ has no limit at 0 in $\mathbb{R}$. However, if we consider f as a function into the extended real line $\overline{\mathbb{R}}$, then $\lim_{x\to 0^+} f(x) = +\infty$.

1.10.2 Pointwise Continuity

Definition 1.69. A function $f\colon E \to E'$ is *continuous at a point* $a \in E$ if for every neighborhood W of $f(a)$, there exists a neighborhood V of a such that $f(V) \subseteq W$. Equivalently, $\lim_{x\to a} f(x) = f(a)$. The function f is *continuous (on E)* if it is continuous at every point of E.

Example 1.70.

1. Every constant map is continuous.
2. If E has the discrete topology, then every function $f\colon E \to E'$ is continuous.
3. The characteristic function $\mathbf{1}_{\mathbb{Q}}$ is discontinuous at every point of $\mathbb{R}$.
4. The function $\mathbf{1}_{\mathbb{R}_+}$ is continuous at every $x \neq 0$ but discontinuous at 0.

Proposition 1.71. A function $f\colon E \to E'$ is continuous at $a \in E$ if and only if the preimage under f of every neighborhood of $f(a)$ is a neighborhood of a.

Proposition 1.72. Let $f\colon E \to E'$ be a function. The following properties are equivalent:

1. f is continuous.
2. The preimage of every open set in E' is open in E.
3. The preimage of every closed set in E' is closed in E.
4. For every $A \subseteq E$, $f(\overline{A}) \subseteq \overline{f(A)}$.
5. For every $B \subseteq E'$, $\overline{f^{-1}(B)} \subseteq f^{-1}(\overline{B})$.

Proof. The equivalence of (2) and (3) follows by taking complements and using the identity $f^{-1}(E' \setminus B) = E \setminus f^{-1}(B)$.

(1) $\iff$ (2): f is continuous iff for every $a \in E$ and every open neighborhood O of $f(a)$, $f^{-1}(O)$ is a neighborhood of a. This is equivalent to saying that $f^{-1}(O)$ is a neighborhood of each of its points, i.e., $f^{-1}(O)$ is open.

(3) $\implies$ (5): If $F = \overline{B}$ is closed, then $f^{-1}(F)$ is closed and contains $f^{-1}(B)$, so it contains $\overline{f^{-1}(B)}$.

(5) $\implies$ (4): Let $B = f(A)$. Then $A \subseteq f^{-1}(B)$, so $\overline{A} \subseteq \overline{f^{-1}(B)} \subseteq f^{-1}(\overline{B})$, which implies $f(\overline{A}) \subseteq \overline{B} = \overline{f(A)}$.

(4) $\implies$ (3): Let $F \subseteq E'$ be closed and set $A = f^{-1}(F)$. Then $f(A) \subseteq F$, so $\overline{f(A)} \subseteq F$ (since F is closed). By (4), $f(\overline{A}) \subseteq \overline{f(A)} \subseteq F$, so $\overline{A} \subseteq f^{-1}(F) = A$. Thus, A is closed. ∎

Example 1.73. 1. If $f\colon E \to \mathbb{R}$ is continuous, then for any $\alpha, \beta \in \mathbb{R}$:

$$\{x \in E \mid \alpha \leq f(x) \leq \beta\} = f^{-1}([\alpha, \beta]) \text{ is closed,}$$

$$\{x \in E \mid f(x) > \alpha\} = f^{-1}(]\alpha, +\infty[) \text{ is open,}$$
$$\{x \in E \mid f(x) = \alpha\} = f^{-1}(\{\alpha\}) \text{ is closed.}$$

2. For any $a \in \mathbb{R}$, the translation map $t_a\colon \mathbb{R} \to \mathbb{R}$, $t_a(x) = x + a$, is continuous. Indeed, the preimage of an open interval $]\alpha, \beta[$ is $]\alpha - a, \beta - a[$, which is open.

3. The characteristic function $\mathbf{1}_{\mathbb{R}_+}$ is not continuous: the preimage of the open set $]1/2, 3/2[$ is $\mathbb{R}_+$, which is not open in $\mathbb{R}$.

Remark 1.74. The direct image of an open (resp. closed) set under a continuous function is not necessarily open (resp. closed). For example, $\sin(]-10, 13[) = [-1, 1]$.

Definition 1.75. A function is called *open* (resp. *closed*) if the image of every open (resp. closed) set is open (resp. closed).

Example 1.76. Let E_1 and E_2 be topological spaces, and let $E_1 \times E_2$ have the product topology. The projection $\pi_1\colon E_1 \times E_2 \to E_1$, $\pi_1(x_1, x_2) = x_1$, is continuous and open. It is continuous because $\pi_1^{-1}(O) = O \times E_2$ is open for any open $O \subseteq E_1$. It is open because the image of an elementary open set $O_1 \times O_2$ is O_1, and arbitrary unions of open sets are open.

1.10.3 Composition of Continuous Functions

Proposition 1.77. Let $f\colon E \to E'$ and $g\colon E' \to E''$ be functions between topological spaces.

1. If f and g are continuous, then $g \circ f$ is continuous.

2. If f is continuous at $x \in E$ and g is continuous at $f(x) \in E'$, then $g \circ f$ is continuous at x.

Proof. (1) If $O \subseteq E''$ is open, then $(g \circ f)^{-1}(O) = f^{-1}(g^{-1}(O))$ is open in E by the continuity of g and f.

(2) Let V be a neighborhood of $(g \circ f)(x) = g(f(x))$. Since g is continuous at $f(x)$, $g^{-1}(V)$ is a neighborhood of $f(x)$. Since f is continuous at x, $f^{-1}(g^{-1}(V)) = (g \circ f)^{-1}(V)$ is a neighborhood of x. ■

1.11 Homeomorphisms

Homeomorphisms are the isomorphisms of topological spaces; they allow us to identify two topologically equivalent spaces.

Definition 1.78. A *homeomorphism* between two topological spaces $(E, \mathcal{T}_E)$ and $(E', \mathcal{T}_{E'})$ is a bijective continuous function $f\colon E \to E'$ whose inverse f^{-1} is also continuous. Two spaces are said to be *homeomorphic* if there exists a homeomorphism between them.

Remark 1.79.
- The relation "is homeomorphic to" is an equivalence relation.
- The composition of two homeomorphisms is a homeomorphism.

Example 1.80. 1. Translations and homotheties on $\mathbb{R}$ are homeomorphisms. It follows that any two open intervals in $\mathbb{R}$ are homeomorphic. In fact, every open interval is homeomorphic to $\mathbb{R}$ itself.

2. The identity map $\mathrm{Id}\colon (E,\mathcal{T}) \to (E,\mathcal{T}')$ is a homeomorphism if and only if $\mathcal{T} = \mathcal{T}'$.

3. A continuous bijection is not necessarily a homeomorphism. For example, the map $f\colon [0,1[\cup\{2\} \to [0,1]$ defined by $f|_{[0,1[} = \mathrm{Id}$ and $f(2) = 1$ is a continuous bijection, but its inverse is not continuous at 1.

Definition 1.81. A property $\mathcal{P}$ of a topological space is called a *topological property* if it is preserved under homeomorphisms; that is, if a space E has property $\mathcal{P}$, then any space homeomorphic to E also has property $\mathcal{P}$.

Example 1.82. The following are topological properties: being open, closed, or a neighborhood of a point; being Hausdorff; being the closure, interior, or boundary of a set.

1.12 Topology of Metric Spaces

Let E be a non-empty set.

Definition 1.83. A *metric* (or *distance*) on E is a function $d\colon E \times E \to \mathbb{R}_+$ satisfying the following properties for all $x, y, z \in E$:

1. $d(x,x) = 0$.

2. $d(x,y) = d(y,x)$ (symmetry).

3. $d(x,z) \leq d(x,y) + d(y,z)$ (triangle inequality).

4. $d(x,y) = 0 \implies x = y$ (separation axiom).

The number $d(x,y)$ is called the *distance* between x and y. A set E equipped with a metric d is called a *metric space* and is denoted by (E,d).

If a function d satisfies properties (1), (2), and (3) but not necessarily (4), it is called a *pseudo-metric* (or *semi-metric*), and the pair (E,d) is a *pseudo-metric space*.

Example 1.84.

1. **Standard metric on $\mathbb{R}$:** $d(x,y) = |x-y|$.

2. **Standard metric on $\mathbb{C}$:** $d(z_1,z_2) = |z_1 - z_2|$.

3. **Discrete metric on any set E:**

$$d(x,y) = \begin{cases} 0 & \text{if } x = y, \\ 1 & \text{if } x \neq y. \end{cases}$$

Indeed, this function satisfies all the axioms of a metric:

- $d(x,x) = 0$ for all $x \in E$.
- $d(x,y) = d(y,x)$ by symmetry of the definition.
- The triangle inequality holds: for any $x, y, z \in E$, if $x = z$ then $d(x,z) = 0 \leq d(x,y) + d(y,z)$. If $x \neq z$, then either $x \neq y$ or $y \neq z$ (or both), so $d(x,y) + d(y,z) \geq 1 = d(x,z)$.
- $d(x,y) = 0$ implies $x = y$ by definition.

Thus, (E,d) is a metric space.

4. **Euclidean metric on $\mathbb{R}^n$:**

$$d_e(x,y) = \sqrt{\sum_{i=1}^{n} |x_i - y_i|^2}, \quad \text{where } x = (x_1, \ldots, x_n), y = (y_1, \ldots, y_n).$$

5. **Hölder metric of exponent p on $\mathbb{R}^n$ (for $p \geq 1$):**

$$d_p(x,y) = \left(\sum_{i=1}^{n} |x_i - y_i|^p \right)^{1/p}.$$

The triangle inequality follows from Minkowski's inequality. As $p \to \infty$, this converges to the *supremum metric*:

$$d_\infty(x,y) = \max_{1 \leq i \leq n} |x_i - y_i|.$$

6. **Uniform convergence metric on $\mathcal{F}(A,E)$:** Let A be a set and (E,d) a metric space. On the set $\mathcal{F}(A,E)$ of all functions from A to E, the function

$$d_u(f,g) = \sup_{x \in A} \min\{1, d(f(x), g(x))\}$$

defines a metric, called the *uniform convergence metric.*

7. **Metrics on $C([a,b],\mathbb{C})$:** On the space of continuous complex-valued functions on $[a,b]$, the following are metrics:

$$d_1(f,g) = \int_a^b |f(x) - g(x)|\, dx, \quad d_2(f,g) = \left(\int_a^b |f(x) - g(x)|^2\, dx \right)^{1/2}.$$

We verify the metric axioms for both d_1 and d_2.

1. The distance d_1:

- *Non-negativity:* Since $|f(x) - g(x)| \geq 0$ for all x, the integral is non-negative: $d_1(f,g) \geq 0$.
- *Separation:* If $f = g$, then $d_1(f,g) = 0$. Conversely, if $d_1(f,g) = 0$, then $\int_a^b |f(x) - g(x)|\, dx = 0$. Since the integrand is a continuous non-negative function, this implies $|f(x) - g(x)| = 0$ for all $x \in [a,b]$, so $f = g$.
- *Symmetry:* $|f(x) - g(x)| = |g(x) - f(x)|$, so $d_1(f,g) = d_1(g,f)$.
- *Triangle inequality:* For any $f, g, h \in E$, we have the pointwise inequality

$$|f(x) - h(x)| \leq |f(x) - g(x)| + |g(x) - h(x)|.$$

  Integrating both sides over $[a,b]$ yields

$$d_1(f,h) \leq d_1(f,g) + d_1(g,h).$$

Thus, d_1 is a metric on E.

2. The distance d_2:

- *Non-negativity and symmetry* are immediate, as in the case of d_1.
- *Separation:* If $d_2(f,g) = 0$, then $\int_a^b |f(x) - g(x)|^2\, dx = 0$. The integrand is a continuous non-negative function, so it must be identically zero, implying $f = g$.

- *Triangle inequality:* This follows from the *Minkowski inequality* for $p = 2$:

  $$\left(\int_a^b |f(x) - h(x)|^2 \, dx\right)^{1/2} \le \left(\int_a^b |f(x) - g(x)|^2 \, dx\right)^{1/2} + \left(\int_a^b |g(x) - h(x)|^2 \, dx\right)^{1/2}.$$

  A direct proof uses the Cauchy–Schwarz inequality. Let $u = f - g$ and $v = g - h$. Then

  $$\|u + v\|_2^2 = \|u\|_2^2 + 2\,\mathrm{Re}\langle u, v\rangle + \|v\|_2^2 \le \|u\|_2^2 + 2\|u\|_2\|v\|_2 + \|v\|_2^2 = (\|u\|_2 + \|v\|_2)^2,$$

  where $\langle u, v\rangle = \int_a^b u(x)\overline{v(x)}\, dx$ is the standard inner product on $C([a, b], \mathbb{C})$. Taking square roots gives the triangle inequality.

Therefore, d_2 is also a metric on E.

Proposition 1.85. Let E be a non-empty set equipped with a metric d, and let n be a positive integer.

1. (*Generalized triangle inequality*) For all $(x_1, \dots, x_n) \in E^n$,

   $$d(x_1, x_n) \le \sum_{i=1}^{n-1} d(x_i, x_{i+1}).$$

2. (*Reverse triangle inequality*) For all $(x, y, z) \in E^3$,

   $$|d(x, z) - d(y, z)| \le d(x, y).$$

3. For every $\lambda \in \mathbb{R}_+^*$, the function λd is also a metric on E.

Proof. 1. The proof is by induction on n. The case $n = 2$ is trivial, and $n = 3$ is the standard triangle inequality. Assume the property holds for some $n \ge 3$. For a sequence $(x_1, \dots, x_{n+1})$, we have

   $$d(x_1, x_{n+1}) \le d(x_1, x_n) + d(x_n, x_{n+1}) \le \sum_{i=1}^{n-1} d(x_i, x_{i+1}) + d(x_n, x_{n+1}) = \sum_{i=1}^{n} d(x_i, x_{i+1}),$$

   which completes the induction.

2. By the triangle inequality, $d(x, z) \le d(x, y) + d(y, z)$, so $d(x, z) - d(y, z) \le d(x, y)$. Similarly, $d(y, z) \le d(y, x) + d(x, z) = d(x, y) + d(x, z)$, so $d(y, z) - d(x, z) \le d(x, y)$, which is equivalent to $-(d(x, z) - d(y, z)) \le d(x, y)$. Combining these two inequalities yields $|d(x, z) - d(y, z)| \le d(x, y)$.

3. Let $\lambda > 0$. We verify the metric axioms for $d'(x, y) = \lambda d(x, y)$.

   - $d'(x, y) = 0 \iff \lambda d(x, y) = 0 \iff d(x, y) = 0 \iff x = y$.
   - $d'(x, y) = \lambda d(x, y) = \lambda d(y, x) = d'(y, x)$.
   - $d'(x, z) = \lambda d(x, z) \le \lambda(d(x, y) + d(y, z)) = d'(x, y) + d'(y, z)$.

   Thus, λd is a metric.

■

1.12.1 Exercises

Exercise 29.

1. Under what condition on a function $f\colon \mathbb{R} \to \mathbb{R}$ does the mapping $d(x, y) = |f(x) - f(y)|$ define a metric on $\mathbb{R}$?

2. Determine whether the following mappings define metrics on $\mathbb{R}$:

$$\delta_1(x, y) = |\sin x - \sin y|, \quad \delta_2(x, y) = |x^2 - y^2|, \quad \delta_3(x, y) = |x^3 - y^3|, \quad \delta_4(x, y) = \log(1 + |x - y|).$$

3. Prove that the mapping

$$d(x, y) = \left| \frac{x}{1 + |x|} - \frac{y}{1 + |y|} \right|$$

defines a metric on $\mathbb{R}$.

Solution 29. 1. The mapping $d(x, y) = |f(x) - f(y)|$ is a metric if and only if f is *injective*. The metric axioms of non-negativity, symmetry, and the triangle inequality are automatic. The separation axiom $d(x, y) = 0 \iff x = y$ is equivalent to $f(x) = f(y) \iff x = y$, which is injectivity.

2.
   - δ_1: Not a metric, since $f(x) = \sin x$ is not injective ($\sin 0 = \sin \pi$).
   - δ_2: Not a metric, since $f(x) = x^2$ is not injective ($1^2 = (-1)^2$).
   - δ_3: Is a metric, since $f(x) = x^3$ is injective.
   - δ_4: Is a metric. The function $\varphi(t) = \log(1 + t)$ is increasing, $\varphi(0) = 0$, and is subadditive for $t \geq 0$, so $d = \varphi \circ d_{\text{std}}$ is a metric.

3. The function $f(x) = x/(1 + |x|)$ is a continuous, strictly increasing bijection from $\mathbb{R}$ to $(-1, 1)$. Since it is injective, the mapping $d(x, y) = |f(x) - f(y)|$ is a metric.

Exercise 30. Prove that the mapping $d\colon \mathbb{R} \times \mathbb{R} \to \mathbb{R}_+$ defined by

$$d(x, y) = \frac{|x - y|}{1 + |x - y|}$$

is a metric on $\mathbb{R}$.

Solution 30. We verify the four axioms of a metric.

- *Non-negativity:* Since $|x - y| \geq 0$, it follows that $d(x, y) \geq 0$ for all $x, y \in \mathbb{R}$.

- *Separation:* $d(x, y) = 0$ if and only if $\frac{|x-y|}{1+|x-y|} = 0$, which is equivalent to $|x - y| = 0$, i.e., $x = y$.

- *Symmetry:* $d(x, y) = \frac{|x-y|}{1+|x-y|} = \frac{|y-x|}{1+|y-x|} = d(y, x)$.

- *Triangle inequality:* This is the non-trivial part. Let $x, y, z \in \mathbb{R}$. Define the function $f\colon [0, \infty) \to [0, 1)$ by

$$f(t) = \frac{t}{1 + t}.$$

The function f is **increasing** (since $f'(t) = \frac{1}{(1+t)^2} > 0$) and **subadditive**, meaning $f(a+b) \le f(a) + f(b)$ for all $a, b \ge 0$. To see this, note that

$$f(a+b) = \frac{a+b}{1+a+b} = \frac{a}{1+a+b} + \frac{b}{1+a+b} \le \frac{a}{1+a} + \frac{b}{1+b} = f(a) + f(b),$$

since $1+a+b \ge 1+a$ and $1+a+b \ge 1+b$.

Now, by the standard triangle inequality for the absolute value, we have

$$|x-z| \le |x-y| + |y-z|.$$

Since f is increasing,

$$d(x,z) = f(|x-z|) \le f(|x-y| + |y-z|).$$

By the subadditivity of f,

$$f(|x-y| + |y-z|) \le f(|x-y|) + f(|y-z|) = d(x,y) + d(y,z).$$

Combining these, we obtain $d(x,z) \le d(x,y) + d(y,z)$.

Therefore, d is a metric on $\mathbb{R}$.

Exercise 31. Let (E, d) be a metric space. For $\alpha > 0$, define the mapping $d^\alpha \colon E \times E \to \mathbb{R}$ by $d^\alpha(x,y) = (d(x,y))^\alpha$.

1. Show that for any $\alpha \in (0,1]$, the mapping d^α defines a metric on E.
2. Show that there exists a number $\alpha^* = \alpha^*(d) \in [1, \infty]$ such that:
   - If $0 < \alpha < \alpha^*$, then d^α is a metric on E.
   - If $\alpha > \alpha^*$, then d^α is not a metric on E.
3. Determine α^* in the following cases:
   (a) d is the standard metric on $\mathbb{R}$.
   (b) d is the discrete metric on an arbitrary set E.

Solution 31. 1. Let $\varphi(t) = t^\alpha$ for $t \ge 0$. We must show that φ is *subadditive*, i.e., $\varphi(a+b) \le \varphi(a) + \varphi(b)$ for all $a, b \ge 0$.

If $a = 0$ or $b = 0$, the inequality is trivial. Assume $a, b > 0$ and set $t = b/a \ge 0$. The inequality becomes $(1+t)^\alpha \le 1 + t^\alpha$. Define $f(t) = (1+t)^\alpha - (1+t^\alpha)$ for $t \ge 0$. Then $f(0) = 0$, and

$$f'(t) = \alpha\left((1+t)^{\alpha-1} - t^{\alpha-1}\right).$$

Since $\alpha - 1 \le 0$ and $t \le 1+t$, we have $(1+t)^{\alpha-1} \le t^{\alpha-1}$, so $f'(t) \le 0$. Thus, f is non-increasing, and since $f(0) = 0$, we have $f(t) \le 0$ for all $t \ge 0$.

This proves that φ is subadditive. Therefore, for any $x, y, z \in E$,

$$d^\alpha(x,z) = \varphi(d(x,z)) \le \varphi(d(x,y) + d(y,z)) \le \varphi(d(x,y)) + \varphi(d(y,z)) = d^\alpha(x,y) + d^\alpha(y,z),$$

so the triangle inequality holds. The other metric axioms are immediate, so d^α is a metric for all $\alpha \in (0,1]$.

2. Define the set
$$I = \{\alpha > 0 \mid d^\alpha \text{ is a metric on } E\}.$$
From part 1, $(0,1] \subseteq I$, so I is nonempty. We now show that I is an interval. Suppose $\alpha \in I$ and $0 < \beta < \alpha$. Then
$$d^\beta = (d^\alpha)^{\beta/\alpha}.$$
Since $\beta/\alpha \in (0,1]$ and d^α is a metric, part 1 implies that $(d^\alpha)^{\beta/\alpha}$ is also a metric. Hence $\beta \in I$, so I is an interval.

Let $\alpha^* = \sup I$. By the definition of the supremum, if $0 < \alpha < \alpha^*$, then $\alpha \in I$, so d^α is a metric. If $\alpha > \alpha^*$, then $\alpha \notin I$, so d^α is not a metric. This establishes the existence of the desired $\alpha^* \in [1, \infty]$.

3. We now determine α^* for the two specific cases.

   (a) Let d be the standard metric on $\mathbb{R}$, so $d(x,y) = |x-y|$. We claim $\alpha^* = 1$.
   From part 1, d^α is a metric for all $\alpha \in (0,1]$, so $\alpha^* \geq 1$. Now let $\alpha > 1$. Consider the points $x = 0$, $y = 1$, $z = 2$. Then
   $$d^\alpha(x,z) = 2^\alpha > 2 = 1^\alpha + 1^\alpha = d^\alpha(x,y) + d^\alpha(y,z),$$
   so the triangle inequality fails. Thus, $\alpha \notin I$ for all $\alpha > 1$, and therefore $\alpha^* = 1$.

   (b) Let d be the discrete metric on E, defined by $d(x,y) = 0$ if $x = y$ and $d(x,y) = 1$ if $x \neq y$.
   For any $\alpha > 0$, if $x = y$, then $d^\alpha(x,y) = 0^\alpha = 0$. If $x \neq y$, then $d^\alpha(x,y) = 1^\alpha = 1$. Hence $d^\alpha = d$ for all $\alpha > 0$, so $I = (0, \infty)$ and $\alpha^* = \sup I = \infty$.

Exercise 32. Let (E,d) be a metric space, and let $\varphi\colon \mathbb{R}_+ \to \mathbb{R}_+$ be an increasing function such that $\varphi(t) = 0 \iff t = 0$ and $\varphi(a+b) \leq \varphi(a) + \varphi(b)$ for all $a, b \geq 0$.

1. Show that $d' = \varphi \circ d$ is a metric on E.
2. Show that $d_1 = \dfrac{d}{1+d}$ and $d_2 = \log(1+d)$ are metrics on E.

Solution 32. 1. We verify the four axioms of a metric for $d'(x,y) = \varphi(d(x,y))$.

- *Non-negativity:* Since φ maps to $\mathbb{R}_+$, $d'(x,y) \geq 0$ for all $x, y \in E$.
- *Separation:* $d'(x,y) = 0 \iff \varphi(d(x,y)) = 0 \iff d(x,y) = 0 \iff x = y$.
- *Symmetry:* $d'(x,y) = \varphi(d(x,y)) = \varphi(d(y,x)) = d'(y,x)$.
- *Triangle inequality:* Since d is a metric, $d(x,z) \leq d(x,y) + d(y,z)$. As φ is increasing,
  $$\varphi(d(x,z)) \leq \varphi(d(x,y) + d(y,z)).$$
  By the subadditivity of φ,
  $$\varphi(d(x,y) + d(y,z)) \leq \varphi(d(x,y)) + \varphi(d(y,z)).$$
  Combining these, we get $d'(x,z) \leq d'(x,y) + d'(y,z)$.

Thus, d' is a metric on E.

2. We apply the result from part 1 by showing that the corresponding φ functions satisfy the required properties.

- For $d_1 = \dfrac{d}{1+d}$, define $\varphi_1(t) = \dfrac{t}{1+t}$ for $t \geq 0$.
  - $\varphi_1(t) = 0 \iff t = 0$.
  - φ_1 is increasing on $\mathbb{R}_+$ (its derivative $\varphi_1'(t) = \frac{1}{(1+t)^2} > 0$).
  - φ_1 is subadditive: for $a, b \geq 0$,

$$\varphi_1(a+b) = \frac{a+b}{1+a+b} = \frac{a}{1+a+b} + \frac{b}{1+a+b} \leq \frac{a}{1+a} + \frac{b}{1+b} = \varphi_1(a) + \varphi_1(b).$$

  Therefore, $d_1 = \varphi_1 \circ d$ is a metric.

- For $d_2 = \log(1+d)$, define $\varphi_2(t) = \log(1+t)$ for $t \geq 0$.
  - $\varphi_2(t) = 0 \iff t = 0$.
  - φ_2 is increasing on $\mathbb{R}_+$ (its derivative $\varphi_2'(t) = \frac{1}{1+t} > 0$).
  - φ_2 is subadditive: for $a, b \geq 0$,

$$\varphi_2(a+b) = \log(1+a+b) \leq \log(1+a+b+ab) = \log((1+a)(1+b)) = \log(1+a)+\log(1+b),$$

    which implies that

$$\varphi_2(a+b) \leq \varphi_2(a) + \varphi_2(b).$$

  Therefore, $d_2 = \varphi_2 \circ d$ is a metric.

1.12.2 Balls and Spheres

Definition 1.86. Let (E, d) be a metric space, let $x \in E$, and let $r > 0$.

- The set
$$B(x, r) = \{y \in E \mid d(x, y) < r\}$$
is called the *open ball* with center x and radius r.

- The set
$$\overline{B}(x, r) = \{y \in E \mid d(x, y) \leq r\}$$
is called the *closed ball* with center x and radius r.

- The set
$$S(x, r) = \{y \in E \mid d(x, y) = r\}$$
is called the *sphere* with center x and radius r.

Example 1.87. 1. **Standard metric on $\mathbb{R}$.** For the standard metric $d(x, y) = |x - y|$, the open ball is

$$B(x, r) = \{y \in \mathbb{R} \mid |x - y| < r\} =]x - r, x + r[.$$

Conversely, every bounded open interval $]a, b[$ can be written uniquely as an open ball:

$$]a, b[= B\left(\frac{a+b}{2}, \frac{b-a}{2}\right).$$

Thus, the open balls in $\mathbb{R}$ are precisely the bounded open intervals. Similarly, the closed balls are the bounded closed intervals $[a, b]$.

2. **Metric $d(x, y) = |\arctan x - \arctan y|$ on $\mathbb{R}$.** The function $\arctan : \mathbb{R} \to]-\pi/2, \pi/2[$ is a homeomorphism. The open balls in this metric are the preimages under arctan of open intervals in $]-\pi/2, \pi/2[$. They are:

$$]a, b[, \quad]a, +\infty[, \quad]-\infty, b[, \quad \text{and } \mathbb{R},$$

for $a, b \in \mathbb{R}$. Note that the unbounded intervals (e.g., $]a, +\infty[$) can be expressed as open balls, but the center and radius are not unique.

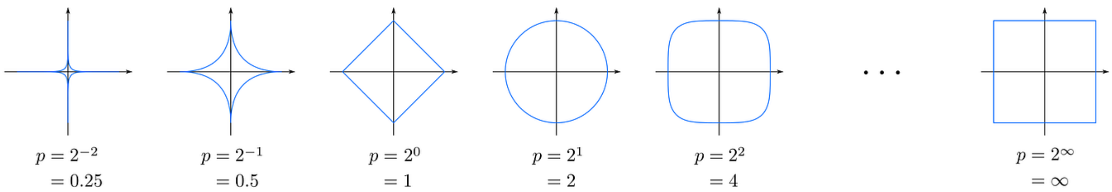
3. **Minkowski metrics on $\mathbb{R}^n$.** For $p \geq 1$, the Minkowski distance of order p between $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ is

$$d_p(x, y) = \left(\sum_{i=1}^{n} |x_i - y_i|^p \right)^{1/p}.$$

Important special cases include:

- *Manhattan distance* ($p = 1$): $d_1(x, y) = \sum_{i=1}^{n} |x_i - y_i|$.
- *Euclidean distance* ($p = 2$): $d_2(x, y) = \sqrt{\sum_{i=1}^{n} |x_i - y_i|^2}$.
- *Chebyshev distance* ($p \to \infty$): $d_\infty(x, y) = \max_{1 \leq i \leq n} |x_i - y_i|$.

The shapes of the open balls $B(0, 1)$ in these metrics are very different: a diamond for $p = 1$, a circle for $p = 2$, and a square for $p = \infty$.



1.12.3 Characterization of Neighborhoods and Open Sets in a Metric Space

Definition 1.88. Let (E, d) be a metric space. A subset $O \subseteq E$ is said to be *open* if for every $x \in O$, there exists a radius $r_x > 0$ such that the open ball $B(x, r_x) \subseteq O$.

Proposition 1.89. Let (E, d) be a metric space.

1. Every open ball in E is an open set.
2. Every closed ball in E is a closed set.

Proof. 1. Let $B(x, r)$ be an open ball, and let $y \in B(x, r)$. Then $d(x, y) < r$. Choose a radius $r_y > 0$ such that $r_y < r - d(x, y)$. For any $z \in B(y, r_y)$, the triangle inequality gives

$$d(z, x) \leq d(z, y) + d(y, x) < r_y + d(x, y) < (r - d(x, y)) + d(x, y) = r.$$

Therefore, $z \in B(x, r)$, so $B(y, r_y) \subseteq B(x, r)$. This shows that $B(x, r)$ is open.

2. Let $\overline{B}(x, r)$ be a closed ball. We will show its complement is open. Let $y \in E \setminus \overline{B}(x, r)$, so $d(x, y) > r$. Define $\delta = d(x, y) - r > 0$. For any $z \in B(y, \delta)$, the reverse triangle inequality gives

$$d(z, x) \geq d(y, x) - d(z, y) > d(x, y) - \delta = r.$$

Hence, $z \in E \setminus \overline{B}(x,r)$, so $B(y,\delta) \subseteq E \setminus \overline{B}(x,r)$. Thus, the complement is open, and $\overline{B}(x,r)$ is closed. ■

Proposition 1.90. Let (E,d) be a metric space, let $x \in E$, and let $V \subseteq E$. Then V is a neighborhood of x if and only if there exists $r > 0$ such that $B(x,r) \subseteq V$.

Proof. ($\Rightarrow$) If V is a neighborhood of x, there exists an open set O such that $x \in O \subseteq V$. By the definition of an open set in a metric space, there exists $r > 0$ such that $B(x,r) \subseteq O \subseteq V$.

($\Leftarrow$) Conversely, if $B(x,r) \subseteq V$ for some $r > 0$, then since $B(x,r)$ is an open set containing x, its superset V is a neighborhood of x by definition. ■

Proposition 1.91. Let (E,d) be a metric space.

1. For every $x \in E$, the countable family $\{B(x,1/n) \mid n \in \mathbb{N}^*\}$ is a neighborhood basis at x.

2. The family $\mathcal{B} = \{B(x,1/n) \mid x \in E, n \in \mathbb{N}^*\}$ is a basis for the topology of (E,d).

Proof.

1. Let V be an arbitrary neighborhood of x. Then there exists $r > 0$ such that $B(x,r) \subseteq V$. By the Archimedean property, there exists $n \in \mathbb{N}^*$ such that $1/n < r$. It follows that $B(x,1/n) \subseteq B(x,r) \subseteq V$, which proves the claim.

2. To show $\mathcal{B}$ is a basis, we must verify that every open set in E is a union of elements of $\mathcal{B}$. Let $O \subseteq E$ be open and let $x \in O$. There exists $r > 0$ such that $B(x,r) \subseteq O$. Choose n such that $1/n < r$. Then $x \in B(x,1/n) \subseteq B(x,r) \subseteq O$. Thus, $O = \bigcup_{x \in O} B(x,1/n_x)$ for suitable n_x, which is a union of elements of $\mathcal{B}$.

■

1.12.4 Distance from a Point to a Set, from a Set to Another, and Diameter

Definition 1.92. Let (E,d) be a metric space, and let $A, B \subseteq E$ be non-empty subsets. For a point $x \in E$, we define:

1. The *distance between the sets* A and B is
$$d(A,B) = \inf\{d(a,b) \mid a \in A,\ b \in B\}.$$

2. The *distance from the point x to the set A* is
$$d(x,A) = \inf\{d(x,a) \mid a \in A\}.$$

3. The *diameter* of A is the element of $\mathbb{R}_+ \cup \{+\infty\}$ defined by
$$\delta(A) = \sup\{d(x,y) \mid x,y \in A\}.$$

   The set A is said to be *bounded* if $\delta(A) < +\infty$. Equivalently, A is bounded if there exists a point $a \in E$ and a radius $r > 0$ such that $A \subseteq B(a,r)$.

Remark 1.93.

1. The "distance" between sets is not a metric on the power set of E. For example, in $\mathbb{R}$ with the standard metric, let $A = \{2\}$ and $B = \{2 + 1/(n+1) \mid n \in \mathbb{N}\}$. Then $d(A,B) = 0$ even though $A \neq B$.

2. The diameter of an open ball is at most twice its radius: $\delta(B(a,r)) \leq 2r$.

3. The distance between sets can be expressed as

$$d(A,B) = \inf_{x\in A} d(x,B) = \inf_{y\in B} d(y,A).$$

Proposition 1.94. Let A and B be subsets of a metric space (E,d). Then:

1. If $A \subseteq B$, then $\delta(A) \leq \delta(B)$.
2. $\delta(A) = \delta(\overline{A})$, where $\overline{A}$ is the closure of A.

Proof.

1. If $A \subseteq B$, then every pair of points in A is also a pair of points in B. Therefore, the set of distances $\{d(x,y) \mid x,y \in A\}$ is a subset of $\{d(x,y) \mid x,y \in B\}$. The supremum of a subset is less than or equal to the supremum of the whole set, so $\delta(A) \leq \delta(B)$.
2. Since $A \subseteq \overline{A}$, we have $\delta(A) \leq \delta(\overline{A})$ by part 1. For the reverse inequality, let $x,y \in \overline{A}$. By definition of the closure, there exist sequences $(x_n),(y_n) \subseteq A$ such that $x_n \to x$ and $y_n \to y$. The metric d is continuous, so $d(x_n,y_n) \to d(x,y)$. Since $d(x_n,y_n) \leq \delta(A)$ for all n, it follows that $d(x,y) \leq \delta(A)$. As this holds for all $x,y \in \overline{A}$, we have $\delta(\overline{A}) \leq \delta(A)$.

■

Definition 1.95. Let Λ be a non-empty set and (E,d) a metric space. A function $f\colon \Lambda \to E$ is said to be *bounded* if its image $f(\Lambda)$ is a bounded subset of E.

1.12.5 Exercises

Exercise 33. Let (E,d) be the metric space $\mathbb{R}^2$ with the Euclidean distance

$$d((x_1,y_1),(x_2,y_2)) = \sqrt{(x_1-x_2)^2 + (y_1-y_2)^2}.$$

Let $\Lambda = \{1,2,3,4\}$ and define $f\colon \Lambda \to E$ by

$$f(1) = (2,3), \quad f(2) = (5,7), \quad f(3) = (-1,-2), \quad f(4) = (0,0).$$

1. Verify that the set $f(\Lambda) = \{(2,3),(5,7),(-1,-2),(0,0)\}$ is bounded in $\mathbb{R}^2$.
2. Calculate the maximum distance between any two points in $f(\Lambda)$.
3. Find a bound M such that for all $x,y \in f(\Lambda)$, $d(x,y) \leq M$.

Solution 33.

1. A finite subset of a metric space is always bounded, since the maximum of a finite number of real numbers is finite.
2. We compute the Euclidean distance for each pair of distinct points:

$$\begin{aligned}
d(f(1),f(2)) &= \sqrt{(5-2)^2+(7-3)^2} = \sqrt{9+16} = \sqrt{25} = 5,\\
d(f(1),f(3)) &= \sqrt{(-1-2)^2+(-2-3)^2} = \sqrt{9+25} = \sqrt{34} \approx 5.83,\\
d(f(1),f(4)) &= \sqrt{(0-2)^2+(0-3)^2} = \sqrt{4+9} = \sqrt{13} \approx 3.61,\\
d(f(2),f(3)) &= \sqrt{(-1-5)^2+(-2-7)^2} = \sqrt{36+81} = \sqrt{117} \approx 10.82,\\
d(f(2),f(4)) &= \sqrt{(0-5)^2+(0-7)^2} = \sqrt{25+49} = \sqrt{74} \approx 8.60,\\
d(f(3),f(4)) &= \sqrt{(0+1)^2+(0+2)^2} = \sqrt{1+4} = \sqrt{5} \approx 2.24.
\end{aligned}$$

The maximum distance is $\sqrt{117}$, between the points $(5,7)$ and $(-1,-2)$.

3. We can take $M = \sqrt{117}$ as a bound. Thus, for all $x, y \in f(\Lambda)$, $d(x, y) \leq \sqrt{117}$.

Exercise 34. Metrics on $\mathbb{R}_+^*$ Let $E = \mathbb{R}_+^* = (0, \infty)$ and define $d\colon E \times E \to \mathbb{R}_+$ by

$$d(x, y) = \left| \frac{1}{x} - \frac{1}{y} \right|.$$

1. Verify that d is a metric on E.
2. Compute the open balls $B_d(1, 1)$ and $B_d\left(\frac{1}{2}, 3\right)$.
3. Determine a general description of all open balls for this metric.
4. Determine whether the segment $(0, 1]$ (viewed as a subset of E) is bounded with respect to this metric. Is it closed?
5. Give an example of a set that is bounded for the metric d but unbounded for the standard metric on $\mathbb{R}$.

Solution 34. 1. The map $\psi\colon E \to \mathbb{R}_+^*$ defined by $\psi(x) = 1/x$ is a bijection. The metric d is the pullback of the standard metric on $\mathbb{R}$ via ψ:

$$d(x, y) = |\psi(x) - \psi(y)|.$$

Since the standard metric is a metric and ψ is a bijection, d is a metric on E.

2. The open ball is defined as $B_d(a, r) = \{x \in E \mid d(x, a) < r\}$.
   - For $B_d(1, 1)$: Solve $\left|\frac{1}{x} - 1\right| < 1$. This gives $0 < \frac{1}{x} < 2$, so $x > \frac{1}{2}$.
     Thus, $B_d(1, 1) = \left(\frac{1}{2}, \infty\right)$.
   - For $B_d\left(\frac{1}{2}, 3\right)$: Solve $\left|\frac{1}{x} - 2\right| < 3$. This gives $-1 < \frac{1}{x} < 5$.
     Since $x > 0$, we have $0 < \frac{1}{x} < 5$, so $x > \frac{1}{5}$. Thus, $B_d\left(\frac{1}{2}, 3\right) = \left(\frac{1}{5}, \infty\right)$.

3. For a fixed center $x_0 \in E$ and radius $r > 0$, the condition $x \in B_d(x_0, r)$ is

$$\left|\frac{1}{x} - \frac{1}{x_0}\right| < r \quad \iff \quad \frac{1}{x_0} - r < \frac{1}{x} < \frac{1}{x_0} + r.$$

We distinguish two cases.

Case 1: $r \geq \frac{1}{x_0}$**.** The left inequality is automatically satisfied since $\frac{1}{x} > 0$. The right inequality gives $\frac{1}{x} < \frac{1}{x_0} + r$, so

$$x > \frac{x_0}{1 + r x_0}.$$

Hence, $B_d(x_0, r) = \left(\frac{x_0}{1+rx_0}, \infty\right)$.

Case 2: $r < \frac{1}{x_0}$**.** Both bounds are positive, and taking reciprocals (which reverses inequalities) yields

$$\frac{x_0}{1 + r x_0} < x < \frac{x_0}{1 - r x_0}.$$

Hence, $B_d(x_0, r) = \left(\frac{x_0}{1+rx_0}, \frac{x_0}{1-rx_0}\right)$.

In summary,

$$B_d(x_0, r) = \begin{cases} \left(\dfrac{x_0}{1 + r x_0}, \infty\right) & \text{if } r \geq \dfrac{1}{x_0}, \\ \left(\dfrac{x_0}{1 + r x_0}, \dfrac{x_0}{1 - r x_0}\right) & \text{if } r < \dfrac{1}{x_0}. \end{cases}$$

4. The set $(0,1]$ is **unbounded** in (E,d). Consider the sequence $x_n = 1/n \in (0,1]$. Then $d(x_n,1) = |n-1| \to \infty$ as $n \to \infty$.
   Closedness : Let (x_n) be a sequence in $A = (0,1]$ that converges to a limit $\ell \in \mathbb{R}_+^*$ with respect to the metric d. By definition of d, this means
   $$d(x_n,\ell) = \left|\frac{1}{x_n} - \frac{1}{\ell}\right| \xrightarrow[n\to\infty]{} 0.$$
   Thus, the sequence $(1/x_n)$ converges to $1/\ell$ in the standard topology of $\mathbb{R}$.

   Since $x_n \in (0,1]$ for all n, it follows that $1/x_n \in [1,\infty)$ for all n. The set $[1,\infty)$ is closed in $\mathbb{R}$ with the standard topology, so its limit $1/\ell$ must also belong to $[1,\infty)$. Therefore, $1/\ell \geq 1$, which implies $\ell \leq 1$. As $\ell \in \mathbb{R}_+^*$, we have $\ell > 0$, so $\ell \in (0,1] = A$.

   This shows that A contains all its limit points in the metric space $(\mathbb{R}_+^*, d)$, and hence A is closed.

5. The set $A = (1,\infty)$ is bounded in (E,d) but unbounded in the standard metric. For any $x,y \in A$, we have
   $$d(x,y) = \left|\frac{1}{x} - \frac{1}{y}\right| < 1,$$
   so $\operatorname{diam}_d(A) \leq 1$. However, A is clearly unbounded in the standard metric on $\mathbb{R}$.

Exercise 35. Let E be a set, and define $d\colon E \times E \to \mathbb{R}$ by

$$d(x,y) = \begin{cases} 0 & \text{if } x = y, \\ 1 & \text{if } x \neq y. \end{cases}$$

1. Show that d is a metric on E.
2. Determine the open ball $B(x,r)$ for $x \in E$ and $r > 0$.
3. Determine the open and closed sets in the metric space (E,d).

Solution 35. 1. We verify the axioms of a metric.

- *Non-negativity:* By definition, $d(x,y) \in \{0,1\}$, so $d(x,y) \geq 0$ for all $x,y \in E$.
- *Identity of indiscernibles:* $d(x,y) = 0$ if and only if $x = y$, which is true by the definition of d.
- *Symmetry:* If $x = y$, then $d(x,y) = d(y,x) = 0$. If $x \neq y$, then $d(x,y) = d(y,x) = 1$. Thus, $d(x,y) = d(y,x)$ for all $x,y \in E$.
- *Triangle inequality:* Let $x,y,z \in E$. If $x = z$, then $d(x,z) = 0 \leq d(x,y) + d(y,z)$. If $x \neq z$, then $d(x,z) = 1$. In this case, it is impossible for both $x = y$ and $y = z$ to hold (as this would imply $x = z$), so at least one of $d(x,y)$ or $d(y,z)$ is equal to 1. Hence, $d(x,y) + d(y,z) \geq 1 = d(x,z)$.

Therefore, d is a metric on E, called the *discrete metric*.

2. The open ball of center $x \in E$ and radius $r > 0$ is defined as
   $$B(x,r) = \{y \in E \mid d(x,y) < r\}.$$
   - If $0 < r \leq 1$, then $d(x,y) < r$ implies $d(x,y) = 0$, so $y = x$. Thus, $B(x,r) = \{x\}$.

- If $r > 1$, then for all $y \in E$, $d(x, y) \leq 1 < r$, so $B(x, r) = E$.

3. Let $A \subseteq E$ be an arbitrary subset. For any $x \in A$, the open ball $B(x, 1/2) = \{x\}$ is contained in A. Therefore, A is a neighborhood of each of its points, which by Theorem 1.8 implies that A is open.

   Since every subset of E is open, the complement of any subset is also open. By definition, a set is closed if its complement is open, so every subset of E is also closed. The topology induced by the discrete metric is the *discrete topology.*

Exercise 36. Let $\mathbb{R}^{\mathbb{N}}$ denote the set of all real sequences. For $u = (u_n)_{n\in\mathbb{N}}$ and $v = (v_n)_{n\in\mathbb{N}}$ in $\mathbb{R}^{\mathbb{N}}$, define

$$d(u, v) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{|u_n - v_n|}{|u_n - v_n| + 1}.$$

1. Show that d is a metric on $\mathbb{R}^{\mathbb{N}}$.
2. Show that the metric space $(\mathbb{R}^{\mathbb{N}}, d)$ is bounded.

Solution 36. 1. We verify the axioms of a metric.

Well-definedness: For each n, the term $\frac{|u_n - v_n|}{|u_n - v_n|+1}$ is in $[0, 1)$. Since $\sum_{n=1}^{\infty} \frac{1}{2^n} = 1$, the series for $d(u, v)$ converges by the comparison test, so d is well-defined and $d(u, v) \in [0, 1)$.

Non-negativity and separation: Each term in the sum is non-negative, so $d(u, v) \geq 0$. Moreover, $d(u, v) = 0$ if and only if every term in the sum is zero, which happens if and only if $|u_n - v_n| = 0$ for all n, i.e., $u = v$.

Symmetry: The expression $|u_n - v_n|$ is symmetric in u and v, so $d(u, v) = d(v, u)$.

Triangle inequality: Consider the function $\varphi\colon [0, \infty) \to [0, 1)$ defined by $\varphi(t) = \frac{t}{t+1}$. This function is increasing and subadditive (as shown in previous exercises). For each n, the standard metric on $\mathbb{R}$ gives $|u_n - w_n| \leq |u_n - v_n| + |v_n - w_n|$. Applying the increasing function φ and its subadditivity, we get

$$\varphi(|u_n - w_n|) \leq \varphi(|u_n - v_n| + |v_n - w_n|) \leq \varphi(|u_n - v_n|) + \varphi(|v_n - w_n|).$$

Multiplying by $1/2^n$ and summing over n yields $d(u, w) \leq d(u, v) + d(v, w)$.

Therefore, d is a metric on $\mathbb{R}^{\mathbb{N}}$.

2. For any $u, v \in \mathbb{R}^{\mathbb{N}}$, we have

$$d(u, v) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{|u_n - v_n|}{|u_n - v_n| + 1} < \sum_{n=1}^{\infty} \frac{1}{2^n} = 1.$$

   Thus, the diameter of the space is at most 1, and the metric space $(\mathbb{R}^{\mathbb{N}}, d)$ is bounded.

Exercise 37. Let (E, d) be a metric space. For two subsets $A, B \subseteq E$, define

$$d(A, B) = \inf\{d(a, b) \mid a \in A,\, b \in B\}, \quad \delta(A) = \sup\{d(a, a') \mid a, a' \in A\}.$$

1. Show that if $A \subseteq B$, then $\delta(A) \leq \delta(B)$.
2. Show that $\delta(A) = \delta(\overline{A})$.
3. Show that $\delta(A \cup B) \leq \delta(A) + \delta(B) + d(A, B)$.

Solution 37. 1. If $A \subseteq B$, then every pair of points in A is also a pair of points in B. Therefore, the set $\{d(a,a') \mid a,a' \in A\}$ is a subset of $\{d(b,b') \mid b,b' \in B\}$. The supremum of a subset cannot exceed the supremum of the whole set, so $\delta(A) \leq \delta(B)$.

2. Since $A \subseteq \overline{A}$, part 1 gives $\delta(A) \leq \delta(\overline{A})$. For the reverse inequality, let $x, y \in \overline{A}$. By the definition of closure, there exist sequences $(x_n), (y_n) \subseteq A$ such that $x_n \to x$ and $y_n \to y$. The metric d is continuous, so $d(x_n, y_n) \to d(x,y)$. Since $d(x_n,y_n) \leq \delta(A)$ for all n, it follows that $d(x,y) \leq \delta(A)$. As this holds for all $x, y \in \overline{A}$, we conclude $\delta(\overline{A}) \leq \delta(A)$. Hence, $\delta(A) = \delta(\overline{A})$.

3. Let $x, y \in A \cup B$. We consider three cases.

   - If $x, y \in A$, then $d(x,y) \leq \delta(A) \leq \delta(A) + \delta(B) + d(A,B)$.
   - If $x, y \in B$, then $d(x,y) \leq \delta(B) \leq \delta(A) + \delta(B) + d(A,B)$.
   - If $x \in A$ and $y \in B$ (or vice versa), then for any $\varepsilon > 0$, there exist $a \in A$ and $b \in B$ such that $d(a,b) < d(A,B) + \varepsilon$. By the triangle inequality,

   $$d(x,y) \leq d(x,a) + d(a,b) + d(b,y) \leq \delta(A) + (d(A,B) + \varepsilon) + \delta(B).$$

     Since $\varepsilon > 0$ is arbitrary, we get $d(x,y) \leq \delta(A) + \delta(B) + d(A,B)$.

   In all cases, the distance between any two points in $A \cup B$ is bounded above by $\delta(A) + \delta(B) + d(A,B)$. Taking the supremum over all such pairs yields the desired inequality.

Exercise 38. Let (E,d) be a metric space and let $A \subseteq E$.

1. For $r > 0$, define the *r-open neighborhood* of A by

   $$V_r(A) = \{x \in E \mid d(x,A) < r\}.$$

   (a) Show that $V_r(A)$ is open in (E,d).

   (b) Show that if A is closed, then $A = \bigcap_{n=1}^{\infty} V_{1/n}(A)$.

2. Let $\widetilde{A} = \bigcap_{n=1}^{\infty} \left\{x \in E \mid d(x,A) < \frac{1}{n}\right\}$.

   (a) Show that $\widetilde{A}$ is closed.

   (b) Show that $\widetilde{A}$ is the smallest closed set containing A (i.e., $\widetilde{A} = \overline{A}$).

3. Show that if (E,d) is complete, then every nonempty closed subset is a countable intersection of open sets (i.e., every closed set is a G_δ set).

4. Let $B \neq E$ be a closed subset of (E,d). Show that B is a countable intersection of open sets (even if E is not complete).

5. Let $\widehat{B} = \bigcup_{n=1}^{\infty} \left\{x \in E \mid d(x,B^c) \geq \frac{1}{n}\right\}$.

   (a) Show that $\widehat{B}$ is open in (E,d).

   (b) Show that $\widehat{B}$ is the largest open set contained in B (i.e., $\widehat{B} = \mathring{B}$).

Solution 38. 1. (a) Let $x \in V_r(A)$, so $d(x,A) < r$. Choose $\varepsilon > 0$ such that $d(x,A) + \varepsilon < r$. For any $y \in B(x,\varepsilon)$, the triangle inequality gives $d(y,A) \leq d(y,x) + d(x,A) < \varepsilon + d(x,A) < r$. Thus, $B(x,\varepsilon) \subseteq V_r(A)$, so $V_r(A)$ is open.

(b) Suppose A is closed. Clearly, $A \subseteq V_{1/n}(A)$ for all n, so $A \subseteq \bigcap_{n=1}^{\infty} V_{1/n}(A)$. For the reverse inclusion, let $x \in \bigcap_{n=1}^{\infty} V_{1/n}(A)$. Then $d(x, A) < 1/n$ for all n, so $d(x, A) = 0$. This means there is a sequence in A converging to x. Since A is closed, $x \in A$. Hence, the equality holds.

2. Note that $\widetilde{A} = \bigcap_{n=1}^{\infty} V_{1/n}(A)$.

   (a) Each $V_{1/n}(A)$ is open by part 1(a), but $\widetilde{A}$ is an intersection of open sets, which is not necessarily open. However, we will show it is closed by proving it equals $\overline{A}$. Alternatively, one can show its complement is open: if $x \notin \widetilde{A}$, then $d(x, A) \geq 1/n$ for some n, so $B(x, 1/(2n))$ is disjoint from $\widetilde{A}$.

   (b) We claim $\widetilde{A} = \overline{A}$. Let $x \in \overline{A}$. Then $d(x, A) = 0$, so $x \in V_{1/n}(A)$ for all n, hence $x \in \widetilde{A}$. Conversely, if $x \in \widetilde{A}$, then $d(x, A) < 1/n$ for all n, so $d(x, A) = 0$, which implies $x \in \overline{A}$. Since $\overline{A}$ is the smallest closed set containing A, the result follows.

3. Let $F \subseteq E$ be closed and nonempty. By part 2, $F = \bigcap_{n=1}^{\infty} V_{1/n}(F)$. Each $V_{1/n}(F)$ is open by part 1(a). This representation holds in any metric space; completeness is not required. (Thus, the hypothesis of completeness is superfluous for this conclusion).

4. The proof is the same as in part 3. For any closed set B in a metric space, $B = \bigcap_{n=1}^{\infty} V_{1/n}(B)$, a countable intersection of open sets. The condition $B \neq E$ ensures the sets $V_{1/n}(B)$ are proper, but the equality holds regardless.

5. (a) Let $x \in \widehat{B}$, so $d(x, B^c) \geq 1/n$ for some n. Then the open ball $B(x, 1/(2n))$ is contained in $\widehat{B}$, because for any y in this ball, $d(y, B^c) \geq d(x, B^c) - d(x, y) > 1/n - 1/(2n) = 1/(2n) > 0$, so $y \in \widehat{B}$. Thus, $\widehat{B}$ is open.

   (b) First, $\widehat{B} \subseteq B$ because if $x \in \widehat{B}$, then $d(x, B^c) > 0$, so $x \notin B^c$, hence $x \in B$. Now let U be any open set with $U \subseteq B$. For any $x \in U$, there exists $\varepsilon > 0$ such that $B(x, \varepsilon) \subseteq B$, which implies $d(x, B^c) \geq \varepsilon$. Choosing n such that $1/n \leq \varepsilon$, we get $x \in \{y \mid d(y, B^c) \geq 1/n\} \subseteq \widehat{B}$. Thus, $U \subseteq \widehat{B}$, so $\widehat{B}$ is the largest open set contained in B, i.e., $\widehat{B} = \mathring{B}$.

Exercise 39. **Part 1:** Consider the metric space $(\mathbb{R}^2, d_2)$, where for all $x = (x_1, x_2)$, $y = (y_1, y_2) \in \mathbb{R}^2$, the Euclidean distance is

$$d_2(x, y) = \left(\sum_{i=1}^{2} (x_i - y_i)^2 \right)^{1/2}.$$

1. Show that the half-plane $F = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_2 \leq 0\}$ is closed in $\mathbb{R}^2$.

2. Show that any straight line in $\mathbb{R}^2$ is closed.

Part 2: Define a new metric δ on $\mathbb{R}^2$ by

$$\delta(x, y) = \max_{1 \leq i \leq 2} |x_i - y_i|.$$

1. Show that δ defines a metric on $\mathbb{R}^2$.

2. Describe the open balls of center $C = (1, 1)$ and radius $r > 0$.

Part 3: Define the function $\rho \colon \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}_+$ by

$$\rho(x, y) = \sum_{i=1}^{2} |x_i - y_i|.$$

1. Show that ρ defines a metric on $\mathbb{R}^2$.

2. Draw the open ball of center $O = (0, 0)$ and radius 2.

Solution 39. Part 1: The Euclidean metric d_2.

1. The complement of F is $F^c = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_2 > 0\}$. Let $p = (p_1, p_2) \in F^c$, so $p_2 > 0$. Consider the open ball $B_{d_2}(p, r)$ with radius $r = p_2/2 > 0$. For any $x = (x_1, x_2) \in B_{d_2}(p, r)$, we have
$$|x_2 - p_2| \leq d_2(x, p) < r = \frac{p_2}{2},$$
which implies $x_2 > p_2 - p_2/2 = p_2/2 > 0$. Thus, $x \in F^c$, so $B_{d_2}(p, r) \subseteq F^c$. Hence, F^c is open, and F is closed.

2. Let $L = \{(x_1, x_2) \in \mathbb{R}^2 \mid ax_1 + bx_2 + c = 0\}$ be a straight line, where $(a, b) \neq (0, 0)$. The distance from a point $p = (p_1, p_2)$ to the line L is given by
$$\text{dist}(p, L) = \frac{|ap_1 + bp_2 + c|}{\sqrt{a^2 + b^2}}.$$
If $p \notin L$, then $\text{dist}(p, L) = d_0 > 0$. The open ball $B_{d_2}(p, d_0/2)$ is disjoint from L, so the complement of L is open. Therefore, L is closed.

Part 2: The Chebyshev metric δ.

1. The function $\delta(x, y) = \max\{|x_1 - y_1|, |x_2 - y_2|\}$ is the ℓ^∞-metric on $\mathbb{R}^2$. It is a standard result that this is a metric: non-negativity, symmetry, and the identity of indiscernibles are immediate. The triangle inequality follows from the fact that the maximum of a sum is less than or equal to the sum of the maxima.

2. The open ball of center $C = (1, 1)$ and radius $r > 0$ is
$$B_\delta(C, r) = \{(x, y) \in \mathbb{R}^2 \mid \max\{|x - 1|, |y - 1|\} < r\} =]1 - r, 1 + r[\times]1 - r, 1 + r[,$$
which is an open square (with sides parallel to the axes) centered at $(1, 1)$.

Part 3: The Manhattan metric ρ.

1. The function $\rho(x, y) = |x_1 - y_1| + |x_2 - y_2|$ is the ℓ^1-metric on $\mathbb{R}^2$. It satisfies all the metric axioms: non-negativity, symmetry, and the identity of indiscernibles are clear. The triangle inequality holds because the absolute value satisfies the triangle inequality, and the sum of inequalities is an inequality.

2. The open ball of center $O = (0, 0)$ and radius 2 is
$$B_\rho(O, 2) = \{(x, y) \in \mathbb{R}^2 \mid |x| + |y| < 2\}.$$
This is a diamond-shaped region (a square rotated by 45°) with vertices at $(2, 0)$, $(0, 2)$, $(-2, 0)$, and $(0, -2)$.

1.12.6 Topology Associated with a Metric

Proposition 1.96. Let (E, d) be a metric space, and let $O \subseteq E$ be a non-empty subset. The following properties are equivalent:

1. O is a union of open balls.
2. For every $x \in O$, there exists $r > 0$ such that $B(x, r) \subseteq O$.

Proof. $(2 \Rightarrow 1)$ For each $x \in O$, choose $r_x > 0$ such that $B(x, r_x) \subseteq O$. Then

$$O = \bigcup_{x \in O} B(x, r_x),$$

since every x is in its own ball, and every ball is contained in O.

$(1 \Rightarrow 2)$ Suppose $O = \bigcup_{i \in I} B(x_i, r_i)$. For any $x \in O$, there exists an index i such that $x \in B(x_i, r_i)$. Let $r = r_i - d(x, x_i) > 0$. By the triangle inequality, $B(x, r) \subseteq B(x_i, r_i) \subseteq O$, which establishes the claim. ■

Theorem 1.97. The collection of all subsets of E satisfying the equivalent properties of the previous proposition forms a topology on E. This topology is called the *topology induced by the metric d*.

Proof. Let $\mathcal{T}_d$ be the collection of all subsets of E that are unions of open balls.

1. *The empty set and the whole space are in $\mathcal{T}_d$.* The empty set is the empty union of balls, so $\emptyset \in \mathcal{T}_d$. The whole space is $E = \bigcup_{x \in E} B(x, 1)$, so $E \in \mathcal{T}_d$.

2. *$\mathcal{T}_d$ is closed under finite intersections.* Let $O_1, \ldots, O_k \in \mathcal{T}_d$ and let $x \in \bigcap_{i=1}^k O_i$. For each i, there exists $r_i > 0$ such that $B(x, r_i) \subseteq O_i$. Let $r = \min\{r_1, \ldots, r_k\} > 0$. Then $B(x, r) \subseteq \bigcap_{i=1}^k O_i$. By the proposition, the intersection is a union of open balls, so it belongs to $\mathcal{T}_d$.

3. *$\mathcal{T}_d$ is closed under arbitrary unions.* Let $\{O_i\}_{i \in I} \subseteq \mathcal{T}_d$. Each O_i is a union of open balls, so their total union $\bigcup_{i \in I} O_i$ is also a union of open balls, hence in $\mathcal{T}_d$.

Therefore, $\mathcal{T}_d$ is a topology on E. ■

Definition 1.98 (Metrizable Spaces). A topological space $(E, \mathcal{T})$ is said to be *metrizable* if there exists a metric d on E such that the topology $\mathcal{T}_d$ induced by d coincides with $\mathcal{T}$.

Example 1.99.

1. **The discrete metric induces the discrete topology.** Let d be the discrete metric on E. For any $x \in E$, the open ball $B(x, 1/2) = \{x\}$. Thus, every singleton is open, and since any subset is a union of singletons, every subset of E is open. Hence, the induced topology is the discrete topology.

2. **The standard metric induces the usual topology on $\mathbb{R}$.** The open balls for the standard metric $d(x, y) = |x - y|$ are the open intervals $]x - r, x + r[$. Since the usual topology on $\mathbb{R}$ is generated by these intervals, the two topologies coincide.

   There are other metrics that induce the same topology. For example, the metric $d(x, y) = |\arctan x - \arctan y|$ is topologically equivalent to the standard metric, as the function $\arctan\colon \mathbb{R} \to]-\pi/2, \pi/2[$ is a homeomorphism.

1.12.7 Lipschitz and Contraction Mappings

Definition 1.100. Let (E, d) and (E', d') be two metric spaces, and let $f\colon E \to E'$ be a function. The map f is said to be *k-Lipschitz* (or *Lipschitz continuous with constant k*) for some real number $k \geq 0$ if

$$\forall x, y \in E, \quad d'(f(x), f(y)) \leq k\, d(x, y).$$

If such a constant k exists, the smallest such constant is called the *Lipschitz constant* of f (or the *Lipschitz ratio*).

- The function f is *bilipschitz* if it is bijective and both f and its inverse f^{-1} are Lipschitz.
- The function f is a *contraction mapping* if it is Lipschitz with a constant $k < 1$.

Example 1.101. Consider the real line $\mathbb{R}$ with its standard metric $d(x, y) = |x-y|$, and define $f\colon \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \frac{2}{7}|x|.$$

For all $x, y \in \mathbb{R}$, we have

$$|f(x) - f(y)| = \frac{2}{7}\Big||x| - |y|\Big| \leq \frac{2}{7}|x - y|,$$

where the inequality follows from the reverse triangle inequality. Thus, f is $\frac{2}{7}$-Lipschitz, and its Lipschitz constant is $k = \frac{2}{7}$.

Definition 1.102 (Isometries)**.** Let (E, d) and (E', d') be metric spaces. A function $f\colon E \to E'$ is an *isometry* if

$$\forall x, y \in E, \quad d'(f(x), f(y)) = d(x, y).$$

In other words, an isometry is a distance-preserving map. Every isometry is injective, 1-Lipschitz, and a homeomorphism onto its image. If it is also surjective, it is a global homeomorphism.

Example 1.103 (Isometry between $\mathbb{R}^2$ and $\mathbb{C}$)**.** Equip $\mathbb{C}$ with the standard metric $d_{\mathbb{C}}(z_1, z_2) = |z_1 - z_2|$, and $\mathbb{R}^2$ with the Euclidean metric $d_2((x_1, y_1), (x_2, y_2)) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$. The map

$$f\colon \mathbb{R}^2 \to \mathbb{C}, \quad f(x, y) = x + iy$$

is an isometry, since

$$d_{\mathbb{C}}(f(x_1, y_1), f(x_2, y_2)) = |(x_1 - x_2) + i(y_1 - y_2)| = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} = d_2((x_1, y_1), (x_2, y_2)).$$

Proposition 1.104. Let (E, d) be a metric space and let $A \subseteq E$ be a non-empty subset. The function

$$\varphi_A\colon E \to \mathbb{R}, \quad \varphi_A(x) = d(x, A) = \inf_{a \in A} d(x, a)$$

is 1-Lipschitz. That is, for all $x, y \in E$,

$$|d(x, A) - d(y, A)| \leq d(x, y).$$

Proof. Fix $x, y \in E$. For any $a \in A$, the triangle inequality gives

$$d(x, a) \leq d(x, y) + d(y, a).$$

Taking the infimum over $a \in A$ on both sides yields

$$d(x, A) \leq d(x, y) + d(y, A).$$

Rearranging, we obtain $d(x, A) - d(y, A) \leq d(x, y)$. By symmetry, interchanging x and y gives $d(y, A) - d(x, A) \leq d(x, y)$. Combining these two inequalities,

$$|d(x, A) - d(y, A)| \leq d(x, y),$$

as required. ■

1.12.8 Topologically Equivalent and Equivalent Metrics

Definition 1.105. Let E be a set, and let d and d' be two metrics on E.

- The metrics d and d' are *topologically equivalent* if they induce the same topology on E, i.e., $\mathcal{T}_d = \mathcal{T}_{d'}$. This means that a subset of E is open with respect to d if and only if it is open with respect to d'.
- The metrics d and d' are (*metrically*) *equivalent* if there exist positive constants $\alpha, \beta > 0$ such that for all $x, y \in E$,

$$\alpha\, d'(x, y) \leq d(x, y) \leq \beta\, d'(x, y).$$

Example 1.106 (Topological equivalence of the Euclidean and Manhattan metrics)**.** On $\mathbb{R}^n$, consider the following two metrics:

- The *Euclidean metric*:

$$d(x, y) = \sqrt{\sum_{i=1}^{n} |x_i - y_i|^2},$$

- The *Manhattan metric* (or ℓ^1-metric):

$$d'(x, y) = \sum_{i=1}^{n} |x_i - y_i|,$$

where $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$.

The open ball of radius $\varepsilon > 0$ centered at a point x is:

- For the Euclidean metric: $B_d(x, \varepsilon) = \{y \in \mathbb{R}^n \mid d(x, y) < \varepsilon\}$, which is a spherical (round) ball.
- For the Manhattan metric: $B_{d'}(x, \varepsilon) = \{y \in \mathbb{R}^n \mid d'(x, y) < \varepsilon\}$, which is a diamond-shaped (polyhedral) ball.

These two metrics are *metrically equivalent*, and hence topologically equivalent. Indeed, for all $x, y \in \mathbb{R}^n$, the following inequalities hold:

$$\frac{1}{\sqrt{n}}\, d'(x, y) \leq d(x, y) \leq d'(x, y).$$

The right-hand inequality follows from $|x_i - y_i| \leq d(x, y)$ and summing over i. The left-hand inequality is a consequence of the Cauchy–Schwarz inequality:

$$d'(x,y) = \sum_{i=1}^{n} |x_i - y_i| \cdot 1 \leq \left(\sum_{i=1}^{n} |x_i - y_i|^2\right)^{1/2} \left(\sum_{i=1}^{n} 1^2\right)^{1/2} = \sqrt{n}\, d(x,y).$$

This mutual bounding by constant multiples implies that every open ball in one metric contains an open ball of the other metric, and vice versa. Consequently, the two metrics induce the exact same topology on $\mathbb{R}^n$.

Remark 1.107.

1. If two metrics are metrically equivalent, then they are topologically equivalent.
2. The converse is false: there exist metrics that are topologically equivalent but not metrically equivalent.

Example 1.108 (Topologically and metrically equivalent metrics on $\mathbb{R}^n$)**.** On $\mathbb{R}^n$, the following metrics are metrically (hence topologically) equivalent:

- *Manhattan metric:* $d_1(x,y) = \sum_{i=1}^{n} |x_i - y_i|$,
- *Euclidean metric:* $d_2(x,y) = \left(\sum_{i=1}^{n} |x_i - y_i|^2\right)^{1/2}$,
- *Chebyshev metric:* $d_\infty(x,y) = \max_{1 \leq i \leq n} |x_i - y_i|$.

They satisfy the inequalities

$$d_\infty(x,y) \leq d_2(x,y) \leq d_1(x,y) \leq \sqrt{n}\, d_2(x,y) \leq n\, d_\infty(x,y),$$

which establish their metric equivalence.

Example 1.109 (Topologically but not metrically equivalent metrics on $\mathbb{R}$)**.** Let $d(x,y) = |x-y|$ be the standard metric on $\mathbb{R}$, and let

$$d'(x,y) = \frac{|x-y|}{1+|x-y|}.$$

The metrics d and d' are topologically equivalent because $d(x_n, x) \to 0$ if and only if $d'(x_n, x) \to 0$. However, they are not metrically equivalent. For metric equivalence, there would need to exist $\beta > 0$ such that $d(x,y) \leq \beta\, d'(x,y)$ for all x, y. But as $|x-y| \to \infty$, $d'(x,y) \to 1$ while $d(x,y) \to \infty$, making this impossible.

Proposition 1.110. Let E be a non-empty set, and let d and d' be two metrics on E.

1. The metrics d and d' are *metrically equivalent* if and only if the identity map $\mathrm{Id}_E \colon (E,d) \to (E,d')$ is bilipschitz.
2. The metrics d and d' are *topologically equivalent* if and only if the identity map $\mathrm{Id}_E \colon (E,d) \to (E,d')$ is a homeomorphism.

Proof. 1. The identity map $\mathrm{Id}_E \colon (E,d) \to (E,d')$ is Lipschitz if and only if there exists a constant $k > 0$ such that

$$d'(x,y) \leq k\, d(x,y) \quad \text{for all } x, y \in E.$$

Its inverse, which is also the identity map $\mathrm{Id}_E \colon (E,d') \to (E,d)$, is Lipschitz if and only if there exists a constant $k' > 0$ such that

$$d(x,y) \leq k'\, d'(x,y) \quad \text{for all } x, y \in E.$$

Combining these two inequalities, we obtain

$$\frac{1}{k'}\, d(x,y) \leq d'(x,y) \leq k\, d(x,y) \quad \text{for all } x,y \in E,$$

which is precisely the definition of metric equivalence. Therefore, Id_E is bilipschitz if and only if d and d' are metrically equivalent.

2. The identity map $\mathrm{Id}_E \colon (E,d) \to (E,d')$ is continuous if and only if the preimage of every d'-open set is d-open. Since the map is the identity, this means that every d'-open set is also d-open, i.e., $\mathcal{T}_{d'} \subseteq \mathcal{T}_d$.

   Similarly, the inverse map $\mathrm{Id}_E \colon (E,d') \to (E,d)$ is continuous if and only if $\mathcal{T}_d \subseteq \mathcal{T}_{d'}$.

   The identity map is a homeomorphism if and only if it is continuous and its inverse is continuous, which is equivalent to $\mathcal{T}_d = \mathcal{T}_{d'}$. This is exactly the definition of topological equivalence.

■

Exercise 40. Let (E,d) be a metric space, and define a new function $d' \colon E \times E \to \mathbb{R}_+$ by

$$d'(x,y) = \frac{d(x,y)}{1+d(x,y)}, \quad \text{for all } x,y \in E.$$

1. Show that d' is a valid metric on E and that it is bounded.
2. Show that d and d' are topologically equivalent.

Solution 40. 1. We verify the axioms of a metric for d'.

- *Non-negativity and separation:* Since $d(x,y) \geq 0$, we have $d'(x,y) \geq 0$. Moreover, $d'(x,y) = 0$ if and only if $d(x,y) = 0$, which is equivalent to $x = y$.
- *Symmetry:* $d'(x,y) = \frac{d(x,y)}{1+d(x,y)} = \frac{d(y,x)}{1+d(y,x)} = d'(y,x)$.
- *Triangle inequality:* Consider the function $\varphi \colon [0,\infty) \to [0,1)$ defined by $\varphi(t) = \frac{t}{1+t}$. This function is increasing and *subadditive*, meaning $\varphi(a+b) \leq \varphi(a) + \varphi(b)$ for all $a,b \geq 0$. Since d is a metric, $d(x,z) \leq d(x,y) + d(y,z)$. As φ is increasing and subadditive, we have

  $$d'(x,z) = \varphi(d(x,z)) \leq \varphi(d(x,y)+d(y,z)) \leq \varphi(d(x,y))+\varphi(d(y,z)) = d'(x,y)+d'(y,z).$$

Therefore, d' is a metric. It is bounded because $d'(x,y) < 1$ for all $x,y \in E$.

2. To prove topological equivalence, we show that the identity map $\mathrm{Id} \colon (E,d) \to (E,d')$ is a homeomorphism. It suffices to show that a sequence (x_n) converges to x in (E,d) if and only if it converges to x in (E,d').

   Since the function $\varphi(t) = t/(1+t)$ is continuous at 0 with $\varphi(0) = 0$, and its inverse $\varphi^{-1}(s) = s/(1-s)$ is also continuous at 0, we have:

   $$d(x_n,x) \xrightarrow[n\to\infty]{} 0 \quad \Longleftrightarrow \quad d'(x_n,x) = \varphi(d(x_n,x)) \xrightarrow[n\to\infty]{} 0.$$

   This shows that the two metrics have the same convergent sequences, and therefore induce the same topology on E.

1.12.9 Sequential Characterizations

Let (E, d) be a metric space. Recall that for any point $x \in E$, the family of open balls $\{B(x, \varepsilon) \mid \varepsilon > 0\}$ forms a neighborhood basis at x. Using this, the definition of the limit of a sequence in a general topological space specializes to the following precise analytical formulation in metric spaces.

Proposition 1.111. Let (E, d) be a metric space and let $(x_n)_{n\in\mathbb{N}}$ be a sequence of elements in E. Then the sequence (x_n) converges to a limit $\ell \in E$ if and only if

$$\forall \varepsilon > 0,\ \exists N \in \mathbb{N} \text{ such that } \forall n \in \mathbb{N},\ n \geq N \implies d(x_n, \ell) < \varepsilon.$$

Proof. ($\Rightarrow$) Assume (x_n) converges to ℓ in the topological sense. Let $\varepsilon > 0$ be given. The open ball $B(\ell, \varepsilon)$ is a neighborhood of ℓ. By the definition of convergence, there exists $N \in \mathbb{N}$ such that for all $n \geq N$, $x_n \in B(\ell, \varepsilon)$, which is equivalent to $d(x_n, \ell) < \varepsilon$.

($\Leftarrow$) Conversely, suppose the ε–N condition holds. Let V be an arbitrary neighborhood of ℓ. By the definition of the metric topology, there exists $r > 0$ such that $B(\ell, r) \subseteq V$. Choose $\varepsilon = r/2 > 0$. By hypothesis, there exists $N \in \mathbb{N}$ such that for all $n \geq N$, $d(x_n, \ell) < \varepsilon$. This implies $x_n \in B(\ell, \varepsilon) \subseteq B(\ell, r) \subseteq V$. Since this holds for every neighborhood V of ℓ, the sequence (x_n) converges to ℓ.

Finally, since every metric space is Hausdorff, the limit of a convergent sequence is unique. ■

1.12.10 Adherent Points and Closed Sets

Definition 1.112. Let (E, d) be a metric space, let $A \subseteq E$, and let $a \in E$. The point a is said to be *adherent* to A if

$$\forall \varepsilon > 0, \quad B(a, \varepsilon) \cap A \neq \emptyset.$$

Equivalently, a is adherent to A if and only if

$$\forall \varepsilon > 0,\ \exists x \in A \text{ such that } d(x, a) < \varepsilon.$$

The set of all points adherent to A is precisely the *closure* of A, denoted $\overline{A}$.

Example 1.113. Consider the set $A =]0, 1[\subset \mathbb{R}$, where $\mathbb{R}$ is equipped with its standard metric. The points 0 and 1 are adherent to A.

Indeed, for any $\varepsilon > 0$, the open ball (interval) $B(0, \varepsilon) =]-\varepsilon, \varepsilon[$ intersects A because, for instance, $\varepsilon/2 \in A \cap B(0, \varepsilon)$. Similarly, $B(1, \varepsilon) =]1 - \varepsilon, 1 + \varepsilon[$ intersects A since $1 - \varepsilon/2 \in A \cap B(1, \varepsilon)$. Hence, both 0 and 1 are adherent points of A, and in fact $\overline{A} = [0, 1]$.

Proposition 1.114. (**Characterization of the Closure**)
Let (E, d) be a metric space, let $A \subseteq E$ be a non-empty subset, and let $x \in E$. The following three statements are equivalent:

1. $x \in \overline{A}$ (the closure of A).
2. $d(x, A) = 0$.
3. There exists a sequence $(a_n)_{n\in\mathbb{N}}$ of elements of A that converges to x.

Proof. We prove the equivalence by showing the implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (1).

(1) $\Rightarrow$ (2): Assume $x \in \overline{A}$. By definition of the closure, for every $\varepsilon > 0$, the open ball $B(x, \varepsilon)$ intersects A. In particular, for every $n \in \mathbb{N}^*$, there exists a point $a_n \in A$ such that

$d(x, a_n) < 1/n$. This implies that $d(x, A) \leq 1/n$ for all n. Taking the limit as $n \to \infty$, we obtain $d(x, A) = 0$.

(2) ⇒ (3): Assume $d(x, A) = 0$. By the definition of the infimum, for every $n \in \mathbb{N}^*$, there exists a point $a_n \in A$ such that $d(x, a_n) < 1/n$. The sequence $(a_n)_{n\in\mathbb{N}}$ is therefore a sequence in A that converges to x.

(3) ⇒ (1): Assume there is a sequence $(a_n) \subseteq A$ such that $a_n \to x$. Let $\varepsilon > 0$ be arbitrary. By the definition of convergence, there exists $N \in \mathbb{N}$ such that for all $n \geq N$, $d(a_n, x) < \varepsilon$. In particular, $a_N \in B(x, \varepsilon) \cap A$, so this intersection is non-empty. Since this holds for every $\varepsilon > 0$, x is adherent to A, which means $x \in \overline{A}$. ■

Remark 1.115. The set of adherent points of a subset $A \subseteq E$ can be partitioned into two distinct types:

1. A point $x \in E$ is an *accumulation point* (or *limit point*) of A if every neighborhood of x contains at least one point of A different from x itself. Equivalently, for all $r > 0$,

$$(B(x, r) \setminus \{x\}) \cap A \neq \emptyset.$$

(Note: Some authors include x itself in the intersection, but the standard definition requires points *other than* x).

2. A point $x \in A$ is an *isolated point* of A if there exists a radius $r > 0$ such that

$$B(x, r) \cap A = \{x\}.$$

That is, x is in A but is not an accumulation point of A.

The closure of A is the disjoint union of its set of accumulation points and its set of isolated points.

Proposition 1.116. (**Sequential Characterization of Closed Sets**)
Let (E, d) be a metric space, and let $A \subseteq E$. The set A is closed if and only if it is *sequentially closed*, that is, for every sequence $(a_n)_{n\in\mathbb{N}}$ of elements of A, if (a_n) converges to a limit $x \in E$, then $x \in A$.

Proof. (⇒) Suppose A is closed, so $A = \overline{A}$. Let $(a_n) \subseteq A$ be a sequence such that $a_n \to x \in E$. By Proposition **??**, $x \in \overline{A}$. Since $A = \overline{A}$, it follows that $x \in A$.

(⇐) Conversely, assume that A is sequentially closed. We will show that $A = \overline{A}$. The inclusion $A \subseteq \overline{A}$ is always true. To prove the reverse inclusion, let $x \in \overline{A}$. By Proposition **??**, there exists a sequence $(a_n) \subseteq A$ such that $a_n \to x$. By the sequential closedness of A, we must have $x \in A$. Thus, $\overline{A} \subseteq A$, and so $A = \overline{A}$, which means A is closed. ■

Exercise 41. (**Distance to a Closed Set**)
Let F be a closed subset of a metric space (E, d). Assume that $d(x, F) = 0$. Prove that $x \in F$.

Solution 41. Since $d(x, F) = 0$, by the definition of the infimum, for every $\varepsilon > 0$ there exists a point $y \in F$ such that $d(x, y) < \varepsilon$.

In particular, for each $n \in \mathbb{N}^*$, choose a point $x_n \in F$ such that

$$d(x, x_n) < \frac{1}{n}.$$

The sequence $(x_n)_{n\in\mathbb{N}}$ is then a sequence of elements in F that converges to x, since $d(x, x_n) \to 0$ as $n \to \infty$.

Because F is closed, it contains the limit of every convergent sequence of its points. Therefore, $x \in F$.

1.12.11 Uniform Continuity

Definition 1.117. Let (E, d) and (E', δ) be metric spaces, and let $f: E \to E'$ be a function.

1. The function f is *continuous at a point* $x_0 \in E$ if

$$\forall \varepsilon > 0,\ \exists \eta > 0,\ \forall x \in E,\ d(x, x_0) < \eta \implies \delta(f(x), f(x_0)) < \varepsilon.$$

2. The function f is *uniformly continuous on* E if

$$\forall \varepsilon > 0,\ \exists \eta > 0,\ \forall x, x' \in E,\ d(x, x') < \eta \implies \delta(f(x), f(x')) < \varepsilon.$$

Remark 1.118. Every uniformly continuous function is continuous at every point of its domain. However, the converse is not true in general. A classic example is the function $f: \mathbb{R} \to \mathbb{R}$, $f(x) = x^2$, which is continuous everywhere but not uniformly continuous on $\mathbb{R}$. In fact, among real polynomial functions, only the affine functions (i.e., functions of the form $f(x) = ax + b$) are uniformly continuous on the entire real line.

Example 1.119. (Uniform Continuity and Lipschitz Maps)

1. The function $f: \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2$ is continuous on $\mathbb{R}$ but *not uniformly continuous*.

   To see this, fix $\varepsilon = 1$. For any $\eta > 0$, choose $x = \frac{1}{\eta} + \frac{\eta}{2}$ and $y = \frac{1}{\eta}$. Then

$$|x - y| = \frac{\eta}{2} < \eta,$$

   but

$$|f(x) - f(y)| = \left| \left(\frac{1}{\eta} + \frac{\eta}{2}\right)^2 - \left(\frac{1}{\eta}\right)^2 \right| = 1 + \frac{\eta^2}{4} > 1 = \varepsilon.$$

   Hence, the uniform continuity condition fails.

2. Any Lipschitz continuous mapping is uniformly continuous. Specifically, let $f: (E, d) \to (E', \delta)$ be a k-Lipschitz function for some $k \geq 0$, i.e.,

$$\delta(f(x), f(y)) \leq k\, d(x, y) \quad \text{for all } x, y \in E.$$

   Then for any $\varepsilon > 0$, choosing $\eta = \varepsilon / k$ (with η arbitrary if $k = 0$) ensures that $d(x, y) < \eta$ implies $\delta(f(x), f(y)) < \varepsilon$. Therefore, f is uniformly continuous on E.

Proposition 1.120. (**Composition of Uniformly Continuous Maps**)
Let (E, d), (E', d'), and (E'', d'') be metric spaces, and let $f: E \to E'$ and $g: E' \to E''$ be mappings. If f is uniformly continuous on E and g is uniformly continuous on E', then the composition $g \circ f: E \to E''$ is uniformly continuous on E.

Proof. Let $\varepsilon > 0$ be given.
Since g is uniformly continuous on E', there exists $\eta > 0$ such that for all $u, v \in E'$,

$$d'(u, v) < \eta \quad \implies \quad d''(g(u), g(v)) < \varepsilon.$$

Since f is uniformly continuous on E, there exists $\delta > 0$ such that for all $x, y \in E$,

$$d(x, y) < \delta \quad \implies \quad d'(f(x), f(y)) < \eta.$$

Now, let $x, y \in E$ with $d(x, y) < \delta$. Then $d'(f(x), f(y)) < \eta$, and by the uniform continuity of g, it follows that

$$d''((g \circ f)(x), (g \circ f)(y)) = d''(g(f(x)), g(f(y))) < \varepsilon.$$

Thus, for every $\varepsilon > 0$, we have found a $\delta > 0$ such that $d(x, y) < \delta$ implies $d''((g \circ f)(x), (g \circ f)(y)) < \varepsilon$. Therefore, $g \circ f$ is uniformly continuous on E. ■

Exercise 42. (**Uniformly Continuous Maps Preserve Cauchy Sequences**)
Let $f: (E, d) \to (F, d')$ be a uniformly continuous function between two metric spaces. Show that f maps every Cauchy sequence in E to a Cauchy sequence in F.

Solution 42. Let $(x_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in (E, d). We must show that the image sequence $(f(x_n))_{n\in\mathbb{N}}$ is Cauchy in (F, d').

Let $\varepsilon > 0$ be arbitrary. Since f is uniformly continuous, there exists $\delta > 0$ such that for all $x, y \in E$,

$$d(x, y) < \delta \quad \Longrightarrow \quad d'(f(x), f(y)) < \varepsilon.$$

Because (x_n) is a Cauchy sequence, there exists an integer $N \in \mathbb{N}$ such that for all $n, p \geq N$,

$$d(x_n, x_p) < \delta.$$

Applying the uniform continuity of f, we obtain for all $n, p \geq N$,

$$d'(f(x_n), f(x_p)) < \varepsilon.$$

This shows that $(f(x_n))$ is a Cauchy sequence in (F, d'), as required.

Exercise 43. Let (X, d_X) and (Y, d_Y) be metric spaces, let $f: X \to Y$ be a mapping, and let $a \in X$. Show that f is continuous at a if and only if for every sequence $(x_n)_{n\in\mathbb{N}}$ of elements of X that converges to a, the sequence $(f(x_n))_{n\in\mathbb{N}}$ converges to $f(a)$ in Y.

Solution 43. We prove both implications.

($\Rightarrow$) Assume that f is continuous at a, and let $(x_n)_{n\in\mathbb{N}}$ be a sequence in X with $x_n \to a$. We must show that $f(x_n) \to f(a)$.

Let $\varepsilon > 0$. By continuity of f at a, there exists $\eta > 0$ such that for all $x \in X$,

$$d_X(x, a) < \eta \quad \Longrightarrow \quad d_Y(f(x), f(a)) < \varepsilon.$$

Since $x_n \to a$, there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$,

$$d_X(x_n, a) < \eta.$$

For these n, the above implication gives $d_Y(f(x_n), f(a)) < \varepsilon$. This proves that $f(x_n) \to f(a)$.

($\Leftarrow$) We prove the contrapositive. Suppose f is *not* continuous at a. Then there exists $\varepsilon_0 > 0$ such that for every $\eta > 0$, there is some $x \in X$ with

$$d_X(x, a) < \eta \quad \text{and} \quad d_Y(f(x), f(a)) \geq \varepsilon_0.$$

In particular, for each $n \in \mathbb{N}^*$, choose $\eta = 1/n$. There exists a point $x_n \in X$ such that

$$d_X(x_n, a) < \frac{1}{n} \quad \text{and} \quad d_Y(f(x_n), f(a)) \geq \varepsilon_0.$$

The sequence $(x_n)_{n\in\mathbb{N}^*}$ thus satisfies $x_n \to a$, but the sequence $(f(x_n))$ does not converge to $f(a)$ because its terms are always at least ε_0 away from $f(a)$.

This completes the proof of the equivalence.

1.13 Separable Metric Spaces

Definition 1.121. A metric space (E, d) is said to be *separable* if it contains a countable subset $D \subseteq E$ that is dense in E, i.e., $\overline{D} = E$.

Remark 1.122. It is important not to confuse *separable* with *separated* (Hausdorff). Every metric space is Hausdorff (i.e., for any $x \neq y$, there exist disjoint open balls around x and y), but not every metric space is separable. Separability is a property concerning the *size* (cardinality) of a dense subset, not the separation of points.

Example 1.123. .

1. The real line $\mathbb{R}$ with its standard metric is separable, since the set of rational numbers $\mathbb{Q}$ is countable and dense.
2. More generally, $\mathbb{R}^n$ is separable, with the countable dense subset $\mathbb{Q}^n$.
3. Any discrete metric space is separable if and only if it is countable. Indeed, in a discrete space, the only dense subset is the entire space itself.

Proposition 1.124. Every metric space is Hausdorff.

Proof. Let (E, d) be a metric space, and let $x, y \in E$ with $x \neq y$. Define $r = d(x, y) > 0$. Consider the open balls

$$V = B\left(x, \frac{r}{3}\right) \quad \text{and} \quad W = B\left(y, \frac{r}{3}\right).$$

Suppose, for contradiction, that there exists a point $z \in V \cap W$. Then

$$d(x, z) < \frac{r}{3} \quad \text{and} \quad d(z, y) < \frac{r}{3}.$$

By the triangle inequality,

$$r = d(x, y) \leq d(x, z) + d(z, y) < \frac{r}{3} + \frac{r}{3} = \frac{2r}{3},$$

which is a contradiction since $r > 0$. Therefore, $V \cap W = \emptyset$, and the space is Hausdorff. ■

Proposition 1.125. In a Hausdorff space, the limit of a convergent sequence is unique.

Proof. Let $(u_n)_{n\in\mathbb{N}}$ be a sequence in a Hausdorff space E, and suppose it has two distinct limits ℓ and ℓ' with $\ell \neq \ell'$. Since E is Hausdorff, there exist disjoint open neighborhoods $V \in \mathcal{V}(\ell)$ and $W \in \mathcal{V}(\ell')$ such that $V \cap W = \emptyset$.

By the definition of convergence, there exist $n_1, n_2 \in \mathbb{N}$ such that for all $n \geq n_1$, $u_n \in V$, and for all $n \geq n_2$, $u_n \in W$. For any $n \geq \max(n_1, n_2)$, we have $u_n \in V \cap W = \emptyset$, a contradiction. Hence, the limit must be unique. ■

Chapter 2

Compact Spaces

2.1 Compact Topological Spaces

Definition 2.1 (Open Cover, Finite Subcover)**.** Let E be a topological space.

- An *open cover* of E is a family $(O_i)_{i\in I}$ of open subsets of E such that
$$\bigcup_{i\in I} O_i = E.$$
- A *finite subcover* of an open cover $(O_i)_{i\in I}$ is a finite subfamily $(O_j)_{j\in J}$ (where $J \subseteq I$ is finite) that is itself an open cover of E.

Definition 2.2 (Compact Space)**.** A topological space E is said to be *compact* if every open cover of E admits a finite subcover.

Example 2.3. .

1. The family of open intervals $\{]-n, n[\mid n \in \mathbb{N}^*\}$ is an open cover of $\mathbb{R}$ (with its usual topology). Similarly, the family $\{]n-1, n+1[\mid n \in \mathbb{Z}\}$ is also an open cover of $\mathbb{R}$.
2. In $\mathbb{R}^2$ with the Euclidean topology, consider the open unit disk $D = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 < 1\}$. The family of open sets
$$\left\{ \left\{ (x, y) \in \mathbb{R}^2 \mid x^2 + y^2 < 1 - \frac{1}{n} \right\} \,\middle|\, n \in \mathbb{N},\ n \geq 2 \right\}$$
is an open cover of D. However, it does *not* admit a finite subcover, which is consistent with the fact that D is not compact.

Definition 2.4 (Compact Space, Compact Subset)**.**
- A topological space E is said to be *compact* if it is *Hausdorff* and if every open cover of E admits a finite subcover.
- A subset $A \subseteq E$ is said to be *compact* if the subspace A, equipped with the induced topology, is a compact space.

Remark 2.5. In many modern texts in general topology, the term "compact" refers only to the finite subcover property (without the Hausdorff assumption). A space that is compact in that sense is called *quasi-compact* in contexts where the Hausdorff condition is required for compactness. The requirement that a compact space be Hausdorff is common in analysis and differential geometry.

Example 2.6. .

1. If a set is equipped with a topology that has only finitely many open sets, then the space is compact. In particular:
   - Any set with the *trivial topology* $\{\emptyset, E\}$ is compact.
   - Any *finite* topological space is compact, regardless of its topology.
2. A set X equipped with the *discrete topology* is compact if and only if X is finite.

   Proof. If X is finite, then it is compact by the previous point. Conversely, if X is infinite, consider the open cover $\mathcal{U} = \{\{x\} \mid x \in X\}$. Since every singleton is open in the discrete topology, $\mathcal{U}$ is an open cover of X. However, no finite subfamily of $\mathcal{U}$ can cover X, because X is infinite. Therefore, X is not compact. ■

Remark 2.7 (Borel–Lebesgue Property)**.** The property that "every open cover admits a finite subcover" is known as the *Borel–Lebesgue property.* A topological space (not necessarily Hausdorff) that satisfies this property is called *quasi-compact.* In the terminology used in this course, a *compact* space is a quasi-compact *Hausdorff* space.

Proposition 2.8. Every compact subset of a Hausdorff space is closed.

Proof. Let E be a Hausdorff topological space, and let $A \subseteq E$ be a compact subset. We will show that A is closed by proving that its complement $E \setminus A$ is open.

Let $x \in E \setminus A$. Since E is Hausdorff, for each $y \in A$ there exist disjoint open neighborhoods U_y of y and V_y of x, i.e.,

$$U_y \cap V_y = \emptyset.$$

The family $\{U_y\}_{y\in A}$ is an open cover of A. Because A is compact, there exists a finite subcover $\{U_{y_1}, \dots, U_{y_n}\}$ such that

$$A \subseteq \bigcup_{i=1}^{n} U_{y_i}.$$

Now consider the corresponding open neighborhoods of x:

$$V = \bigcap_{i=1}^{n} V_{y_i}.$$

Since the intersection is finite and each V_{y_i} is open, V is an open neighborhood of x.

Moreover, for each i, $U_{y_i} \cap V_{y_i} = \emptyset$, and since $V \subseteq V_{y_i}$, it follows that $U_{y_i} \cap V = \emptyset$ for all i. Therefore,

$$V \cap \left(\bigcup_{i=1}^{n} U_{y_i} \right) = \emptyset.$$

As $A \subseteq \bigcup_{i=1}^{n} U_{y_i}$, we conclude that $V \cap A = \emptyset$, which means $V \subseteq E \setminus A$.

Thus, every point $x \in E \setminus A$ has an open neighborhood contained in $E \setminus A$, so $E \setminus A$ is open. Hence, A is closed. ■

Proposition 2.9 (prop:closed-in-compact)**.** Every closed subset of a compact space is compact.

Proof. Let E be a compact topological space, and let $A \subseteq E$ be a closed subset. We will show that A is compact.

Let $\{O_i\}_{i\in I}$ be an open cover of A in the subspace topology. By definition of the induced topology, for each $i \in I$, there exists an open set $U_i \subseteq E$ such that $O_i = U_i \cap A$. The family $\{U_i\}_{i\in I}$ is a collection of open sets in E such that

$$A \subseteq \bigcup_{i\in I} U_i.$$

Since A is closed in E, its complement $E \setminus A$ is open. Therefore, the family

$$\{E \setminus A\} \cup \{U_i\}_{i\in I}$$

is an open cover of the entire space E.

Because E is compact, this open cover admits a finite subcover. Thus, there exists a finite subset $J \subseteq I$ such that

$$E = (E \setminus A) \cup \left(\bigcup_{i\in J} U_i\right).$$

Intersecting both sides with A, and noting that $A \cap (E \setminus A) = \emptyset$, we obtain

$$A = A \cap \left(\bigcup_{i\in J} U_i\right) = \bigcup_{i\in J}(A \cap U_i) = \bigcup_{i\in J} O_i.$$

Hence, $\{O_i\}_{i\in J}$ is a finite subcover of the original open cover of A. This proves that A is compact. ■

Exercise 44. Let E be a Hausdorff topological space. Show that:

1. Every finite union of compact subsets of E is compact.

2. Every intersection of a non-empty family of compact subsets of E is compact.

Solution 44. .

1. Let $A_1, A_2, \ldots, A_n$ be compact subsets of E, and let $A = \bigcup_{k=1}^{n} A_k$. Let $\{O_i\}_{i\in I}$ be an open cover of A in E. Then for each k, $\{O_i\}_{i\in I}$ is also an open cover of A_k. Since A_k is compact, there exists a finite subset $J_k \subseteq I$ such that $A_k \subseteq \bigcup_{i\in J_k} O_i$.

   Let $J = \bigcup_{k=1}^{n} J_k$. Since this is a finite union of finite sets, J is finite. Moreover,

   $$A = \bigcup_{k=1}^{n} A_k \subseteq \bigcup_{k=1}^{n}\left(\bigcup_{i\in J_k} O_i\right) = \bigcup_{i\in J} O_i.$$

   Thus, $\{O_i\}_{i\in J}$ is a finite subcover of A, proving that A is compact.

2. Let $\{K_\alpha\}_{\alpha\in A}$ be a non-empty family of compact subsets of E, and let $K = \bigcap_{\alpha\in A} K_\alpha$. Since E is Hausdorff, every compact subset is closed (by Proposition **??**). Therefore, each K_α is closed, and so their intersection K is also closed.

   Choose any $\alpha_0 \in A$. Then $K \subseteq K_{\alpha_0}$. Since K is a closed subset of the compact space K_{α_0}, it is itself compact.

2.2 Compact Metric Spaces

In the context of metric spaces, several notions related to compactness are particularly important.

Definition 2.10. Let (E, d) be a metric space.

1. The space (E, d) is called *totally bounded* (or *precompact*) if for every $\varepsilon > 0$, there exists a finite cover of E by subsets of diameter less than ε. Equivalently, for every $\varepsilon > 0$, there exists a finite set of points $\{x_1, \ldots, x_n\} \subseteq E$ such that
$$E \subseteq \bigcup_{i=1}^{n} B(x_i, \varepsilon).$$

2. The space (E, d) is called *sequentially compact* if every sequence in E has a convergent subsequence (whose limit is in E).

Remark 2.11. For metric spaces, the following three properties are equivalent:

1. Compactness (in the topological sense: every open cover has a finite subcover).
2. Sequential compactness.
3. Completeness and total boundedness.

This equivalence is a fundamental result in metric space theory and is often used to characterize compact metric spaces.

Theorem 2.12 (Bolzano–Weierstrass)**.** Let (E, d) be a metric space. Then E is compact (in the topological sense) if and only if every sequence in E has a convergent subsequence (i.e., E is sequentially compact).

Proof. We outline the two implications.

($\Rightarrow$) Suppose E is compact. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in E. If the set $\{x_n \mid n \in \mathbb{N}\}$ is finite, then some value is repeated infinitely often, and we obtain a constant (hence convergent) subsequence.

If the set is infinite, then it has a limit point $x \in E$ (since every infinite subset of a compact metric space has a limit point). Using this limit point, one can construct a subsequence (x_{n_k}) converging to x by choosing $x_{n_k} \in B(x, 1/k)$ with $n_k > n_{k-1}$.

($\Leftarrow$) Suppose E is sequentially compact. First, E is totally bounded: if not, there would exist $\varepsilon > 0$ such that no finite number of ε-balls covers E. One could then construct a sequence (x_n) with $d(x_n, x_m) \geq \varepsilon$ for all $n \neq m$, which cannot have a Cauchy subsequence, contradicting sequential compactness.

Second, E is complete: every Cauchy sequence has a convergent subsequence (by sequential compactness), and a Cauchy sequence with a convergent subsequence must itself converge to the same limit.

Since E is complete and totally bounded, it is compact. ■

Remark 2.13. Every compact metric space is sequentially compact. The converse is also true, though less immediate: in metric spaces, sequential compactness implies compactness. This equivalence is the content of Theorem 2.12.

Lemma 2.14. If a metric space (E, d) is sequentially compact, then it is totally bounded. That is, for every $\varepsilon > 0$, there exists a finite set of points $\{x_1, \dots, x_n\} \subseteq E$ such that

$$E = \bigcup_{i=1}^{n} B(x_i, \varepsilon).$$

Proof. We prove the contrapositive. Suppose that for some $\varepsilon > 0$, the space E cannot be covered by finitely many open balls of radius ε. We construct a sequence $(x_n)_{n\in\mathbb{N}}$ inductively as follows.

Choose $x_0 \in E$ arbitrarily. Having chosen $x_0, \dots, x_{n-1}$, the set $\bigcup_{k=0}^{n-1} B(x_k, \varepsilon)$ does not cover E by hypothesis, so we can pick

$$x_n \in E \setminus \bigcup_{k=0}^{n-1} B(x_k, \varepsilon).$$

By construction, for all $m < n$, we have $d(x_n, x_m) \geq \varepsilon$. Hence, for all $m \neq n$,

$$d(x_n, x_m) \geq \varepsilon.$$

This sequence (x_n) has no Cauchy subsequence, because any two distinct terms are at least ε apart. Consequently, it has no convergent subsequence. Therefore, E is not sequentially compact.

This proves the contrapositive, and hence the lemma. ■

Lemma 2.15 (Lebesgue Number Lemma)**.** Let (E, d) be a sequentially compact metric space, and let $(O_i)_{i\in I}$ be an open cover of E. Then there exists a number $r > 0$, called a *Lebesgue number* for the cover, such that every open ball of radius r in E is contained in at least one of the sets O_i.

Proof. We argue by contradiction. Suppose that no such $r > 0$ exists. Then for every $n \in \mathbb{N}^*$, there exists a point $x_n \in E$ such that the open ball $B\left(x_n, \frac{1}{n}\right)$ is not contained in any O_i, i.e.,

$$\forall i \in I, \quad B\left(x_n, \frac{1}{n}\right) \not\subseteq O_i.$$

Since E is sequentially compact, the sequence (x_n) has a convergent subsequence, which we denote again by (x_n) for simplicity. Let $x = \lim_{n\to\infty} x_n$. As $(O_i)_{i\in I}$ is a cover of E, there exists some index $i_0 \in I$ such that $x \in O_{i_0}$. Since O_{i_0} is open, there exists $\varepsilon > 0$ such that $B(x, \varepsilon) \subseteq O_{i_0}$.

Choose $N \in \mathbb{N}$ large enough so that $\frac{1}{N} < \frac{\varepsilon}{2}$ and $d(x_N, x) < \frac{\varepsilon}{2}$. Then for any $y \in B\left(x_N, \frac{1}{N}\right)$, the triangle inequality gives

$$d(y, x) \leq d(y, x_N) + d(x_N, x) < \frac{1}{N} + \frac{\varepsilon}{2} < \varepsilon,$$

so $y \in B(x, \varepsilon) \subseteq O_{i_0}$. This implies $B\left(x_N, \frac{1}{N}\right) \subseteq O_{i_0}$, which contradicts the construction of x_N.

Hence, a Lebesgue number $r > 0$ must exist. ■

Conclusion of the proof of Theorem 2.12. It remains to show that a sequentially compact metric space E is compact (in the open-cover sense).

Let $(O_i)_{i\in I}$ be an arbitrary open cover of E. By Lemma 2.15, there exists a Lebesgue number $r > 0$ such that every open ball of radius r is contained in some O_i.

By Lemma 2.14, E is totally bounded. Therefore, there exists a finite set $F = \{x_1, \ldots, x_n\} \subseteq E$ such that

$$E = \bigcup_{k=1}^{n} B(x_k, r).$$

For each k, choose an index $i_k \in I$ such that $B(x_k, r) \subseteq O_{i_k}$ (possible by the definition of the Lebesgue number). Then the finite subfamily $\{O_{i_1}, \ldots, O_{i_n}\}$ covers E.

Since every open cover of E admits a finite subcover, E is compact. ∎

2.2.1 Exercises

Exercise 45. Let (E, d) be a compact metric space, and let $(x_n)_{n\in\mathbb{N}}$ be a sequence in E that has exactly one adherent value (i.e., one cluster point) $a \in E$. Show that (x_n) converges to a.

Solution 45. Assume, for contradiction, that (x_n) does not converge to a. Then there exists $\varepsilon > 0$ and a subsequence $(x_{n_k})_{k\in\mathbb{N}}$ such that

$$x_{n_k} \notin B(a, \varepsilon) \quad \text{for all } k \in \mathbb{N},$$

i.e., $x_{n_k} \in E \setminus B(a, \varepsilon)$ for all k.

Since E is compact, the subsequence (x_{n_k}) has a convergent subsequence (again denoted (x_{n_k}) for simplicity), say $x_{n_k} \to b \in E$. By construction, each $x_{n_k} \in E \setminus B(a, \varepsilon)$, and since $E \setminus B(a, \varepsilon)$ is closed (as the complement of an open ball), the limit b must also lie in $E \setminus B(a, \varepsilon)$. In particular, $b \neq a$.

Thus, b is a second adherent value of the original sequence (x_n), distinct from a. This contradicts the hypothesis that a is the *unique* adherent value of (x_n).

Therefore, the assumption that (x_n) does not converge to a is false, and we conclude that $x_n \to a$.

Exercise 46. Let (E, d) be a compact metric space, and let $(x_n)_{n\in\mathbb{N}}$ be a sequence in E that has exactly one adherent value (i.e., one cluster point) $a \in E$. Show that (x_n) converges to a.

Solution 46. Assume, for contradiction, that (x_n) does not converge to a. Then there exists $\varepsilon > 0$ and a subsequence $(x_{n_k})_{k\in\mathbb{N}}$ such that

$$x_{n_k} \notin B(a, \varepsilon) \quad \text{for all } k \in \mathbb{N},$$

i.e., $x_{n_k} \in E \setminus B(a, \varepsilon)$ for all k.

Since E is compact, the subsequence (x_{n_k}) has a convergent subsequence (again denoted (x_{n_k}) for simplicity), say $x_{n_k} \to b \in E$. By construction, each $x_{n_k} \in E \setminus B(a, \varepsilon)$, and since $E \setminus B(a, \varepsilon)$ is closed (as the complement of an open ball), the limit b must also lie in $E \setminus B(a, \varepsilon)$. In particular, $b \neq a$.

Thus, b is a second adherent value of the original sequence (x_n), distinct from a. This contradicts the hypothesis that a is the *unique* adherent value of (x_n).

Therefore, the assumption that (x_n) does not converge to a is false, and we conclude that $x_n \to a$.

Exercise 47. Let (E, d) be a metric space, and let $(x_n)_{n\in\mathbb{N}}$ be a convergent sequence in E with limit $\ell \in E$. Define the set

$$F = \{\ell\} \cup \{x_n \mid n \in \mathbb{N}\}.$$

Endow F with the subspace topology induced by E. Show that F is compact.

Solution 47. Let $\{O_i\}_{i\in I}$ be an arbitrary open cover of F in the subspace topology. Then for each $i \in I$, there exists an open set $U_i \subseteq E$ such that $O_i = U_i \cap F$, and

$$F \subseteq \bigcup_{i\in I} U_i.$$

Since $\ell \in F$, there exists an index $i_0 \in I$ such that $\ell \in U_{i_0}$. Because U_{i_0} is open in E and $x_n \to \ell$, there exists an integer $N \in \mathbb{N}$ such that for all $n \geq N$,

$$x_n \in U_{i_0}.$$

The remaining points $x_0, x_1, \dots, x_{N-1}$ are finitely many. For each $k = 0, 1, \dots, N-1$, choose an index $i_k \in I$ such that $x_k \in U_{i_k}$ (possible since $\{U_i\}_{i\in I}$ covers F).

Now consider the finite subfamily $\{U_{i_0}, U_{i_1}, \dots, U_{i_{N-1}}\}$. This family covers all of F: - $x_n \in U_{i_0}$ for all $n \geq N$, - $x_k \in U_{i_k}$ for $k = 0, \dots, N-1$, - $\ell \in U_{i_0}$.

Therefore, the corresponding sets $\{O_{i_0}, O_{i_1}, \dots, O_{i_{N-1}}\}$ form a finite subcover of F. Since every open cover of F admits a finite subcover, F is compact.

Exercise 48. Let (E, d) be a metric space. Show that the following statements are equivalent:

1. E is compact.
2. Every decreasing sequence $(F_n)_{n\in\mathbb{N}}$ of non-empty closed subsets of E has non-empty intersection, i.e.,

$$F_0 \supseteq F_1 \supseteq F_2 \supseteq \cdots \quad \text{with } F_n \neq \emptyset \text{ and closed} \quad \Longrightarrow \bigcap_{n=0}^{\infty} F_n \neq \emptyset.$$

Solution 48. We prove the two implications.

(1 $\Rightarrow$ 2) Assume E is compact, and let $(F_n)_{n\in\mathbb{N}}$ be a decreasing sequence of non-empty closed subsets of E. For each n, choose a point $x_n \in F_n$. The sequence (x_n) lies in the compact space E, so it has a convergent subsequence; let x be its limit.

We claim that $x \in \bigcap_{n=0}^{\infty} F_n$. Fix $k \in \mathbb{N}$. Since the sequence (F_n) is decreasing, for all $n \geq k$ we have $x_n \in F_k$. The tail $(x_n)_{n\geq k}$ is therefore a sequence in the closed set F_k, and its limit x must also belong to F_k. As this holds for every k, we conclude that $x \in \bigcap_{n=0}^{\infty} F_n$, so the intersection is non-empty.

(2 $\Rightarrow$ 1) Assume that every decreasing sequence of non-empty closed sets in E has non-empty intersection. To show that E is compact, it suffices (by Theorem 2.12) to prove that every sequence in E has a convergent subsequence.

Let $(x_n)_{n\in\mathbb{N}}$ be an arbitrary sequence in E. For each n, define the tail set

$$E_n = \{x_k \mid k \geq n\},$$

and let $F_n = \overline{E_n}$ be its closure. Each F_n is closed and non-empty, and the sequence (F_n) is decreasing: $F_0 \supseteq F_1 \supseteq F_2 \supseteq \cdots$.

By hypothesis, $\bigcap_{n=0}^{\infty} F_n \neq \emptyset$. Let x be a point in this intersection. We claim that x is an adherent (cluster) point of the sequence (x_n).

Indeed, for every n, $x \in F_n = \overline{E_n}$, so every neighborhood of x intersects E_n, i.e., contains some x_k with $k \geq n$. This is precisely the definition of x being a cluster point of (x_n). Therefore, there exists a subsequence of (x_n) converging to x.

Thus, every sequence in E has a convergent subsequence, so E is sequentially compact, and hence compact.

Exercise 49. Let (E, d) be a metric space, and let $A \subseteq E$ be a compact subset. Show that there exist points $a, b \in A$ such that

$$d(a, b) = \delta(A),$$

where $\delta(A) = \sup\{d(x, y) \mid x, y \in A\}$ is the diameter of A.

Solution 49. Since A is compact, it is in particular bounded, so $\delta(A) < \infty$.

By the definition of the supremum, there exist sequences $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ in A such that

$$\lim_{n\to\infty} d(x_n, y_n) = \delta(A).$$

Because A is compact, the sequence (x_n) has a convergent subsequence. Denote this subsequence by (x_{n_k}) and let its limit be $a \in A$. (The limit lies in A because a compact subset of a metric space is closed.)

Now consider the corresponding subsequence (y_{n_k}) in A. Again, by compactness of A, it has a convergent subsequence $(y_{n_{k_j}})$ with limit $b \in A$.

Define the associated subsequence of (x_{n_k}) by $x_{n_{k_j}}$; it still converges to a, since it is a subsequence of a convergent sequence.

We now have two convergent sequences:

$$x_{n_{k_j}} \to a \quad \text{and} \quad y_{n_{k_j}} \to b,$$

with $a, b \in A$. By continuity of the metric d, we obtain

$$\lim_{j\to\infty} d\Big(x_{n_{k_j}}, y_{n_{k_j}}\Big) = d(a, b).$$

But the left-hand side is also equal to $\delta(A)$, as it is a subsequence of a sequence converging to $\delta(A)$. Therefore,

$$d(a, b) = \delta(A),$$

which completes the proof.

2.3 Product of Compact Metric Spaces

2.3.1 Product of Metric Spaces

Let $\{(E_i, d_i)\}_{1\leq i\leq n}$ be a finite family of metric spaces. We aim to define a metric on the Cartesian product

$$E = E_1 \times E_2 \times \cdots \times E_n = \prod_{i=1}^{n} E_i$$

that is compatible with the individual metrics d_i.

Proposition 2.16. Let $\{(E_i, d_i)\}_{1\leq i\leq n}$ be a finite family of metric spaces, and let $E = \prod_{i=1}^{n} E_i$. For $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_n)$ in E, the following formulas define metrics on E:

$$\delta_1(x, y) = \sum_{i=1}^{n} d_i(x_i, y_i),$$

$$\delta_2(x, y) = \left(\sum_{i=1}^{n} d_i(x_i, y_i)^2\right)^{1/2},$$

$$\delta_\infty(x, y) = \max_{1\leq i\leq n} d_i(x_i, y_i).$$

Moreover, these three metrics are equivalent (and therefore induce the same topology on E, called the *product topology*).

Proof. The verification that each δ_p satisfies the axioms of a metric is straightforward: - Non-negativity, symmetry, and the identity of indiscernibles follow immediately from the corresponding properties of each d_i. - The triangle inequality for δ_1 and δ_∞ follows from the triangle inequality for each d_i and basic properties of sums and maxima. - For δ_2, the triangle inequality is a consequence of the Minkowski inequality.

Equivalence of the metrics follows from the standard inequalities:

$$\delta_\infty(x,y) \leq \delta_2(x,y) \leq \delta_1(x,y) \leq n\,\delta_\infty(x,y),$$

which hold for all $x, y \in E$. ∎

Remark 2.17. It is also possible to define a product metric for a countable product of metric spaces. For example, if (E_i, d_i) are metric spaces with $d_i \leq 1$ (which can always be arranged by replacing d_i with $\frac{d_i}{1+d_i}$, an equivalent bounded metric), then the function

$$d(x,y) = \sum_{i=1}^{\infty} \frac{1}{2^i} d_i(x_i, y_i)$$

defines a metric on $\prod_{i=1}^{\infty} E_i$ that induces the product topology.

Proposition 2.18. Let $\{(E_i, d_i)\}_{i\in\mathbb{N}^*}$ be a countable family of metric spaces. Define the set

$$E = \prod_{i=1}^{\infty} E_i.$$

For $x = (x_1, x_2, \dots)$ and $y = (y_1, y_2, \dots)$ in E, the function $d\colon E \times E \to \mathbb{R}_+$ given by

$$d(x,y) = \sum_{i=1}^{\infty} \frac{1}{2^i} \min\Big(1, d_i(x_i, y_i)\Big)$$

defines a metric on E.

Proof. We verify the metric axioms.

Well-definedness: For each i, $0 \leq \min(1, d_i(x_i, y_i)) \leq 1$, so the general term of the series is bounded by $1/2^i$. Since $\sum_{i=1}^{\infty} 1/2^i = 1$ converges, the series for $d(x,y)$ converges absolutely for all $x, y \in E$. Thus, d is well-defined and $d(x,y) \in [0,1]$.

Non-negativity and separation: Each term in the sum is non-negative, so $d(x,y) \geq 0$. Moreover, $d(x,y) = 0$ if and only if $\min(1, d_i(x_i, y_i)) = 0$ for all i, which is equivalent to $d_i(x_i, y_i) = 0$ for all i, i.e., $x_i = y_i$ for all i, so $x = y$.

Symmetry: Since each d_i is symmetric, $d_i(x_i, y_i) = d_i(y_i, x_i)$, so $d(x,y) = d(y,x)$.

Triangle inequality: For each i, the function $t \mapsto \min(1,t)$ is subadditive on $[0,\infty)$, i.e.,

$$\min(1, a+b) \leq \min(1,a) + \min(1,b) \quad \text{for all } a, b \geq 0.$$

Since d_i satisfies the triangle inequality, we have

$$d_i(x_i, z_i) \leq d_i(x_i, y_i) + d_i(y_i, z_i),$$

and hence

$$\min(1, d_i(x_i, z_i)) \leq \min(1, d_i(x_i, y_i) + d_i(y_i, z_i)) \leq \min(1, d_i(x_i, y_i)) + \min(1, d_i(y_i, z_i)).$$

Multiplying by $1/2^i$ and summing over i yields $d(x,z) \leq d(x,y) + d(y,z)$.

Therefore, d is a metric on E. ∎

2.3.2 Product of Compact Spaces

Theorem 2.19 (Tychonoff's Theorem — Finite Case)**.** Let $\{(E_i, d_i)\}_{1\leq i\leq n}$ be a finite family of non-empty metric spaces. Then the product space

$$E = E_1 \times E_2 \times \cdots \times E_n,$$

equipped with any of the standard product metrics (e.g., δ_1, δ_2, or δ_∞ from Proposition **??**), is compact if and only if each factor E_i is compact.

Proof. ($\Rightarrow$) If E is compact, then for each i, the projection map $\pi_i\colon E \to E_i$ is continuous. Since the continuous image of a compact set is compact, each $E_i = \pi_i(E)$ is compact.

($\Leftarrow$) We prove the converse for $n = 2$; the general finite case follows by induction.

Let (E_1, d_1) and (E_2, d_2) be compact metric spaces, and consider the product space $E = E_1 \times E_2$ equipped with the metric $\delta_\infty((x_1, x_2),(y_1, y_2)) = \max\{d_1(x_1, y_1), d_2(x_2, y_2)\}$ (which induces the product topology).

Let $\{(x_n, y_n)\}_{n\in\mathbb{N}}$ be an arbitrary sequence in E. Since E_1 is compact, the sequence (x_n) has a convergent subsequence (x_{n_k}) with limit $x \in E_1$. Now consider the corresponding subsequence (y_{n_k}) in E_2. Since E_2 is compact, this subsequence has a further convergent subsequence $(y_{n_{k_j}})$ with limit $y \in E_2$.

The associated subsequence $(x_{n_{k_j}})$ still converges to x, as it is a subsequence of a convergent sequence. Therefore, the subsequence $\{(x_{n_{k_j}}, y_{n_{k_j}})\}$ converges to (x, y) in E, because

$$\delta_\infty\Big((x_{n_{k_j}}, y_{n_{k_j}}),(x, y)\Big) = \max\{d_1(x_{n_{k_j}}, x),\ d_2(y_{n_{k_j}}, y)\} \to 0.$$

Thus, every sequence in E has a convergent subsequence, so E is sequentially compact. Since E is a metric space, sequential compactness implies compactness.

By induction, the result extends to any finite product. ∎

Theorem 2.20 (Tychonoff's Theorem — Countable Case)**.** Let $\{(E_i, d_i)\}_{i\in\mathbb{N}^*}$ be a countable family of non-empty metric spaces. Equip the product space

$$E = \prod_{i=1}^{\infty} E_i$$

with the metric

$$d(x, y) = \sum_{i=1}^{\infty} \frac{1}{2^i} \min\Big(1, d_i(x_i, y_i)\Big),$$

which induces the product topology on E.

Then E is compact if and only if each factor E_i is compact.

Proof. ($\Rightarrow$) If E is compact, then for each i, the projection map $\pi_i\colon E \to E_i$ is continuous. Since the continuous image of a compact space is compact, each $E_i = \pi_i(E)$ is compact.

($\Leftarrow$) Suppose each E_i is compact. Since each E_i is a compact metric space, it is sequentially compact. We will show that E is sequentially compact; since E is metrizable, this implies compactness.

Let $(x^{(n)})_{n\in\mathbb{N}}$ be a sequence in E, where $x^{(n)} = (x_1^{(n)}, x_2^{(n)}, x_3^{(n)}, \dots)$.

We construct a convergent subsequence by a diagonal argument:

- Since E_1 is sequentially compact, the sequence $(x_1^{(n)})_{n\in\mathbb{N}}$ has a convergent subsequence. Denote the corresponding indices by $n_k^{(1)}$, so that $(x_1^{(n_k^{(1)})})_k$ converges in E_1.

- The sequence $(x_2^{(n_k^{(1)})})_k$ in E_2 has a convergent subsequence. Denote the new indices by $n_k^{(2)}$, so that $(x_2^{(n_k^{(2)})})_k$ converges in E_2, and note that $(x_1^{(n_k^{(2)})})_k$ still converges (as a subsequence of a convergent sequence).
- Proceeding inductively, for each $m \geq 1$, we obtain a subsequence $(n_k^{(m)})_k$ such that $(x_i^{(n_k^{(m)})})_k$ converges in E_i for all $i = 1, \dots, m$.

Now define the diagonal subsequence $m_k = n_k^{(k)}$. Then for each fixed i, the sequence $(x_i^{(m_k)})_{k\geq i}$ is a subsequence of $(x_i^{(n_k^{(i)})})_k$, and hence converges in E_i. Let $x_i = \lim_{k\to\infty} x_i^{(m_k)}$, and define $x = (x_1, x_2, \dots) \in E$.

We claim that $x^{(m_k)} \to x$ in the metric d. Let $\varepsilon > 0$. Choose N such that $\sum_{i=N+1}^{\infty} \frac{1}{2^i} < \frac{\varepsilon}{2}$. For each $i = 1, \dots, N$, since $x_i^{(m_k)} \to x_i$, there exists K_i such that for all $k \geq K_i$,

$$d_i(x_i^{(m_k)}, x_i) < \frac{\varepsilon}{2}.$$

Let $K = \max\{K_1, \dots, K_N\}$. Then for all $k \geq K$,

$$d(x^{(m_k)}, x) = \sum_{i=1}^{\infty} \frac{1}{2^i} \min(1, d_i(x_i^{(m_k)}, x_i)) \leq \sum_{i=1}^{N} \frac{1}{2^i} \cdot \frac{\varepsilon}{2} + \sum_{i=N+1}^{\infty} \frac{1}{2^i} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Thus, $x^{(m_k)} \to x$ in E, so E is sequentially compact, and therefore compact. ■

2.4 Compact Subsets of the Real Line

Definition 2.21. A subset $A \subseteq \mathbb{R}$ is said to be *compact* if every open cover of A by open intervals admits a finite subcover. Precisely, for every family $\mathcal{F} = \{I_\lambda\}_{\lambda\in\Lambda}$ of open intervals such that

$$A \subseteq \bigcup_{\lambda\in\Lambda} I_\lambda,$$

there exists a finite subset $\{\lambda_1, \dots, \lambda_k\} \subseteq \Lambda$ such that

$$A \subseteq \bigcup_{i=1}^{k} I_{\lambda_i}.$$

The following fundamental result characterizes compact subsets of $\mathbb{R}$.

Theorem 2.22 (Heine–Borel)**.** A subset $A \subseteq \mathbb{R}$ is compact if and only if it is closed and bounded.

Proof. ($\Rightarrow$) Suppose A is compact. - *Boundedness:* Consider the open cover $\{(-n, n)\}_{n\in\mathbb{N}^*}$. By compactness, there exists N such that $A \subseteq (-N, N)$, so A is bounded. - *Closedness:* Since $\mathbb{R}$ is Hausdorff and A is compact, Proposition 2.8 implies that A is closed.

($\Leftarrow$) Suppose A is closed and bounded. Then $A \subseteq [-M, M]$ for some $M > 0$. The interval $[-M, M]$ is compact (this is the classical Heine–Borel theorem for closed intervals, provable via the least upper bound property). Since A is a closed subset of the compact set $[-M, M]$, it follows from Proposition 2.9 that A is compact. ■

Example 2.23. The following sets illustrate the theorem:

- $[0, 1]$ is compact (closed and bounded).

- $(0,1)$ is not compact (not closed); the open cover $\{(1/n,1)\}_{n\geq 2}$ has no finite subcover.
- $[0,\infty)$ is not compact (not bounded).
- The Cantor set is compact (closed and bounded).

Theorem 2.24 (Borel–Lebesgue (Heine–Borel))**.** *Let A be a non-empty subset of $\mathbb{R}$ equipped with the standard metric. The following statements are equivalent:*

1. *A is compact.*
2. *A is closed and bounded.*

Proof. We prove the equivalence by showing that a closed interval $[a,b]$ is compact, and then deducing the general result.

Step 1: $[a,b]$ is compact. Assume $a<b$ (the case $a=b$ is trivial). Let $\{O_i\}_{i\in I}$ be an open cover of $[a,b]$. Define the set

$$F=\left\{x\in[a,b]\;\middle|\;\exists\text{ a finite subset }J\subseteq I\text{ such that }[a,x]\subseteq\bigcup_{j\in J}O_j\right\}.$$

We will show that $b\in F$, which implies that $[a,b]$ admits a finite subcover.

- F is non-empty: since $a\in[a,b]$, and the cover is open, there exists some $i_0\in I$ such that $a\in O_{i_0}$, so $a\in F$.

- Let $\alpha=\sup F$. Since $F\subseteq[a,b]$, we have $\alpha\in[a,b]$.

- Because $\{O_i\}$ covers $[a,b]$, there exists $i_0\in I$ such that $\alpha\in O_{i_0}$. Since O_{i_0} is open, there exists $\varepsilon>0$ such that

$$(\alpha-\varepsilon,\alpha+\varepsilon)\cap[a,b]\subseteq O_{i_0}.$$

- By definition of supremum, there exists $x\in F$ with $\alpha-\varepsilon<x\leq\alpha$. Since $x\in F$, there is a finite subfamily $\{O_j\}_{j\in J}$ covering $[a,x]$. Then

$$[a,\alpha]\subseteq[a,x]\cup(\alpha-\varepsilon,\alpha+\varepsilon)\subseteq\left(\bigcup_{j\in J}O_j\right)\cup O_{i_0},$$

so $\alpha\in F$.

- If $\alpha<b$, then choose $y\in(\alpha,\min\{\alpha+\varepsilon,b\})$. Then $[a,y]\subseteq[a,\alpha]\cup(\alpha,y)\subseteq\left(\bigcup_{j\in J}O_j\right)\cup O_{i_0}$, so $y\in F$, contradicting that $\alpha=\sup F$.

Therefore, $\alpha=b$, and $b\in F$. Hence, $[a,b]$ is compact.

Step 2: General case. ($1\Rightarrow 2$) If A is compact, then it is bounded (since $\{(-n,n)\}_{n\in\mathbb{N}}$ is an open cover, and a finite subcover bounds A), and closed (as a compact subset of the Hausdorff space $\mathbb{R}$).

($2\Rightarrow 1$) If A is closed and bounded, then $A\subseteq[-M,M]$ for some $M>0$. The interval $[-M,M]$ is compact by Step 1. Since A is a closed subset of a compact set, it is itself compact. ■

Example 2.25. Let $\mathbb{R}$ be equipped with its usual metric.

1. The closed interval $A=[0,1]$ is compact. Consider the open cover

$$\mathcal{U}=\left\{\left(x-\tfrac{1}{5},\,x+\tfrac{1}{5}\right)\;\middle|\;0<x<1\right\}.$$

Although $\mathcal{U}$ does not contain neighborhoods of the endpoints 0 and 1, we can still extract a finite subcover of $[0,1]$ by choosing, for example,

$$x_k = \frac{k}{10}, \quad k = 0, 1, \dots, 10.$$

Then the finite family

$$\left\{ \left(\tfrac{k}{10} - \tfrac{1}{5}, \tfrac{k}{10} + \tfrac{1}{5} \right) \;\middle|\; k = 0, 1, \dots, 10 \right\}$$

covers $[0,1]$, confirming compactness.

2. The set $B = \{1, 2, 3, \dots\} = \mathbb{N}^*$ is *not* compact. Consider the open cover

$$\mathcal{V} = \left\{ \left(n - \tfrac{1}{4}, n + \tfrac{1}{4} \right) \;\middle|\; n \in \mathbb{N}^* \right\}.$$

Each interval contains exactly one integer, and the intervals are pairwise disjoint. Therefore, no finite subfamily of $\mathcal{V}$ can cover the infinite set B.

Alternatively, in the subspace topology on B, every singleton $\{n\}$ is open (since $\{n\} = B \cap (n - \frac{1}{4}, n + \frac{1}{4})$). Thus,

$$B = \bigcup_{n=1}^{\infty} \{n\}$$

is an open cover with no finite subcover, confirming that B is not compact.

3. Consider the union

$$\bigcup_{n=1}^{\infty} \left[1, 2 - \tfrac{1}{n}\right] = [1, 2).$$

Each set $[1, 2 - \frac{1}{n}]$ is compact (closed and bounded), but their union is the half-open interval $[1,2)$, which is not compact (it is not closed in $\mathbb{R}$). This shows that an infinite union of compact sets need not be compact.

Theorem 2.26 (Generalized Heine–Borel Theorem)**.** Let F be a subset of $\mathbb{R}^n$ equipped with any of the standard metrics (e.g., d_1, d_2, or d_∞). Then F is compact if and only if it is closed and bounded.

Proof. All standard metrics on $\mathbb{R}^n$ are equivalent, so they induce the same topology and the same notion of compactness. We work with the Euclidean metric d_2.

($\Rightarrow$) If F is compact, then it is bounded (since the open cover $\{B(0,k)\}_{k \in \mathbb{N}}$ admits a finite subcover) and closed (as a compact subset of the Hausdorff space $\mathbb{R}^n$).

($\Leftarrow$) Suppose F is closed and bounded. Since F is bounded, there exists $M > 0$ such that

$$F \subseteq [-M, M]^n.$$

We first show that the closed cube $[-M, M]^n$ is compact. By Theorem 2.19, the finite product of compact spaces is compact. Since each interval $[-M, M] \subset \mathbb{R}$ is compact (by the Heine–Borel theorem in $\mathbb{R}$), their product $[-M, M]^n$ is compact in $\mathbb{R}^n$.

Now, F is a closed subset of the compact set $[-M, M]^n$. Because every closed subset of a compact space is compact. Therefore, F is compact. ∎

2.5 Continuous Functions on Compact Spaces

Theorem 2.27. Let (X, d) and (Y, δ) be metric spaces, and let $f\colon X \to Y$ be a continuous function. If $K \subseteq X$ is compact, then $f(K)$ is a compact subset of Y.

Proof. Let $\{O_i\}_{i\in I}$ be an arbitrary open cover of $f(K)$ in Y, i.e.,

$$f(K) \subseteq \bigcup_{i\in I} O_i.$$

Since f is continuous, for each $i \in I$, the preimage $f^{-1}(O_i)$ is an open subset of X. Therefore, the family $\{f^{-1}(O_i)\}_{i\in I}$ is an open cover of K, because

$$K \subseteq f^{-1}(f(K)) \subseteq f^{-1}\left(\bigcup_{i\in I} O_i\right) = \bigcup_{i\in I} f^{-1}(O_i).$$

As K is compact, there exists a finite subset $J \subseteq I$ such that

$$K \subseteq \bigcup_{i\in J} f^{-1}(O_i).$$

Applying f to both sides and using the fact that $f(A \cup B) = f(A) \cup f(B)$, we obtain

$$f(K) \subseteq f\left(\bigcup_{i\in J} f^{-1}(O_i)\right) = \bigcup_{i\in J} f\big(f^{-1}(O_i)\big) \subseteq \bigcup_{i\in J} O_i.$$

Thus, $\{O_i\}_{i\in J}$ is a finite subcover of $f(K)$. Since every open cover of $f(K)$ admits a finite subcover, $f(K)$ is compact. ■

Proposition 2.28. Let $f\colon X \to Y$ be a continuous bijection, where X is a compact topological space and Y is a Hausdorff space. Then f is a homeomorphism.

Proof. To show that f is a homeomorphism, it suffices to prove that the inverse map $f^{-1}\colon Y \to X$ is continuous. By the topological characterization of continuity, this is equivalent to showing that the image under f of every closed subset of X is closed in Y.

Let $F \subseteq X$ be closed. Since X is compact and F is closed, Proposition **??** implies that F is compact. As f is continuous, Theorem 2.27 ensures that $f(F)$ is compact in Y. Finally, since Y is Hausdorff, every compact subset of Y is closed (Proposition 2.9. Therefore, $f(F)$ is closed in Y.

This holds for every closed set $F \subseteq X$, so f^{-1} is continuous. Hence, f is a homeomorphism. ■

2.5.1 Exercises

Exercise 50. Let (E, d) be a metric space, and let $f\colon E \to \mathbb{R}$ be a continuous function. Let $K \subseteq E$ be a compact subset. Show that $f(K)$ is bounded in $\mathbb{R}$ and that f attains its supremum and infimum on K; that is, there exist points $x_{\max}, x_{\min} \in K$ such that

$$f(x_{\max}) = \sup_{x\in K} f(x), \quad f(x_{\min}) = \inf_{x\in K} f(x).$$

Solution 50. Since f is continuous and K is compact, Theorem 2.27 implies that $f(K)$ is a compact subset of $\mathbb{R}$. In $\mathbb{R}$ (with the usual metric), a set is compact if and only if it is closed and bounded (Heine–Borel theorem). Therefore, $f(K)$ is bounded and closed.

Being bounded, $f(K)$ has a supremum $M = \sup f(K)$ and an infimum $m = \inf f(K)$ in $\mathbb{R}$. Since $f(K)$ is closed, it contains all its limit points; in particular, it contains its supremum and infimum. Hence, there exist $x_{\max}, x_{\min} \in K$ such that

$$f(x_{\max}) = M, \quad f(x_{\min}) = m.$$

For completeness, we can also argue constructively: Let $M = \sup f(K)$. For each $n \in \mathbb{N}^*$, choose $x_n \in K$ such that

$$M - \frac{1}{n} < f(x_n) \leq M.$$

The sequence (x_n) lies in the compact set K, so it has a convergent subsequence (x_{n_k}) with limit $x \in K$. By continuity of f,

$$f(x) = \lim_{k\to\infty} f(x_{n_k}) = M.$$

Thus, f attains its maximum at $x \in K$. A similar argument applies to the infimum.

Exercise 51. Let (E, d) and (E', δ) be metric spaces, and assume that E is compact. Let $f\colon E \to E'$ be a continuous function. Show that f is uniformly continuous.

Solution 51. We argue by contradiction. Suppose that f is **not** uniformly continuous. Then there exists $\varepsilon_0 > 0$ such that for every $\eta > 0$, there exist points $x, y \in E$ with

$$d(x, y) < \eta \quad \text{but} \quad \delta(f(x), f(y)) \geq \varepsilon_0.$$

In particular, for each $n \in \mathbb{N}^*$, choose $\eta = \frac{1}{n}$. Then there exist points $x_n, y_n \in E$ such that

$$d(x_n, y_n) < \frac{1}{n} \quad \text{and} \quad \delta(f(x_n), f(y_n)) \geq \varepsilon_0.$$

Since E is compact, the sequence (x_n) has a convergent subsequence (x_{n_k}) with limit $x \in E$. Consider the corresponding subsequence (y_{n_k}). Because

$$d(y_{n_k}, x) \leq d(y_{n_k}, x_{n_k}) + d(x_{n_k}, x) < \frac{1}{n_k} + d(x_{n_k}, x) \xrightarrow{k\to\infty} 0,$$

the sequence (y_{n_k}) also converges to x.

Now, since f is continuous at x, we have

$$f(x_{n_k}) \to f(x) \quad \text{and} \quad f(y_{n_k}) \to f(x).$$

The metric δ is continuous, so

$$\delta(f(x_{n_k}), f(y_{n_k})) \longrightarrow \delta(f(x), f(x)) = 0.$$

But this contradicts the fact that $\delta(f(x_{n_k}), f(y_{n_k})) \geq \varepsilon_0 > 0$ for all k. Hence, our assumption was false, and f must be uniformly continuous.

Exercise 52. Let E be a normed vector space. For subsets $A, B \subseteq E$, define their Minkowski sum as

$$A + B = \{a + b \mid a \in A,\ b \in B\}.$$

1. Show that if A is compact and B is closed, then $A+B$ is closed.

2. Give an example of two closed subsets of $\mathbb{R}^2$ whose sum is not closed.

Solution 52. 1. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in $A+B$ such that $x_n \to x \in E$. For each n, write $x_n = a_n + b_n$ with $a_n \in A$ and $b_n \in B$.

Since A is compact, there exists a subsequence (a_{n_k}) converging to some $a \in A$. Consider the corresponding subsequence $(b_{n_k}) = (x_{n_k} - a_{n_k})$. As $x_{n_k} \to x$ and $a_{n_k} \to a$, we have

$$b_{n_k} \to x - a.$$

Let $b = x - a$. Since B is closed and $(b_{n_k}) \subseteq B$ converges to b, it follows that $b \in B$. Therefore, $x = a + b \in A + B$.

This shows that $A+B$ contains all its limit points, hence it is closed.

2. Consider the following closed subsets of $\mathbb{R}^2$:

$$A = \left\{(x,y) \in \mathbb{R}^2 \;\middle|\; xy = 1,\; x > 0\right\}, \quad B = \left\{(x,y) \in \mathbb{R}^2 \;\middle|\; x = 0\right\}.$$

The set A is the graph of $y = 1/x$ for $x > 0$, which is closed in $\mathbb{R}^2$ (it is the preimage of $\{1\}$ under the continuous map $(x,y) \mapsto xy$ restricted to $\{x > 0\}$, and its complement is open). The set B is the y-axis, which is closed.

Their sum is

$$A + B = \{(x, y_1 + y_2) \mid x > 0,\; y_1 = 1/x,\; y_2 \in \mathbb{R}\} = \{(x,y) \in \mathbb{R}^2 \mid x > 0\}.$$

This is the open right half-plane, which is **not closed** in $\mathbb{R}^2$ (for instance, the sequence $(1/n, 0) \in A+B$ converges to $(0,0) \notin A+B$).

Thus, the sum of two closed sets need not be closed.

2.6 Locally Compact Spaces

Definition 2.29. A topological space E is said to be *locally compact* if it is Hausdorff and every point of E has a compact neighborhood. Equivalently, for every $x \in E$, there exists an open set U and a compact set K such that

$$x \in U \subseteq K \subseteq E.$$

This is equivalent to saying that every point admits a neighborhood basis consisting of compact sets.

Remark 2.30. Every compact Hausdorff space is locally compact, but the converse is false. Thus, local compactness is strictly weaker than compactness.

Proposition 2.31. Let E be a locally compact Hausdorff space.

1. **Open subsets are locally compact.**

2. **Closed subsets are locally compact.**

3. More generally, a subspace $F \subseteq E$ is locally compact if and only if $F = F_1 \setminus F_2$ for some closed subsets $F_1, F_2 \subseteq E$.

4. Local compactness is preserved under homeomorphisms.

Theorem 2.32 (Baire Category Theorem – Locally Compact Version)**.** Every locally compact Hausdorff space is a Baire space. That is, the countable intersection of dense open subsets is dense.

Proof. Let $\{O_n\}_{n\in\mathbb{N}^*}$ be a sequence of dense open subsets of E, and let $U \subseteq E$ be a non-empty open set. We must show that

$$U \cap \left(\bigcap_{n=1}^{\infty} O_n\right) \neq \emptyset.$$

Since E is locally compact and Hausdorff, there exists a non-empty open set U_0 such that $\overline{U_0} \subseteq U$ and $\overline{U_0}$ is compact.

Because O_1 is dense, $U_0 \cap O_1 \neq \emptyset$. Choose a non-empty open set U_1 such that $\overline{U_1} \subseteq U_0 \cap O_1$ and $\overline{U_1}$ is compact.

Proceeding inductively, having constructed U_{n-1} with $\overline{U_{n-1}}$ compact, use the density of O_n to find a non-empty open set U_n such that

$$\overline{U_n} \subseteq U_{n-1} \cap O_n \quad \text{and} \quad \overline{U_n} \text{ compact.}$$

The sequence $\{\overline{U_n}\}_{n\in\mathbb{N}}$ is a decreasing sequence of non-empty compact sets. By the finite intersection property (Proposition **??**), their intersection is non-empty:

$$\bigcap_{n=1}^{\infty} \overline{U_n} \neq \emptyset.$$

But $\overline{U_n} \subseteq O_n$ for all n, and $\overline{U_1} \subseteq U$, so

$$\emptyset \neq \bigcap_{n=1}^{\infty} \overline{U_n} \subseteq U \cap \left(\bigcap_{n=1}^{\infty} O_n\right),$$

as required. ■

Example 2.33. .

1. The spaces $\mathbb{R}^n$ and $\mathbb{C}^n$ (with their usual topologies) are locally compact. Consequently, every open or closed subset (e.g., the open interval $]0,1[$, the open unit disk in $\mathbb{C}$) is locally compact but not necessarily compact.

2. The space $\mathbb{Q}$ of rational numbers (with the subspace topology from $\mathbb{R}$) is *not* locally compact. No point of $\mathbb{Q}$ has a compact neighborhood in $\mathbb{Q}$.

3. Infinite-dimensional normed vector spaces (e.g., ℓ^2, $C([0,1])$) are not locally compact. This is a consequence of Riesz's lemma.

4. The set $\Gamma = \{(0,0)\} \cup \{(x,y) \in \mathbb{R}^2 \mid x > 0\}$ is not locally compact: the origin has no compact neighborhood in Γ.

Exercise 53. Let (E,d) be a locally compact metric space, and let $\Omega \subseteq E$ be an open subset. Show that Ω is locally compact.

Solution 53. Let $x \in \Omega$. Since E is locally compact, there exists a compact neighborhood K of x in E; that is, there is an open set $U \subseteq E$ such that

$$x \in U \subseteq K.$$

Now consider $V = U \cap \Omega$. Then V is open in Ω and contains x. Moreover, the closure of V in E satisfies

$$\overline{V}^E \subseteq \overline{U}^E \subseteq K,$$

so $\overline{V}^E$ is a closed subset of the compact set K, hence compact.

Since Ω is open in E, the closure of V in Ω is $\overline{V}^\Omega = \overline{V}^E \cap \Omega$, which is a closed subset of the compact set $\overline{V}^E$, and therefore compact in Ω.

Thus, x has a compact neighborhood in Ω, and since x was arbitrary, Ω is locally compact.

Chapter 3

Complete Spaces

3.1 Cauchy Sequences

Definition 3.1. Let (E, d) be a metric space. A sequence $(x_n)_{n\in\mathbb{N}}$ in E is called a *Cauchy sequence* if

$$\forall \varepsilon > 0,\ \exists n_0 \in \mathbb{N},\ \forall p, q \geq n_0, \quad d(x_p, x_q) < \varepsilon.$$

Equivalently, this means that

$$\lim_{p,q\to\infty} d(x_p, x_q) = 0.$$

Remark 3.2. .

1. If d_1 and d_2 are two Lipschitz-equivalent metrics on a set E (i.e., there exist constants $\alpha, \beta > 0$ such that $\alpha d_1 \leq d_2 \leq \beta d_1$), then a sequence is Cauchy with respect to d_1 if and only if it is Cauchy with respect to d_2.
2. The image of a Cauchy sequence under a uniformly continuous map is again a Cauchy sequence (see Exercise **??**).

Proposition 3.3. Every convergent sequence in a metric space is a Cauchy sequence.

Proof. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in (E, d) such that $x_n \to x \in E$. Let $\varepsilon > 0$. By convergence, there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$,

$$d(x_n, x) < \frac{\varepsilon}{2}.$$

Then, for all $p, q \geq n_0$, the triangle inequality gives

$$d(x_p, x_q) \leq d(x_p, x) + d(x_q, x) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Hence, (x_n) is Cauchy. ∎

Proposition 3.4. Every Cauchy sequence in a metric space is bounded.

Proof. Let $(x_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in a metric space (E, d). Choose $\varepsilon = 1$. By the Cauchy property, there exists $n_0 \in \mathbb{N}$ such that for all $n, m \geq n_0$,

$$d(x_n, x_m) < 1.$$

In particular, for all $n \geq n_0$, we have $d(x_n, x_{n_0}) < 1$, so

$$x_n \in B(x_{n_0}, 1).$$

The entire sequence is therefore contained in the ball $B(x_{n_0}, R)$, where

$$R = \max\{d(x_0, x_{n_0}),\ d(x_1, x_{n_0}),\ \dots,\ d(x_{n_0-1}, x_{n_0}),\ 1\} + 1.$$

Hence, (x_n) is bounded. ■

Proposition 3.5. Let $(x_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in a metric space (E, d). If the sequence has a limit point $x \in E$, then the entire sequence converges to x.

Proof. Let x be a limit point of (x_n). Then there exists a subsequence $(x_{n_k})_{k\in\mathbb{N}}$ such that $x_{n_k} \to x$.

Let $\varepsilon > 0$. Since (x_n) is Cauchy, there exists $N_1 \in \mathbb{N}$ such that

$$\forall p, q \geq N_1, \quad d(x_p, x_q) < \frac{\varepsilon}{2}.$$

Since $x_{n_k} \to x$, there exists $K \in \mathbb{N}$ such that

$$\forall k \geq K, \quad d(x_{n_k}, x) < \frac{\varepsilon}{2}.$$

Let $N = \max\{N_1, n_K\}$. For any $n \geq N$, choose $k \geq K$ such that $n_k \geq N_1$ (possible because $n_k \to \infty$). Then both n and n_k are $\geq N_1$, so

$$d(x_n, x) \leq d(x_n, x_{n_k}) + d(x_{n_k}, x) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Thus, $x_n \to x$. In particular, a Cauchy sequence can have **at most one** limit point, and if it has one, the whole sequence converges to it. ■

3.2 Completeness

Definition 3.6. A metric space (E, d) is said to be *complete* if every Cauchy sequence in E converges to a point of E.

Remark 3.7. .

1. Completeness is a *metric* property: it depends on the specific distance d, not only on the topology it induces. Two metrics that define the same topology may differ in completeness. For example, $(0, 1)$ with the standard metric is not complete, but it is homeomorphic (hence topologically identical) to $\mathbb{R}$, which is complete.

2. If d_1 and d_2 are *Lipschitz-equivalent* metrics on a set E—that is, there exist constants $c, C > 0$ such that

$$c\, d_1(x, y) \leq d_2(x, y) \leq C\, d_1(x, y) \quad \text{for all } x, y \in E,$$

then (E, d_1) is complete if and only if (E, d_2) is complete. This follows because Lipschitz-equivalent metrics have the same Cauchy sequences.

Proposition 3.8. Let (E, d) be a metric space, and let $F \subseteq E$ be a subspace equipped with the induced metric. If (F, d) is complete, then F is closed in E.

Proof. Assume that (F, d) is complete. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in F that converges to some point $x \in E$. We will show that $x \in F$.

Since (x_n) converges in E, it is a Cauchy sequence in E. Because the metric on F is the restriction of the metric on E, the sequence (x_n) is also Cauchy in F.

By completeness of F, the sequence (x_n) converges to some point $y \in F$. But limits in metric spaces are unique, so $x = y \in F$.

Thus, every limit of a convergent sequence from F lies in F, which means F is closed in E. ■

Proposition 3.9. Let (E_1, d_1) and (E_2, d_2) be complete metric spaces. Then the product space $E_1 \times E_2$, equipped with either the *sum metric*

$$d_{\text{sum}}\Big((x_1, y_1), (x_2, y_2)\Big) = d_1(x_1, x_2) + d_2(y_1, y_2),$$

or the *maximum metric*

$$d_{\max}\Big((x_1, y_1), (x_2, y_2)\Big) = \max\{d_1(x_1, x_2),\ d_2(y_1, y_2)\},$$

is a complete metric space.

Proof. Let $\{(x_n, y_n)\}_{n\in\mathbb{N}}$ be a Cauchy sequence in $E_1 \times E_2$ with respect to either d_{sum} or $d_{\max}$.

We first show that (x_n) and (y_n) are Cauchy in their respective spaces.

- If the sequence is Cauchy for d_{sum}, then for every $\varepsilon > 0$, there exists N such that for all $m, n \geq N$,

$$d_1(x_n, x_m) + d_2(y_n, y_m) < \varepsilon,$$

which implies $d_1(x_n, x_m) < \varepsilon$ and $d_2(y_n, y_m) < \varepsilon$. Hence, both component sequences are Cauchy.

- If the sequence is Cauchy for $d_{\max}$, then for every $\varepsilon > 0$, there exists N such that for all $m, n \geq N$,

$$\max\{d_1(x_n, x_m),\ d_2(y_n, y_m)\} < \varepsilon,$$

so again $d_1(x_n, x_m) < \varepsilon$ and $d_2(y_n, y_m) < \varepsilon$, and both component sequences are Cauchy.

Since E_1 and E_2 are complete, there exist $x \in E_1$ and $y \in E_2$ such that $x_n \to x$ and $y_n \to y$.

Now consider convergence in the product: - For the sum metric:

$$d_{\text{sum}}\Big((x_n, y_n), (x, y)\Big) = d_1(x_n, x) + d_2(y_n, y) \xrightarrow{n\to\infty} 0.$$

- For the maximum metric:

$$d_{\max}\Big((x_n, y_n), (x, y)\Big) = \max\{d_1(x_n, x),\ d_2(y_n, y)\} \xrightarrow{n\to\infty} 0.$$

Thus, $(x_n, y_n) \to (x, y)$ in either metric. Since every Cauchy sequence converges, the product space is complete. ■

Proposition 3.10. Every compact metric space is complete.

Proof. Let (E, d) be a compact metric space, and let $(x_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in E. We will show that (x_n) converges to a point in E.

Since E is compact, every sequence in E has a convergent subsequence. Thus, there exist a subsequence $(x_{n_k})_{k\in\mathbb{N}}$ and a point $x \in E$ such that $x_{n_k} \to x$.

We now prove that the entire sequence (x_n) converges to x. Let $\varepsilon > 0$.

- Because (x_n) is Cauchy, there exists $N_1 \in \mathbb{N}$ such that for all $m, n \geq N_1$,

$$d(x_m, x_n) < \frac{\varepsilon}{2}.$$

- Because $x_{n_k} \to x$, there exists $K \in \mathbb{N}$ such that for all $k \geq K$,

$$d(x_{n_k}, x) < \frac{\varepsilon}{2}.$$

Let $N = \max\{N_1, n_K\}$. For any $n \geq N$, choose $k \geq K$ such that $n_k \geq N_1$ (possible since $n_k \to \infty$). Then both n and n_k are $\geq N_1$, so

$$d(x_n, x) \leq d(x_n, x_{n_k}) + d(x_{n_k}, x) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Thus, $x_n \to x \in E$, and since the Cauchy sequence (x_n) converges in E, the space is complete. ■

Proposition 3.11. The real line $\mathbb{R}$, equipped with the standard metric $d(x, y) = |x - y|$, is a complete metric space.

Proof. Let $(x_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in $\mathbb{R}$. We will show that it converges to a real number.

Step 1: (x_n) is bounded. Since (x_n) is Cauchy, there exists $N \in \mathbb{N}$ such that for all $m, n \geq N$,

$$|x_n - x_m| < 1.$$

Fixing $n = N$, we have $|x_m - x_N| < 1$ for all $m \geq N$, so $x_m \in [x_N - 1, x_N + 1]$. The finite set $\{x_1, \ldots, x_{N-1}\}$ is also bounded. Hence, the entire sequence (x_n) is bounded.

Step 2: Existence of a convergent subsequence. By the Bolzano–Weierstrass theorem, every bounded sequence in $\mathbb{R}$ has a convergent subsequence. Thus, there exist a subsequence $(x_{n_k})_{k\in\mathbb{N}}$ and a point $x \in \mathbb{R}$ such that

$$x_{n_k} \longrightarrow x \quad \text{as } k \to \infty.$$

Step 3: The whole sequence converges to x. Let $\varepsilon > 0$.

- Since (x_n) is Cauchy, there exists $N_1 \in \mathbb{N}$ such that for all $m, n \geq N_1$,

$$|x_n - x_m| < \frac{\varepsilon}{2}.$$

- Since $x_{n_k} \to x$, there exists $K \in \mathbb{N}$ such that for all $k \geq K$,

$$|x_{n_k} - x| < \frac{\varepsilon}{2}.$$

Let $N = \max\{N_1, n_K\}$. For any $n \geq N$, choose $k \geq K$ such that $n_k \geq N_1$ (possible because $n_k \to \infty$). Then

$$|x_n - x| \leq |x_n - x_{n_k}| + |x_{n_k} - x| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Thus, $x_n \to x \in \mathbb{R}$, and every Cauchy sequence in $\mathbb{R}$ converges. Therefore, $\mathbb{R}$ is complete. ■

Corollary 3.12. For every integer $n \geq 1$, the Euclidean space $\mathbb{R}^n$, equipped with any of the standard metrics (e.g., the Euclidean metric d_2, the taxicab metric d_1, or the maximum metric d_∞), is a complete metric space.

Proof. We proceed by induction on n.

- The case $n = 1$ is precisely Proposition 3.11: $\mathbb{R}$ is complete.

- Assume $\mathbb{R}^{n-1}$ is complete. Then $\mathbb{R}^n = \mathbb{R}^{n-1} \times \mathbb{R}$ is the product of two complete metric spaces. By Proposition 3.9, the product space $\mathbb{R}^n$ is complete with respect to the sum or maximum metric. Since all standard metrics on $\mathbb{R}^n$ are Lipschitz-equivalent, completeness holds for any of them .

Alternatively, one may argue directly: a sequence $(x^{(k)})_{k\in\mathbb{N}}$ in $\mathbb{R}^n$ is Cauchy if and only if each of its coordinate sequences is Cauchy in $\mathbb{R}$. Since $\mathbb{R}$ is complete, each coordinate converges, and hence the sequence converges in $\mathbb{R}^n$. ■

3.2.1 Exercises

Exercise 54. Let (X,d) be a metric space and $(x_n)_{n\geq 1}$ a sequence in X. Show that if

$$\sum_{n=1}^{\infty} d(x_n, x_{n+1}) < +\infty,$$

then (x_n) is a Cauchy sequence. Is the converse true?

Solution 54. Part 1: The series condition implies Cauchy. Assume that

$$\sum_{n=1}^{\infty} d(x_n, x_{n+1}) < \infty.$$

Let $S_n = \sum_{k=1}^{n} d(x_k, x_{k+1})$. Then (S_n) is a convergent (hence Cauchy) sequence of real numbers. Therefore, for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $q > p \geq N$,

$$\sum_{k=p}^{q-1} d(x_k, x_{k+1}) < \varepsilon.$$

By the triangle inequality,

$$d(x_p, x_q) \leq \sum_{k=p}^{q-1} d(x_k, x_{k+1}) < \varepsilon.$$

Thus, (x_n) is a Cauchy sequence in (X,d).

Part 2: The converse is false. Consider $X = \mathbb{R}$ with the usual metric, and define

$$x_n = \sum_{k=1}^{n} \frac{(-1)^k}{k}.$$

The sequence (x_n) converges (by the alternating series test), so it is Cauchy. However,

$$d(x_n, x_{n+1}) = |x_{n+1} - x_n| = \left|\frac{(-1)^{n+1}}{n+1}\right| = \frac{1}{n+1}.$$

The series $\sum_{n=1}^{\infty} d(x_n, x_{n+1}) = \sum_{n=1}^{\infty} \frac{1}{n+1}$ is the harmonic series (minus its first term), which diverges. Hence, the converse does not hold.

Conclusion: If the sum of successive distances converges, the sequence is Cauchy. However, a Cauchy sequence need not satisfy this summability condition.

Exercise 55. Let $E = (0, +\infty)$. For $x, y \in E$, define

$$d(x,y) = \left|\frac{1}{x} - \frac{1}{y}\right|.$$

.

1. Show that d is a metric on E.
2. Is the metric space (E,d) complete?

Solution 55. .

1. We verify the metric axioms.
   - *Non-negativity and identity of indiscernibles:* For all $x, y \in E$,
   $$d(x,y) = \left|\frac{1}{x} - \frac{1}{y}\right| \geq 0,$$
   and $d(x,y) = 0$ if and only if $\frac{1}{x} = \frac{1}{y}$, i.e., $x = y$.
   - *Symmetry:* Clearly,
   $$d(x,y) = \left|\frac{1}{x} - \frac{1}{y}\right| = \left|\frac{1}{y} - \frac{1}{x}\right| = d(y,x).$$
   - *Triangle inequality:* For all $x, y, z \in E$,
   $$d(x,z) = \left|\frac{1}{x} - \frac{1}{z}\right| \leq \left|\frac{1}{x} - \frac{1}{y}\right| + \left|\frac{1}{y} - \frac{1}{z}\right| = d(x,y) + d(y,z).$$

   Hence, d is a metric on E.
2. The space (E,d) is **not** complete.

   Consider the sequence $(x_n)_{n\in\mathbb{N}^*}$ defined by $x_n = n$. For $m, n \in \mathbb{N}^*$ with $m \geq n$,
   $$d(x_n, x_m) = \left|\frac{1}{n} - \frac{1}{m}\right| \leq \frac{1}{n}.$$
   Given $\varepsilon > 0$, choose N such that $1/N < \varepsilon$. Then for all $m, n \geq N$, $d(x_n, x_m) < \varepsilon$. Thus, (x_n) is a Cauchy sequence in (E,d).

   Suppose, for contradiction, that (x_n) converges to some $\ell \in E$. Then
   $$\lim_{n\to\infty} d(x_n, \ell) = \lim_{n\to\infty} \left|\frac{1}{n} - \frac{1}{\ell}\right| = \left|0 - \frac{1}{\ell}\right| = \frac{1}{\ell} > 0,$$
   since $\ell > 0$. This contradicts convergence. Therefore, (x_n) does not converge in (E,d), and the space is not complete.

   Remark: The map $\varphi\colon (E,d) \to (\mathbb{R}, |\cdot|)$, $\varphi(x) = 1/x$, is an isometry onto the open interval $(0,\infty) \subset \mathbb{R}$, which is not closed, hence not complete. This explains the incompleteness of (E,d).

Exercise 56. Determine whether the space $(\mathbb{R}, d)$ is complete for each of the following metrics:

1. $d(x,y) = |x^3 - y^3|$,
2. $d(x,y) = |e^x - e^y|$,
3. $d(x,y) = \log(1 + |x - y|)$.

Solution 56. .

1. **Complete.**

   Consider the map $\varphi\colon \mathbb{R} \to \mathbb{R}$, $\varphi(x) = x^3$. This map is a bijection, and
   $$d(x,y) = |\varphi(x) - \varphi(y)|.$$
   Hence, φ is an isometry from $(\mathbb{R}, d)$ onto $(\mathbb{R}, |\cdot|)$. Since $(\mathbb{R}, |\cdot|)$ is complete, so is $(\mathbb{R}, d)$.

   Equivalently: if (x_n) is Cauchy for d, then (x_n^3) is Cauchy in $\mathbb{R}$, hence converges to some $\alpha \in \mathbb{R}$. Then $x_n \to \alpha^{1/3}$ in the metric d, so the space is complete.

2. **Not complete.**

   Consider the sequence $x_n = -n$. Then
   $$d(x_n, x_m) = |e^{-n} - e^{-m}| \leq e^{-n} + e^{-m}.$$
   Given $\varepsilon > 0$, choose N such that $e^{-N} < \varepsilon/2$. Then for all $n, m \geq N$, $d(x_n, x_m) < \varepsilon$, so (x_n) is Cauchy.

   Suppose $x_n \to \ell \in \mathbb{R}$ in this metric. Then
   $$d(x_n, \ell) = |e^{-n} - e^{\ell}| \xrightarrow{n\to\infty} |0 - e^{\ell}| = e^{\ell}.$$
   For convergence, this limit must be 0, so $e^{\ell} = 0$, which is impossible for real ℓ. Hence, (x_n) does not converge, and the space is not complete.

3. **Complete.**

   The function $f(t) = \log(1+t)$ satisfies, for $t \geq 0$,
   $$\frac{t}{1+t} \leq \log(1+t) \leq t.$$
   In particular, for $t \in [0,1]$, we have $\frac{1}{2}t \leq \log(1+t) \leq t$.

   Let (x_n) be a Cauchy sequence for d. Then for any $\varepsilon \in (0,1)$, there exists N such that for all $m, n \geq N$,
   $$\log(1 + |x_n - x_m|) < \varepsilon \quad \Rightarrow \quad |x_n - x_m| \leq 2\log(1 + |x_n - x_m|) < 2\varepsilon.$$
   Thus, (x_n) is Cauchy in the standard metric, so $x_n \to x \in \mathbb{R}$ for $|\cdot|$.

   Now, since $|x_n - x| \to 0$, we have $\log(1 + |x_n - x|) \to \log(1) = 0$, so $x_n \to x$ in the metric d. Therefore, $(\mathbb{R}, d)$ is complete.

Exercise 57. Let $E = \{x_n \mid n \in \mathbb{N}^*\}$ be a countable set. Define a function $d : E \times E \to \mathbb{R}$ by
$$d(x_n, x_m) = \begin{cases} 0 & \text{if } n = m, \\ 1 + \dfrac{1}{n} + \dfrac{1}{m} & \text{if } n \neq m. \end{cases}$$

1. Verify that d is a metric on E.
2. Show that (E, d) is a complete metric space.

Solution 57. .

1. We check the metric axioms.
   - *Non-negativity and identity of indiscernibles:* By definition, $d(x_n, x_m) \geq 0$ and $d(x_n, x_m) = 0$ if and only if $n = m$, i.e., $x_n = x_m$.
   - *Symmetry:* Clearly, $d(x_n, x_m) = d(x_m, x_n)$ because the expression $1 + \frac{1}{n} + \frac{1}{m}$ is symmetric in n and m.
   - *Triangle inequality:* Let $x_n, x_m, x_k \in E$. If any two indices coincide, the inequality is immediate. Suppose n, m, k are pairwise distinct. Then
   $$d(x_n, x_k) = 1 + \frac{1}{n} + \frac{1}{k} \leq \left(1 + \frac{1}{n} + \frac{1}{m}\right) + \left(1 + \frac{1}{m} + \frac{1}{k}\right) = d(x_n, x_m) + d(x_m, x_k),$$
   since each term on the right is at least 1. The triangle inequality also holds in mixed cases (e.g., $n = m \neq k$), as both sides reduce appropriately. Thus, d is a metric.

2. **Completeness.** Let $(y_k)_{k\in\mathbb{N}}$ be a Cauchy sequence in (E,d). Since all points of E are of the form x_n, we may write $y_k = x_{n_k}$ for some sequence (n_k) in $\mathbb{N}^*$.

   Choose $\varepsilon = 1$. Because (y_k) is Cauchy, there exists $K \in \mathbb{N}$ such that for all $k,\ell \geq K$,
$$d(y_k, y_\ell) < 1.$$
   But if $y_k \neq y_\ell$, then $d(y_k,y_\ell) = 1 + \frac{1}{n_k} + \frac{1}{n_\ell} > 1$, a contradiction. Hence, for all $k,\ell \geq K$, we must have $y_k = y_\ell$.

   Therefore, the sequence (y_k) is eventually constant, hence convergent in E. Since every Cauchy sequence converges, (E,d) is complete.

Exercise 58. Let $E = C([a,b])$ be the space of real-valued continuous functions on the compact interval $[a,b]$. Let $h \in E$ be a function such that $h(t) \neq 0$ for all $t \in [a,b]$. Define a metric d_h on E by
$$d_h(f,g) = \sup_{t\in[a,b]} \Big| h(t)\,(f(t) - g(t)) \Big|.$$
Is the metric space (E,d_h) complete?

Solution 58. Yes, (E,d_h) is complete.

Let $(f_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in (E,d_h). We will show that it converges to a function $f \in C([a,b])$ with respect to d_h.

Step 1: Pointwise convergence. For each fixed $t \in [a,b]$, we have
$$|f_n(t) - f_m(t)| = \frac{1}{|h(t)|} \Big| h(t)(f_n(t) - f_m(t)) \Big| \leq \frac{1}{|h(t)|}\, d_h(f_n,f_m).$$
Since (f_n) is Cauchy in d_h, the right-hand side tends to 0 as $n,m \to \infty$. Thus, $(f_n(t))_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathbb{R}$, which is complete. Define
$$f(t) := \lim_{n\to\infty} f_n(t), \quad t \in [a,b].$$

Step 2: Uniform convergence with respect to d_h. Given $\varepsilon > 0$, choose N such that for all $n,m \geq N$,
$$d_h(f_n,f_m) = \sup_{t\in[a,b]} \Big| h(t)(f_n(t) - f_m(t)) \Big| < \varepsilon.$$
Fix $n \geq N$ and let $m \to \infty$. By pointwise convergence, $f_m(t) \to f(t)$ for each t, and the inequality is preserved in the limit (by continuity of absolute value and supremum over a compact set):
$$\sup_{t\in[a,b]} \Big| h(t)(f_n(t) - f(t)) \Big| \leq \varepsilon.$$
Hence, $d_h(f_n,f) \leq \varepsilon$ for all $n \geq N$, so $f_n \to f$ in (E,d_h).

Step 3: Continuity of the limit function f. Since h is continuous and never vanishes on the compact set $[a,b]$, it attains a positive minimum:
$$\exists\, \alpha > 0 \quad \text{such that} \quad |h(t)| \geq \alpha \quad \text{for all } t \in [a,b].$$
Then, for all n,
$$\|f_n - f\|_\infty = \sup_{t\in[a,b]} |f_n(t) - f(t)| \leq \frac{1}{\alpha} \sup_{t\in[a,b]} |h(t)(f_n(t) - f(t))| = \frac{1}{\alpha}\, d_h(f_n,f).$$
Since $d_h(f_n,f) \to 0$, it follows that $f_n \to f$ uniformly on $[a,b]$. As the uniform limit of continuous functions, f is continuous, i.e., $f \in C([a,b])$.

Thus, every Cauchy sequence in (E,d_h) converges to an element of E, and the space is complete.

Exercise 59. Soit $f\colon \mathbb{R} \to \mathbb{R}$ une fonction continue telle qu'il existe une constante $a > 0$ vérifiant, pour tous $x, y \in \mathbb{R}$,

$$|f(x+y) - f(x) - f(y)| \leq a.$$

1. Montrer que, pour tout $y \in \mathbb{R}$ et tout entier $k \geq 1$,
$$\left|f(2^k y) - 2^k f(y)\right| \leq 2^k a.$$
2. En déduire que, pour tout $x \in \mathbb{R}$, la suite $\left(\dfrac{f(2^n x)}{2^n}\right)_{n\geq 0}$ est de Cauchy.
3. En déduire que la fonction $g\colon \mathbb{R} \to \mathbb{R}$ définie par
$$g(x) = \lim_{n\to\infty} \frac{f(2^n x)}{2^n}$$
est bien définie et vérifie $g(x+y) = g(x) + g(y)$ pour tous $x, y \in \mathbb{R}$.

Solution 59. .

1. **Preuve par récurrence.** Pour $k = 1$, on a, en prenant $x = y$ dans l'hypothèse,
$$|f(2y) - 2f(y)| \leq a = 2^1 \cdot \frac{a}{2} \leq 2^1 a,$$
donc la propriété est vraie.
Supposons que, pour un certain $k \geq 1$,
$$|f(2^k y) - 2^k f(y)| \leq 2^k a.$$
Appliquons l'hypothèse à $x = y = 2^k y$:
$$|f(2^{k+1} y) - 2f(2^k y)| \leq a.$$
Alors, en utilisant l'inégalité triangulaire,
$$\begin{aligned}|f(2^{k+1}y) - 2^{k+1}f(y)| &\leq |f(2^{k+1}y) - 2f(2^k y)| + |2f(2^k y) - 2^{k+1}f(y)| \\ &\leq a + 2\cdot |f(2^k y) - 2^k f(y)| \\ &\leq a + 2\cdot 2^k a = a + 2^{k+1}a \leq 2^{k+1}a,\end{aligned}$$
ce qui achève la récurrence.
2. **La suite est de Cauchy.** Soient $m > n \geq 0$. Posons $x \in \mathbb{R}$ fixé. On écrit :
$$\left|\frac{f(2^m x)}{2^m} - \frac{f(2^n x)}{2^n}\right| = \left|\frac{f(2^n \cdot 2^{m-n}x)}{2^n \cdot 2^{m-n}} - \frac{f(2^n x)}{2^n}\right| = \frac{1}{2^n}\left|\frac{f(2^{m-n}(2^n x))}{2^{m-n}} - f(2^n x)\right|.$$
D'après la question précédente appliquée à $y = 2^n x$ et $k = m - n$, on a
$$\left|f(2^m x) - 2^{m-n} f(2^n x)\right| \leq 2^{m-n} a.$$
En divisant par $2^m = 2^n \cdot 2^{m-n}$, on obtient
$$\left|\frac{f(2^m x)}{2^m} - \frac{f(2^n x)}{2^n}\right| \leq \frac{a}{2^n}.$$
Comme $\frac{a}{2^n} \to 0$ lorsque $n \to \infty$, la suite $\left(\frac{f(2^n x)}{2^n}\right)_{n\geq 0}$ est de Cauchy dans $\mathbb{R}$.

3. **Définition et additivité de g.** Puisque $\mathbb{R}$ est complet, la limite

$$g(x) := \lim_{n\to\infty} \frac{f(2^n x)}{2^n}$$

existe pour tout $x \in \mathbb{R}$.

Montrons que g est additive. Soient $x, y \in \mathbb{R}$. On a, pour tout n,

$$|f(2^n(x+y)) - f(2^n x) - f(2^n y)| \leq a,$$

d'après l'hypothèse initiale appliquée à $x' = 2^n x$, $y' = 2^n y$. En divisant par 2^n, on obtient

$$\left|\frac{f(2^n(x+y))}{2^n} - \frac{f(2^n x)}{2^n} - \frac{f(2^n y)}{2^n}\right| \leq \frac{a}{2^n}.$$

En passant à la limite quand $n \to \infty$, le membre de droite tend vers 0, et on obtient

$$g(x+y) = g(x) + g(y).$$

Ainsi, g est une fonction additive sur $\mathbb{R}$. (Remarque : grâce à la continuité de f, on peut montrer que g est en fait linéaire, mais cela n'est pas demandé ici.)

3.3 Extension of Uniformly Continuous Functions

Theorem 3.13. Let (E, d_E) and (F, d_F) be metric spaces, with F complete. Let $A \subseteq E$ be a dense subset, and let $f\colon A \to F$ be a uniformly continuous function. Then there exists a unique continuous function $\widetilde{f}\colon E \to F$ such that

$$\widetilde{f}|_A = f.$$

Moreover, $\widetilde{f}$ is uniformly continuous on E.

Proof. We prove existence and uniqueness separately.

1. Existence. Let $x \in E$. Since A is dense in E, there exists a sequence $(x_n)_{n\in\mathbb{N}} \subseteq A$ such that $x_n \to x$ in E. Because (x_n) converges, it is Cauchy in E, and hence Cauchy in A. Since f is uniformly continuous, the image sequence $(f(x_n))$ is Cauchy in F. As F is complete, $(f(x_n))$ converges to some point in F.

We must verify that this limit depends only on x, not on the choice of the approximating sequence. Suppose $(y_n) \subseteq A$ is another sequence with $y_n \to x$. Consider the interlaced sequence (z_n) defined by $z_{2n} = x_n$, $z_{2n+1} = y_n$. Then $z_n \to x$, so (z_n) is Cauchy in A, and $(f(z_n))$ is Cauchy in F, hence convergent. But $(f(x_n))$ and $(f(y_n))$ are subsequences of $(f(z_n))$, so they converge to the same limit. Denote this common limit by $\widetilde{f}(x)$.

Thus, the map $\widetilde{f}\colon E \to F$ is well-defined and extends f.

We now show that $\widetilde{f}$ is uniformly continuous. Let $\varepsilon > 0$. Since f is uniformly continuous on A, there exists $\delta > 0$ such that for all $u, v \in A$,

$$d_E(u, v) < \delta \quad \Rightarrow \quad d_F(f(u), f(v)) < \varepsilon.$$

Let $x, y \in E$ with $d_E(x, y) < \delta$. Choose sequences $(x_n), (y_n) \subseteq A$ such that $x_n \to x$ and $y_n \to y$. For sufficiently large n, we have

$$d_E(x_n, y_n) \leq d_E(x_n, x) + d_E(x, y) + d_E(y, y_n) < \delta.$$

Hence, $d_F(f(x_n), f(y_n)) < \varepsilon$ for all such n. Taking limits as $n \to \infty$ and using the definition of $\widetilde{f}$, we obtain

$$d_F(\widetilde{f}(x), \widetilde{f}(y)) \leq \varepsilon.$$

Therefore, $\widetilde{f}$ is uniformly continuous (hence continuous) on E.

2. Uniqueness. Suppose $\widetilde{f}_1, \widetilde{f}_2 \colon E \to F$ are two continuous extensions of f. Consider the set

$$A_0 = \{x \in E \mid \widetilde{f}_1(x) = \widetilde{f}_2(x)\}.$$

Since $\widetilde{f}_1$ and $\widetilde{f}_2$ are continuous and F is Hausdorff, A_0 is closed in E. Moreover, $A \subseteq A_0$ because both extensions agree with f on A. As A is dense in E and A_0 is closed, it follows that $A_0 = E$. Hence, $\widetilde{f}_1 = \widetilde{f}_2$.

This proves uniqueness, and the theorem is established. ■

3.3.1 Exercises

Exercise 60. Let $\varphi \colon \mathbb{N} \to \mathbb{Q} \cap [0,1]$ be a bijection (which exists because $\mathbb{Q} \cap [0,1]$ is countable). Define a function $f \colon [0,1] \to \mathbb{R}$ by

$$f(x) = \sum_{\substack{n \in \mathbb{N} \\ \varphi(n) < x}} \frac{1}{2^n}.$$

Show that the restriction of f to the set $\Omega = [0,1] \setminus \mathbb{Q}$ of irrational numbers is continuous, but that f cannot be extended to a continuous function on all of $[0,1]$.

Solution 60. First observe that f is well-defined: the series $\sum_{n=0}^{\infty} 2^{-n} = 2$ converges, so every subseries (including those indexed by $\{n \mid \varphi(n) < x\}$) converges absolutely. Moreover, f is non-decreasing: if $x < y$, then $\{n \mid \varphi(n) < x\} \subseteq \{n \mid \varphi(n) < y\}$, so $f(x) \leq f(y)$.

1. Continuity on the irrationals. Let $x \in \Omega$ (so x is irrational), and let $\varepsilon > 0$. Since $\sum_{n=0}^{\infty} 2^{-n}$ converges, there exists $N \in \mathbb{N}$ such that

$$\sum_{n=N+1}^{\infty} \frac{1}{2^n} < \varepsilon.$$

The set $\{\varphi(0), \varphi(1), \ldots, \varphi(N)\}$ is finite. Because x is irrational, none of these rational numbers equals x. Therefore, we can choose $\delta > 0$ so small that the interval $(x - \delta, x + \delta)$ contains none of $\varphi(0), \ldots, \varphi(N)$.

Now let $y \in \Omega$ with $|x - y| < \delta$. Without loss of generality, assume $y > x$ (the case $y < x$ is similar). Then

$$|f(y) - f(x)| = f(y) - f(x) = \sum_{\substack{n \in \mathbb{N} \\ x \leq \varphi(n) < y}} \frac{1}{2^n}.$$

But any $\varphi(n)$ in $[x, y)$ must satisfy $n > N$ (since no $\varphi(n)$ with $n \leq N$ lies in $(x - \delta, x + \delta)$). Hence,

$$|f(y) - f(x)| \leq \sum_{n=N+1}^{\infty} \frac{1}{2^n} < \varepsilon.$$

Thus, f is continuous at every irrational point.

2. No continuous extension to $[0,1]$**.** Let $b \in \mathbb{Q} \cap [0,1]$. Then $b = \varphi(m)$ for a unique $m \in \mathbb{N}$. Consider sequences of irrationals $(x_k) \uparrow b$ and $(y_k) \downarrow b$. Then: - For all k, $f(x_k) = \sum_{\varphi(n) < x_k} 2^{-n} \leq \sum_{\varphi(n) < b,\, n \neq m} 2^{-n}$, - While $f(y_k) \geq \sum_{\varphi(n) < b} 2^{-n} = f(x_k) + 2^{-m}$.

Hence,

$$\limsup_{x\to b^-,\, x\in\Omega} f(x) \le f(b) - \frac{1}{2^m}, \quad \liminf_{x\to b^+,\, x\in\Omega} f(x) \ge f(b).$$

In particular, the left and right limits at b (along irrationals) differ by at least $2^{-m} > 0$. Therefore, f has a jump discontinuity at every rational point.

Consequently, there is no way to redefine (or extend) f at rational points to make it continuous on all of $[0,1]$. Even though $f|_\Omega$ is continuous, it does **not** admit a continuous extension to the whole compact interval $[0,1]$.

3.4 Fixed Points of Contractions

3.4.1 Contractive Mappings

Definition 3.14. Let (E,d) and (E',d') be metric spaces, and let $f\colon E\to E'$ be a mapping. We say that f is *contractive* (or a *contraction*) if there exists a constant $k\in[0,1)$ such that for all $x,y\in E$,

$$d'(f(x),f(y)) \le k\, d(x,y).$$

In this case, f is also said to be k-*contractive*.

Remark 3.15. .

1. Every contractive mapping is uniformly continuous (indeed, Lipschitz continuous), and hence continuous.
2. The condition $k<1$ is essential: if $k=1$, the map is merely non-expansive and need not have a fixed point (e.g., $f(x)=x+1$ on $\mathbb{R}$).

3.4.2 Banach Fixed Point Theorem

Theorem 3.16 (Banach Fixed Point Theorem)**.** Let (E,d) be a non-empty complete metric space, and let $f\colon E\to E$ be a k-contractive mapping for some $k\in[0,1)$. Then f has a unique fixed point $x\in E$, i.e., there exists a unique $x\in E$ such that

$$f(x)=x.$$

Moreover, for any initial point $x_0\in E$, the Picard iterates defined by $x_{n+1}=f(x_n)$ converge to this fixed point.

Proof. **Existence.** Fix an arbitrary $x_0\in E$, and define a sequence $(x_n)_{n\in\mathbb{N}}$ recursively by

$$x_{n+1}=f(x_n), \quad n\in\mathbb{N}.$$

We first show that (x_n) is Cauchy. Since f is k-contractive,

$$d(x_{n+1},x_n)=d(f(x_n),f(x_{n-1}))\le k\,d(x_n,x_{n-1})\le\cdots\le k^n d(x_1,x_0).$$

For integers $m>n\ge 0$, the triangle inequality yields:

$$d(x_m,x_n)\le\sum_{j=n}^{m-1} d(x_{j+1},x_j)\le\sum_{j=n}^{m-1} k^j d(x_1,x_0)=d(x_1,x_0)\,k^n\frac{1-k^{m-n}}{1-k}\le\frac{k^n}{1-k}\,d(x_1,x_0).$$

Since $0 \le k < 1$, we have $k^n \to 0$ as $n \to \infty$. Thus, for every $\varepsilon > 0$, there exists N such that $d(x_m, x_n) < \varepsilon$ for all $m > n \ge N$, so (x_n) is Cauchy.

Because E is complete, there exists $x \in E$ such that $x_n \to x$. As f is contractive, it is continuous. Passing to the limit in $x_{n+1} = f(x_n)$ gives

$$x = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} f(x_n) = f\Big(\lim_{n\to\infty} x_n\Big) = f(x),$$

so x is a fixed point.

Uniqueness. Suppose $x, x' \in E$ satisfy $f(x) = x$ and $f(x') = x'$. Then

$$d(x, x') = d(f(x), f(x')) \le k\, d(x, x').$$

Since $k < 1$, this implies $d(x, x') = 0$, so $x = x'$.

Hence, the fixed point exists and is unique. Moreover, the convergence of (x_n) to x holds for every initial $x_0 \in E$. ■

3.4.3 Exercises

Exercise 61. Let $E = (0, +\infty)$. For $x, y \in E$, define

$$\delta(x, y) = |\log x - \log y|.$$

1. Verify that δ is a metric on E.
2. Let $d(x, y) = |x - y|$ be the usual metric on E. Show that d and δ are topologically equivalent, i.e., they induce the same topology on E.
3. Show that (E, d) is not complete.
4. Consider the sequence $(x_n) = (1/n)_{n\ge1}$. Is it convergent in (E, δ)? Is it Cauchy in (E, δ)?
5. Show that (E, δ) is complete.
6. Let $f\colon E \to E$ be a differentiable function such that there exists $k \in [0, 1)$ satisfying

$$x|f'(x)| \le k f(x) \quad \text{for all } x \in E.$$

Show that f has a unique fixed point in E.

Solution 61. .

1. **δ is a metric.**
   - $\delta(x, y) = 0 \iff \log x = \log y \iff x = y.$
   - $\delta(x, y) = |\log x - \log y| = |\log y - \log x| = \delta(y, x).$
   - $\delta(x, z) = |\log x - \log z| = |(\log x - \log y) + (\log y - \log z)| \le \delta(x, y) + \delta(y, z).$

   Hence, δ is a metric.

2. **Topological equivalence.** . By continuity of the function $x \mapsto \log x$ on $(0, +\infty)$, for any $x_0 \in (0, +\infty)$ and for every $\varepsilon > 0$, there exists $\alpha > 0$ such that

$$\forall x \in (0, +\infty), \quad |x - x_0| < \alpha \implies |\log x - \log x_0| < \varepsilon,$$

that is,

$$B_d(x_0, \alpha) \subset B_\delta(x_0, \varepsilon),$$

and therefore $\mathcal{T}_\delta \subset \mathcal{T}_d$.

Conversely, by continuity of the function $x \mapsto \exp(x)$ on $\mathbb{R}$ (or equivalently, the continuity of the inverse of ln on $(0, +\infty)$), for any $x_0 \in (0, +\infty)$ and for every $\varepsilon > 0$, there exists $\alpha > 0$ such that

$$\forall x \in (0, +\infty), \quad |\log x - \log x_0| < \alpha \implies |x - x_0| < \varepsilon,$$

that is,

$$B_\delta(x_0, \alpha) \subset B_d(x_0, \varepsilon),$$

and hence $\mathcal{T}_d \subset \mathcal{T}_\delta$. Consequently, $\mathcal{T}_d = \mathcal{T}_\delta$.

3. (E, d) **is not complete.** The sequence $x_n = 1/n$ is Cauchy in (E, d) because $|x_n - x_m| \leq 1/n + 1/m \to 0$. However, $x_n \to 0 \notin E$, so the space is not complete.

4. **Behavior of** $(1/n)$ **in** (E, δ)**.** We have

$$\delta(x_n, x_m) = |\log(1/n) - \log(1/m)| = |\log m - \log n|.$$

The sequence $(\log n)$ diverges to $+\infty$ in $\mathbb{R}$, so it is **not** Cauchy. Hence, (x_n) is **not** Cauchy in (E, δ), and therefore **does not converge** in (E, δ).

5. (E, δ) **is complete.** Let (x_n) be a Cauchy sequence in (E, δ). Then $(\log x_n)$ is Cauchy in $(\mathbb{R}, |\cdot|)$. Since $\mathbb{R}$ is complete, $\log x_n \to \ell \in \mathbb{R}$. Let $x = e^\ell > 0$. Then

$$\delta(x_n, x) = |\log x_n - \log x| = |\log x_n - \ell| \to 0,$$

so $x_n \to x$ in (E, δ). Thus, (E, δ) is complete.

6. **Fixed point of** f**.** Note that the condition $x|f'(x)| \leq kf(x)$ implies $f(x) > 0$ for all x (since $f: E \to E$). Consider the function $g(x) = \log f(x)$. Then

$$|g'(x)| = \left| \frac{f'(x)}{f(x)} \right| \leq \frac{k}{x}.$$

For $0 < x < y$, by the mean value theorem,

$$|g(y) - g(x)| \leq \int_x^y |g'(t)| \, dt \leq \int_x^y \frac{k}{t} \, dt = k|\log y - \log x| = k\delta(x, y).$$

Therefore,

$$\delta(f(x), f(y)) = |\log f(x) - \log f(y)| = |g(x) - g(y)| \leq k\delta(x, y),$$

so f is a k-contraction on the complete metric space (E, δ). By the Banach Fixed Point Theorem (Theorem 3.16), f has a unique fixed point in E.

Chapter 4

Connected Spaces

4.1 Connectedness

Definition 4.1. Let E be a topological space. We say that E is *connected* if any (and hence all) of the following equivalent conditions hold:

1. There do not exist two non-empty, disjoint open subsets $U, V \subseteq E$ such that $E = U \cup V$.

2. There do not exist two non-empty, disjoint closed subsets $F, G \subseteq E$ such that $E = F \cup G$.

3. The only subsets of E that are both open and closed (clopen) are $\emptyset$ and E itself.

4. Every continuous function $f : E \to \{0, 1\}$ (where $\{0, 1\}$ is equipped with the discrete topology) is constant.

Definition 4.2. A subset $A \subseteq E$ is said to be *connected* (or a *connected subspace* of E) if it is connected with respect to the subspace topology induced by E. Equivalently, A is connected if it satisfies any of the four conditions above when viewed as a topological space in its own right.

Remark 4.3. Connectedness is a topological property: it is preserved under homeomorphisms. Moreover, the image of a connected space under a continuous map is connected.

Example 4.4. .

1. Intuitively, a space like a single interval or a solid disk is connected, whereas a space consisting of two disjoint pieces (e.g., two separated intervals) is not. (A schematic illustration may accompany this point in lecture notes.)

2. The set $\mathbb{R}^* = \mathbb{R} \setminus \{0\}$ is **not connected**. Indeed, the subsets $(-\infty, 0)$ and $(0, +\infty)$ are both open and closed in $\mathbb{R}^*$ (they are complements of each other and each is open in the subspace topology). Thus, $\mathbb{R}^*$ is the disjoint union of two non-empty clopen sets.

3. In $\mathbb{R}$:

   - Any singleton $\{a\}$ is connected (it cannot be partitioned into two non-empty disjoint subsets).
   - The sets $\mathbb{N}$ and $\mathbb{Z}$ are **not connected** in $\mathbb{R}$. For example, $\mathbb{N} = \{0\} \cup \{1, 2, 3, \dots\}$ is a separation into two non-empty disjoint sets that are open in the subspace topology (since every subset of a discrete subspace is open).

4. The set $\mathbb{Q}$ of rational numbers is **not connected** in $\mathbb{R}$. Indeed, fix an irrational number $r \in \mathbb{R} \setminus \mathbb{Q}$, and define

$$A = \{q \in \mathbb{Q} \mid q < r\}, \quad B = \{q \in \mathbb{Q} \mid q > r\}.$$

   Then A and B are non-empty, disjoint, relatively open in $\mathbb{Q}$, and satisfy $\mathbb{Q} = A \cup B$. Hence, $\mathbb{Q}$ is disconnected.

5. A topological space equipped with the *trivial (indiscrete) topology* $\{\emptyset, E\}$ is always connected, since the only open sets are $\emptyset$ and E itself non trivial separation is possible.

6. A topological space with the *discrete topology* is connected **if and only if** it contains at most one point. Indeed, if E has two or more points, then any singleton $\{x\}$ is both open and closed, providing a non trivial clopen subset, so the space is disconnected.

Proposition 4.5. Let A be a connected subset of a topological space (E, τ). Then every subset B satisfying

$$A \subseteq B \subseteq \overline{A}$$

is connected.

Proof. Suppose, for contradiction, that B is disconnected. Then there exist non-empty disjoint open subsets O_1, O_2 in the subspace topology of B such that $B = O_1 \cup O_2$.

By definition of the subspace topology, there exist open sets $U_1, U_2 \in \tau$ such that

$$O_1 = B \cap U_1, \quad O_2 = B \cap U_2.$$

Since $A \subseteq B$, we have

$$A = A \cap B = A \cap (O_1 \cup O_2) = (A \cap U_1) \cup (A \cap U_2).$$

The sets $A \cap U_1$ and $A \cap U_2$ are open in the subspace topology on A, and they are disjoint because

$$(A \cap U_1) \cap (A \cap U_2) = A \cap (U_1 \cap U_2) \subseteq B \cap (U_1 \cap U_2) = O_1 \cap O_2 = \emptyset.$$

Since A is connected, one of these sets must be empty. Assume without loss of generality that $A \cap U_1 = \emptyset$. Then $A \subseteq E \setminus U_1$, so $\overline{A} \subseteq E \setminus U_1$ (as $E \setminus U_1$ is closed). But $B \subseteq \overline{A}$, hence $B \subseteq E \setminus U_1$, which implies

$$O_1 = B \cap U_1 = \emptyset,$$

contradicting the assumption that O_1 is non-empty.

Therefore, B cannot be disconnected, and so B is connected. ■

Proposition 4.6. Let A and B be connected subsets of a topological space E. In general, neither the union $A \cup B$ nor the intersection $A \cap B$ need be connected.

Proof. We provide counterexamples in $\mathbb{R}^2$ with the usual topology.

Union may be disconnected. Let

$$A = \{(x, 0) \mid x \in [-2, -1]\}, \quad B = \{(x, 0) \mid x \in [1, 2]\}.$$

Both A and B are homeomorphic to closed intervals, hence connected. However, $A \cup B$ is the disjoint union of two non-empty separated sets, so it is disconnected.

Note: if $A \cap B \neq \emptyset$, then $A \cup B$ *is* connected. But without this condition, the union can fail to be connected.

Intersection may be disconnected (or empty). Let

$$A = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\} \quad \text{(the unit circle)},$$

$$B = \{(x, y) \in \mathbb{R}^2 \mid (x-1)^2 + y^2 = 1\} \quad \text{(a circle of radius 1 centered at } (1, 0)\text{)}.$$

Both A and B are connected (as continuous images of $[0, 2\pi]$). Their intersection consists of exactly two points:

$$A \cap B = \left\{ \left(\tfrac{1}{2}, \tfrac{\sqrt{3}}{2}\right),\ \left(\tfrac{1}{2}, -\tfrac{\sqrt{3}}{2}\right) \right\},$$

which is a discrete two-point set hence disconnected (in fact, totally disconnected).

Thus, without additional hypotheses (e.g., $A \cap B \neq \emptyset$ for unions, or convexity in $\mathbb{R}^n$), neither operation preserves connectedness. ■

Proposition 4.7. Let (E, τ) be a topological space, and let $\{A_i\}_{i\in I}$ be a family of connected subsets of E such that

$$\bigcap_{i\in I} A_i \neq \emptyset.$$

Then the union

$$A = \bigcup_{i\in I} A_i$$

is connected.

Proof. Let $x \in \bigcap_{i\in I} A_i$ (such a point exists by hypothesis). Suppose, for contradiction, that A is disconnected. Then there exist non-empty disjoint open subsets O_1, O_2 of A (in the subspace topology) such that

$$A = O_1 \cup O_2.$$

Since $x \in A$, it belongs to either O_1 or O_2; assume without loss of generality that $x \in O_1$.

Now, for each $i \in I$, the set A_i is connected and satisfies $A_i \subseteq A = O_1 \cup O_2$. Moreover, $x \in A_i \cap O_1$, so $A_i \cap O_1 \neq \emptyset$.

If for some $i_0 \in I$ we had $A_{i_0} \cap O_2 \neq \emptyset$, then A_{i_0} would be the union of two non-empty disjoint relatively open subsets:

$$A_{i_0} = (A_{i_0} \cap O_1) \cup (A_{i_0} \cap O_2),$$

contradicting the connectedness of A_{i_0}.

Therefore, for every $i \in I$, we must have $A_i \cap O_2 = \emptyset$, which implies $A \cap O_2 = \emptyset$. But this contradicts the assumption that O_2 is non-empty.

Hence, A cannot be disconnected, and so A is connected. ■

Proposition 4.8. Let $f\colon (E, \tau) \to (E', \tau')$ be a continuous map between topological spaces. If $A \subseteq E$ is connected, then its image $f(A) \subseteq E'$ is also connected.

Proof. Suppose, for contradiction, that $f(A)$ is disconnected. Then there exist non-empty disjoint subsets $U, V \subseteq f(A)$ that are open in the subspace topology of $f(A)$ and satisfy

$$f(A) = U \cup V.$$

By definition of the subspace topology, there exist open sets $U', V' \in \tau'$ such that

$$U = f(A) \cap U', \quad V = f(A) \cap V'.$$

Since f is continuous, the preimages $f^{-1}(U')$ and $f^{-1}(V')$ are open in E. Consider their intersections with A:

$$A_1 = A \cap f^{-1}(U'), \quad A_2 = A \cap f^{-1}(V').$$

These sets are open in the subspace topology on A. Moreover: - $A_1 \neq \emptyset$ and $A_2 \neq \emptyset$ (because U and V are non-empty and contained in $f(A)$), - $A_1 \cap A_2 = A \cap f^{-1}(U' \cap V') = \emptyset$ (since $U \cap V = \emptyset$), - $A = A_1 \cup A_2$, because for every $x \in A$, $f(x) \in f(A) = U \cup V$, so $x \in f^{-1}(U') \cup f^{-1}(V')$.

Thus, A is the union of two non-empty disjoint open subsets, contradicting the connectedness of A.

Therefore, $f(A)$ must be connected. ∎

4.2 Locally Connected Spaces

Definition 4.9. A topological space E is said to be *locally connected* if every point $x \in E$ admits a neighborhood basis consisting of connected open sets. Equivalently, for every $x \in E$ and every neighborhood U of x, there exists a connected open set V such that $x \in V \subseteq U$.

Definition 4.10. Let (E, τ) be a topological space and let $x \in E$. The *connected component* of x is the union of all connected subsets of E that contain x. It is denoted by $C(x)$.

Remark 4.11. The connected component $C(x)$ is itself a connected subset of E, and it is the largest connected subset containing x (with respect to inclusion). Moreover, the connected components of E form a partition of E into disjoint maximal connected subsets.

In a locally connected space, every connected component is open. Consequently, if a space is both locally connected and connected, it is *path-connected* in many familiar settings (e.g., open subsets of $\mathbb{R}^n$), although local connectedness alone does not imply path-connectedness in general.

Example 4.12. Let $X = [0,1) \cup [2,3) \subseteq \mathbb{R}$. The connected components of X are the two intervals $[0,1)$ and $[2,3)$. Indeed, each is connected (as intervals in $\mathbb{R}$), and they are maximal: no larger connected subset of X contains either.

Moreover, both components are **both open and closed** in the subspace topology of X:

$$[0,1) = (-1,1) \cap X, \quad [2,3) = \left(\tfrac{3}{2}, 3\right) \cap X,$$

so they are relatively open; and since they are complements of each other in X, they are also relatively closed.

Proposition 4.13. Let E be a topological space and let $x \in E$. Denote by $C(x)$ the connected component of x. Then:

1. $C(x)$ is the largest (with respect to inclusion) connected subset of E containing x.

2. $C(x)$ is a closed subset of E.

Proof. .

This follows directly from the definition: $C(x)$ is the union of all connected subsets of E that contain x, so it contains every such set and is itself connected (by Proposition 4.7, applied to the family of all connected sets containing x, which all intersect at x).

2. Let $C = C(x)$. By Proposition 4.5, the closure $\overline{C}$ is connected (since $C \subseteq \overline{C} \subseteq \overline{C}$). Moreover, $\overline{C}$ contains x, so by maximality of C, we must have $\overline{C} \subseteq C$. Hence, $\overline{C} = C$, which means C is closed.

∎

Remark 4.14. .

1. Local connectedness is **not preserved** under continuous maps. For example, consider the topologist's sine curve

$$S = \left\{ \left(x, \sin\left(\tfrac{1}{x}\right)\right) \,\middle|\, 0 < x \leq 1 \right\} \cup \{(0, y) \mid -1 \leq y \leq 1\}.$$

   The domain $(0, 1]$ is locally connected, but its continuous image under $x \mapsto (x, \sin(1/x))$, when combined with the limit segment, yields a space that is connected but not locally connected at points on the vertical segment $\{0\} \times [-1, 1]$.

2. A topological space E is locally connected **if and only if** for every open subset $U \subseteq E$, each connected component of U is open in E. In particular, in a locally connected space, the connected components of *any* open set form a partition of that set into open (hence also closed-in-U) connected subsets.

Example 4.15. .

1. The most classical examples of locally connected spaces are the Euclidean spaces $\mathbb{R}^n$ (including $\mathbb{R}$ and $\mathbb{C} \cong \mathbb{R}^2$). In fact, every open subset of $\mathbb{R}^n$ is locally connected, because open balls form a basis of connected (even convex) neighborhoods.

2. A standard example of a space that is **connected but not locally connected** is the *topologist's sine curve*:

$$T = \left\{ \left(x, \sin\left(\tfrac{1}{x}\right)\right) \in \mathbb{R}^2 \,\middle|\, 0 < x \leq 1 \right\} \cup \Big(\{0\} \times [-1, 1]\Big).$$

   The space T is connected (it is the closure of the graph of $\sin(1/x)$ on $(0, 1]$, which is connected). However, it is **not locally connected** at any point on the vertical segment $\{0\} \times [-1, 1]$. Indeed, any small neighborhood of such a point intersects infinitely many oscillations of the sine curve, and no sufficiently small neighborhood is connected.

   Remark: The set $A = (\{0\} \times \mathbb{R}) \cup (\mathbb{R} \times \mathbb{Q}) \subset \mathbb{R}^2$, sometimes proposed as an example, is actually **disconnected**. For instance, the sets

$$A_+ = A \cap \Big(\mathbb{R} \times (0, \infty)\Big), \quad A_- = A \cap \Big(\mathbb{R} \times (-\infty, 0]\Big)$$

   are non-empty, disjoint, relatively open in A, and cover A. Thus, A is not connected. The topologist's sine curve remains the canonical correct example.

Proposition 4.16. (**Connected Subsets of** $\mathbb{R}$) The connected subsets of $\mathbb{R}$ (with the standard topology) are precisely the intervals (including unbounded intervals and singletons). In particular, $\mathbb{R}$ is connected and locally connected.

Proof. ($\Rightarrow$) Let $A \subseteq \mathbb{R}$ be connected. We show that A is an interval. Suppose, for contradiction, that there exist $x, y \in A$ with $x < y$, and some $a \in \mathbb{R}$ such that $x < a < y$ but $a \notin A$. Then

$$A = \Big(A \cap (-\infty, a)\Big) \cup \Big(A \cap (a, +\infty)\Big).$$

Both sets on the right are non-empty (they contain x and y, respectively), open in the subspace topology on A, and disjoint. This contradicts the connectedness of A. Hence, A must be an interval.

($\Leftarrow$) Conversely, let $I \subseteq \mathbb{R}$ be an interval. Suppose $f\colon I \to \{0, 1\}$ is continuous, where $\{0, 1\}$ has the discrete topology. By the Intermediate Value Theorem, any continuous real-valued function on an interval that takes only the values 0 and 1 must be constant (otherwise it would have to take all intermediate values). Thus, f is constant, so I is connected.

Since $\mathbb{R}$ itself is an interval, it is connected. Moreover, for any $x \in \mathbb{R}$, the open intervals $(x - \varepsilon, x + \varepsilon)$ with $\varepsilon > 0$ form a neighborhood basis of connected (indeed, convex) sets. Hence, $\mathbb{R}$ is locally connected. ∎

Theorem 4.17 (Intermediate Value Theorem). Let $f\colon (a,b) \to \mathbb{R}$ be a continuous function, where $-\infty \le a < b \le +\infty$. Let $\alpha, \beta \in (a,b)$ with $\alpha < \beta$, and let C be a real number strictly between $f(\alpha)$ and $f(\beta)$; that is,

$$C \in \Big(\min\{f(\alpha), f(\beta)\}, \max\{f(\alpha), f(\beta)\}\Big).$$

Then there exists $c \in (\alpha, \beta)$ such that $f(c) = C$.

Proof. The interval (α, β) is a connected subset of $\mathbb{R}$ (Proposition 4.16). Since f is continuous, the image $f\big((\alpha,\beta)\big)$ is a connected subset of $\mathbb{R}$ (Proposition 4.8). But the connected subsets of $\mathbb{R}$ are precisely the intervals. Therefore, $f\big((\alpha,\beta)\big)$ is an interval containing both $f(\alpha)$ and $f(\beta)$, and hence it contains every real number C between them.

In particular, for any such C, there exists $c \in (\alpha, \beta)$ with $f(c) = C$, which proves the theorem. ■

4.3 Exercises

Exercise 62. Let (E, d) be a metric space. Prove the equivalence of the following statements:

1. The only subsets of E that are both open and closed (clopen) are $\emptyset$ and E.
2. E cannot be written as the union of two non-empty disjoint open subsets.
3. E cannot be written as the union of two non-empty disjoint closed subsets.
4. There is no continuous surjection $f\colon E \to \{0,1\}$, where $\{0,1\}$ is equipped with the discrete topology (as a subspace of $\mathbb{R}$).

Solution 62. We prove the cyclic implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (1).

(1) $\Rightarrow$ (2). Assume (2) fails: there exist non-empty disjoint open sets $U, V \subseteq E$ such that $E = U \cup V$. Then $U = E \setminus V$ is also closed. Thus, U is a non-trivial clopen subset ($U \neq \emptyset$, $U \neq E$), contradicting (1).

(2) $\Rightarrow$ (3). Assume (3) fails: there exist non-empty disjoint closed sets $F, G \subseteq E$ with $E = F \cup G$. Then $F = E \setminus G$ and $G = E \setminus F$ are also open. Hence, E is the union of two non-empty disjoint open sets, contradicting (2).

(3) $\Rightarrow$ (4). Assume (4) fails: there exists a continuous surjection $f\colon E \to \{0,1\}$. Since $\{0\}$ and $\{1\}$ are closed in the discrete space $\{0,1\}$, their preimages

$$F = f^{-1}(\{0\}), \quad G = f^{-1}(\{1\})$$

are closed in E. They are non-empty (by surjectivity), disjoint, and satisfy $E = F \cup G$, contradicting (3).

(4) $\Rightarrow$ (1). Assume (1) fails: there exists a clopen set $A \subseteq E$ with $A \neq \emptyset$ and $A \neq E$. Define $f\colon E \to \{0,1\}$ by

$$f(x) = \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{if } x \in E \setminus A. \end{cases}$$

Since both A and $E \setminus A$ are open (and closed), the preimage of any subset of $\{0,1\}$ is open in E. Hence, f is continuous. Moreover, f is surjective because both A and $E \setminus A$ are non-empty. This contradicts (4).

Therefore, all four statements are equivalent.

Exercise 63. Let $E = \{a, b, c\}$. Determine whether E is connected under the following topologies:

$$\tau_1 = \Big\{\emptyset,\ \{b\},\ \{b, c\},\ E\Big\},\quad \tau_2 = \Big\{\emptyset,\ \{b\},\ \{c\},\ \{a, c\},\ \{b, c\},\ E\Big\}.$$

In each case, find the connected components of E.

Solution 63. Topology τ_1. The open sets are $\emptyset$, $\{b\}$, $\{b, c\}$, and E. Suppose $E = U \cup V$ with U, V non-empty, disjoint, and open in τ_1. The only non-empty proper open sets are $\{b\}$ and $\{b, c\}$, and neither has an open complement: - The complement of $\{b\}$ is $\{a, c\} \notin \tau_1$, - The complement of $\{b, c\}$ is $\{a\} \notin \tau_1$.

Thus, no such separation exists, and E is **connected** under τ_1. Consequently, the only connected component is E itself:

$$C(a) = C(b) = C(c) = E.$$

Topology τ_2. Here, both $\{b\}$ and $\{a, c\}$ are open, non-empty, disjoint, and satisfy

$$E = \{b\} \cup \{a, c\}.$$

Hence, E is **disconnected** under τ_2.

Now consider the subspace $\{a, c\}$. The induced topology is

$$\{\emptyset,\ \{c\},\ \{a, c\}\},$$

which has no non-trivial clopen subsets (the only open sets are $\emptyset$, $\{c\}$, and $\{a, c\}$; but $\{c\}$ is not closed in the subspace, since its complement $\{a\}$ is not open). Therefore, $\{a, c\}$ is connected.

On the other hand, $\{b\}$ is a singleton, hence connected.

Thus, the connected components are:

$$C(a) = C(c) = \{a, c\},\quad C(b) = \{b\}.$$

Exercise 64. Determine the connected subsets (in particular, the connected components) of the following sets:

$$A = \{(x, y) \in \mathbb{R}^2 \mid x \neq y\},\quad B = \{(z, w) \in \mathbb{C}^2 \mid z \neq w\}.$$

Solution 64. We analyze each space separately.

1. The set $A \subseteq \mathbb{R}^2$. Consider the continuous map $f\colon \mathbb{R}^2 \to \mathbb{R}$, $f(x, y) = x - y$. Then

$$A = f^{-1}(\mathbb{R} \setminus \{0\}) = f^{-1}((-\infty, 0) \cup (0, \infty)).$$

Since f is continuous and $\mathbb{R} \setminus \{0\}$ has two connected components, the preimage A is the disjoint union of the two open sets

$$A_1 = \{(x, y) \in \mathbb{R}^2 \mid x < y\},\quad A_2 = \{(x, y) \in \mathbb{R}^2 \mid x > y\}.$$

Each of these is convex (hence path-connected, and thus connected). Moreover, they are open in A and disjoint, so they are precisely the **connected components** of A. Therefore, A has **two** connected components.

2. The set $B \subseteq \mathbb{C}^2$. Identify $\mathbb{C}^2$ with $\mathbb{R}^4$. The diagonal

$$\Delta = \{(z, z) \in \mathbb{C}^2 \mid z \in \mathbb{C}\}$$

is a real 2-dimensional linear subspace of $\mathbb{R}^4$. Its complement $B = \mathbb{C}^2 \setminus \Delta$ is the complement of a proper linear subspace of real codimension 2.

In $\mathbb{R}^n$ with $n \geq 2$, the complement of a linear subspace of codimension ≥ 2 is **path-connected**. Indeed, given two points $p, q \in B$, the straight line segment joining them may intersect Δ in at most one point (since Δ is affine-linear), and one can always perturb the path slightly to avoid Δ. More formally, $\mathbb{C}^2 \setminus \Delta$ is homeomorphic to $\mathbb{C} \times (\mathbb{C} \setminus \{0\})$ via the change of variables $(z, w) \mapsto (z, w - z)$, and $\mathbb{C} \setminus \{0\}$ is path-connected. Hence, B is path-connected, and therefore **connected**.

Thus, B has **exactly one** connected component: itself.

Exercise 65. Let A and B be subsets of a topological space E. Assume that B is connected, $B \cap A \neq \emptyset$, and $B \cap (E \setminus A) \neq \emptyset$. Show that $B \cap \partial A \neq \emptyset$, where ∂A denotes the boundary of A.

Solution 65. Recall that the boundary of A is defined by

$$\partial A = \overline{A} \cap \overline{E \setminus A},$$

and that the space E admits the disjoint decomposition

$$E = \operatorname{int}(A) \sqcup \partial A \sqcup \operatorname{int}(E \setminus A).$$

Suppose, for contradiction, that $B \cap \partial A = \emptyset$. Then

$$B \subseteq \operatorname{int}(A) \cup \operatorname{int}(E \setminus A).$$

Both $\operatorname{int}(A)$ and $\operatorname{int}(E \setminus A)$ are open in E, disjoint, and their union contains B. Moreover:
- Since $B \cap A \neq \emptyset$ and $B \cap \partial A = \emptyset$, we have $B \cap \operatorname{int}(A) \neq \emptyset$ (because $A = \operatorname{int}(A) \cup (\partial A \cap A)$).
- Similarly, $B \cap (E \setminus A) \neq \emptyset$ and $B \cap \partial A = \emptyset$ imply $B \cap \operatorname{int}(E \setminus A) \neq \emptyset$.

Thus, B is the union of two non-empty disjoint sets that are open in the subspace topology on B. This contradicts the assumption that B is connected.

Therefore, our supposition is false, and we must have

$$B \cap \partial A \neq \emptyset.$$

Exercise 66. Let $X = [0, 1]$, and define $d\colon X \times X \to \mathbb{R}$ by

$$d(x, y) = \begin{cases} |x - y| & \text{if } x - y \in \mathbb{Q}, \\ 1 & \text{if } x - y \notin \mathbb{Q}. \end{cases}$$

1. Show that d is a metric on X.
2. Verify that (X, d) is bounded and determine its diameter.
3. Compute the diameter of the interval $\left[\frac{1}{2026}, \frac{1}{2025}\right]$.
4. Let $A = [0, 1] \setminus \mathbb{Q}$ (the irrationals in $[0, 1]$).
   (a) Let $(x_n) \subseteq A$ converge to $x \in X$ in the metric d. Show that x is irrational.
   (b) Deduce that A is closed in (X, d).
5. Let $B = [0, 1] \cap \mathbb{Q}$ (the rationals in $[0, 1]$).
   (a) Let $(x_n) \subseteq B$ converge to $x \in X$ in the metric d. Show that x is rational.

(b) Deduce that B is closed in (X, d).

6. Conclude that (X, d) is not connected.

7. Define $f\colon (X, d) \to (\mathbb{R}, |\cdot|)$ by

$$f(x) = \begin{cases} 0 & \text{if } x \in \mathbb{Q}, \\ 1 & \text{if } x \notin \mathbb{Q}. \end{cases}$$

Show that f is continuous.

Solution 66. .

1. **d is a metric.**
   - *Non-negativity*: $d(x, y) \geq 0$ by definition.
   - *Identity of indiscernibles*: $d(x, y) = 0$ iff $x - y \in \mathbb{Q}$ and $|x - y| = 0$, i.e., $x = y$.
   - *Symmetry*: $x - y \in \mathbb{Q} \iff y - x \in \mathbb{Q}$, and $|x - y| = |y - x|$, so $d(x, y) = d(y, x)$.
   - *Triangle inequality*: Let $x, y, z \in X$.
     - If $x - y \in \mathbb{Q}$, then $d(x, y) = |x - y| \leq |x - z| + |z - y|$. If both $x - z, z - y \in \mathbb{Q}$, then RHS $= d(x, z) + d(z, y)$. Otherwise, at least one of $d(x, z), d(z, y)$ equals 1, so $d(x, z) + d(z, y) \geq 1 \geq |x - y| = d(x, y)$.
     - If $x - y \notin \mathbb{Q}$, then $d(x, y) = 1$. But $d(x, z) + d(z, y) \geq 1$ always (since each term is ≥ 0 and at least one is ≥ 1 unless both differences are rational—yet if both $x - z$ and $z - y$ are rational, then $x - y = (x - z) + (z - y) \in \mathbb{Q}$, a contradiction). Hence, $d(x, z) + d(z, y) \geq 1 = d(x, y)$.

   Thus, d satisfies all axioms of a metric.

2. **Boundedness and diameter.** For all $x, y \in X$, $d(x, y) \leq 1$. Moreover, $d(0, x) = 1$ for any irrational $x \in (0, 1]$, so $\operatorname{diam}(X) = \sup_{x,y \in X} d(x, y) = 1$.

3. **Diameter of $\left[\frac{1}{2026}, \frac{1}{2025}\right]$.** This interval contains both rational and irrational numbers. If x is rational and y is irrational in the interval, then $x - y \notin \mathbb{Q}$, so $d(x, y) = 1$. Hence, the diameter is 1.

4. **The set $A = [0, 1] \setminus \mathbb{Q}$.**
   (a) Suppose $(x_n) \subseteq A$ and $x_n \to x$ in (X, d). If x were rational, then $x_n - x \notin \mathbb{Q}$ for all n, so $d(x_n, x) = 1$ for all n, contradicting convergence. Thus, x is irrational.
   (b) Since every convergent sequence in A converges to a point of A, the set A is closed in (X, d).

5. **The set $B = [0, 1] \cap \mathbb{Q}$.**
   (a) Suppose $(x_n) \subseteq B$ and $x_n \to x$ in (X, d). If x were irrational, then $x_n - x \notin \mathbb{Q}$, so $d(x_n, x) = 1$ for all n, contradicting convergence. Thus, x is rational.
   (b) Hence, B is closed in (X, d).

6. **Disconnectedness.** We have $X = A \cup B$, with $A \cap B = \emptyset$. Both A and B are non-empty, closed, and (since complements of closed sets) also open. Thus, X is the disjoint union of two non-empty clopen subsets, so (X, d) is **not connected**.

7. **Continuity of f.** Note that for any $x, y \in X$:

$$|f(x) - f(y)| = \begin{cases} 0 & \text{if } x, y \in \mathbb{Q} \text{ or } x, y \notin \mathbb{Q}, \\ 1 & \text{otherwise.} \end{cases}$$

But $|f(x) - f(y)| = 1$ occurs exactly when $x - y \notin \mathbb{Q}$, in which case $d(x, y) = 1$. If $x - y \in \mathbb{Q}$, then $f(x) = f(y)$, so $|f(x) - f(y)| = 0 \leq d(x, y)$. Thus, in all cases,

$$|f(x) - f(y)| \leq d(x, y),$$

so f is 1-Lipschitz, hence continuous.

Chapter 5

Normed Vector Spaces

5.1 Norms

Definition 5.1. Let E be a vector space over the field $\mathbb{K}$, where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. A *norm* on E is a function $\|\cdot\|: E \to \mathbb{R}$ satisfying the following properties for all $x, y \in E$ and all $\lambda \in \mathbb{K}$:

[label=1., leftmargin=*]

1. **Positivity:** $\|x\| \geq 0$.
2. **Definiteness (separation):** $\|x\| = 0$ if and only if $x = 0$.
3. **Absolute homogeneity:** $\|\lambda x\| = |\lambda| \, \|x\|$.
4. **Triangle inequality:** $\|x + y\| \leq \|x\| + \|y\|$.

A vector space equipped with a norm is called a *normed vector space*, and is denoted $(E, \|\cdot\|)$.

Remark 5.2 (Notation and terminology)**.** • The value $\|x\|$ is called the *norm* of the vector x.

- A vector x with $\|x\| = 1$ is called a *unit vector* (or *normalized vector*).
- For any non-zero $x \in E$, the vectors $\pm \dfrac{x}{\|x\|}$ are unit vectors colinear with x.

Remark 5.3 (Seminorms)**.** If a function $p: E \to \mathbb{R}$ satisfies properties (1), (3), and (4) above, but not necessarily (2), it is called a *seminorm*. In this case, $p(x) = 0$ may occur for non-zero vectors x.

Example 5.4 (Standard Norms)**.** [label=4., leftmargin=*]

1. **Norms on $\mathbb{K}^n$ (finite-dimensional spaces).** Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, and let $x = (x_1, x_2, \ldots, x_n) \in \mathbb{K}^n$. The following are standard norms on $\mathbb{K}^n$:

   (a) The ℓ^1-norm (taxicab norm): $$\|x\|_1 = \sum_{i=1}^{n} |x_i|,$$

   (b) The ℓ^2-norm (Euclidean norm): $$\|x\|_2 = \left(\sum_{i=1}^{n} |x_i|^2 \right)^{1/2},$$

   (c) The ℓ^∞-norm (maximum norm): $$\|x\|_\infty = \max_{1 \leq i \leq n} |x_i|.$$

   All three norms are equivalent on $\mathbb{K}^n$ (a consequence of finite dimensionality).

2. **Norms on spaces of continuous functions.** Let $C([a, b]; \mathbb{K})$ be the vector space of continuous $\mathbb{K}$-valued functions on the compact interval $[a, b]$. For $f \in C([a, b]; \mathbb{K})$, define:

   (a) The L^1-norm: $$\|f\|_1 = \int_a^b |f(t)|\, dt,$$

   (b) The L^2-norm: $$\|f\|_2 = \left(\int_a^b |f(t)|^2\, dt \right)^{1/2},$$

   (c) The sup-norm (uniform norm): $$\|f\|_\infty = \sup_{t \in [a,b]} |f(t)|.$$

   Note that $\|\cdot\|_\infty$ makes $C([a, b])$ a Banach space, whereas $\|\cdot\|_1$ and $\|\cdot\|_2$ do not (their completions are the Lebesgue spaces L^1 and L^2).

3. **Norms on polynomial spaces.** Let $\mathbb{K}_n[X]$ be the vector space of polynomials of degree at most n with coefficients in $\mathbb{K}$. Every $P \in \mathbb{K}_n[X]$ can be written uniquely as $P(X) = \sum_{i=0}^{n} a_i X^i$. Define:

$$\|P\|_1 = \sum_{i=0}^{n} |a_i|,$$
$$\|P\|_2 = \left(\sum_{i=0}^{n} |a_i|^2 \right)^{1/2},$$
$$\|P\|_\infty = \max_{0 \le i \le n} |a_i|.$$

   These are the ℓ^1, ℓ^2, and ℓ^∞ norms on the coefficient vector $(a_0, \ldots, a_n) \in \mathbb{K}^{n+1}$, and are therefore equivalent.

5.1.1 Exercise

Exercise 67. Define $N \colon \mathbb{R}^2 \to \mathbb{R}$ by

$$N(x, y) = \sup_{t \in \mathbb{R}} \frac{|x + ty|}{\sqrt{1 + t^2}}.$$

Show that N is a norm on $\mathbb{R}^2$.

Solution 67. We verify the four norm axioms.

1. **Positivity:** For all $(x, y) \in \mathbb{R}^2$, the numerator and denominator are non-negative, so $N(x, y) \ge 0$.

2. **Absolute homogeneity:** For $\lambda \in \mathbb{R}$,

$$N(\lambda x, \lambda y) = \sup_{t \in \mathbb{R}} \frac{|\lambda x + t\lambda y|}{\sqrt{1 + t^2}} = |\lambda| \sup_{t \in \mathbb{R}} \frac{|x + ty|}{\sqrt{1 + t^2}} = |\lambda|\, N(x, y).$$

3. **Definiteness:** Suppose $N(x, y) = 0$. Then for all $t \in \mathbb{R}$,

$$|x + ty| = 0 \quad \Rightarrow \quad x + ty = 0.$$

   Taking $t = 0$ gives $x = 0$; then taking $t = 1$ gives $y = 0$. Hence, $(x, y) = (0, 0)$.

4. **Triangle inequality:** For $(x, y), (x', y') \in \mathbb{R}^2$, we have for all $t \in \mathbb{R}$:

$$|(x + x') + t(y + y')| \leq |x + ty| + |x' + ty'|.$$

Dividing by $\sqrt{1 + t^2}$ and taking the supremum over t yields

$$N((x, y) + (x', y')) \leq N(x, y) + N(x', y').$$

Thus, N is a norm on $\mathbb{R}^2$.

Exercise 68. Determine whether the following statement is true or false:

$$N(x, y) = |5x + 3y|$$

defines a norm on $\mathbb{R}^2$.

Solution 68. The statement is **false**. Consider $(x, y) = (3, -5)$. Then

$$N(3, -5) = |5 \cdot 3 + 3 \cdot (-5)| = |15 - 15| = 0,$$

but $(3, -5) \neq (0, 0)$. Hence, definiteness fails, and N is not a norm.

Exercise 69. Let $\alpha_1, \ldots, \alpha_n \in \mathbb{R}$, and define $N \colon \mathbb{R}^n \to \mathbb{R}$ by

$$N(x_1, \ldots, x_n) = \alpha_1|x_1| + \cdots + \alpha_n|x_n|.$$

Find a necessary and sufficient condition on the coefficients α_i for N to be a norm on $\mathbb{R}^n$.

Solution 69. N is a norm on $\mathbb{R}^n$ if and only if

$$\alpha_i > 0 \quad \text{for all } i = 1, \ldots, n.$$

Necessity: If N is a norm, then for each canonical basis vector e_i, we have

$$N(e_i) = \alpha_i > 0,$$

since $e_i \neq 0$.

Sufficiency: Assume $\alpha_i > 0$ for all i.

- *Positivity:* $N(x) \geq 0$ as a sum of non-negative terms.
- *Definiteness:* If $N(x) = 0$, then $\sum \alpha_i |x_i| = 0$. Since each $\alpha_i > 0$, this implies $|x_i| = 0$ for all i, so $x = 0$.
- *Homogeneity:* For $\lambda \in \mathbb{R}$,

$$N(\lambda x) = \sum \alpha_i |\lambda x_i| = |\lambda| \sum \alpha_i |x_i| = |\lambda| N(x).$$

- *Triangle inequality:* Using $|x_i + y_i| \leq |x_i| + |y_i|$,

$$N(x + y) = \sum \alpha_i |x_i + y_i| \leq \sum \alpha_i (|x_i| + |y_i|) = N(x) + N(y).$$

Thus, N is a norm.

Exercise 70. Soit $n \in \mathbb{N}$ et $a_0, a_1, \ldots, a_k$ des réels deux à deux distincts. On définit l'application $\|\cdot\|_k : \mathbb{R}_n[X] \to \mathbb{R}_+$ par

$$\|P\|_k = \sum_{i=0}^{k} |P(a_i)|.$$

À quelle condition sur k (et n) cette application définit-elle une norme sur $\mathbb{R}_n[X]$?

Solution 70. L'application $\|\cdot\|_k$ satisfait clairement les propriétés d'homogénéité et de sous-additivité (inégalité triangulaire) pour tout $k \geq 0$, comme le montre un calcul direct :

- $\|\lambda P\|_k = |\lambda|\,\|P\|_k$ pour tout $\lambda \in \mathbb{R}$,
- $\|P+Q\|_k \leq \|P\|_k + \|Q\|_k$ pour tous $P, Q \in \mathbb{R}_n[X]$.

La question cruciale est donc la *définitude* : a-t-on $\|P\|_k = 0 \implies P = 0$?

Supposons $\|P\|_k = 0$. Alors $\sum_{i=0}^{k} |P(a_i)| = 0$, donc $P(a_i) = 0$ pour tout $i = 0, 1, \ldots, k$. Ainsi, P admet $k+1$ racines distinctes.

Or, tout polynôme non nul de $\mathbb{R}_n[X]$ est de degré au plus n, et ne peut donc avoir plus de n racines distinctes. Par conséquent :

- Si $k+1 > n$, c'est-à-dire $k \geq n$, alors P ayant $k+1 \geq n+1$ racines distinctes, on en déduit que $P = 0$.
- Si $k+1 \leq n$, c'est-à-dire $k \leq n-1$, alors il existe des polynômes non nuls de degré $\leq n$ s'annulant en les $k+1$ points $a_0, \ldots, a_k$. Par exemple, le polynôme

  $$P(X) = (X-a_0)(X-a_1)\cdots(X-a_k)$$

  est non nul, de degré $k+1 \leq n$, et satisfait $\|P\|_k = 0$.

Ainsi, $\|\cdot\|_k$ est une norme sur $\mathbb{R}_n[X]$ **si et seulement si** $k \geq n$ (c'est-à-dire qu'on évalue le polynôme en **au moins** $n+1$ points distincts).

Exercise 71. Let (E, d) be a vector space over $\mathbb{R}$ equipped with a distance function d satisfying the following properties:

1. For all $x, y \in E$ and all $\lambda \in \mathbb{R}$,

   $$d(\lambda x, \lambda y) = |\lambda|\, d(x, y).$$

2. For all $x, y, z \in E$,

   $$d(x+z, y+z) = d(x, y).$$

Show that d is induced by a norm; that is, prove that there exists a norm N on E such that

$$d(x, y) = N(x-y) \quad \text{for all } x, y \in E.$$

Solution 71. Define a map $N : E \to [0, \infty)$ by

$$N(x) := d(x, 0) \quad \text{for all } x \in E.$$

We verify that N satisfies the three axioms of a norm.

1. **Positive definiteness.** $N(x) = 0 \iff d(x, 0) = 0 \iff x = 0$, since d is a metric.

2. **Absolute homogeneity.** Let $\lambda \in \mathbb{R}$ and $x \in E$. Using property (i) and the fact that $\lambda \cdot 0 = 0$, we have

$$N(\lambda x) = d(\lambda x, 0) = d(\lambda x, \lambda \cdot 0) = |\lambda|\, d(x, 0) = |\lambda|\, N(x).$$

3. **Triangle inequality.** Let $x, y \in E$. Using property (ii) with $z = -y$, we obtain

$$N(x + y) = d(x + y, 0) = d\Big((x + y) + (-y),\, 0 + (-y)\Big) = d(x, -y).$$

Since d is a metric, it satisfies the triangle inequality:

$$d(x, -y) \leq d(x, 0) + d(0, -y).$$

By property (i), $d(0, -y) = d(-y, 0) = |-1|\, d(y, 0) = d(y, 0)$. Hence,

$$N(x + y) \leq d(x, 0) + d(y, 0) = N(x) + N(y).$$

Thus, N is a norm on E.

Finally, the distance induced by this norm is given by

$$\widetilde{d}(x, y) := N(x - y) = d(x - y, 0).$$

Applying property (ii) with $z = y$, we have

$$d(x - y, 0) = d\Big((x - y) + y,\, 0 + y\Big) = d(x, y).$$

Therefore, $\widetilde{d}(x, y) = d(x, y)$ for all $x, y \in E$, which shows that d is precisely the metric induced by the norm N.

5.2 Metric Induced by a Norm

Definition 5.5. Let $(E, \|\cdot\|)$ be a normed vector space over the field $\mathbb{K}$, where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. The function $d\colon E \times E \to \mathbb{R}_+$ defined by

$$d(x, y) = \|x - y\|, \quad \text{for all } x, y \in E,$$

is a metric on E, called the *metric induced by the norm* $\|\cdot\|$.

Remark 5.6. The induced metric d satisfies two additional properties reflecting the linear structure of E:

1. **Absolute homogeneity:** For all $\lambda \in \mathbb{K}$ and all $x, y \in E$,

$$d(\lambda x, \lambda y) = |\lambda|\, d(x, y).$$

2. **Translation invariance:** For all $x, y, z \in E$,

$$d(x + z, y + z) = d(x, y).$$

(Equivalently, $d(x - z, y - z) = d(x, y)$.)

These properties are direct consequences of the homogeneity and triangle inequality of the norm.

Verification that d is a metric.

- *Non-negativity and separation:* $d(x,y) = \|x-y\| \geq 0$, and $d(x,y) = 0 \iff x-y=0 \iff x=y$.
- *Symmetry:* $d(x,y) = \|x-y\| = \|-(y-x)\| = \|y-x\| = d(y,x)$.
- *Triangle inequality:* $d(x,z) = \|x-z\| = \|(x-y)+(y-z)\| \leq \|x-y\| + \|y-z\| = d(x,y) + d(y,z)$.

Hence, d is indeed a metric. ■

Proposition 5.7. Let (E,d) be a metric space such that E is a vector space over $\mathbb{K}$ (where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$) and the metric d satisfies the following two properties:

1. **Absolute homogeneity:** $d(\lambda x, \lambda y) = |\lambda|\, d(x,y)$ for all $\lambda \in \mathbb{K}$ and all $x, y \in E$.
2. **Translation invariance:** $d(x+z, y+z) = d(x,y)$ for all $x,y,z \in E$.

Then the map $\|\cdot\|\colon E \to \mathbb{R}$ defined by

$$\|x\| := d(0,x), \quad x \in E,$$

is a norm on E, and the metric induced by this norm coincides with d; that is,

$$d(x,y) = \|x-y\| \quad \text{for all } x,y \in E.$$

Proof. We verify the three axioms of a norm.

1. **Positivity:** Since d is a metric, $d(0,x) \geq 0$ for all $x \in E$, so $\|x\| \geq 0$.
2. **Absolute homogeneity:** For any $\lambda \in \mathbb{K}$ and $x \in E$, using (P1),
$$\|\lambda x\| = d(0, \lambda x) = d(\lambda \cdot 0, \lambda x) = |\lambda|\, d(0,x) = |\lambda|\, \|x\|.$$
3. **Triangle inequality:** For any $x, y \in E$, using translation invariance (P2),
$$\|x+y\| = d(0, x+y) = d(-x, y) \leq d(-x, 0) + d(0, y) = d(0,x) + d(0,y) = \|x\| + \|y\|,$$
where the inequality follows from the triangle inequality for the metric d.

Thus, $\|\cdot\|$ is a norm on E.
Finally, for any $x, y \in E$, we have by translation invariance:

$$d(x,y) = d(x-y, 0) = d(0, x-y) = \|x-y\|.$$

Hence, d is precisely the metric induced by the norm $\|\cdot\|$. ■

Remark 5.8. The discrete metric $d\colon \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}_+$ defined by

$$d(x,y) = \begin{cases} 0 & \text{if } x = y, \\ 1 & \text{if } x \neq y, \end{cases} \qquad \text{for all } x, y \in \mathbb{R}^2,$$

is **not** induced by any norm on $\mathbb{R}^2$. Indeed, it fails to satisfy the absolute homogeneity property (P1): for any $\lambda \neq 0, 1$ and any $x \neq y$, we would need

$$d(\lambda x, \lambda y) = |\lambda|\, d(x,y) = |\lambda| > 1,$$

but by definition $d(\lambda x, \lambda y) \leq 1$. Hence, no norm can induce the discrete metric.

More generally, on any vector space of dimension ≥ 1, the discrete metric is never translation-invariant and homogeneous, and therefore cannot arise from a norm.

Example 5.9. Let $n \in \mathbb{N}^*$. Define the maps $\|\cdot\|_1, \|\cdot\|_2, \|\cdot\|_\infty : \mathbb{R}^n \to \mathbb{R}_+$ by, for all $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$,

$$\|x\|_1 = \sum_{i=1}^{n} |x_i|, \qquad \|x\|_2 = \left(\sum_{i=1}^{n} |x_i|^2\right)^{1/2}, \qquad \|x\|_\infty = \max_{1\le i\le n} |x_i|.$$

Each of these is a norm on $\mathbb{R}^n$. The metrics induced by these norms are respectively:

$$d_1(x,y) = \|x-y\|_1 = \sum_{i=1}^{n} |x_i - y_i|,$$
$$d_2(x,y) = \|x-y\|_2 = \left(\sum_{i=1}^{n} |x_i - y_i|^2\right)^{1/2},$$
$$d_\infty(x,y) = \|x-y\|_\infty = \max_{1\le i\le n} |x_i - y_i|.$$

- d_1 is called the *Manhattan distance* (or taxicab distance).
- d_2 is the standard *Euclidean distance*.
- d_∞ is known as the *uniform distance* (or Chebyshev distance).

All three norms—and hence their induced metrics—are equivalent on $\mathbb{R}^n$, a consequence of the finite dimensionality of the space.

Definition 5.10. Let $(E, \|\cdot\|)$ be a normed vector space over the field $\mathbb{K}$, where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. For $x_0 \in E$ and $r > 0$, we define:

1. The *open ball* of center x_0 and radius r:
$$B(x_0, r) = \{\, x \in E \mid \|x - x_0\| < r \,\}.$$
2. The *closed ball* of center x_0 and radius r:
$$\overline{B}(x_0, r) = \{\, x \in E \mid \|x - x_0\| \le r \,\}.$$
3. The *sphere* of center x_0 and radius r:
$$S(x_0, r) = \{\, x \in E \mid \|x - x_0\| = r \,\}.$$

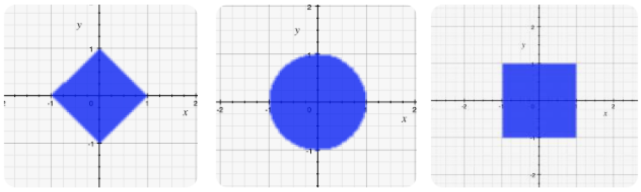
Example 5.11. In the two-dimensional space $\mathbb{R}^2$, the unit spheres (i.e., $S(0,1)$) associated with the three standard norms have distinct geometric shapes:

- For the ℓ^1-norm (Manhattan norm):
$$\|x\|_1 = |x_1| + |x_2|,$$
the unit sphere is a diamond (a square rotated by 45°) with vertices at $(\pm1, 0)$ and $(0, \pm1)$.
- For the ℓ^2-norm (Euclidean norm):
$$\|x\|_2 = \sqrt{x_1^2 + x_2^2},$$
the unit sphere is the usual unit circle.

- For the ℓ^∞-norm (uniform or max norm):

$$\|x\|_\infty = \max\{|x_1|, |x_2|\},$$

the unit sphere is a square with sides parallel to the coordinate axes and vertices at $(\pm 1, \pm 1)$.



These illustrations highlight how the geometry of normed spaces depends on the choice of norm, even though all norms on $\mathbb{R}^2$ are topologically equivalent.

5.2.1 Exercises

Exercise 72. For $(x, y) \in \mathbb{R}^2$, define

$$N(x, y) = \max\Big\{|x|,\ |y|,\ |x - y|\Big\}.$$

1. Show that N is a norm on $\mathbb{R}^2$.
2. Sketch the closed unit ball centered at the origin, i.e.,

$$\overline{B}_N\Big((0,0), 1\Big) = \{(x, y) \in \mathbb{R}^2 \mid N(x, y) \leq 1\}.$$

Solution 72. .

1. We verify the three axioms of a norm.

(a) **Definiteness:** Suppose $N(x, y) = 0$. Then

$$|x| = 0, \quad |y| = 0, \quad |x - y| = 0,$$

which implies $x = 0$ and $y = 0$. Hence, $(x, y) = (0, 0)$.

(b) **Absolute homogeneity:** For any $\lambda \in \mathbb{R}$,

$$N(\lambda x, \lambda y) = \max\Big\{|\lambda x|,\ |\lambda y|,\ |\lambda x - \lambda y|\Big\} = |\lambda| \max\Big\{|x|,\ |y|,\ |x - y|\Big\} = |\lambda|\, N(x, y).$$

(c) **Triangle inequality:** Let $(x, y), (u, v) \in \mathbb{R}^2$. Then

$$\begin{aligned} N\Big((x, y) + (u, v)\Big) &= \max\Big\{|x + u|,\ |y + v|,\ |(x + u) - (y + v)|\Big\} \\ &\leq \max\Big\{|x| + |u|,\ |y| + |v|,\ |x - y| + |u - v|\Big\} \\ &\leq \max\Big\{|x|, |y|, |x - y|\Big\} + \max\Big\{|u|, |v|, |u - v|\Big\} \\ &= N(x, y) + N(u, v). \end{aligned}$$

Therefore, N is a norm on $\mathbb{R}^2$.

2. **Closed unit ball.** By definition,

$$\overline{B}_N\big((0,0),1\big) = \{(x,y) \in \mathbb{R}^2 \mid |x| \leq 1,\ |y| \leq 1,\ |x-y| \leq 1\}.$$

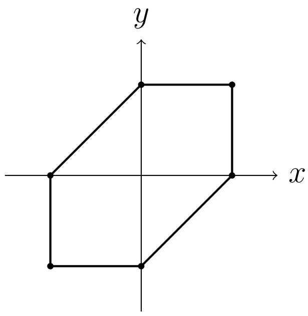
The first two conditions restrict (x,y) to the square $[-1,1]\times[-1,1]$. The third condition, $|x-y| \leq 1$, is equivalent to

$$-1 \leq x - y \leq 1 \quad \Longleftrightarrow \quad x - 1 \leq y \leq x + 1.$$

Intersecting the square with this horizontal strip yields a convex hexagon with vertices at

$$(1,0),\ (1,1),\ (0,1),\ (-1,0),\ (-1,-1),\ (0,-1).$$

Thus, the closed unit ball is a centrally symmetric hexagon (see figure below).



5.3 Equivalent Norms

Definition 5.12. Let E be a vector space over $\mathbb{K}$ (where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$), and let $\|\cdot\|_1$ and $\|\cdot\|_2$ be two norms on E. We say that $\|\cdot\|_1$ and $\|\cdot\|_2$ are *equivalent*, and write $\|\cdot\|_1 \sim \|\cdot\|_2$, if there exist positive constants $\alpha > 0$ and $\beta > 0$ such that

$$\alpha\,\|x\|_1 \ \leq\ \|x\|_2 \ \leq\ \beta\,\|x\|_1 \quad \text{for all } x \in E.$$

Remark 5.13. Equivalent norms induce the same topology on E: they define the same open sets, the same convergent sequences, and the same Cauchy sequences. In particular, a sequence converges (or is Cauchy) with respect to one norm if and only if it does so with respect to any equivalent norm.

Proposition 5.14. On any finite-dimensional vector space over $\mathbb{K}$, all norms are equivalent.

Proposition 5.15. The relation of norm equivalence is an equivalence relation on the set of all norms on a vector space E; that is, it is reflexive, symmetric, and transitive.

Proof. Let $\sim$ denote the relation of norm equivalence.

- **Reflexivity.** Let $\|\cdot\|$ be any norm on E. Taking $\alpha = \beta = 1$, we have

$$1\cdot\|x\| \leq \|x\| \leq 1\cdot\|x\| \quad \text{for all } x \in E.$$

  Hence, $\|\cdot\| \sim \|\cdot\|$.

- **Symmetry.** Suppose $\|\cdot\|_1 \sim \|\cdot\|_2$. Then there exist constants $\alpha, \beta > 0$ such that
$$\alpha\|x\|_1 \leq \|x\|_2 \leq \beta\|x\|_1 \quad \text{for all } x \in E.$$
Dividing the inequalities by $\alpha\beta > 0$, we obtain
$$\frac{1}{\beta}\|x\|_2 \leq \|x\|_1 \leq \frac{1}{\alpha}\|x\|_2 \quad \text{for all } x \in E,$$
which shows that $\|\cdot\|_2 \sim \|\cdot\|_1$.
- **Transitivity.** Suppose $\|\cdot\|_1 \sim \|\cdot\|_2$ and $\|\cdot\|_2 \sim \|\cdot\|_3$. Then there exist constants $\alpha, \beta, \gamma, \delta > 0$ such that
$$\alpha\|x\|_1 \leq \|x\|_2 \leq \beta\|x\|_1, \qquad \gamma\|x\|_2 \leq \|x\|_3 \leq \delta\|x\|_2 \quad \text{for all } x \in E.$$
Combining these inequalities yields
$$(\alpha\gamma)\|x\|_1 \leq \|x\|_3 \leq (\beta\delta)\|x\|_1 \quad \text{for all } x \in E.$$
Since $\alpha\gamma > 0$ and $\beta\delta > 0$, it follows that $\|\cdot\|_1 \sim \|\cdot\|_3$.

Therefore, norm equivalence is an equivalence relation. ■

Proposition 5.16. On $\mathbb{R}^n$ ($n \geq 1$), the standard norms
$$\|x\|_1 = \sum_{i=1}^{n} |x_i|, \quad \|x\|_2 = \left(\sum_{i=1}^{n} |x_i|^2\right)^{1/2}, \quad \|x\|_\infty = \max_{1\leq i\leq n} |x_i|$$
are pairwise equivalent.

Proof. Let $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$.

1. Equivalence of $\|\cdot\|_1$ and $\|\cdot\|_\infty$. Let i_0 be an index such that $\|x\|_\infty = |x_{i_0}|$. Then
$$\|x\|_\infty = |x_{i_0}| \leq \sum_{i=1}^{n} |x_i| = \|x\|_1.$$
Conversely, for each i, $|x_i| \leq \|x\|_\infty$, so
$$\|x\|_1 = \sum_{i=1}^{n} |x_i| \leq \sum_{i=1}^{n} \|x\|_\infty = n\|x\|_\infty.$$
Hence,
$$\|x\|_\infty \leq \|x\|_1 \leq n\|x\|_\infty,$$
which shows that $\|\cdot\|_1 \sim \|\cdot\|_\infty$.

2. Equivalence of $\|\cdot\|_2$ and $\|\cdot\|_\infty$. Again, with $\|x\|_\infty = |x_{i_0}|$, we have
$$\|x\|_\infty = |x_{i_0}| \leq \left(\sum_{i=1}^{n} |x_i|^2\right)^{1/2} = \|x\|_2.$$
Conversely, since $|x_i| \leq \|x\|_\infty$ for all i,
$$\|x\|_2 = \left(\sum_{i=1}^{n} |x_i|^2\right)^{1/2} \leq \left(\sum_{i=1}^{n} \|x\|_\infty^2\right)^{1/2} = \sqrt{n}\,\|x\|_\infty.$$
Thus,
$$\|x\|_\infty \leq \|x\|_2 \leq \sqrt{n}\,\|x\|_\infty,$$
so $\|\cdot\|_2 \sim \|\cdot\|_\infty$.

3. Conclusion. Since $\|\cdot\|_1 \sim \|\cdot\|_\infty$ and $\|\cdot\|_2 \sim \|\cdot\|_\infty$, transitivity of norm equivalence implies $\|\cdot\|_1 \sim \|\cdot\|_2$. Therefore, all three standard norms on $\mathbb{R}^n$ are pairwise equivalent. ■

5.3.1 Exercises

Exercise 73. On the space $E = C([0,1];\mathbb{R})$ of real-valued continuous functions on $[0,1]$, consider the two norms

$$\|f\|_1 = \int_0^1 |f(t)|\,dt \quad \text{and} \quad \|f\|_\infty = \sup_{t\in[0,1]} |f(t)|.$$

Show that these two norms are **not equivalent**.

Solution 73. It is immediate that for every $f \in E$,

$$\|f\|_1 = \int_0^1 |f(t)|\,dt \le \int_0^1 \|f\|_\infty\,dt = \|f\|_\infty,$$

so $\|f\|_1 \le \|f\|_\infty$.

Assume, for contradiction, that the norms are equivalent. Then there exists a constant $\alpha > 0$ such that

$$\|f\|_\infty \le \alpha\,\|f\|_1 \quad \text{for all } f \in E.$$

For each integer $n \ge 1$, define $f_n \in E$ by $f_n(t) = t^n$. Then:

$$\|f_n\|_1 = \int_0^1 t^n\,dt = \frac{1}{n+1}, \qquad \|f_n\|_\infty = \max_{t\in[0,1]} t^n = 1.$$

The assumed inequality would give

$$1 = \|f_n\|_\infty \le \alpha\,\|f_n\|_1 = \frac{\alpha}{n+1}.$$

Letting $n \to \infty$, the right-hand side tends to 0, while the left-hand side remains 1—a contradiction.

Therefore, no such constant α exists, and the norms $\|\cdot\|_1$ and $\|\cdot\|_\infty$ are **not equivalent** on $C([0,1];\mathbb{R})$.

Exercise 74. Let $E = C^1([0,1];\mathbb{R})$ be the space of continuously differentiable real-valued functions on $[0,1]$. Define two norms on E by

$$\|f\|_1 = \int_0^1 |f(t)|\,dt, \qquad \|f\| = |f(0)| + \int_0^1 |f'(t)|\,dt.$$

1. Show that $\|\cdot\|$ is a norm on E.
2. Compare $\|\cdot\|_1$ and $\|\cdot\|$, and in particular show that they are **not equivalent**.

Solution 74. .

1. **$\|\cdot\|$ is a norm.**
   - *Well-definedness:* Since $f \in C^1([0,1])$, f' is continuous, hence $|f'|$ is integrable, so $\|f\| \in \mathbb{R}$.
   - *Positivity:* $\|f\| = |f(0)| + \int_0^1 |f'(t)|\,dt \ge 0$.
   - *Definiteness:* If $\|f\| = 0$, then $|f(0)| = 0$ and $\int_0^1 |f'(t)|\,dt = 0$. Since $|f'|$ is continuous and non-negative, the integral being zero implies $f'(t) = 0$ for all $t \in [0,1]$. Hence, f is constant, and since $f(0) = 0$, we get $f \equiv 0$.

- *Homogeneity:* For $\lambda \in \mathbb{R}$,

$$\|\lambda f\| = |\lambda f(0)| + \int_0^1 |\lambda f'(t)|\, dt = |\lambda| \left(|f(0)| + \int_0^1 |f'(t)|\, dt \right) = |\lambda|\, \|f\|.$$

- *Triangle inequality:* For $f, g \in E$,

$$\begin{aligned} \|f+g\| &= |f(0)+g(0)| + \int_0^1 |f'(t)+g'(t)|\, dt \\ &\leq |f(0)| + |g(0)| + \int_0^1 |f'(t)|\, dt + \int_0^1 |g'(t)|\, dt \\ &= \|f\| + \|g\|. \end{aligned}$$

Hence, $\|\cdot\|$ is a norm on E.

2. **Comparison and non-equivalence.** First, we show that $\|f\|_1 \leq \|f\|$ for all $f \in E$. For any $x \in [0,1]$, by the fundamental theorem of calculus,

$$f(x) = f(0) + \int_0^x f'(t)\, dt,$$

so

$$|f(x)| \leq |f(0)| + \int_0^x |f'(t)|\, dt \leq |f(0)| + \int_0^1 |f'(t)|\, dt = \|f\|.$$

Integrating over $x \in [0,1]$ yields

$$\|f\|_1 = \int_0^1 |f(x)|\, dx \leq \int_0^1 \|f\|\, dx = \|f\|.$$

Thus, $\|f\|_1 \leq \|f\|$ for all f.

To show that the norms are **not equivalent**, suppose for contradiction that there exists $\alpha > 0$ such that

$$\alpha \|f\| \leq \|f\|_1 \quad \text{for all } f \in E.$$

For each $n \in \mathbb{N}^*$, define $f_n(x) = x^n$. Then $f_n \in C^1([0,1])$, and

$$\|f_n\|_1 = \int_0^1 x^n\, dx = \frac{1}{n+1}, \qquad \|f_n\| = |f_n(0)| + \int_0^1 |f_n'(x)|\, dx = 0 + \int_0^1 n x^{n-1}\, dx = 1.$$

The assumed inequality would give

$$\alpha \cdot 1 \leq \frac{1}{n+1} \quad \text{for all } n \geq 1.$$

Letting $n \to \infty$ yields $\alpha \leq 0$, contradicting $\alpha > 0$.

Therefore, no such α exists, and the norms $\|\cdot\|_1$ and $\|\cdot\|$ are **not equivalent**.

Exercise 75. Define $N\colon \mathbb{R}^2 \to \mathbb{R}_+$ by

$$N(x,y) = \max\Big\{|x|,\ |2x+y|\Big\}.$$

1. Show that N is a norm on $\mathbb{R}^2$. Describe the open ball of center $a \in \mathbb{R}^2$ and radius $r > 0$.

2. Show that N is equivalent to the ℓ^1-norm $\|(x,y)\|_1 = |x| + |y|$, and find explicit positive constants α, β such that

$$\alpha \|(x,y)\|_1 \leq N(x,y) \leq \beta \|(x,y)\|_1 \quad \text{for all } (x,y) \in \mathbb{R}^2.$$

3. Describe the interior of the set

$$A = \Big\{(x,y) \in \mathbb{R}^2 \mid |2x-y| < 1\Big\}.$$

Solution 75. .

1. N **is a norm.**

- *Positive definiteness:* If $N(x,y)=0$, then $|x|=0$ and $|2x+y|=0$, so $x=0$ and $y=0$.
- *Absolute homogeneity:* For $\lambda \in \mathbb{R}$,

$$N(\lambda x, \lambda y) = \max\{|\lambda x|, |\lambda(2x+y)|\} = |\lambda| \max\{|x|, |2x+y|\} = |\lambda| N(x,y).$$

- *Triangle inequality:* Let $(x_1,y_1),(x_2,y_2) \in \mathbb{R}^2$. Then

$$\begin{aligned}|x_1+x_2| &\le |x_1|+|x_2| \le N(x_1,y_1)+N(x_2,y_2),\\ |2(x_1+x_2)+(y_1+y_2)| &\le |2x_1+y_1|+|2x_2+y_2| \le N(x_1,y_1)+N(x_2,y_2).\end{aligned}$$

Taking the maximum gives

$$N(x_1+x_2, y_1+y_2) \le N(x_1,y_1)+N(x_2,y_2).$$

Hence, N is a norm.

The open ball of center $a=(x_0,y_0)$ and radius $r>0$ is

$$B_N(a,r) = \Big\{(x,y)\in\mathbb{R}^2 \mid N(x-x_0, y-y_0) < r\Big\},$$

which is equivalent to the system

$$\begin{cases}|x-x_0| < r,\\ |2(x-x_0)+(y-y_0)| < r,\end{cases}$$

or, explicitly,

$$\begin{cases}x_0 - r < x < x_0 + r,\\ y_0 - r - 2(x-x_0) < y < y_0 + r - 2(x-x_0).\end{cases}$$

Geometrically, this is an open parallelogram centered at (x_0,y_0).

2. **Equivalence with the ℓ^1-norm.** For any $(x,y)\in\mathbb{R}^2$,

$$N(x,y) = \max\{|x|,|2x+y|\} \le \max\{|x|, 2|x|+|y|\} \le 2(|x|+|y|) = 2\|(x,y)\|_1.$$

Conversely,

$$\|(x,y)\|_1 = |x|+|y| = |x|+|(2x+y)-2x| \le |x|+|2x+y|+2|x| \le 3|x|+|2x+y| \le 4N(x,y).$$

Therefore,

$$\frac{1}{4}\|(x,y)\|_1 \le N(x,y) \le 2\|(x,y)\|_1,$$

so the norms are equivalent with $\alpha = \frac{1}{4}$ and $\beta = 2$.

3. **Interior of A.** Consider the linear map $f\colon \mathbb{R}^2 \to \mathbb{R}$, $f(x, y) = 2x - y$. Since all linear maps on finite-dimensional normed spaces are continuous, f is continuous with respect to the norm N (and any norm). The set $A = f^{-1}\big((-1, 1)\big)$ is the preimage of an open set under a continuous map, hence **open**. Therefore, the interior of A is A itself:

$$\operatorname{int}(A) = A.$$

Exercise 76. Let $E = C^1([a, b]; \mathbb{R})$ be the space of continuously differentiable real-valued functions on the compact interval $[a, b]$. Define the norm

$$N(f) = \|f\|_\infty + \|f'\|_\infty, \quad \text{where } \|f\|_\infty = \sup_{t\in[a,b]} |f(t)|.$$

Is the normed space (E, N) complete? In other words, is it a Banach space?

Solution 76. Yes, (E, N) is a Banach space.

Let $(f_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in (E, N). Then, for every $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that for all $p, q \geq n_0$,

$$N(f_p - f_q) = \|f_p - f_q\|_\infty + \|f_p' - f_q'\|_\infty < \varepsilon.$$

In particular,

$$\|f_p - f_q\|_\infty < \varepsilon \quad \text{and} \quad \|f_p' - f_q'\|_\infty < \varepsilon,$$

so both (f_n) and (f_n') are Cauchy sequences in the Banach space $(C([a, b]; \mathbb{R}), \|\cdot\|_\infty)$. Hence, there exist functions $f, g \in C([a, b]; \mathbb{R})$ such that

$$f_n \xrightarrow[n\to\infty]{} f \quad \text{and} \quad f_n' \xrightarrow[n\to\infty]{} g \quad \text{uniformly on } [a, b].$$

Now, for each n and for all $t \in [a, b]$, the fundamental theorem of calculus gives

$$f_n(t) = f_n(a) + \int_a^t f_n'(s)\, ds.$$

Passing to the limit as $n \to \infty$, and using the uniform convergence of $f_n \to f$ and $f_n' \to g$, we obtain

$$f(t) = f(a) + \int_a^t g(s)\, ds.$$

Since g is continuous, the right-hand side is differentiable on $[a, b]$ with derivative $g(t)$. Therefore, $f \in C^1([a, b]; \mathbb{R})$ and $f' = g$.

Finally,

$$N(f_n - f) = \|f_n - f\|_\infty + \|f_n' - f'\|_\infty \xrightarrow[n\to\infty]{} 0,$$

so $f_n \to f$ in the norm N. Thus, every Cauchy sequence in (E, N) converges in E, and (E, N) is complete.

Conclusion: $(C^1([a, b]; \mathbb{R}), \|\cdot\|_\infty + \|\cdot'\|_\infty)$ is a Banach space.

Exercise 77. Let $E = C([a, b]; \mathbb{R})$ be the space of real-valued continuous functions on the compact interval $[a, b]$. Let $\varphi \in E$ be a function such that $\varphi(t) \neq 0$ for all $t \in [a, b]$. Define a norm on E by

$$\|f\|_\varphi = \sup_{t\in[a,b]} |\varphi(t)f(t)|, \quad f \in E.$$

Is the normed space $(E, \|\cdot\|_\varphi)$ complete?

Solution 77. Yes, $(E, \|\cdot\|_\varphi)$ is a Banach space.

Let $(f_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in $(E, \|\cdot\|_\varphi)$. Then for every $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that for all $p, q \geq n_0$,

$$\|f_p - f_q\|_\varphi = \sup_{t\in[a,b]} |\varphi(t)(f_p(t) - f_q(t))| < \varepsilon.$$

Define $g_n = \varphi f_n$. Then (g_n) is a Cauchy sequence in $(C([a, b]), \|\cdot\|_\infty)$, which is a Banach space. Hence, there exists $g \in C([a, b])$ such that $g_n \to g$ uniformly, i.e.,

$$\|g_n - g\|_\infty \xrightarrow[n\to\infty]{} 0.$$

Since φ is continuous and never vanishes on the compact set $[a, b]$, it is bounded away from zero: there exists $\alpha > 0$ such that $|\varphi(t)| \geq \alpha$ for all $t \in [a, b]$. Therefore, the function

$$f(t) := \frac{g(t)}{\varphi(t)}, \quad t \in [a, b],$$

is well-defined and continuous (as the quotient of continuous functions with non-vanishing denominator), so $f \in E$.

Now observe that

$$\|f_n - f\|_\varphi = \sup_{t\in[a,b]} |\varphi(t)(f_n(t) - f(t))| = \|g_n - g\|_\infty \xrightarrow[n\to\infty]{} 0.$$

Thus, $f_n \to f$ in the norm $\|\cdot\|_\varphi$, and the limit f belongs to E.

Conclusion: The space $(C([a, b]; \mathbb{R}), \|\cdot\|_\varphi)$ is complete; hence, it is a Banach space.

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