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Ministère de l'Enseignement Supérieur et de la Recherche Scientifique

Université Djilali Bounaama, Khemis Miliana
Faculté des Sciences de la Matière et d'informatique
Conseil Scientifique de la Faculté

جامعة جيلالي بونعامة خميس مليانة
كليّة علوم المادة والإعلام الآلي
المجلس العلمي للكلية

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EXTRAIT DU PV
DE LA REUNION ORDINAIRE DU CONSEIL SCIENTIFIQUE
Du 02/05/2026

Objet : : Expertise de polycopié pédagogique

En l'an deux mille vingt-six (2026), le deux (02) mai 09 h 30, une réunion ordinaire du Conseil Scientifique de la Faculté des Sciences de la Matière et de l'Informatique s'est tenue dans la salle de réunion de la faculté (Bloc B).

Suite aux rapports favorables reçus de la part des experts cités ci-après concernant l'expertise du polycopié pédagogique, le CSF a prononcé favorablement pour la conformité du polycopié pédagogique en vue de préparer son professorat.

- **Auteur du polycopié** : Dr. AYADI Souad (MCA)
- **Intitulé du polycopié** : Functional of a complex variable : Course and Exercises
- **Destiné aux étudiants de** : L2 Physique
- **Experts du polycopié** :
  - BENALI Abdelkader Pr UHB - Chlef
  - KELLECHE Abdelkarim MCA UDB - Khemis Miliana

Président du
Conseil Scientifique de la Faculté SMI
Dr. BOUDERBALA Mihoub

رئيس المجلس العلمي
بودربالة ميهوب

University of Khemis Miliana - Djilali Bounaama
Faculty of Science, Materials and Computer Science
Department of Physics

Function of a Complex Variable

Course & Exercises

2^{nd} Year Licence (LMD System)
Physics Track – Material Sciences

Second Semester

Author: **AYADI SOUAD**

Academic Year: **2025–2026**

Contents

Introduction

This course manual entitled *"Function of a Complex Variable"* is intended for second-year undergraduate students in physics, within the field of material sciences. It is part of the fundamental teaching unit UEF4 and is scheduled for the second semester. The study of functions of a complex variable occupies an essential place in the scientific training of students in both theoretical and applied physics. Indeed, complex analysis provides a powerful mathematical tool for addressing and solving a wide variety of problems, ranging from the study of oscillatory phenomena, the resolution of differential equations, and the evaluation of integrals, to concrete applications in quantum mechanics, electromagnetism, signal processing, and the analysis of dynamical systems.

The course begins with an in-depth study of complex numbers in order to provide students with a solid foundation for the subsequent material. General properties of complex numbers are recalled, along with their geometric interpretation in the complex plane, and their different representations: algebraic, trigonometric, and exponential forms. This chapter also introduces the study of complex functions and develops the notions of limits and continuity in the complex plane, including the particular role played by the point at infinity. These fundamental concepts allow a rigorous understanding of the behavior of complex functions and lay the groundwork for the analysis developed in later chapters.

The second chapter of the course is devoted to elementary functions. We start with the study of simple algebraic transformations such as translations $z + a$, homotheties az, the inversion $1/z$. These functions are elementary but fundamental because of their remarkable geometric properties in the complex plane. The study continues with functions defined by power series, in particular exponential, trigonometric, and hyperbolic functions, which play a central role in analysis and have direct applications in physics. This part also introduces the notion of multivalued functions such as the argument, powers, and logarithm, which raise important questions regarding multivaluation and the necessity of introducing branch cuts in the complex plane to ensure coherent definitions. These considerations pave the way for a deeper reflection on the structure of the complex plane and the analytical complexity of definitions.

The third chapter of the course introduces holomorphic functions, which form the true core of complex analysis. To understand the specificity of complex differentiability, we begin by recalling the framework of real functions of two variables. It is then shown that complex differentiability requires stronger conditions than real differentiability, conditions which are expressed through the celebrated Cauchy-Riemann equations. These equations provide a necessary and sufficient criterion for a function to be holomorphic, i.e., differentiable at every point of a domain in the complex plane. The study of holomorphic functions also reveals their connection to harmonic functions, which appear as their real or imaginary parts. This relationship opens the way to numerous applications of complex analysis to domains such as electrostatics, fluid mechanics, or heat propagation, where harmonic-type equations naturally arise.

The course then devotes significant attention to integration in the complex plane, which represents one of the cornerstones of complex analysis. The notions of paths and curves are introduced, followed by the definition of the line integral of a complex function along a contour. From this, Cauchy's theorem is developed, leading to Cauchy's integral formula, which constitutes a fundamental tool of complex analysis. This chapter highlights profound results such as the existence of primitives for holomorphic functions, Morera's theorem, and remarkable properties such as the identity principle, the mean-value property, and the maximum modulus principle. Classical theorems such as those of Liouville, D'Alembert, and Rouché illustrate the power of the theory and its direct applications to concrete problems.

Subsequently, the course examines power series and their central role in the definition of analytic functions. The study of complex power series explains why holomorphic functions are also called analytic functions. It is shown that these functions can be locally expanded as Taylor series within their domain of regularity, which provides an essential tool for local analysis. The notion of analytic continuation is then introduced to extend the domain of validity of these expansions beyond their circle of convergence. Laurent series are also presented for the study of functions with singularities, allowing for a rigorous classification of isolated singular points. This part of the course demonstrates the richness of possible behaviors of complex functions and sets the stage for the introduction of residues.

The residue theorem represents a decisive step in complex analysis. This theorem offers an elegant and highly efficient method for the computation of integrals and the summation of series. Students learn how to exploit the poles and residues of complex functions to obtain results that would otherwise be very difficult to establish using the methods of real analysis. The practical applications of this theorem emphasize the elegance and power of the complex approach, showing how it often far surpasses classical real-variable tools.

The last chapter illustrates the main applications of complex analysis, using fundamental theorems to compute integrals and study the zeros of analytic functions.

Finally, I would be very grateful to all readers-both students and instructors-who wish to share their feedback, comments, or suggestions regarding the content or the form of this manuscript. You may contact me at: `souad.ayadi@univ-dbkm.dz`

Chapter 1

Generalities on Complex Numbers

1.1 General Review of Complex Numbers

1.1.1 Properties of Complex Numbers

We define the imaginary unit i by the property

$$i^2 = -1.$$

This definition allowed mathematicians to propose **imaginary solutions** to quadratic equations with a negative discriminant, that is, equations of the form

$$ax^2 + bx + c = 0, \quad a \neq 0.$$

Recall that the discriminant of a quadratic equation is given by

$$\Delta = b^2 - 4ac.$$

Definition 1.1 (Complex Number) *A **complex number** is any number that can be written in the form*

$$z = a + ib,$$

where $a, b \in \mathbb{R}$ and i is the imaginary unit defined by $i^2 = -1$.

We denote

$$a = \Re(z) \quad \text{(the real part of } z\text{)}, \qquad b = \Im(z) \quad \text{(the imaginary part of } z\text{)}.$$

Remark 1.1 *Any multiple of the imaginary unit i is called a **purely imaginary number**.*

Definition 1.2 (Zero Complex Number) *The **zero complex number** is defined as*

$$0 + i0.$$

It is usually denoted simply by 0, and it is the additive identity in $\mathbb{C}$, that is,

$$z + 0 = z, \quad \forall z \in \mathbb{C}.$$

Definition 1.3 (Conjugate of a Complex Number) *Let $z = a + ib \in \mathbb{C}$, with $a, b \in \mathbb{R}$. The **conjugate** of z is defined as*

$$\overline{z} = \overline{(a + ib)} = a - ib.$$

Note that

$$\overline{\overline{z}} = z.$$

Definition 1.4 (Equality) *Two complex numbers $z_1 = a_1 + ib_1$ and $z_2 = a_2 + ib_2$ are said to be equal if and only if*

$$a_1 = \Re(z_1) = a_2 = \Re(z_2) \quad \text{and} \quad b_1 = \Im(z_1) = b_2 = \Im(z_2).$$

Definition 1.5 (Arithmetic Operations) *Let $z_1 = a_1 + ib_1$ and $z_2 = a_2 + ib_2$. The following operations are defined:*

- ***Addition:***

$$z_1 + z_2 = (a_1 + ib_1) + (a_2 + ib_2) = (a_1 + a_2) + i(b_1 + b_2).$$

$$z + \overline{z} = 2\Re(z)$$

- ***Subtraction:***

$$z_1 - z_2 = (a_1 + ib_1) - (a_2 + ib_2) = (a_1 - a_2) + i(b_1 - b_2).$$

$$z - \overline{z} = 2i\Im(z)$$

- ***Multiplication:***

$$z_1 \cdot z_2 = (a_1 + ib_1)(a_2 + ib_2) = (a_1a_2 - b_1b_2) + i(a_1b_2 + a_2b_1).$$

$$z \cdot \overline{z} = |z|^2 = a^2 + b^2$$

.

Exercise 1.1 *Let $z = a + ib \in \mathbb{C}$ with $a, b \in \mathbb{R}$. Calculate the product $z \cdot \overline{z}$.*

Solution. *We have*

$$z \cdot \overline{z} = (a + ib)(a - ib) = a^2 - iab + iab - i^2b^2.$$

Since $i^2 = -1$, this simplifies to

$$z \cdot \overline{z} = a^2 + b^2.$$

Therefore,

$$z \cdot \overline{z} = |z|^2.$$

- ***Division:***

$$\frac{z_1}{z_2} = \frac{z_1\,\overline{z_2}}{z_2\,\overline{z_2}}$$

$$\frac{z_1}{z_2} = \frac{(a_1 + ib_1)\,(a_2 - ib_2)}{(a_2 + ib_2)\,(a_2 - ib_2)} = \frac{a_1a_2 + b_1b_2}{a_2^2 + b_2^2} + i\,\frac{b_1a_2 - a_1b_2}{a_2^2 + b_2^2}, \qquad (a_2, b_2) \neq (0, 0).$$

Remark 1.2 *These operations satisfy the usual algebraic rules: commutativity, associativity, and distributivity.*

- **Distributive Properties of Conjugation:**
  For all $z_1, z_2 \in \mathbb{C}$, the conjugation satisfies:

$$\overline{(z_1 + z_2)} = \overline{z_1} + \overline{z_2},$$

$$\overline{(z_1 - z_2)} = \overline{z_1} - \overline{z_2},$$

$$\overline{(z_1 \cdot z_2)} = \overline{z_1} \cdot \overline{z_2},$$

$$\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\overline{z_1}}{\overline{z_2}}, \qquad z_2 \neq 0,$$

Example 1.1 *Let $z_1 = 3 + 4i$ and $z_2 = 1 - 2i$. Then:*

$$z_1 + z_2 = (3 + 1) + (4 - 2)i = 4 + 2i,$$

$$z_1 \cdot z_2 = (3 \cdot 1 - 4 \cdot (-2)) + i(3 \cdot (-2) + 1 \cdot 4) = 11 - 2i,$$

$$\frac{z_1}{z_2} = \frac{3 + 4i}{1 - 2i} = \frac{3 + 4i}{1 - 2i} \cdot \frac{1 + 2i}{1 + 2i} = \frac{11 + 2i}{5} = \frac{11}{5} + \frac{2}{5}i.$$

Example 1.2 *Let*

$$z_1 = 3 + 4i, \qquad z_2 = 1 - 2i.$$

$$z_1 + z_2 = (3 + 4i) + (1 - 2i) = 4 + 2i, \qquad \overline{z_1 + z_2} = 4 - 2i,$$

$$\overline{z_1} = 3 - 4i, \qquad \overline{z_2} = 1 + 2i, \qquad \overline{z_1} + \overline{z_2} = (3 - 4i) + (1 + 2i) = 4 - 2i.$$

Hence

$$\overline{(z_1 + z_2)} = \overline{z_1} + \overline{z_2}.$$

For subtraction:

$$z_1 - z_2 = (3 + 4i) - (1 - 2i) = 2 + 6i, \qquad \overline{z_1 - z_2} = 2 - 6i,$$

$$\overline{z_1} - \overline{z_2} = (3 - 4i) - (1 + 2i) = 2 - 6i.$$

Thus

$$\overline{(z_1 - z_2)} = \overline{z_1} - \overline{z_2}.$$

For multiplication:

$$z_1 z_2 = (3+4i)(1-2i) = 3 - 6i + 4i - 8i^2 = 11 - 2i, \qquad \overline{z_1 z_2} = 11 + 2i,$$

$$\overline{z_1}\,\overline{z_2} = (3-4i)(1+2i) = 3 + 6i - 4i - 8i^2 = 11 + 2i.$$

Therefore

$$\overline{(z_1 z_2)} = \overline{z_1}\,\overline{z_2}.$$

For division (use conjugates to rationalize):

$$\frac{z_1}{z_2} = \frac{3+4i}{1-2i} = \frac{(3+4i)(1+2i)}{(1-2i)(1+2i)} = \frac{-5+10i}{5} = -1 + 2i,$$

so

$$\overline{\frac{z_1}{z_2}} = \overline{-1+2i} = -1 - 2i.$$

On the other hand

$$\frac{\overline{z_1}}{\overline{z_2}} = \frac{3-4i}{1+2i} = \frac{(3-4i)(1-2i)}{(1+2i)(1-2i)} = \frac{-5-10i}{5} = -1 - 2i.$$

Hence

$$\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\overline{z_1}}{\overline{z_2}}.$$

Finally, double conjugation:

$$\overline{\overline{z_1}} = \overline{3-4i} = 3 + 4i = z_1, \qquad \overline{\overline{z_2}} = \overline{1+2i} = 1 - 2i = z_2.$$

So $\overline{\overline{z}} = z$ *is verified.*

Definition 1.6 (Additive Inverse) *In the system of complex numbers, every number* $z \in \mathbb{C}$ *has a unique additive inverse* z' *such that*

$$z + z' = 0.$$

This inverse is given by

$$z' = -z.$$

Definition 1.7 (Multiplicative Inverse) *In the system of complex numbers, every nonzero number* $z \in \mathbb{C}$ *has a unique multiplicative inverse* z'' *such that*

$$z \cdot z'' = 1.$$

This inverse is given by

$$z'' = \frac{1}{z} = z^{-1}.$$

The number z^{-1} *is also called the reciprocal of* z.

Thus

$$\overline{(z_1 - z_2)} = \overline{z_1} - \overline{z_2}.$$

For multiplication:

$$z_1 z_2 = (3+4i)(1-2i) = 3 - 6i + 4i - 8i^2 = 11 - 2i, \qquad \overline{z_1 z_2} = 11 + 2i,$$

$$\overline{z_1}\,\overline{z_2} = (3-4i)(1+2i) = 3 + 6i - 4i - 8i^2 = 11 + 2i.$$

Therefore

$$\overline{(z_1 z_2)} = \overline{z_1}\,\overline{z_2}.$$

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This inverse is given by

$$z'' = \frac{1}{z} = z^{-1}.$$

The number z^{-1} *is also called the reciprocal of* z*.*

Application **Compute the following powers of i:**

$$i^3,\ i^4,\ i^5,\ i^6,\ i^7,\ i^8,\ i^9,\ i^{10},\ i^{11},\ i^{12}.$$

Then, deduce a general rule for i^n depending on the parity of n.

Solution 1.1 ***Recall that***

$$i^1 = i, \qquad i^2 = -1.$$

Step 1: Compute the powers successively.

$$i^3 = i^2 \cdot i = -1 \cdot i = -i,$$

$$\begin{aligned} i^4 &= i^2 \cdot i^2 = (-1)(-1) = 1,\\ i^5 &= i^4 \cdot i = 1 \cdot i = i,\\ i^6 &= i^5 \cdot i = i \cdot i = i^2 = -1,\\ i^7 &= i^6 \cdot i = (-1) \cdot i = -i,\\ i^8 &= i^4 \cdot i^4 = 1 \cdot 1 = 1,\\ i^9 &= i^8 \cdot i = 1 \cdot i = i,\\ i^{10} &= i^9 \cdot i = i \cdot i = i^2 = -1,\\ i^{11} &= i^{10} \cdot i = (-1) \cdot i = -i,\\ i^{12} &= i^8 \cdot i^4 = 1 \cdot 1 = 1. \end{aligned}$$

Step 2: Observe the pattern.
The powers of i repeat every four steps:

$$i^1 = i, \quad i^2 = -1, \quad i^3 = -i, \quad i^4 = 1.$$

Step 3: General rule. For any integer $n \geq 1$, there exists an integer $k \in \mathbb{N}$ such that:

$$i^n = \begin{cases} 1, & \textbf{\textit{if }} n = 4k,\\ i, & \textbf{\textit{if }} n = 4k+1,\\ -1, & \textbf{\textit{if }} n = 4k+2,\\ -i, & \textbf{\textit{if }} n = 4k+3. \end{cases}$$

1.1.2 The Complex Plane

Definition 1.8 (Complex Plane) ***Since every complex number z can be represented by a pair of real numbers (a, b), we use a plane to represent complex numbers.***

This plane is defined by two perpendicular axes:

- ***the real axis, denoted by $\Re$ or x-axis,***
- ***the imaginary axis, denoted by $\Im$ or y-axis.***

Thus, any complex variable can be written as

$$z = x + iy, \qquad x, y \in \mathbb{R},$$

where $x = \Re(z)$ ***and*** $y = \Im(z)$***.***

We can also interpret each pair of real numbers (x, y) as the components of a 2D vector. Thus, a complex number

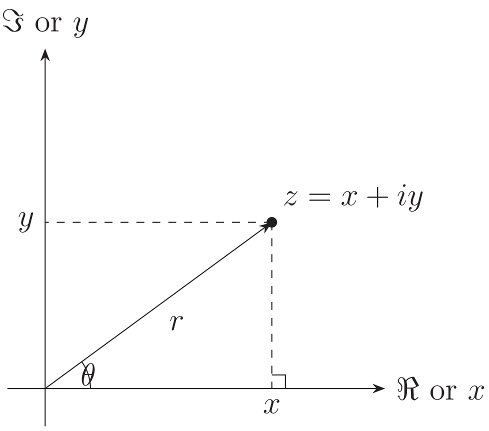
$$z = x + iy$$

can also be seen as a two-dimensional vector whose initial point is at the origin of the axes, and whose terminal point is the point (x, y).

Definition 1.9 ***The vector representation of a complex number*** $z = x + iy$ ***is the directed vector***

$$\vec{z} = (x, y),$$

where $x = \Re(z)$ ***is the projection on the real axis, and*** $y = \Im(z)$ ***is the projection on the imaginary axis.***



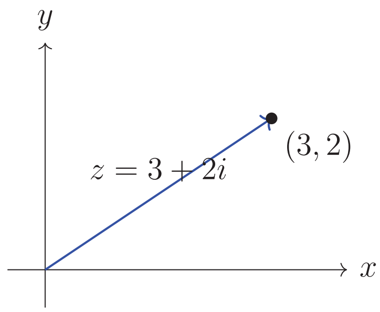
$$r = \sqrt{x^2 + y^2}$$
$$\theta = \arg(z) = \arctan\left(\frac{y}{x}\right)$$

Figure 1.1: Complex plane (Argand plane): representation of $z = x + iy$, modulus r and argument θ.

Example 1.3 ***Let*** $z = 3 + 2i$***. Its vector representation is***

$$\vec{z} = (3, 2),$$

which is a vector starting from the origin $(0, 0)$ ***and ending at the point*** $(3, 2)$ ***in the plane.***



Definition 1.10 (Modulus of a Complex Number) ***Let $z = x + iy$ with $x, y \in \mathbb{R}$. The modulus of z is the positive real number defined by***

$$|z| = \sqrt{x^2 + y^2}.$$

Proposition 1.1 (Properties of the Modulus) ***For any $z, z_1, z_2 \in \mathbb{C}$ with $z_2 \neq 0$, the modulus satisfies the following properties:***

- $|z|^2 = z\,\overline{z}$,
- $|z| = \sqrt{z\,\overline{z}}$,
- $|z_1 z_2| = |z_1|\,|z_2|$,
- $\left|\dfrac{z_1}{z_2}\right| = \dfrac{|z_1|}{|z_2|}$,
- $|z^2| = |z|^2$.

Definition 1.11 (Distance Between Two Complex Numbers) ***Let $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ be two complex numbers. The distance between z_1 and z_2 in the complex plane is defined as***

$$d(z_1, z_2) = |z_2 - z_1| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

1.1.3 Polar and Exponential Form of Complex Numbers

Polar Form

From a simple geometric viewpoint, the components x and y of a complex number

$$z = x + iy$$

can be expressed in terms of the polar parameters r and θ as

$$\begin{cases} x = r\cos\theta, \\ y = r\sin\theta, \end{cases} \qquad \Longrightarrow \qquad z = x + iy = r\cos\theta + ir\sin\theta = r(\cos\theta + i\sin\theta).$$

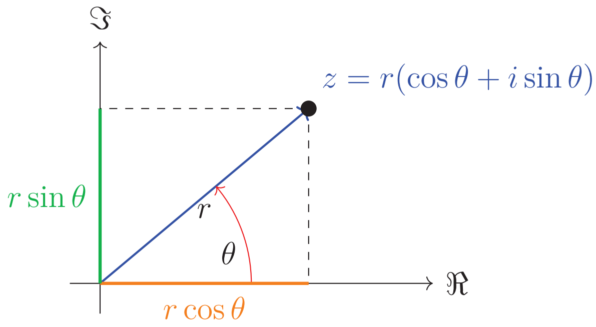
Here:

$$r = |z| = \sqrt{x^2 + y^2} \in \mathbb{R}^+, \qquad -\pi \leq \theta < \pi.$$

The angle θ of the vector z is measured in radians, positive in the counter-clockwise direction. This angle θ is called the argument of z and is denoted by

$$\theta = \arg(z).$$

Remark 1.3 *The argument* $\arg(z)$ *is not unique because of the* 2π*-periodicity of the trigonometric functions* $\cos\theta$ *and* $\sin\theta$*.*



Exercise 1.2 *Write the complex number*

$$z = -\sqrt{3} - i$$

in polar form.

Solution 1.2 *We are given the complex number*

$$z = -\sqrt{3} - i.$$

Step 1: Compute the modulus r*.*

$$r = |z| = \sqrt{(-\sqrt{3})^2 + (-1)^2} = \sqrt{3+1} = \sqrt{4} = 2.$$

Step 2: Determine the argument θ*. We know that*

$$\tan\theta = \frac{y}{x} = \frac{-1}{-\sqrt{3}} = \frac{1}{\sqrt{3}}.$$

Thus, the reference angle is

$$\theta_0 = \arctan\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6}.$$

Since $x < 0$ *and* $y < 0$*, the point lies in the third quadrant. Therefore,*

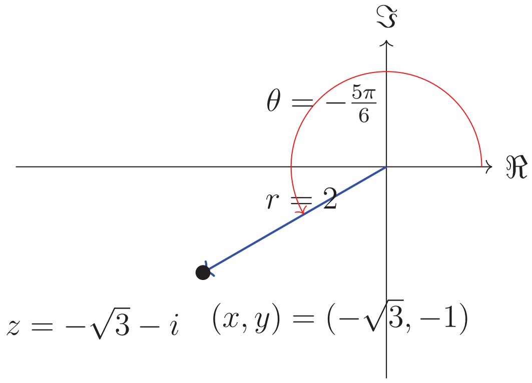
$$\theta = -\pi + \frac{\pi}{6} = -\frac{5\pi}{6}.$$

Step 3: Write z *in polar form.*

$$z = r(\cos\theta + i\sin\theta) = 2\Big(\cos\Big(-\tfrac{5\pi}{6}\Big) + i\sin\Big(-\tfrac{5\pi}{6}\Big)\Big).$$

Final Answer:

$$\boxed{z = 2\left(\cos\left(-\tfrac{5\pi}{6}\right) + i\sin\left(-\tfrac{5\pi}{6}\right)\right).}$$



1. Principal Argument

Definition 1.12 (Argument and Principal Argument) ***Let*** $z \in \mathbb{C} \setminus \{0\}$ ***and write*** $z = r(\cos\theta + i\sin\theta)$ ***with*** $r > 0$ ***and*** $\theta \in \mathbb{R}$. ***The set of all possible values of*** θ ***is called the*** argument ***of*** z ***and is denoted by*** $\arg(z)$.

The principal argument $\mathrm{Arg}(z)$ ***is the unique argument chosen in the interval***

$$-\pi \leq \mathrm{Arg}(z) < \pi.$$

Remark 1.4 ***Because*** $\cos(\theta)$ ***and*** $\sin(\theta)$ ***are*** 2π***-periodic, the argument is not unique: if*** θ ***is an argument of*** z ***then*** $\theta + 2k\pi$ ***is also an argument for every*** $k \in \mathbb{Z}$. ***The principal argument selects the representative in*** $[-\pi, \pi)$.

2.Multiplication and Division in Polar Form

Let

$$z_1 = r_1\Big(\cos\theta_1 + i\sin\theta_1\Big), \qquad z_2 = r_2\Big(\cos\theta_2 + i\sin\theta_2\Big),$$

with $r_1, r_2 > 0$.

Proposition 1.2 (Product and Quotient in Polar Form) ***The product and the quotient are given by***

$$z_1 z_2 = r_1 r_2\Big(\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)\Big),$$

and, for $z_2 \neq 0$,

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}\Big(\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)\Big).$$

Equivalently,

$$\arg(z_1 z_2) = \arg(z_1) + \arg(z_2), \qquad \arg\left(\frac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2),$$

up to addition of integer multiples of 2π ***(and taking principal arguments when needed).***

3. Integer Powers and De Moivre's Formula

Proposition 1.3 (Integer Powers) ***Let*** $z = r(\cos\theta + i\sin\theta)$ ***and*** $n \in \mathbb{N}$***. Then***

$$z^n = r^n\Big(\cos(n\theta) + i\sin(n\theta)\Big).$$

In particular, if $r = 1$ ***then the classical De Moivre formula holds:***

$$\Big(\cos\theta + i\sin\theta\Big)^n = \cos(n\theta) + i\sin(n\theta).$$

Remark 1.5 ***The formula extends to*** $n \in \mathbb{Z}$ ***(negative integers) by using*** $z^{-n} = (z^{-1})^n$ ***and the polar expression of*** z^{-1}***.***

4. Nth Roots of a Complex Number

Let $z = r(\cos\theta + i\sin\theta)$ **with** $r > 0$**. We seek all** w **such that** $w^n = z$**.**

Proposition 1.4 (Formula for the n**th Roots)** ***All solutions*** w ***of*** $w^n = z$ ***are given by***

$$w_k = \rho\Big(\cos\varphi_k + i\sin\varphi_k\Big), \qquad k = 0, 1, \ldots, n-1,$$

where

$$\rho = r^{1/n}, \qquad \varphi_k = \frac{\theta + 2k\pi}{n}.$$

Thus there are exactly n ***distinct*** n***th roots:***

$$w_k = r^{1/n}\left(\cos\Big(\frac{\theta + 2k\pi}{n}\Big) + i\sin\Big(\frac{\theta + 2k\pi}{n}\Big)\right), \quad k = 0, \ldots, n-1.$$

The root corresponding to $k = 0$ ***(with argument taken in the principal branch) is called the*** principal nth root ***of*** z***.***

Exercise 1.3 ***Find all fourth roots of***

$$1 + i.$$

(That is, compute w ***such that*** $w^4 = 1 + i$***.)***

Solution 1.3 ***We seek all complex numbers*** w ***such that***

$$w^4 = 1 + i.$$

Step 1: Put the right-hand side in polar form. ***Write*** $1 + i$ ***in polar (exponential) form. Its modulus is***

$$|1 + i| = \sqrt{1^2 + 1^2} = \sqrt{2},$$

and a principal argument is

$$\mathrm{Arg}(1 + i) = \frac{\pi}{4}.$$

Hence

$$1 + i = \sqrt{2}\, e^{i\pi/4}.$$

Step 2: Apply the formula for n-th roots. If $w^4 = 1 + i$ and we set $w = \rho e^{i\varphi}$, then

$$\rho^4 = \sqrt{2}, \qquad 4\varphi = \frac{\pi}{4} + 2k\pi, \quad k \in \mathbb{Z}.$$

Thus

$$\rho = (\sqrt{2})^{1/4} = 2^{1/8}, \qquad \varphi = \frac{\pi/4 + 2k\pi}{4} = \frac{\pi}{16} + \frac{k\pi}{2}, \quad k \in \mathbb{Z}.$$

Step 3: List the four distinct roots. Taking $k = 0, 1, 2, 3$ gives the four distinct fourth roots:

$$w_k = 2^{1/8}\, e^{i\left(\frac{\pi}{16} + \frac{k\pi}{2}\right)} = 2^{1/8}\left(\cos\left(\frac{\pi}{16} + \frac{k\pi}{2}\right) + i \sin\left(\frac{\pi}{16} + \frac{k\pi}{2}\right)\right), \quad k = 0, 1, 2, 3.$$

Remark (principal root). The principal fourth root corresponds to $k = 0$:

$$w_0 = 2^{1/8}\, e^{i\pi/16}.$$

Check. For any k,

$$w_k^4 = \left(2^{1/8}\right)^4 e^{i\left(4(\pi/16 + k\pi/2)\right)} = 2^{1/2} e^{i(\pi/4 + 2k\pi)} = \sqrt{2}\, e^{i\pi/4} = 1 + i,$$

as required.

Exponential Form and Euler's Formula

Definition 1.13 (Exponential Form) ***Using Euler's formula (see below), any complex number can be written in exponential form as***

$$z = re^{i\theta},$$

where $r = |z|$ and $\theta = \arg(z)$ (or $\operatorname{Arg}(z)$ for the principal value).

Proposition 1.5 (Euler's Formula) ***For every real θ,***

$$e^{i\theta} = \cos\theta + i \sin\theta.$$

Remark 1.6 ***From Euler's formula the polar and exponential forms are equivalent:***

$$z = r(\cos\theta + i\sin\theta) = re^{i\theta}.$$

Multiplication then becomes particularly simple:

$$r_1 e^{i\theta_1} \cdot r_2 e^{i\theta_2} = (r_1 r_2) e^{i(\theta_1 + \theta_2)},$$

and similarly for powers and roots. Endeed, $w = \rho(\cos\varphi + i\sin\varphi) = \rho e^{i\varphi}$ and $w^n = \rho^n e^{in\varphi}$. Hence, when $w^n = z = re^{i\theta}$ identify moduli $\rho^n = r$ and arguments $n\varphi = \theta + 2k\pi$, then solve for ρ and φ.

1.1.4 Application: Quadratic Formula in Complex Numbers

Consider a quadratic equation with complex coefficients:

$$az^2 + bz + c = 0, \quad a \neq 0.$$

We can solve it in a manner similar to real numbers. First, compute the discriminant:

$$\Delta = b^2 - 4ac.$$

In general, the roots are given by:

$$z_i = \frac{-b \pm \sqrt{\Delta}}{2a}, \quad i = 1, 2.$$

Exercise 1.4 ***Solve the following quadratic equations in the set of complex numbers:***

1. $z^2 + 4z + 5 = 0$

2. $z^2 + (1-i)z - 3i = 0$

Solution 1.4 ***1. Solve*** $z^2 + 4z + 5 = 0$***:***
The discriminant is

$$\Delta = b^2 - 4ac = 4^2 - 4 \cdot 1 \cdot 5 = 16 - 20 = -4.$$

Since $\Delta < 0$***, the roots are complex:***

$$z = \frac{-b \pm \sqrt{\Delta}}{2a} = \frac{-4 \pm \sqrt{-4}}{2} = \frac{-4 \pm 2i}{2} = -2 \pm i.$$

So the solutions are:

$$\boxed{z_1 = -2 + i, \quad z_2 = -2 - i}.$$

2. Solve $z^2 + (1-i)z - 3i = 0$***:***
The discriminant is

$$\Delta = b^2 - 4ac = (1-i)^2 - 4 \cdot 1 \cdot (-3i).$$

Compute $(1-i)^2$***:***

$$(1-i)^2 = 1 - 2i + i^2 = 1 - 2i - 1 = -2i.$$

Then:

$$\Delta = -2i - 4(-3i) = -2i + 12i = 10i.$$

Now, compute the square root of $10i$***:***

$$10i = 10\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right) \implies \sqrt{10i} = \sqrt{10}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right) = \sqrt{5}(1+i).$$

The roots are:

$$z = \frac{-b \pm \sqrt{\Delta}}{2a} = \frac{-(1-i) \pm \sqrt{5}(1+i)}{2} = \frac{-1+i \pm \sqrt{5}(1+i)}{2}.$$

Separating the two roots:

$$z_1 = \frac{-1+\sqrt{5}}{2} + i\frac{1+\sqrt{5}}{2}, \qquad z_2 = \frac{-1-\sqrt{5}}{2} + i\frac{1-\sqrt{5}}{2}.$$

So the solutions are:

$$\boxed{z_1 = \frac{-1+\sqrt{5}}{2} + i\frac{1+\sqrt{5}}{2}, \qquad z_2 = \frac{-1-\sqrt{5}}{2} + i\frac{1-\sqrt{5}}{2}}.$$

1.1.5 Sets of Points in the Complex Plane

In this section, we introduce the essential definitions and terminology for sets of points in the complex plane.

1. The Circle

Let

$$z_0 = x_0 + iy_0 \in \mathbb{C}.$$

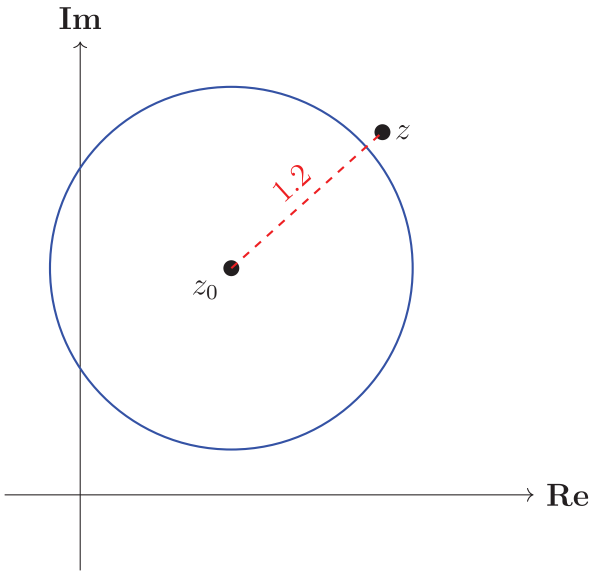
Recall that the distance between any point $z = x + iy$ and z_0 is

$$|z - z_0| = \sqrt{(x-x_0)^2 + (y-y_0)^2}.$$

Definition 1.14 (Circle in the Complex Plane) ***The set of all points z that satisfy***

$$|z - z_0| = \rho, \qquad \rho > 0,$$

is called a circle with center z_0 and radius ρ. In other words, it is the set of all points at a fixed distance ρ from z_0.



Center: $z_0 = x_0 + iy_0$, **Radius:** $|z - z_0| = 1.2$

2. Disk and Neighborhood

Definition 1.15 (Disk) ***Let*** $z_0 \in \mathbb{C}$ ***and*** $\rho > 0$***. The disk of radius*** ρ ***centered at*** z_0 ***is the set of all points*** z ***satisfying***

$$|z - z_0| \leq \rho.$$

This includes all points on the circle $|z - z_0| = \rho$ ***as well as all points inside the circle.***

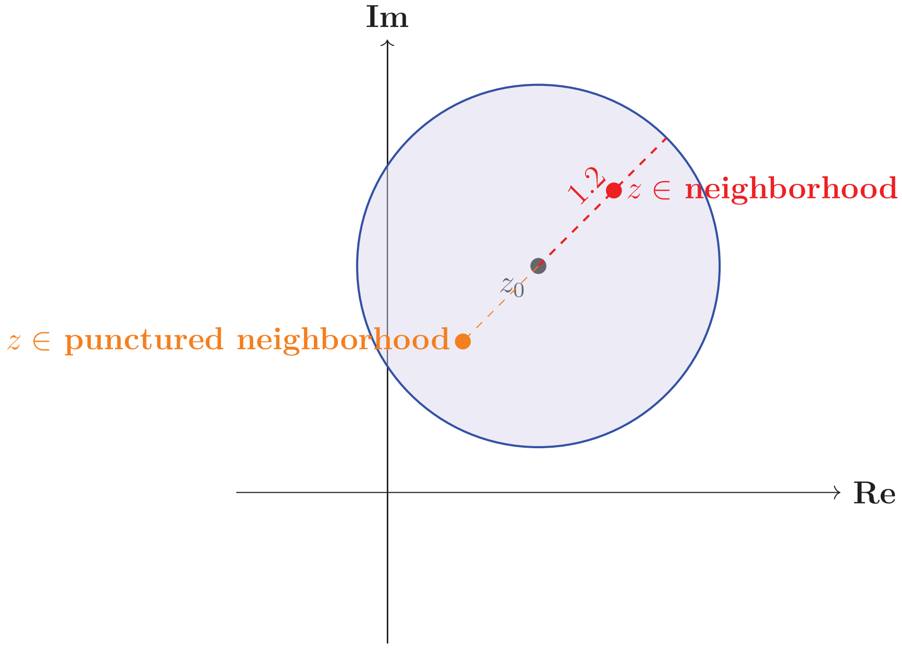
Definition 1.16 (Neighborhood of a Point) ***Let*** $z_0 \in \mathbb{C}$ ***and*** $\rho > 0$***. The neighborhood of*** z_0 ***or open disk with radius*** ρ ***and center*** z_0 ***is the set of points***

$$|z - z_0| < \rho.$$

These points lie strictly inside the disk, excluding the boundary circle.

A punctured neighborhood (or deleted neighborhood) excludes the center itself and is defined by the strict inequalities

$$0 < |z - z_0| < \rho.$$



Disk: $|z - z_0| \leq 1.2$, **Neighborhood:** $|z - z_0| < 1.2$, **Punctured Neighborhood:** $0 < |z - z_0| < 1.2$

Remark 1.7 *"Punctured neighborhood" mean neighborhood with the center punctured/excluded.*

3. Open Sets, Interior, Boundary, and Exterior Points

Definition 1.17 (Interior Point) *A point $z \in \mathbb{C}$ is called an interior point of a set S in the complex plane if there exists a neighborhood of z that is entirely contained in S.*

Definition 1.18 (Open Set) *A set S is called open if every point of S is an interior point.*

Definition 1.19 (Limit Point and Boundary) *A point z_0 is called a limit point of a set S if every neighborhood of z_0 contains at least one point in S and at least one point outside S.*

The set of all limit points of S forms the boundary of S, denoted by ∂S.

Definition 1.20 (Exterior Point) *A point $z \in \mathbb{C}$ is called an exterior point of S if it is neither an interior point nor a boundary point of S.*

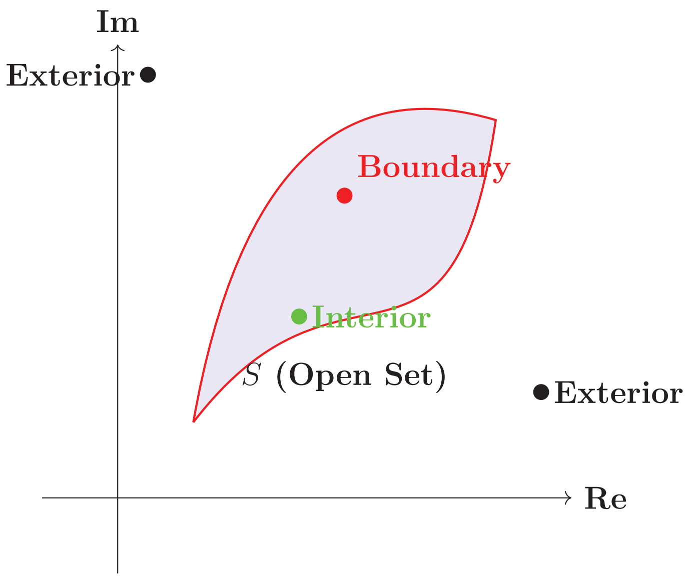


Figure: Open set S, interior points (green), boundary points (red), exterior points (black).

4. Annulus (Circular Ring)

Definition 1.21 (Annulus) ***Consider two sets of points around a center $z_0 \in \mathbb{C}$:***

- $S_1 = \{z \in \mathbb{C} \mid |z - z_0| > \rho_1\}$, ***representing points outside the circle of radius ρ_1 centered at z_0.***
- $S_2 = \{z \in \mathbb{C} \mid |z - z_0| < \rho_2\}$, ***representing points inside the circle of radius ρ_2 centered at z_0.***

If $0 < \rho_1 < \rho_2$, then the set of points satisfying the strict inequalities simultaneously,

$$\rho_1 < |z - z_0| < \rho_2,$$

is the intersection of S_1 and S_2, and it is called a circular annulus.

Remark 1.8 ***If we take $\rho_1 = 0$, we recover the definition of an open disk with the center excluded, i.e., a punctured neighborhood.***

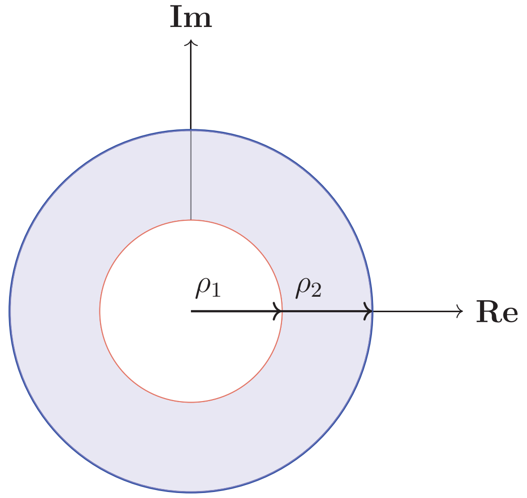
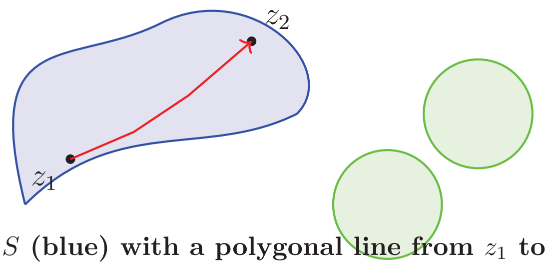


Figure: Circular annulus (blue area) with inner radius ρ_1 (red) and outer radius ρ_2 (blue), centered at z_0.

5. Domain

Definition 1.22 (Domain) ***Let S be a set of points in the complex plane. If any two points $z_1, z_2 \in S$ can be connected by a polygonal line composed of a finite sequence of line segments joined end-to-end, and this line lies entirely within S, then the set S is said to be connected.***

An open connected set is called a domain.



Connected domain S (blue) with a polygonal line from z_1 to z_2

Non-connected set (green) example

Region

Definition 1.23 (Region) ***A region in the complex plane is any set of points that may include all, some, or none of its boundary points.***

According to this definition:

- ***An open set, which contains none of its boundary points, is a region.***
- ***A closed region contains all its boundary points. For example, the closed disk***

$$|z - z_0| \leq \rho$$

is a closed region.

- ***An open region such as the disk***

$$|z - z_0| < \rho$$

is called an open disk.

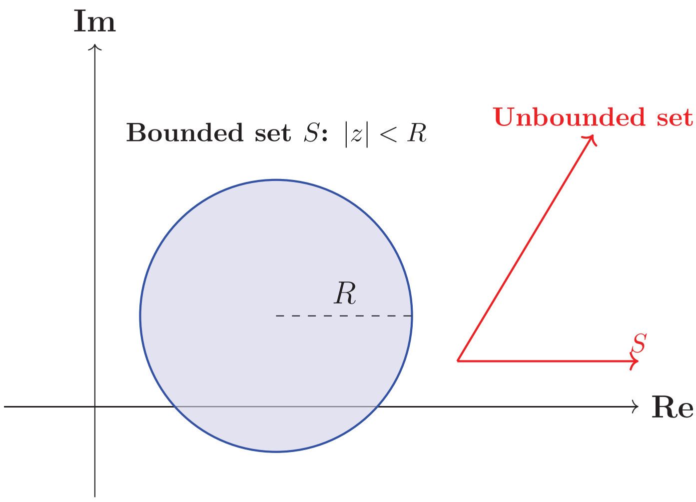
- ***If the center*** z_0 ***is excluded, the set is called a punctured region or punctured disk.***

Bounded Set

Definition 1.24 (Bounded Set) ***A set*** S ***in the complex plane is said to be bounded if there exists a real number*** $R > 0$ ***such that***

$$|z| < R \quad \textit{for all } z \in S.$$

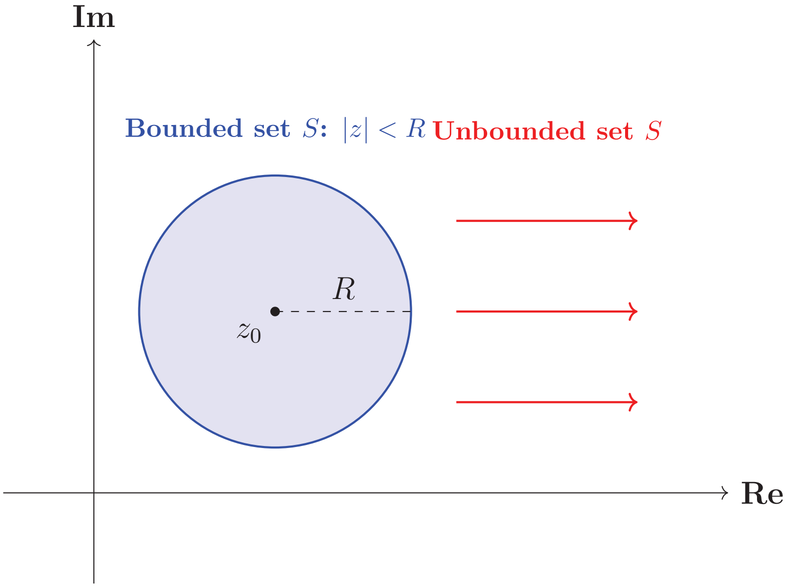
Otherwise, the set is called unbounded.



In the figure above:

- The blue disk represents a bounded set S. Its center is $z_0 = 2 + i$ and all points inside satisfy
$$|z - z_0| < R.$$
The dashed line shows the radius R from the center z_0 to a point on the boundary. All points of this set are within a finite distance from z_0.

- The arrows indicate an unbounded set S. They illustrate that the set extends beyond the visible portion of the plane. Since the points can continue indefinitely in some directions, there is no finite R that contains all points, making it unbounded.



In the figure above:

- The blue disk represents a bounded set S. Its center is z_0 and all points inside satisfy $|z - z_0| < R$, so there exists a finite radius R containing all points.
- The red arrows represent an unbounded set S. The arrows indicate that the points in this set can extend indefinitely in the complex plane. Therefore, no finite R can contain all points, making the set unbounded.
- The dashed line in the blue disk shows the radius R from the center z_0 to a point on the boundary.

1.2 Complex Functions, Limits, Continuity, and the Complex Infinity

1.2.1 Definition of a Complex Function

Definition 1.25 ***A complex function f is a mapping from a subset D of the complex plane $\mathbb{C}$ to $\mathbb{C}$:***

$$f : D \subset \mathbb{C} \to \mathbb{C}, \quad z \mapsto f(z),$$

where $z = x + iy \in D$ and

$$f(z) = u(x, y) + iv(x, y), \quad \textit{with } u(x, y), v(x, y) \in \mathbb{R}.$$

-The set D is called the domain of definition of f.
- $u(x, y)$ is the real part of $f(z)$
- $v(x, y)$ is the imaginary part of $f(z)$
- $f(z) = u(x, y) + iv(x, y)$ is the Algebric form of $f(z)$.

Example 1.4 *Consider the function*

$$f(z) = z^2, \quad z \in \mathbb{C}.$$

Its domain of definition is the entire complex plane $\mathbb{C}$. If $z = x + iy$, then

$$f(z) = (x + iy)^2 = x^2 - y^2 + i(2xy),$$

where

$$u(x, y) = x^2 - y^2, \quad v(x, y) = 2xy.$$

Example 1.5 *Consider the function*

$$f(z) = \frac{1}{z(z^2 + 1)}.$$

This function is not defined for the values of z that make the denominator zero. Solving

$$z(z^2 + 1) = 0 \implies z = 0,\ z = i,\ z = -i.$$

Hence, the domain of definition of f is

$$D = \mathbb{C} \setminus \{0,\, i,\, -i\}.$$

1.2.2 Uniform and Multiform Complex Functions

Definition 1.26 ***In the strict mathematical sense, a function f defined on a domain $D \subset \mathbb{C}$ is a mapping that assigns to each point $z \in D$ one and only one value $w = f(z) \in \mathbb{C}$.***

A complex function is said to be uniform (or single-valued) if it satisfies this property throughout its domain.

However, in complex analysis, we sometimes encounter expressions such as $\sqrt{z}$, $\log z$, or $z^{1/n}$, which can take several possible values for the same z. These are not functions in the strict sense, but are called multiform functions (or multi-valued functions). Each distinct determination (or branch) of such an expression defines a true single-valued function on a suitable subset of $\mathbb{C}$.

Example 1.6 (Uniform function) *The function*

$$f(z) = z^2$$

is uniform, because for each $z \in \mathbb{C}$ there exists exactly one value of $f(z)$.

Example 1.7 (Multiform expression) *The expression*

$$f(z) = \sqrt{z}$$

is multiform because for each $z \neq 0$, there are two possible values:

$$\sqrt{z} = \pm\sqrt{r}\, e^{i\theta/2}, \quad \textit{where } z = re^{i\theta}.$$

To obtain a true function, one must choose a branch - for example, by restricting the argument to $-\pi < \theta \leq \pi$, defining the principal branch of $\sqrt{z}$.

1.2.3 Limit of a Complex Function

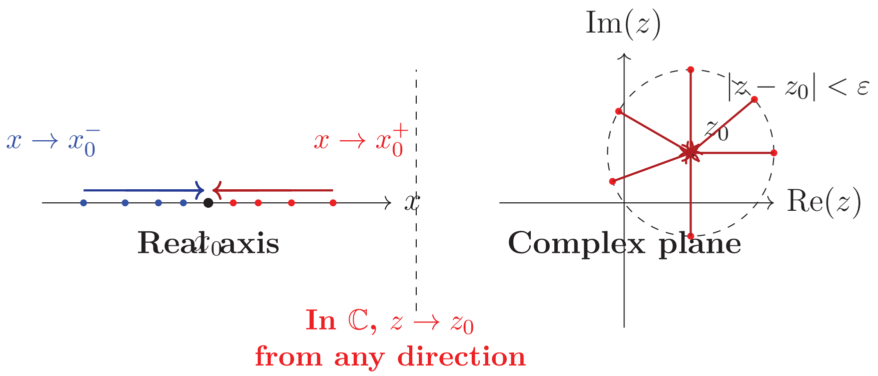
Definition 1.27 *Let $f : D \subset \mathbb{C} \to \mathbb{C}$ be a complex function, and let z_0 be a limit point of D. We say that $f(z)$ tends to a limit $w_0 \in \mathbb{C}$ when $z \to z_0$ if, for every $\varepsilon > 0$, there exists $\delta > 0$ such that*

$$0 < |z - z_0| < \delta \quad \Rightarrow \quad |f(z) - w_0| < \varepsilon.$$

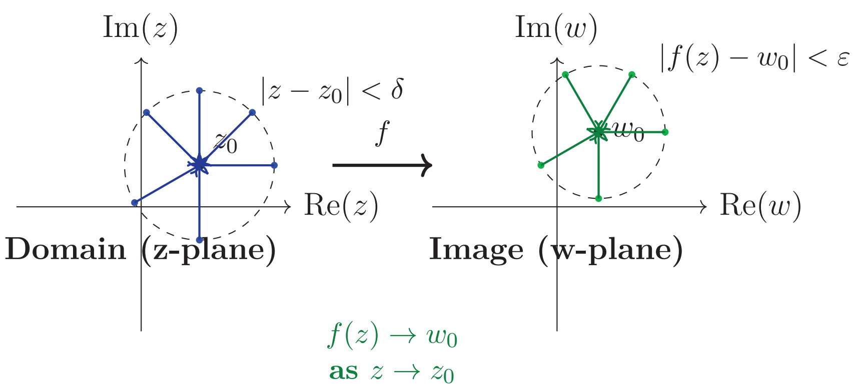
Symbolically, we write

$$\lim_{z \to z_0} f(z) = w_0.$$

Remark 1.9 *Geometrically, this means that when the point z approaches z_0 from any direction in the complex plane, the image point $f(z)$ approaches w_0 in the image plane. This definition extends the concept of limit from real functions (where z and $f(z)$ lie on $\mathbb{R}$) to complex functions (where both belong to $\mathbb{C}$).*



Approach of the variable: real vs. complex



Meaning of $\lim_{z \to z_0} f(z) = w_0$

Example 1.8

$$f(z) = z^2, \quad z_0 = 1 + i$$

$$\lim_{z \to 1+i} f(z) = (1 + i)^2 = 2i.$$

Criterion for the Non-Existence of a Limit

If a function f approaches two distinct complex numbers $L_1 \neq L_2$ along two different paths passing through z_0, then

$$\lim_{z \to z_0} f(z)$$

does not exist.

Example 1.9 ***Show that*** $\lim_{z\to 0} \frac{z}{\bar{z}}$ ***does not exist.***

Solution 1.5 ***Let us examine the limit along two different paths approaching the origin.***

1. ***Along the real axis:*** ***If*** $z = x$ ***(with*** $y = 0$***), then***

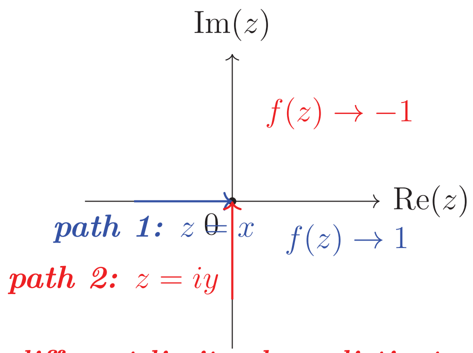
$$\frac{z}{\bar{z}} = \frac{x}{x} = 1 \quad \Rightarrow \quad \lim_{x\to 0} \frac{z}{\bar{z}} = 1$$

2. ***Along the imaginary axis:*** ***If*** $z = iy$ ***(with*** $x = 0$***), then***

$$\frac{z}{\bar{z}} = \frac{iy}{-iy} = -1 \quad \Rightarrow \quad \lim_{y\to 0} \frac{z}{\bar{z}} = -1$$

Since the two limits obtained by approaching the origin along different paths are distinct $(1 \neq -1)$***, the overall limit does not exist.***

$$\boxed{\lim_{z\to 0} \frac{z}{\bar{z}} \ \textit{does not exist.}}$$



Two different limits along distinct paths
⇒ limit does not exist.

In the case where a complex function $f(z)$ can be expressed in terms of its real and imaginary parts as

$$f(z) = f(x, y) = u(x, y) + iv(x, y),$$

then, by using the properties of real limits, one can derive a rule that allows the computation of the limit of $f(z)$ when the limits of $u(x, y)$ and $v(x, y)$ are known. In this situation, the real functions u and v are considered as functions of two real variables.

Theorem 1.1 (Real and Imaginary Parts of a Limit) ***Let***

$$f(z) = u(x,y) + iv(x,y), \quad z_0 = x_0 + iy_0, \quad \textbf{\textit{and}} \quad L = u_0 + iv_0.$$

Then,

$$\lim_{z\to z_0} f(z) = L \quad \textbf{\textit{if and only if}} \quad \begin{cases} \lim\limits_{(x,y)\to(x_0,y_0)} u(x,y) = u_0, \\ \lim\limits_{(x,y)\to(x_0,y_0)} v(x,y) = v_0. \end{cases}$$

Example 1.10 ***Calculate, using Theorem 1.1, the following limit:***

$$\lim_{z\to 1+i} (z^2 + i).$$

Solution 1.6 ***We first write*** $z = x + iy$***. Then,***

$$f(z) = z^2 + i = (x+iy)^2 + i = (x^2 - y^2) + i(2xy + 1).$$

Hence we can identify:

$$u(x,y) = x^2 - y^2, \qquad v(x,y) = 2xy + 1.$$

According to Theorem 1.1, we compute the two real limits:

$$\lim_{(x,y)\to(1,1)} u(x,y) = 1^2 - 1^2 = 0, \qquad \lim_{(x,y)\to(1,1)} v(x,y) = 2(1)(1) + 1 = 3.$$

Therefore,

$$\lim_{z\to 1+i} (z^2 + i) = 0 + 3i = 3i.$$

Theorem 1.2 (Properties of Complex Limits) ***Let*** f ***and*** g ***be two complex functions. If***

$$\lim_{z\to z_0} f(z) = L \quad \textbf{\textit{and}} \quad \lim_{z\to z_0} g(z) = M,$$

then the following properties hold:

(i) $\lim\limits_{z\to z_0} cf(z) = cL,$ ***where*** c ***is a complex constant;***

(ii) $\lim\limits_{z\to z_0} [f(z) \pm g(z)] = L \pm M;$

(iii) $\lim\limits_{z\to z_0} [f(z)\, g(z)] = L\, M;$

(iv) $\lim\limits_{z\to z_0} \dfrac{f(z)}{g(z)} = \dfrac{L}{M},$ ***provided that*** $M \neq 0.$

Basic results from Theorem 1.2:

$$\lim_{z\to z_0} c = c, \quad \textbf{where } c \textbf{ is a complex constant},$$

$$\lim_{z\to z_0} z = z_0.$$

Example 1.11 ***Using Theorem 1.2 and its two basic results, compute the following limits:***

(a) $\lim_{z \to i} \frac{(3+i)z^4 - z^2 + 2z}{z+1}$

(b) $\lim_{z \to 1+i\sqrt{3}} \frac{z^2 - 2z + 4}{z - 1 - i\sqrt{3}}$

Solution 1.7 ***(a) By the properties of complex limits (Theorem 1.2):***

$$\lim_{z \to i} \frac{(3+i)z^4 - z^2 + 2z}{z+1} = \frac{(3+i)(i)^4 - (i)^2 + 2(i)}{i+1}.$$

We compute step by step:

$$i^2 = -1, \quad i^4 = 1.$$

Then:

$$\textbf{\textit{Numerator:}}\ (3+i)(1) - (-1) + 2i = 3 + i + 1 + 2i = 4 + 3i.$$

$$\textbf{\textit{Denominator:}}\ i + 1 = 1 + i.$$

Hence:

$$\frac{4+3i}{1+i} = \frac{(4+3i)(1-i)}{(1+i)(1-i)} = \frac{(4+3i)(1-i)}{2}.$$

Expanding the numerator:

$$(4+3i)(1-i) = 4 - 4i + 3i - 3i^2 = 4 - i + 3 = 7 - i.$$

Therefore:

$$\boxed{\lim_{z \to i} \frac{(3+i)z^4 - z^2 + 2z}{z+1} = \frac{7-i}{2}}.$$

(b) We apply the same rule:

$$\lim_{z \to 1+i\sqrt{3}} \frac{z^2 - 2z + 4}{z - 1 - i\sqrt{3}} = \frac{(1+i\sqrt{3})^2 - 2(1+i\sqrt{3}) + 4}{(1+i\sqrt{3}) - 1 - i\sqrt{3}}.$$

Compute the numerator:

$$(1+i\sqrt{3})^2 = 1 + 2i\sqrt{3} + (i\sqrt{3})^2 = 1 + 2i\sqrt{3} - 3 = -2 + 2i\sqrt{3}.$$

Then:

$$-2 + 2i\sqrt{3} - 2(1+i\sqrt{3}) + 4 = -2 + 2i\sqrt{3} - 2 - 2i\sqrt{3} + 4 = 0.$$

Denominator:

$$(1+i\sqrt{3}) - 1 - i\sqrt{3} = 0.$$

So we get the indeterminate form $\frac{0}{0}$***. To evaluate the limit, we simplify the expression:***

$$\frac{z^2 - 2z + 4}{z - (1 + i\sqrt{3})}.$$

Factorizing the numerator:

$$z^2 - 2z + 4 = (z - (1 + i\sqrt{3}))(z - (1 - i\sqrt{3})).$$

Hence:

$$\frac{z^2 - 2z + 4}{z - (1 + i\sqrt{3})} = z - (1 - i\sqrt{3}).$$

Now taking the limit:

$$\lim_{z \to 1+i\sqrt{3}} [z - (1 - i\sqrt{3})] = (1 + i\sqrt{3}) - (1 - i\sqrt{3}) = 2i\sqrt{3}.$$

Therefore:

$$\boxed{\lim_{z \to 1+i\sqrt{3}} \frac{z^2 - 2z + 4}{z - 1 - i\sqrt{3}} = 2i\sqrt{3}.}$$

1.2.4 The Complex Infinity

Unlike the real number system, which contains two distinct infinities $+\infty$ and $-\infty$, the complex number system has only one infinity. This distinction arises because the field of complex numbers $\mathbb{C}$ is not an ordered field. Recall that in the real number system, $+\infty$ and $-\infty$ serve respectively as the *upper* and *lower* *bounds* of every subset of the extended real line - that is, the set of all real numbers together with $+\infty$ and $-\infty$.

Example 1.12

$$\lim_{z \to \infty} \frac{1}{z} = 0$$

1. Extended Complex Plane

To simplify the study of complex functions, it is often useful to extend the complex plane by adding a single point at infinity, denoted by ∞. The resulting set

$$\mathbb{C} \cup \{\infty\}$$

is called the extended complex plane.

1. Algebraic rules involving ∞

The basic operations with ∞ are defined as follows:

$$\begin{aligned} z+\infty &= \infty, \quad \text{for all } z \in \mathbb{C}, \\ z \cdot \infty &= \infty, \quad \text{for all } z \in \mathbb{C} \setminus \{0\}, \\ \frac{z}{\infty} &= 0, \quad \text{for all } z \in \mathbb{C}, \\ \frac{z}{0} &= \infty, \quad \text{for all } z \in \mathbb{C} \setminus \{0\}. \end{aligned}$$

In particular, $-1 \cdot \infty = \infty$. However, the expressions $0 \cdot \infty$, $\frac{\infty}{\infty}$, $\infty \pm \infty$, and $\frac{0}{0}$ are undefined.

2. Topological interpretation

From a topological point of view, any set of the form

$$\{\, z : |z| > R \,\}, \quad R \geq 0,$$

is called a neighborhood of ∞.

A set $D \subset \mathbb{C} \cup \{\infty\}$ is said to contain the point at infinity if there exists a real number $M > 0$ such that D includes all points z with $|z| > M$.

Examples 1.1

- ***The open half-plane*** $\mathrm{Re}(z) > 0$ ***does not contain the point at infinity, since it includes no neighborhood of*** ∞.
- ***The open set***

$$D = \{\, z : |z+1| + |z-1| > 1 \,\}$$

does contain the point at infinity.

3. The Structure of Infinity in the Complex Plane

When we extend the complex plane by adding the point at infinity, we obtain the extended complex plane, denoted by

$$\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}.$$

This set is also known as the Riemann sphere.

- **The usual complex numbers are the points $z = x + iy \in \mathbb{C}$.**
- **The symbol ∞ is not a number but an additional point. It cannot be manipulated by ordinary arithmetic operations.**

4. Topological structure

To make sense of the point at infinity, we define a topology on $\widehat{\mathbb{C}}$: a neighborhood of ∞ is any set of the form

$$\{ z \in \mathbb{C} : |z| > R \} \cup \{\infty\},$$

for some $R > 0$. In this sense, saying that z is "close to infinity" means that $|z|$ is very large.

The definition of limit at infinity becomes:

$$\lim_{z \to \infty} f(z) = L \iff \forall \varepsilon > 0,\ \exists R > 0,\ |z| > R \Rightarrow |f(z) - L| < \varepsilon.$$

Geometric structure (Riemann sphere) The extended complex plane can be visualized as a sphere in $\mathbb{R}^3$, called the Riemann sphere. We imagine a sphere of radius 1 centered at $(0, 0, 1)$ in space. Each point $z = x + iy$ of the complex plane (lying on the plane $z = 0$) is mapped to the sphere by ***stereographic projection.*** The point ∞ corresponds to the North Pole of the sphere.

Thus, topologically:

$$\widehat{\mathbb{C}} \cong S^2.$$

5. Partial algebraic structure

Arithmetic operations involving ∞ are defined by convention:

$$\begin{aligned} z + \infty &= \infty, \quad z \in \mathbb{C}, \\ z \cdot \infty &= \infty, \quad z \neq 0, \\ \frac{z}{\infty} &= 0, \quad z \in \mathbb{C}, \\ \frac{z}{0} &= \infty, \quad z \neq 0. \end{aligned}$$

However, the following expressions are undefined:

$$0 \cdot \infty, \quad \infty - \infty, \quad \frac{\infty}{\infty}, \quad \frac{0}{0}.$$

Therefore, $\widehat{\mathbb{C}}$ is not a field but a compact topological space endowed with a useful partial algebra.

6. Summary

Aspect	Description
Set	$\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$
Nature of ∞	A point added, not a number
Topology	Neighborhoods of ∞: $\{z : \lvert z\rvert > R\} \cup \{\infty\}$
Geometry	Topologically equivalent to a sphere (Riemann sphere)
Algebra	Partial arithmetic: $z + \infty = \infty$, $z/\infty = 0$, etc.

1.2.5 Continuity of Complex Functions

The definition of continuity for a complex function is analogous to that of a real function. In other words, a complex function is continuous at a point if its value approaches the function value at that point as the variable approaches it.

Definition 1.28 (Continuity at a point) *Let $f : D \subset \mathbb{C} \to \mathbb{C}$ be a complex function and let $z_0 \in D$. We say that f is continuous at z_0 if*

$$\lim_{z \to z_0} f(z) = f(z_0).$$

Equivalently, using the ε–δ definition:

$$\forall \varepsilon > 0, \exists \delta > 0 \text{ such that if } |z - z_0| < \delta, \text{ then } |f(z) - f(z_0)| < \varepsilon.$$

Remark 1.10 *This definition is identical in form to that of real-valued functions, but the variable z now approaches z_0 from* all possible directions *in the complex plane.*

1. Criteria for Continuity

Just as in the real case, for a complex function f to be continuous at a point, the following three conditions must be satisfied.

Definition 1.29 (Continuity criteria) *A complex function f is continuous at a point $z_0 \in \mathbb{C}$ if and only if the following conditions hold:*

(i) The limit $\lim_{z \to z_0} f(z)$ exists.

(ii) The function f is defined at z_0.

(iii) The limit equals the function value:

$$\lim_{z \to z_0} f(z) = f(z_0).$$

If a complex function does not satisfy one or more of these conditions at a point, it is said to be discontinuous at that point.

2. Branches of Multivalued Functions

A *branch* of a multivalued function $F(z)$ defined on a domain D is any complex function $f_i(z)$ that is defined and continuous on a certain subdomain $D_i \subset D$ and that assigns *exactly one value* of $F(z)$ to each point $z \in D_i$.
In the definition of the branches $f_i(z)$ of a multivalued function, a point that is common to the different cuts used to define the domain D_i of a branch is called a Branch Point.

Example 1.13 ***The function***

$$f(z) = \frac{1}{1+z^2}$$

is discontinuous at $z = i$ ***and*** $z = -i$***, since it is not defined at these points.***

Example 1.14 (Continuity verification) ***(a) Verify whether the function***

$$f(z) = z^2 - iz + 2$$

is continuous at $z_0 = 1 - i$.

(b) Show that the function

$$f(z) = \sqrt{z}$$

is discontinuous at $z_0 = -1$.

Solution 1.8 ***(a) The function*** $f(z) = z^2 - iz + 2$ ***is a polynomial in*** z***. Since polynomials are continuous everywhere in*** $\mathbb{C}$***,*** f ***is continuous at*** $z_0 = 1 - i$***.***

(b) Why $\sqrt{z}$ ***is discontinuous at*** $z_0 = -1$

We must first fix what we mean by $\sqrt{z}$***. The complex square root is naturally*** **multi-valued*****:***

$$\sqrt{z} = \pm\sqrt{r}\, e^{i\theta/2} \quad \textit{when } z = re^{i\theta}\ (r > 0).$$

To obtain a single-valued function we choose a branch. The usual **principal branch** ***of the square root is defined by***

$$\sqrt{z} = \sqrt{r}\, e^{i\theta/2}, \qquad \theta \in (-\pi, \pi],$$

which forces a branch cut along the negative real axis $(-\infty, 0]$***. On this branch the function is single-valued and continuous on*** $\mathbb{C} \setminus (-\infty, 0]$***, but it is*** **not** ***defined on the cut itself; in particular it is not defined at*** $z_0 = -1$***.***

There are two natural ways to conclude non-continuity at $z_0 = -1$***:***

(A) The function is not defined at $z_0 = -1$ ***on the principal branch. By the continuity criteria, a function that is not defined at*** z_0 ***cannot be continuous there.***

(B) The one-sided limits (approaching from two sides of the cut) are different. This shows there is no possible single-limit value at -1 ***even if we try to take limits from different directions.***

We illustrate (B) by taking two explicit approach paths that tend to -1 ***from opposite sides of the negative real axis.***

Path 1 (approach from above). Take

$$z_1(t) = e^{i(\pi - t)}, \qquad t > 0,\ t \to 0^+.$$

Then $z_1(t) \to -1$ ***as*** $t \to 0^+$***. Using the principal branch (argument*** $\theta = \pi - t \in (-\pi, \pi]$***) we get***

$$\sqrt{z_1(t)} = \sqrt{1}\, e^{i(\pi - t)/2} = e^{i\pi/2}\, e^{-it/2} \xrightarrow[t \to 0^+]{} e^{i\pi/2} = i.$$

Path 2 (approach from below). Take

$$z_2(t) = e^{i(-\pi+t)}, \qquad t > 0,\ t \to 0^+.$$

Then $z_2(t) \to -1$ ***as*** $t \to 0^+$***. On the principal branch the argument*** $\theta = -\pi + t$ ***(note*** $-\pi + t$ ***is just inside the interval*** $(-\pi, \pi]$***), so***

$$\sqrt{z_2(t)} = \sqrt{1}\, e^{i(-\pi+t)/2} = e^{-i\pi/2}\, e^{it/2} \xrightarrow[t\to 0^+]{} e^{-i\pi/2} = -i.$$

Thus the limits along the two paths exist but are different:

$$\lim_{z\to -1 \text{ from above}} \sqrt{z} = i, \qquad \lim_{z\to -1 \text{ from below}} \sqrt{z} = -i.$$

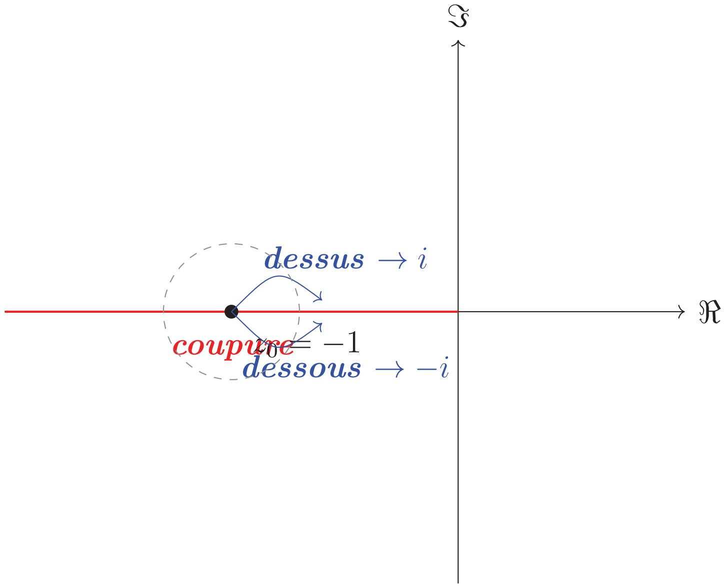
Because these one-sided limits are unequal, the two-dimensional limit $\lim_{z\to -1} \sqrt{z}$ ***does not exist. Hence*** $\sqrt{z}$ ***is discontinuous at*** $z_0 = -1$ ***(on the principal branch).***

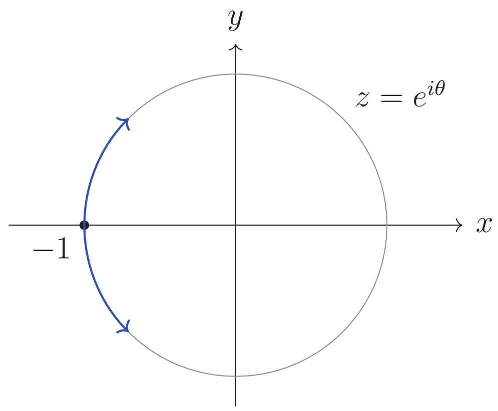
Remark 1.11

- ***The non-existence of a single-valued, continuous square root on any domain that encircles the origin is a consequence of the branch point at*** 0***: analytic continuation around*** 0 ***changes the sign of*** $\sqrt{z}$***. Any branch cut must connect*** 0 ***to*** ∞***; if the chosen cut passes through*** -1 ***(as the usual negative-real-axis cut does), then*** -1 ***lies on the cut and the principal branch is not defined there and shows the discontinuity described above.***

- ***If one chooses a different branch cut that*** does not ***pass through*** -1***, then one can define a single-valued branch of*** $\sqrt{z}$ ***that is continuous in a neighborhood of*** -1***. So the discontinuity at*** -1 ***is not an intrinsic property of the point*** -1 ***alone but depends on the chosen branch (equivalently: on the domain where the single-valued branch is defined). The usual statement -*** $\sqrt{z}$ ***is discontinuous at*** -1***- refers to the common principal branch with cut along*** $(-\infty, 0]$***.***

Conclusion. On the principal branch (argument in $(-\pi, \pi]$***) the square root is not defined at*** -1 ***and the limits from the two sides of the negative real axis give*** i ***and*** $-i$ ***respectively. Therefore*** $\sqrt{z}$ ***is discontinuous at*** $z_0 = -1$***.***





3. Properties of Continuous Functions

As for limits, we study the continuity of a complex function expressed in terms of its real and imaginary parts:

$$f(z) = f(x, y) = u(x, y) + iv(x, y)$$

Theorem 1.3 (Real and Imaginary Parts of a Continuous Function)

$$\textit{Let}\ \ f(z) = u(x, y) + iv(x, y) \quad \textit{and}\ \ z_0 = x_0 + iy_0.$$

Then the complex function f ***is continuous at*** z_0 ***if and only if the two real functions*** u ***and*** v ***are continuous at the point*** (x_0, y_0)***.***

Example 1.15 ***Using Theorem 1.3, show that the function***

$$f(z) = \overline{z}$$

is continuous on $\mathbb{C}$***.***

Theorem 1.4 (Properties of Continuous Functions) ***If*** f ***and*** g ***are two complex functions continuous at*** z_0***, then the following functions are also continuous at the same point:***

1. $cf(z)$, ***where*** c ***is a complex constant,***
2. $f(z) \pm g(z)$,
3. $f(z) \cdot g(z)$,
4. $\dfrac{f(z)}{g(z)}$, ***provided*** $g(z_0) \neq 0$.

\- ***Polynomial functions are continuous over the entire complex plane*** $\mathbb{C}$.
\- ***Unlike complex polynomial functions, complex rational functions***

$$f(z) = \frac{p(z)}{q(z)}$$

are not always continuous on the entire complex plane $\mathbb{C}$***, but they are continuous only on their domain of definition, that is, where*** $q(z) \neq 0$.

- Boundedness Property:
If a complex function f is continuous on a closed and bounded region Γ, then f is bounded on Γ. Thus, there exists a real constant $M > 0$ such that

$$|f(z)| \leq M \quad \textbf{for all } z \in \Gamma.$$

1.3 Solved Exercises

Exercise 1.5 ***Let $z_1 = 3 + 4i$ and $z_2 = 1 - 2i$. Compute the following:***
(a) $z_1 + z_2$, ***(b)*** $z_1 - z_2$, ***(c)*** $z_1 \cdot z_2$, ***(d)*** $\dfrac{z_1}{z_2}$, ***(e)*** $\overline{z_1}$, ***(f)*** $|z_1|$, ***(g)*** $\sqrt{z_1}$,
(h) Solve $z^2 + 1 = 0$.

Solution 1.9 ***(a) Sum:***

$$z_1 + z_2 = (3 + 4i) + (1 - 2i) = 4 + 2i.$$

(b) Difference:

$$z_1 - z_2 = (3 + 4i) - (1 - 2i) = 2 + 6i.$$

(c) Product:

$$z_1 \cdot z_2 = (3 + 4i)(1 - 2i) = 11 - 2i.$$

(d) Quotient:

$$\frac{z_1}{z_2} = \frac{3 + 4i}{1 - 2i} \cdot \frac{1 + 2i}{1 + 2i} = -1 + 2i.$$

(e) Conjugate:

$$\overline{z_1} = 3 - 4i.$$

(f) Modulus:

$$|z_1| = \sqrt{3^2 + 4^2} = 5.$$

(g) **Square root of z_1 using the polar identification method:**

Step 1: Write z_1 in polar form:

$$z_1 = 3 + 4i \implies \rho = |z_1| = \sqrt{3^2 + 4^2} = 5, \quad \phi = \arctan\frac{4}{3} \approx 0.927 \textit{ radians}.$$

Step 2: Let $z = r(\cos\theta + i\sin\theta)$ such that $z^2 = z_1$. Then

$$z^2 = r^2(\cos 2\theta + i\sin 2\theta) = \rho(\cos\phi + i\sin\phi).$$

Step 3: Identification gives:

$$r^2 = \rho \implies r = \sqrt{\rho} = \sqrt{5} \approx 2.236,$$

$$2\theta = \phi + 2k\pi \implies \theta = \frac{\phi}{2} + k\pi, \quad k = 0, 1.$$

Step 4: Compute the two roots: For $k = 0$***:***

$$\theta_0 = \frac{\phi}{2} \approx 0.464 \ \textbf{\textit{radians}}, \qquad z_0 = r(\cos\theta_0 + i\sin\theta_0).$$

For $k = 1$***:***

$$\theta_1 = \frac{\phi}{2} + \pi \approx 3.606 \ \textbf{\textit{radians}}, \qquad z_1 = r(\cos\theta_1 + i\sin\theta_1).$$

- Square root of z_1 using the algebraic method with modulus:

Let $z = x + iy$ ***such that*** $z^2 = z_1 = 3 + 4i$***.***
Step 1: Expand z^2***:***

$$z^2 = (x + iy)^2 = x^2 - y^2 + 2xy\, i.$$

Step 2: Identify the real and imaginary parts with $z_1 = 3 + 4i$***:***

$$\begin{cases} x^2 - y^2 = 3, \\ 2xy = 4. \end{cases}$$

Step 3: Add the modulus condition:

$$x^2 + y^2 = |z|^2 = |z_1| = \sqrt{3^2 + 4^2} = 5.$$

Step 4: Solve the system:

$$\begin{cases} x^2 - y^2 = 3, \\ x^2 + y^2 = 5. \end{cases}$$

Add the two equations:

$$2x^2 = 8 \implies x^2 = 4 \implies x = \pm 2.$$

Then from $x^2 + y^2 = 5$***:***

$$y^2 = 5 - 4 = 1 \implies y = \pm 1.$$

Step 5: Determine compatible signs from $2xy = 4$***:***

$$2 \cdot 2 \cdot 1 = 4 \quad \textbf{\textit{and}} \quad 2 \cdot (-2) \cdot (-1) = 4.$$

Step 6: Conclude the two square roots:

$$\boxed{z = 2 + i \quad \textbf{\textit{and}} \quad z = -2 - i}.$$

(h) Solve $z^2 + 1 = 0$***:***

$$z^2 = -1 \implies z = \pm i.$$

Exercise 1.6 ***Express the complex number*** $z = -1 + i\sqrt{3}$ ***in polar form and exponential form.***

Solution 1.10 ***Step 1: Compute the modulus r of z:***

$$r = |z| = \sqrt{(-1)^2 + (\sqrt{3})^2} = \sqrt{1+3} = 2.$$

Step 2: Compute the argument θ of z:

$$\theta = \arctan\left(\frac{\sqrt{3}}{-1}\right).$$

Since z is in the second quadrant, we have

$$\theta = \pi - \frac{\pi}{3} = \frac{2\pi}{3}.$$

Step 3: Write z in polar form:

$$z = r(\cos\theta + i\sin\theta) = 2\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right).$$

Step 4: Write z in exponential form:

$$z = re^{i\theta} = 2e^{i\frac{2\pi}{3}}.$$

Exercise 1.7 ***(a) Given the set of points defined by $|z-1| = 2$, describe the set geometrically.***

(b) Given the set of points defined by $|z-1| \leq 2$, describe the set geometrically.

(c) Given the set of points defined by $1 \leq |z-1| \leq 2$, describe the set geometrically.

Solution 1.11 ***Part (a): Circle***

Step 1: Write z in Cartesian form:

$$z = x + iy, \qquad x, y \in \mathbb{R}.$$

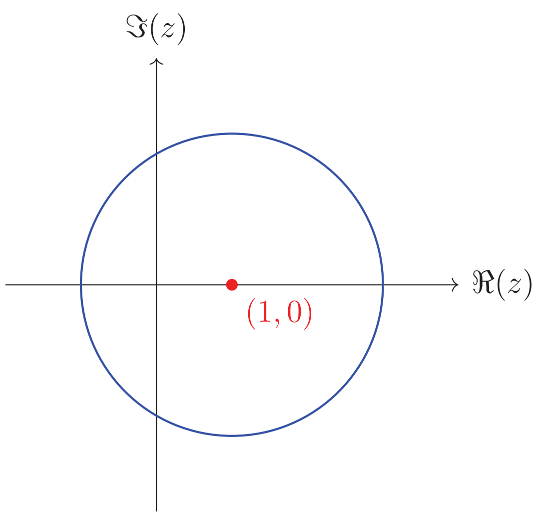
Step 2: Express the modulus:

$$|z-1| = |(x+iy) - 1| = \sqrt{(x-1)^2 + y^2}.$$

Step 3: Apply the condition $|z-1| = 2$:

$$\sqrt{(x-1)^2 + y^2} = 2 \implies (x-1)^2 + y^2 = 4.$$

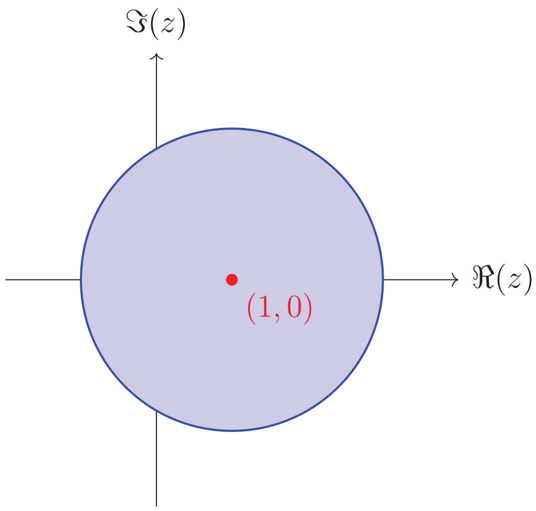
Step 4: Geometric description: - Circle with center $(1,0)$ and radius 2.



Part (b): Disk
Step 1: Condition $|z-1| \leq 2$ ***implies:***

$$(x-1)^2 + y^2 \leq 4.$$

Step 2: Geometric description: - Closed disk (including the boundary) with center $(1,0)$ ***and radius*** 2***.***



Part (c): Ring (Annulus)
Step 1: Condition $1 \leq |z-1| \leq 2$ ***in Cartesian form:***

$$z = x + iy \implies 1 \leq \sqrt{(x-1)^2 + y^2} \leq 2 \implies 1 \leq (x-1)^2 + y^2 \leq 4.$$

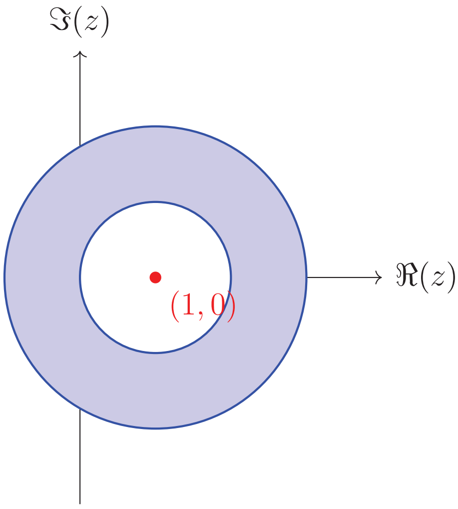
Step 2: Explanation of the left-hand side:

$$1 \leq |z-1| \implies (x-1)^2 + y^2 \geq 1.$$

- This represents all points whose distance from the center $(1,0)$ ***is at least*** 1***. - Geometrically, it is the **exterior of the inner circle** of radius 1.***
Step 3: Explanation of the right-hand side:

$$|z-1| \leq 2 \implies (x-1)^2 + y^2 \leq 4.$$

- Restricts points to **inside the outer circle** of radius 2.
Step 4: Geometric description: - The set is a **ring (annulus)** centered at $(1,0)$***, with inner radius*** 1 ***and outer radius*** 2***.***



Exercise 1.8 ***Complex functions: evaluation and separation of real and imaginary parts***

(A) Evaluate the following complex functions at the given points:

(i) $f(z) = z\overline{z} + 2z - i$, $\quad z = \{1 + 2i, -2 + i, 3 - 3i\}$

(ii) $f(z) = |z|^2 + 3\Re(zi) - z$, $\quad z = \{2 - 3i, 1 + i, -1 + 2i\}$

(iii) $f(z) = \ln|z| + i\,\mathrm{Arg}(z)$, $\quad z = \{2, 2i, -1 + i\}$

(iv) $f(z) = x^2 - y^2 + i(2x + y)$, $\quad z = \{1 + i, 2 - 3i, -2 + 2i\}$, ***where*** $z = x + iy$

(v) $f(z) = e^{2z}$, $\quad z = \{i\pi, 1 - i, \ln 3 + i\pi/4\}$

(B) Find the real part u ***and imaginary part*** v ***of the following functions:***

(i) $f(z) = 5z + 2 - 3i$

(ii) $f(z) = -2z + \overline{z} + i$

(iii) $f(z) = \frac{z}{\overline{z}+1}$

(iv) $f(z) = e^{z+i}$

(C) Express the real part u ***and imaginary part*** v ***in terms of*** r ***and*** θ ***for the following functions:***

(i) $f(z) = \overline{z}$

(ii) $f(z) = z^3$

(iii) $f(z) = e^{2z}$

(iv) $f(z) = z - \frac{1}{z}$

Solution 1.12 ***Part (A): Evaluation of complex functions***

(i) $f(z) = z\overline{z} + 2z - i$

- $z = 1 + 2i$: $z\overline{z} = (1 + 2i)(1 - 2i) = 1 + 4 = 5$, $2z = 2 + 4i$, $-i = -i$ $f(1 + 2i) = 5 + (2 + 4i) - i = 7 + 3i$

- $z = -2+i$: $z\overline{z} = (-2+i)(-2-i) = 4+1 = 5$, $2z = -4+2i$, $-i = -i$ $f(-2+i) = 5 + (-4+2i) - i = 1+i$
- $z = 3-3i$: $z\overline{z} = 9+9 = 18$, $2z = 6-6i$, $-i = -i$ $f(3-3i) = 18+6-6i-i = 24-7i$

(ii) $f(z) = |z|^2 + 3\Re(iz) - z$

- $z = 2-3i$: $|z|^2 = 4+9 = 13$, $iz = i(2-3i) = 2i+3$, $\Re(iz) = 3$, $3\Re(iz) = 9$, $-z = -2+3i$ $f(2-3i) = 13+9-2+3i = 20+3i$
- $z = 1+i$: $|z|^2 = 2$, $iz = i(1+i) = i-1$, $\Re(iz) = -1$, $3\Re(iz) = -3$, $-z = -1-i$ $f(1+i) = 2-3-1-i = -2-i$
- $z = -1+2i$: $|z|^2 = 1+4 = 5$, $iz = i(-1+2i) = -i-2$, $\Re(iz) = -2$, $3\Re(iz) = -6$, $-z = 1-2i$ $f(-1+2i) = 5-6+1-2i = 0-2i = -2i$

(iii) $f(z) = \ln|z| + i\,\mathrm{Arg}(z)$

- $z = 2$: $|z| = 2$, $\mathrm{Arg}(z) = 0$ ***?*** $f(2) = \ln 2 + i0 = \ln 2$
- $z = 2i$: $|z| = 2$, $\mathrm{Arg}(z) = \pi/2$ ***?*** $f(2i) = \ln 2 + i\pi/2$
- $z = -1+i$: $|z| = \sqrt{2}$, $\mathrm{Arg}(z) = 3\pi/4$ ***?*** $f(-1+i) = \frac{1}{2}\ln 2 + i3\pi/4$

(iv) $f(z) = x^2 - y^2 + i(2x+y)$

- $z = 1+i$, $x = 1, y = 1$ ***?*** $f = 1-1+i(2+1) = 0+3i = 3i$
- $z = 2-3i$, $x = 2, y = -3$ ***?*** $f = 4-9+i(4-3) = -5+i1 = -5+i$
- $z = -2+2i$, $x = -2, y = 2$ ***?*** $f = 4-4+i(-4+2) = 0-2i = -2i$

(v) $f(z) = e^{2z}$

- $z = i\pi$: $e^{2i\pi} = \cos(2\pi) + i\sin(2\pi) = 1$
- $z = 1-i$: $e^{2(1-i)} = e^2e^{-2i} = e^2(\cos 2 - i\sin 2)$
- $z = \ln 3 + i\pi/4$: $e^{2(\ln 3 + i\pi/4)} = e^{2\ln 3}e^{i\pi/2} = 9e^{i\pi/2} = 9i$

Part (B): Real and imaginary parts u ***and*** v

(i) $f(z) = 5z+2-3i = 5(x+iy)+2-3i = (5x+2)+i(5y-3) \implies u = 5x+2, v = 5y-3$

(ii) $f(z) = -2z + \overline{z} + i = -2(x+iy) + (x-iy) + i = -x - iy + i = -x + i(-y+1) \implies u = -x, v = -y+1$

(iii) $f(z) = \frac{z}{\overline{z}+1} = \frac{x+iy}{x-iy+1} \cdot \frac{x+1+iy}{x+1+iy} = \frac{x(x+1)+y^2+iy}{(x+1)^2+y^2} \implies u = \frac{x(x+1)+y^2}{(x+1)^2+y^2}, v = \frac{y}{(x+1)^2+y^2}$

(iv) $f(z) = e^{z+i} = e^{x+iy+i} = e^xe^{i(y+1)} = e^x(\cos(y+1) + i\sin(y+1)) \implies u = e^x\cos(y+1), v = e^x\sin(y+1)$

Part (C): Real and imaginary parts in terms of r ***and*** θ

(i) $f(z) = \overline{z} = r(\cos\theta - i\sin\theta) \implies u = r\cos\theta, v = -r\sin\theta$

(ii) $f(z) = z^3 = r^3 e^{i3\theta} = r^3(\cos 3\theta + i \sin 3\theta) \implies u = r^3 \cos 3\theta, v = r^3 \sin 3\theta$

(iii) $f(z) = e^{2z} = e^{2re^{i\theta}} = e^{2r\cos\theta}(\cos(2r\sin\theta) + i\sin(2r\sin\theta)) \implies u = e^{2r\cos\theta}\cos(2r\sin\theta), v = e^{2r\cos\theta}\sin(2r\sin\theta)$

(iv) $f(z) = z - 1/z = re^{i\theta} - (1/r)e^{-i\theta} = (r - 1/r)\cos\theta + i(r + 1/r)\sin\theta \implies u = (r - 1/r)\cos\theta, v = (r + 1/r)\sin\theta$

Chapter 2

Holomorphic Functions

2.1 Derivative of a Complex Function

Definition 2.1 ***Suppose that a complex function f is defined in a neighborhood of z_0. The derivative of f at z_0, denoted by $f'(z_0)$, exists and is given by:***

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}$$

In this case, the function f is said to be differentiable at z_0.

Example 2.1 ***Apply Definition 2.1 to find the derivative of the function***

$$f(z) = z^2 - 5z$$

Solution 2.1 ***We compute the derivative using the definition:***

$$f'(z) = \lim_{\Delta z \to 0} \frac{(z + \Delta z)^2 - 5(z + \Delta z) - (z^2 - 5z)}{\Delta z}$$

Simplify the numerator:

$$(z+\Delta z)^2-5(z+\Delta z)-z^2+5z = z^2+2z\Delta z+(\Delta z)^2-5z-5\Delta z-z^2+5z = 2z\Delta z+(\Delta z)^2-5\Delta z = \Delta z(2z+\Delta z-$$

Divide by Δz:

$$\frac{\Delta z(2z + \Delta z - 5)}{\Delta z} = 2z + \Delta z - 5$$

Take the limit as $\Delta z \to 0$:

$$f'(z) = 2z - 5$$

2.1.1 Rules of Differentiation for Complex Functions

Definition 2.2 (Differentiable Functions) ***Let $f(z)$ and $g(z)$ be two functions defined and differentiable at z. Then the following rules of differentiation hold:***

Theorem 2.1 (Rules of Differentiation) ***$f(z)$ and $g(z)$: two functions defined and differentiable at z***

*1. **Constant Rule:***

$$\frac{d}{dz}c = 0, \qquad \frac{d}{dz}[cf(z)] = cf'(z), \quad c \in \mathbb{C}.$$

*2. **Sum Rule:***

$$\frac{d}{dz}[f(z) \pm g(z)] = f'(z) \pm g'(z).$$

*3. **Product Rule:***

$$\frac{d}{dz}[f(z)g(z)] = f'(z)g(z) + f(z)g'(z).$$

*4. **Quotient Rule:***

$$\frac{d}{dz}\left[\frac{f(z)}{g(z)}\right] = \frac{f'(z)g(z) - f(z)g'(z)}{[g(z)]^2}, \quad g(z) \neq 0.$$

*5. **Chain Rule:***

$$\frac{d}{dz}[f(g(z))] = f'(g(z))g'(z).$$

Example 2.2 ***Use the differentiation rules to compute the derivatives of the following functions:***

(a) $f(z) = 3z^4 - 5z^3 + 2z$

(b) $f(z) = \dfrac{z^2}{4z+1}$

Solution 2.2 ***(a)*** $f(z) = 3z^4 - 5z^3 + 2z$

Using the constant and power rules:

$$f'(z) = 3 \cdot 4z^3 - 5 \cdot 3z^2 + 2 = 12z^3 - 15z^2 + 2$$

(b) $f(z) = \dfrac{z^2}{4z+1}$

Using the quotient rule:

$$f'(z) = \frac{(z^2)'(4z+1) - (z^2)(4z+1)'}{(4z+1)^2} = \frac{2z(4z+1) - z^2(4)}{(4z+1)^2}$$

Simplify the numerator:

$$2z(4z+1) - 4z^2 = 8z^2 + 2z - 4z^2 = 4z^2 + 2z$$

Factorize if desired:

$$f'(z) = \frac{2z(2z+1)}{(4z+1)^2}$$

2.2 Analytic Function

Definition 2.3 (Analyticity at a Point) ***A complex function $w = f(z)$ is said to be analytic at a point z_0 if it is differentiable at z_0 and at every point in a neighborhood of z_0.***

A complex function $f(z)$ is said to be analytic in a domain D if it is analytic at every point of this domain. In this case, the function is also called holomorphic or regular.

A function that is analytic on the entire complex plane $\mathbb{C}$ is called an entire function.

Theorem 2.2 (Polynomial and Rational Functions)

1. ***A complex polynomial function***

$$p(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0,$$

where $n \in \mathbb{N}$ and $a_n, a_{n-1}, \ldots, a_1, a_0 \in \mathbb{C}$, is an entire function.

2. ***A rational function***

$$f(z) = \frac{p(z)}{q(z)}$$

is analytic in any domain D that does not contain any point z_0 for which $q(z_0) = 0$.

Here, $p(z)$ and $q(z)$ are two polynomial functions, and z_0 is called a singular point of $f(z)$.

2.2.1 Differentiability and Cauchy-Riemann Equations

Theorem 2.3 (Differentiability Implies Continuity) ***If f is differentiable at a point z_0 in a domain D, then f is necessarily continuous at z_0.***

Theorem 2.4 (L'Hôpital's Rule for Complex Functions) ***Suppose f and g are two functions analytic at z_0 with $f(z_0) = 0$ and $g(z_0) = 0$ but $g'(z_0) \neq 0$. Then:***

$$\lim_{z \to z_0} \frac{f(z)}{g(z)} = \frac{f'(z_0)}{g'(z_0)}.$$

Example 2.3 ***Use L'Hôpital's Rule to compute the following limit:***

$$\lim_{z \to 2+i} \frac{z^2 - 4z + 5}{z^3 - z - 10i}.$$

Solution 2.3 ***Let $f(z) = z^2 - 4z + 5$ and $g(z) = z^3 - z - 10i$.***

$$f'(z) = 2z - 4, \quad g'(z) = 3z^2 - 1$$

Evaluate at $z_0 = 2 + i$:

$$f'(2+i) = 2(2+i) - 4 = 4 + 2i - 4 = 2i$$

$$g'(2+i) = 3(2+i)^2 - 1 = 3(4+4i-1) - 1 = 3(3+4i) - 1 = 9 + 12i - 1 = 8 + 12i$$

Thus, using L'Hôpital's rule:

$$\lim_{z\to 2+i} \frac{z^2-4z+5}{z^3-z-10i} = \frac{f'(2+i)}{g'(2+i)} = \frac{2i}{8+12i} = \frac{2i(8-12i)}{(8+12i)(8-12i)} = \frac{24+16i}{208} = \frac{3}{26}+\frac{2}{26}i = \frac{3}{26}+\frac{1}{13}i$$

Theorem 2.5 (Cauchy-Riemann Equations) ***Let*** $f(z) = u(x,y) + iv(x,y)$ ***be differentiable at a point*** $z = x + iy$***. Then the first-order partial derivatives of*** u ***and*** v ***exist and satisfy the Cauchy-Riemann equations at*** z***:***

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

Analyticity Criteria

Example 2.4 ***Verify that the function***

$$f(z) = z^2 + z$$

satisfies the Cauchy-Riemann equations.

Solution 2.4 ***Write*** $f(z) = u(x,y) + iv(x,y)$ ***with*** $z = x + iy$***:***

$$f(z) = (x+iy)^2 + (x+iy) = x^2 - y^2 + x + i(2xy + y)$$

So, $u(x,y) = x^2 - y^2 + x$ ***and*** $v(x,y) = 2xy + y$***.***

Compute the partial derivatives:

$$\frac{\partial u}{\partial x} = 2x + 1, \quad \frac{\partial u}{\partial y} = -2y$$

$$\frac{\partial v}{\partial x} = 2y, \quad \frac{\partial v}{\partial y} = 2x + 1$$

Check Cauchy-Riemann equations:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \Rightarrow 2x + 1 = 2x + 1 \quad ?$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \Rightarrow -2y = -2y \quad ?$$

Thus, the Cauchy-Riemann equations are satisfied everywhere, so $f(z)$ ***is analytic in*** $\mathbb{C}$***.***

Theorem 2.6 (Non-Analyticity Criterion) ***If the Cauchy-Riemann equations are not satisfied at every point*** z ***in a domain*** D***, then the function***

$$f(z) = u(x,y) + iv(x,y)$$

cannot be analytic on D***.***

Example 2.5 ***Show that the function***

$$f(z) = 2x^2 + i(y^2 - x)$$

is not analytic at any point.

Solution 2.5 ***Identify*** $u(x, y) = 2x^2$ ***and*** $v(x, y) = y^2 - x$***.***

Compute the partial derivatives:

$$\frac{\partial u}{\partial x} = 4x, \quad \frac{\partial u}{\partial y} = 0$$

$$\frac{\partial v}{\partial x} = -1, \quad \frac{\partial v}{\partial y} = 2y$$

Check Cauchy-Riemann equations:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \Rightarrow 4x = 2y \quad \textbf{\textit{(not true for all }} x, y\textbf{\textit{)}}$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \Rightarrow 0 = 1 \quad \textbf{\textit{(false)}}$$

Since the Cauchy-Riemann equations fail everywhere, $f(z)$ ***is not analytic at any point.***

Theorem 2.7 (Analyticity Criterion) ***Suppose the real functions*** $u(x, y)$ ***and*** $v(x, y)$ ***are continuous and have first-order partial derivatives in a domain*** D***.***

If u ***and*** v ***satisfy the Cauchy-Riemann equations at every point of*** D***, then the complex function***

$$f(z) = u(x, y) + iv(x, y)$$

is analytic in D***.***

Example 2.6 ***Let***

$$f(z) = \frac{x}{x^2 + y^2} - i\frac{y}{x^2 + y^2}, \quad z = x + iy$$

Determine the domain $D = \textbf{Dom}(f)$ ***and show that*** f ***is analytic on*** D***.***

Solution 2.6 ***We can write*** $f(z) = u(x, y) + iv(x, y)$ ***with***

$$u(x, y) = \frac{x}{x^2 + y^2}, \quad v(x, y) = -\frac{y}{x^2 + y^2}.$$

Domain: $x^2 + y^2 \neq 0 \Rightarrow D = \mathbb{C} \setminus \{0\}$***.***

Compute partial derivatives:

$$\frac{\partial u}{\partial x} = \frac{y^2 - x^2}{(x^2 + y^2)^2}, \quad \frac{\partial u}{\partial y} = \frac{-2xy}{(x^2 + y^2)^2}$$

$$\frac{\partial v}{\partial x} = \frac{2xy}{(x^2 + y^2)^2}, \quad \frac{\partial v}{\partial y} = \frac{y^2 - x^2}{(x^2 + y^2)^2}$$

Check Cauchy-Riemann equations:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \textbf{\textit{and}} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Thus, $f(z)$ ***is analytic on*** $D = \mathbb{C} \setminus \{0\}$***.***

2.3 Theorems on Differentiability and Harmonic Functions

2.3.1 Theorems on Differentiability

Theorem 2.8 ***If the real functions*** $u(x, y)$ ***and*** $v(x, y)$ ***are continuous and have first-order partial derivatives in a neighborhood of a point*** z***, and if they satisfy the Cauchy-Riemann equations at*** z***, then the complex function***

$$f(z) = u(x, y) + iv(x, y)$$

is differentiable at z***, and its derivative is given by:***

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}.$$

Theorem 2.9 ***Let*** $f(z) = u(x, y) + iv(x, y)$ ***be an analytic function on a domain*** D***. If*** $f'(z) = 0$ ***on*** D***, then***

$$f(z) = c \quad \textbf{\textit{on}}\ D,$$

where c ***is a complex constant.***

When and Why We Use the Theorem $f'(z) = u_x + iv_x$?

Remark 2.1 ***There are two different ways to compute or study the derivative of a complex function*** $f(z)$***:***

(a) The direct (formal) method, which uses the usual differentiation rules (sum, product, chain rule, etc.);

(b) The analytic (fundamental) method, which starts from the definition

$$f(z) = u(x, y) + iv(x, y), \qquad z = x + iy,$$

and uses the theorem

$$f'(z) = u_x + iv_x,$$

under the condition that the Cauchy-Riemann equations

$$u_x = v_y, \qquad u_y = -v_x$$

are satisfied.

Example 2.7 ***(a) Direct Method (formal differentiation).***

When $f(z)$ ***is written only in terms of*** z ***(for example*** e^z***,*** z^2***,*** z^3***, etc.), we can apply the usual differentiation rules, because such functions are known to be*** **holomorphic everywhere*****.***

For instance,

$$\frac{d}{dz}(z^2) = 2z.$$

In this case, there is no need to express $f(z)$ ***in terms of*** $u(x, y)$ ***and*** $v(x, y)$***.***

Example 2.8 ***(b) Analytic Method (definition via partial derivatives).***

The theorem

$$f'(z) = u_x + iv_x$$

is fundamental, because it allows us to determine whether a function is actually **complex differentiable.** ***That is, it provides a test for holomorphy using the Cauchy-Riemann equations.***

If u ***and*** v ***satisfy***

$$u_x = v_y, \qquad u_y = -v_x,$$

and u, v ***are continuous, then*** $f(z)$ ***is differentiable (holomorphic) at that point, and***

$$f'(z) = u_x + iv_x.$$

Remark 2.2 ***The theorem*** $f'(z) = u_x + iv_x$ ***is particularly useful in the following situations:***

(1) When we do not know whether $f(z)$ ***is holomorphic or not.***

Example: $f(z) = \overline{z}$ ***or*** $f(z) = \overline{z}\, e^z$***. These look simple, but they are not holomorphic. To check this, we must write*** $f = u+iv$ ***and verify the Cauchy-Riemann equations.***

(2) When we need to prove that a function is entire (holomorphic on all of $\mathbb{C}$***).***

In such cases, we show that u, v ***satisfy the Cauchy-Riemann equations everywhere.***

(3) When we study the local geometric behavior of a complex function.

The quantities u_x, v_x, u_y, v_y ***describe how*** f ***transforms small neighborhoods (rotations, dilations, conformality, etc.). This relies directly on*** $f'(z) = u_x + iv_x$***.***

Method	Purpose	Advantage	When to Use It
Usual rules (direct)	Compute $f'(z)$ quickly for known holomorphic functions	Simple and fast	When $f(z)$ is expressed only in terms of z, e.g. $e^z, z^n, \sin z, \cos z$
Analytic definition $f'(z) = u_x + iv_x$	Check if $f(z)$ is holomorphic (Cauchy-Riemann test)	Fundamental for analytic verification	When f depends on $\overline{z}$, or when one must prove holomorphy

Remark 2.3 ***In short:***

The formula $f'(z) = u_x + iv_x$ ***is essential for proving holomorphy,***

while direct rules are used to compute derivatives of already known entire functions.

2.3.2 Harmonic Functions

Definition 2.4 ***A real-valued function*** $\varphi(x, y)$ ***of two real variables is said to be harmonic in a domain*** D ***if it possesses continuous first and second partial derivatives in*** D ***and satisfies the Laplace equation:***

$$\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} = 0.$$

2.3.3 Harmonic Functions and Their Conjugates

Theorem 2.10 ***Suppose the complex function***

$$f(z) = u(x, y) + iv(x, y)$$

is analytic on a domain D***. Then, the real functions*** $u(x, y)$ ***and*** $v(x, y)$ ***are harmonic on*** D***.***

Example 2.9 ***For the entire function***

$$f(z) = z^2,$$

find $u(x, y)$ ***and*** $v(x, y)$***, and show that they are harmonic on*** $\mathbb{C}$***.***

Solution 2.7 ***Let*** $z = x + iy$***. Then:***

$$f(z) = (x + iy)^2 = x^2 - y^2 + 2ixy.$$

Hence,

$$u(x, y) = x^2 - y^2, \quad v(x, y) = 2xy.$$

Compute the second partial derivatives:

$$\frac{\partial^2 u}{\partial x^2} = 2, \quad \frac{\partial^2 u}{\partial y^2} = -2,$$

$$\frac{\partial^2 v}{\partial x^2} = 0, \quad \frac{\partial^2 v}{\partial y^2} = 0.$$

Then:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 2 - 2 = 0,$$

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0 + 0 = 0.$$

Thus, both u ***and*** v ***satisfy Laplace?s equation and are harmonic on*** $\mathbb{C}$***.***

Definition 2.5 (Conjugate Harmonic Function) ***Suppose the real function*** $u(x, y)$ ***is harmonic on a domain*** D***. Then, it is possible to find another real harmonic function*** $v(x, y)$ ***such that*** u ***and*** v ***satisfy the Cauchy-Riemann equations.***

In this case, the complex function

$$f(z) = u(x, y) + iv(x, y)$$

is analytic on D***.***

The function $v(x, y)$ ***is called the harmonic conjugate of*** $u(x, y)$***.***

Example 2.10 ***1. Verify that** $u(x,y) = x^3 - 3xy^2 - 5y$ **is harmonic.***

2. Find the harmonic conjugate** $v(x,y)$ **of** $u(x,y)$**.

Solution 2.8 ***1. Compute the second partial derivatives:***

$$\frac{\partial u}{\partial x} = 3x^2 - 3y^2, \quad \frac{\partial u}{\partial y} = -6xy - 5.$$

$$\frac{\partial^2 u}{\partial x^2} = 6x, \quad \frac{\partial^2 u}{\partial y^2} = -6x.$$

Then:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 6x - 6x = 0,$$

so** $u(x,y)$ **is harmonic. ?

2. To find** $v(x,y)$**, use the Cauchy-Riemann equations:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

Substitute:

$$3x^2 - 3y^2 = \frac{\partial v}{\partial y}, \quad -6xy - 5 = -\frac{\partial v}{\partial x}.$$

Hence,

$$\frac{\partial v}{\partial x} = 6xy + 5, \quad \frac{\partial v}{\partial y} = 3x^2 - 3y^2.$$

Integrate the first equation with respect to** x**:

$$v(x,y) = 3x^2y + 5x + h(y),$$

where** $h(y)$ **is a function of** y **only.

Differentiate with respect to** y**:

$$\frac{\partial v}{\partial y} = 3x^2 + h'(y).$$

Compare with** $\frac{\partial v}{\partial y} = 3x^2 - 3y^2$**:

$$h'(y) = -3y^2 \quad \Rightarrow \quad h(y) = -y^3 + C.$$

Therefore:

$$v(x,y) = 3x^2y - y^3 + 5x + C.$$

Conclusion:

$$u(x,y) = x^3 - 3xy^2 - 5y, \quad v(x,y) = 3x^2y - y^3 + 5x + C,$$

and the analytic function is

$$f(z) = u + iv = (x^3 - 3xy^2 - 5y) + i(3x^2y - y^3 + 5x + C).$$

2.4 Solved Exercises

Exercise 2.1 ***Continuity of a Complex Function at a Point***
Let the complex function

$$f(z) = \frac{z^2 - 1}{z - 1}.$$

1. ***Determine the domain of definition of*** $f(z)$***.***

2. ***Study the continuity of*** f ***at the point*** $z_0 = 1$***.***

3. ***Determine whether*** f ***can be made continuous at*** $z_0 = 1$ ***by defining a suitable value*** $f(1)$***.***

Solution 2.9 ***1. Domain of definition.***

The function is given by:

$$f(z) = \frac{z^2 - 1}{z - 1}.$$

The denominator vanishes at $z = 1$***, hence the function is not defined at this point.***

Therefore, the domain of definition is:

$$D = \mathbb{C} \setminus \{1\}.$$

2. ***Continuity at*** $z_0 = 1$***.***

To check continuity, we compute the limit of $f(z)$ ***as*** $z \to 1$***.***

We have:

$$f(z) = \frac{z^2 - 1}{z - 1} = \frac{(z-1)(z+1)}{z - 1} = z + 1, \quad \textbf{\textit{for }} z \neq 1.$$

Then:

$$\lim_{z \to 1} f(z) = \lim_{z \to 1} (z + 1) = 2.$$

However, $f(1)$ ***is not defined.***

Thus, f ***is discontinuous at*** $z_0 = 1$ ***because it is not defined there.***

3. ***Possible extension of continuity.***

If we define $f(1) = 2$***, then for all*** $z \in \mathbb{C}$***:***

$$f(z) = \begin{cases} \dfrac{z^2 - 1}{z - 1}, & z \neq 1, \\ 2, & z = 1, \end{cases}$$

we obtain a continuous function on $\mathbb{C}$***.***

Indeed, the limit at $z_0 = 1$ ***equals the defined value:***

$$\lim_{z \to 1} f(z) = 2 = f(1).$$

Hence, the extended function is continuous on the whole complex plane.

Exercise 2.2 ***Study of Differentiability at a Point***
Let the complex function

$$f(z) = x^2 + iy^2, \quad \textbf{where } z = x + iy.$$

1. ***Write*** $f(z)$ ***in the form*** $f(z) = u(x, y) + iv(x, y)$ ***and determine the functions*** $u(x, y)$ ***and*** $v(x, y)$***.***
2. ***Study whether*** f ***is differentiable at the point*** $z_0 = 0$ ***by using the definition:***

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}.$$

3. ***Verify differentiability at*** $z_0 = 0$ ***by applying the Cauchy-Riemann equations.***
4. ***Conclude whether*** f ***is analytic on any domain of*** $\mathbb{C}$***.***

Solution 2.10 ***1. We have:***

$$f(z) = x^2 + iy^2,$$

hence

$$u(x, y) = x^2, \qquad v(x, y) = y^2.$$

2. Differentiability by definition.
Let $z_0 = 0$***, and consider the limit***

$$f'(0) = \lim_{\Delta z \to 0} \frac{f(\Delta z) - f(0)}{\Delta z} = \lim_{\Delta z \to 0} \frac{f(\Delta z)}{\Delta z}.$$

Since $\Delta z = \Delta x + i\Delta y$***,***

$$f(\Delta z) = (\Delta x)^2 + i(\Delta y)^2.$$

Thus,

$$\frac{f(\Delta z)}{\Delta z} = \frac{(\Delta x)^2 + i(\Delta y)^2}{\Delta x + i\Delta y}.$$

We test the limit along two different paths:

- ***If we approach along the real axis:*** $\Delta y = 0 \Rightarrow \Delta z = \Delta x$***,***

$$\frac{f(\Delta z)}{\Delta z} = \frac{(\Delta x)^2}{\Delta x} = \Delta x \to 0.$$

- ***If we approach along the imaginary axis:*** $\Delta x = 0 \Rightarrow \Delta z = i\Delta y$***,***

$$\frac{f(\Delta z)}{\Delta z} = \frac{i(\Delta y)^2}{i\Delta y} = \Delta y \to 0.$$

Both give 0***. However, to be sure the limit exists, let us take the general path*** $\Delta y = k\Delta x$***:***

$$\frac{f(\Delta z)}{\Delta z} = \frac{(\Delta x)^2 + i(k\Delta x)^2}{\Delta x(1+ik)} = \Delta x \frac{1+ik^2}{1+ik}.$$

As $\Delta x \to 0$***, the expression tends to*** 0 ***independently of*** k***. Hence, the limit exists and***

$$f'(0) = 0.$$

Therefore, f ***is differentiable at*** $z_0 = 0$***.***

3. ***Using the Cauchy-Riemann equations.***

We have

$$u(x,y) = x^2, \quad v(x,y) = y^2.$$

Compute the partial derivatives:

$$u_x = 2x, \quad u_y = 0, \quad v_x = 0, \quad v_y = 2y.$$

At $z_0 = 0$ ***(i.e.*** $x = 0, y = 0$***):***

$$u_x = 0, \quad v_y = 0, \quad u_y = 0, \quad v_x = 0.$$

Then, at $z_0 = 0$***, the Cauchy-Riemann equations***

$$u_x = v_y \quad \textbf{\textit{and}} \quad u_y = -v_x$$

are satisfied.

Thus, f ***is differentiable at*** $z_0 = 0$***, and its derivative is***

$$f'(0) = u_x + iv_x = 0 + i(0) = 0.$$

4. ***Analyticity on a domain.***

For f ***to be analytic, the Cauchy-Riemann equations must hold at every point.***

In general,

$$u_x = 2x, \quad v_y = 2y,$$

so $u_x = v_y$ ***only when*** $x = y$***. Therefore, the C-R equations are not satisfied everywhere in*** $\mathbb{C}$***.***

Hence f ***is not analytic on any domain of*** $\mathbb{C}$***, but it is differentiable only at the single point*** $z_0 = 0$***.***

Exercise 2.3 ***Let*** $u(x, y) = x^2 - y^2$***.***

1. ***Verify that*** $u(x, y)$ ***is harmonic.***
2. ***Find its harmonic conjugate*** $v(x, y)$***.***
3. ***Construct the analytic function*** $f(z) = u(x, y) + iv(x, y)$***.***

Harmonic and Analytic Functions

Exercise 2.4 ***Let*** $f(z) = z^3$***.***

1. ***Write*** f ***in terms of*** x, y***.***

$$z^3 = (x + iy)^3 = x^3 + 3x^2(iy) + 3x(iy)^2 + (iy)^3 = x^3 - 3xy^2 + i(3x^2y - y^3).$$

So

$$u(x, y) = x^3 - 3xy^2, \qquad v(x, y) = 3x^2y - y^3.$$

2. ***Show*** u ***and*** v ***are harmonic on*** $\mathbb{C}$***.***

Compute second partials for u***:***

$$u_{xx} = \frac{\partial^2}{\partial x^2}(x^3 - 3xy^2) = 6x, \qquad u_{yy} = \frac{\partial^2}{\partial y^2}(x^3 - 3xy^2) = -6x.$$

Hence

$$u_{xx} + u_{yy} = 6x - 6x = 0.$$

For v***:***

$$v_{xx} = \frac{\partial^2}{\partial x^2}(3x^2y - y^3) = 6xy, \qquad v_{yy} = \frac{\partial^2}{\partial y^2}(3x^2y - y^3) = -6xy,$$

so

$$v_{xx} + v_{yy} = 6xy - 6xy = 0.$$

Thus u ***and*** v ***satisfy Laplace's equation everywhere, so they are harmonic on*** $\mathbb{C}$***.***

Exercise 2.5 ***Let*** $u(x, y) = x^2 - y^2$***.***

1. ***Verify*** u ***is harmonic.***

$$u_{xx} = 2, \qquad u_{yy} = -2 \quad \Rightarrow \quad u_{xx} + u_{yy} = 2 - 2 = 0.$$

So u ***is harmonic.***

2. ***Find a harmonic conjugate*** $v(x, y)$***.***

We seek v ***such that the Cauchy-Riemann (C-R) equations hold:***

$$u_x = 2x = v_y, \qquad u_y = -2y = -v_x \Rightarrow v_x = 2y.$$

Integrate $v_y = 2x$ ***with respect to*** y***:***

$$v(x, y) = 2xy + h(x),$$

where $h(x)$ ***is an arbitrary function of*** x***.***

Differentiate this with respect to x***:***

$$v_x = 2y + h'(x),$$

but from C-R we need $v_x = 2y$***. Hence*** $h'(x) = 0$ ***so*** h ***is constant. We may take constant*** 0 ***(it only changes*** f ***by an additive constant).***

Thus one harmonic conjugate is

$$v(x, y) = 2xy + C.$$

3. ***Construct the analytic function*** $f(z) = u + iv$***.***

Choosing $C = 0$***,***

$$f(z) = x^2 - y^2 + i(2xy) = (x + iy)^2 = z^2,$$

which is analytic (indeed entire).

Exercise 2.6 ***Let*** $u(x, y) = e^x \cos y$***.***

1. ***Prove that*** $u(x, y)$ ***is harmonic.***
2. ***Find its harmonic conjugate*** $v(x, y)$***.***
3. ***Write the corresponding analytic function*** $f(z)$***.***

Solution 2.11 ***Let*** $u(x, y) = e^x \cos y$***.***

1. ***Prove*** u ***is harmonic.***

Compute second partials:

$$u_x = e^x \cos y, \quad u_{xx} = e^x \cos y,$$

$$u_y = -e^x \sin y, \quad u_{yy} = -e^x \cos y.$$

Hence

$$u_{xx} + u_{yy} = e^x \cos y - e^x \cos y = 0.$$

So u ***is harmonic.***

2. ***Find a harmonic conjugate*** $v(x, y)$***.***

We need v ***satisfying C-R:***

$$u_x = e^x \cos y = v_y, \qquad u_y = -e^x \sin y = -v_x \Rightarrow v_x = e^x \sin y.$$

Integrate $v_y = e^x \cos y$ ***with respect to*** y***:***

$$v(x, y) = e^x \sin y + h(x).$$

Differentiate this with respect to x***:***

$$v_x = e^x \sin y + h'(x).$$

Equate to required $v_x = e^x \sin y$ ***so*** $h'(x) = 0$***. Thus*** h ***is constant. Take*** $h \equiv 0$***.***
Therefore a harmonic conjugate is

$$v(x, y) = e^x \sin y + C.$$

3. ***Write the analytic function*** $f(z)$***.***

Choosing $C = 0$***,***

$$f(z) = u + iv = e^x \cos y + ie^x \sin y = e^x(\cos y + i \sin y) = e^{x+iy} = e^z,$$

so $f(z) = e^z$ ***(analytic on*** $\mathbb{C}$***).***

Exercise 2.7 ***Let*** $f(z) = \dfrac{1}{z} = \frac{x-iy}{x^2+y^2}$***.***

1. ***Express*** $f(z) = u(x, y) + iv(x, y)$***.***
2. ***Verify that*** u ***and*** v ***satisfy the Cauchy-Riemann equations.***
3. ***Show that*** u ***and*** v ***are harmonic on*** $\mathbb{C} \setminus \{0\}$***.***

Exercise 2.8 ***Let*** $u(x, y) = x^3 - 3xy^2 - 5y$***.***

1. ***Verify that*** $u(x, y)$ ***is harmonic.***
2. ***Determine the harmonic conjugate*** $v(x, y)$***.***
3. ***Deduce the corresponding analytic function*** $f(z)$***.***

Solution 2.12 ***Let*** $f(z) = \dfrac{1}{z} = \dfrac{x - iy}{x^2 + y^2}$***.***

1. ***Express** f **as** $u + iv$.*

$$f(z) = \frac{x}{x^2 + y^2} - i\frac{y}{x^2 + y^2}.$$

So

$$u(x, y) = \frac{x}{x^2 + y^2}, \qquad v(x, y) = -\frac{y}{x^2 + y^2},$$

***defined for** $(x, y) \neq (0, 0)$**, i.e. on** $D = \mathbb{C} \setminus \{0\}$.*

2. ***Verify Cauchy-Riemann equations.***

Compute partial derivatives (for** $(x, y) \neq (0, 0)$**):

$$u_x = \frac{(x^2 + y^2) - 2x^2}{(x^2 + y^2)^2} = \frac{y^2 - x^2}{(x^2 + y^2)^2},$$

$$u_y = \frac{-2xy}{(x^2 + y^2)^2},$$

$$v_x = \frac{2xy}{(x^2 + y^2)^2}, \qquad v_y = \frac{y^2 - x^2}{(x^2 + y^2)^2}.$$

Thus

$$u_x = v_y, \qquad u_y = -v_x,$$

***so the C-R equations hold on** D.*

3. ***Show** u **and** v **are harmonic on** D.*

Compute Laplacians (one gives the idea; the other is analogous):

For** u**:

$$u_{xx} = \frac{\partial}{\partial x}\left(\frac{y^2 - x^2}{(x^2 + y^2)^2}\right), \quad u_{yy} = \frac{\partial}{\partial y}\left(\frac{-2xy}{(x^2 + y^2)^2}\right).$$

After simplification (algebraic but straightforward), one finds

$$u_{xx} + u_{yy} = 0 \quad \textit{for } (x, y) \neq (0, 0).$$

***Likewise for** v**,** $v_{xx} + v_{yy} = 0$**. Hence** u **and** v **are harmonic on** $D = \mathbb{C} \setminus \{0\}$.*

(Alternatively:** $1/z$ **is analytic on** $\mathbb{C} \setminus \{0\}$**; therefore its real and imaginary parts are harmonic there.)

Exercise 2.9 ***Let** $u(x, y) = x^3 - 3xy^2 - 5y$.*

1. ***Verify** u **is harmonic.***

Compute second partials:

$$u_x = 3x^2 - 3y^2, \quad u_{xx} = 6x,$$

$$u_y = -6xy - 5, \quad u_{yy} = -6x.$$

Thus

$$u_{xx} + u_{yy} = 6x - 6x = 0,$$

so** u **is harmonic.

2. ***Determine the harmonic conjugate*** $v(x, y)$***.***

Use C-R:

$$u_x = 3x^2 - 3y^2 = v_y, \qquad u_y = -6xy - 5 = -v_x \Rightarrow v_x = 6xy + 5.$$

Integrate v_x ***w.r.t.*** x***:***

$$v(x, y) = 3x^2y + 5x + h(y),$$

where $h(y)$ ***is a function of*** y ***only.***

Differentiate w.r.t. y***:***

$$v_y = 3x^2 + h'(y).$$

Compare with required $v_y = 3x^2 - 3y^2$ ***to get***

$$h'(y) = -3y^2 \quad \Rightarrow \quad h(y) = -y^3 + C.$$

Therefore a harmonic conjugate is

$$v(x, y) = 3x^2y - y^3 + 5x + C,$$

where C ***is an arbitrary real constant.***

3. ***Deduce the analytic function*** $f(z) = u + iv$***.***

Taking $C = 0$ ***for simplicity,***

$$f(z) = \left(x^3 - 3xy^2 - 5y\right) + i\left(3x^2y - y^3 + 5x\right).$$

One may check this f ***is analytic on the domain where*** u ***is defined (all*** $\mathbb{C}$***).***

Remark 2.4 ***In each exercise the harmonic conjugate is determined up to an additive real constant. When constructing the analytic function*** $f(z) = u + iv$***, that constant adds an imaginary constant*** iC ***to*** f***, which does not affect analyticity.***

Chapter 3

Elementary functions

3.1 Complex Exponential Function

Definition 3.1 ***For any complex number $z = x + iy$, the complex exponential function is defined by***

$$e^z = e^x(\cos y + i \sin y).$$

It is denoted by $\exp(z)$ as well.

Remark 3.1 ***The complex exponential has the same properties as the real exponential function. In particular, it is differentiable everywhere and satisfies***

$$\frac{d}{dx}e^x = e^x.$$

Theorem 3.1 ***The function e^z is entire (holomorphic on the whole complex plane), and its derivative is***

$$\frac{d}{dz}e^z = e^z.$$

Verification of the Cauchy-Riemann equations.
Let

$$e^z = e^x(\cos y + i \sin y) = u(x, y) + iv(x, y),$$

where

$$u(x, y) = e^x \cos y, \qquad v(x, y) = e^x \sin y.$$

Compute the partial derivatives:

$$u_x = e^x \cos y, \qquad u_y = -e^x \sin y,$$
$$v_x = e^x \sin y, \qquad v_y = e^x \cos y.$$

We observe that

$$u_x = v_y \quad \textbf{and} \quad u_y = -v_x,$$

so the Cauchy-Riemann equations are satisfied everywhere. Therefore, e^z is analytic (entire) and

$$f'(z) = \frac{d}{dz}e^z = e^z.$$

Exercise 3.1 ***Compute the complex derivative (where it exists) of each of the following functions:***

$$(1)\ f(z) = e^{2z},$$
$$(2)\ g(z) = e^{z^2},$$
$$(3)\ h(z) = z\,e^z,$$
$$(4)\ p(z) = e^{\overline{z}},$$
$$(5)\ q(z) = \overline{z}\,e^z,$$
$$(6)\ E(z) = e^z.$$

Solution 3.1 ***(1)*** $f(z) = e^{2z}$***.***
Since $2z$ ***is holomorphic with derivative*** 2***, we apply the chain rule:***

$$\frac{d}{dz}e^{2z} = 2e^{2z}.$$

(2) $g(z) = e^{z^2}$***.***
The inner function z^2 ***is holomorphic with derivative*** $2z$***. Hence:***

$$\frac{d}{dz}e^{z^2} = 2z\,e^{z^2}.$$

(3) $h(z) = z\,e^z$***.***
Both z ***and*** e^z ***are entire, so by the product rule:***

$$h'(z) = e^z + z\,e^z = (1+z)e^z.$$

(4) $p(z) = e^{\overline{z}}$***.***
Let $z = x + iy$***. Then*** $\overline{z} = x - iy$***, and***

$$p(z) = e^{\overline{z}} = e^{x-iy} = e^x(\cos y - i \sin y).$$

Thus

$$u(x,y) = e^x \cos y, \qquad v(x,y) = -e^x \sin y.$$

We find

$$u_x = e^x \cos y, \quad u_y = -e^x \sin y, \quad v_x = -e^x \sin y, \quad v_y = -e^x \cos y.$$

The Cauchy-Riemann equations require $u_x = v_y$ ***and*** $u_y = -v_x$***, which give***

$$e^x \cos y = -e^x \cos y \quad \Rightarrow \quad \cos y = 0.$$

These are satisfied only on isolated lines $y = \frac{\pi}{2} + k\pi$, not on any open set. Therefore $p(z)$ is not holomorphic on any open domain, and does not possess a complex derivative.

***(5)** $q(z) = \overline{z}\, e^z$.*

Since $\overline{z}$ is not holomorphic, the product $\overline{z}e^z$ fails to satisfy the Cauchy?Riemann equations. Hence $q(z)$ is not holomorphic and has no complex derivative.

***(6)** $E(z) = e^z$.*

For completeness, we recall that e^z is entire and satisfies

$$\frac{d}{dz}e^z = e^z.$$

Summary of derivatives (where defined):

$$\begin{aligned}
&(1)\ \frac{d}{dz}e^{2z} = 2e^{2z},\\
&(2)\ \frac{d}{dz}e^{z^2} = 2z\, e^{z^2},\\
&(3)\ \frac{d}{dz}(ze^z) = (1+z)e^z,\\
&(4)\ e^{\overline{z}} \ \textbf{\textit{is not holomorphic}},\\
&(5)\ \overline{z}\, e^z \ \textbf{\textit{is not holomorphic}},\\
&(6)\ \frac{d}{dz}e^z = e^z.
\end{aligned}$$

3.1.1 Properties

1. Module, Argument, and Conjugate of e^z

Definition 3.2 ***Let***

$$w = e^z = e^x(\cos y + i\sin y),$$

which can also be written in polar form as

$$w = r(\cos\theta + i\sin\theta).$$

Proposition 3.1 ***By identification, we obtain:***

$$r = e^x, \qquad \theta = y + 2n\pi, \quad n = 0, \pm1, \pm2, \ldots$$

Since $e^x > 0$ for all real x, we have:

$$|e^z| = e^x, \qquad \arg(e^z) = y + 2n\pi, \quad n \in \mathbb{Z}.$$

Remark 3.2 ***An important consequence is that for all $z \in \mathbb{C}$:***

$$\boxed{e^z \neq 0.}$$

Proposition 3.2 (Complex Conjugate) ***The conjugate of e^z is obtained as follows:***

$$\overline{e^z} = \overline{e^x(\cos y + i\sin y)} = e^x(\cos y - i\sin y) = e^x(\cos(-y) + i\sin(-y)) = e^{x-iy} = e^{\overline{z}}.$$

Hence,

$$\boxed{\overline{e^z} = e^{\overline{z}}.}$$

2. Algebraic Properties of e^z

Proposition 3.3 ***For any complex numbers*** z_1, z_2 ***and integer*** $n \in \mathbb{Z}$***:***

$$e^0 = 1,$$
$$e^{z_1} \cdot e^{z_2} = e^{z_1+z_2},$$
$$\frac{e^{z_1}}{e^{z_2}} = e^{z_1-z_2},$$
$$(e^{z_1})^n = e^{nz_1}.$$

3. Periodicity of e^z

Proposition 3.4 ***Since*** $e^z = e^x(\cos y + i \sin y)$ ***involves the trigonometric functions*** $\cos y$ ***and*** $\sin y$***, which are both*** 2π***-periodic, it follows that:***

$$e^{z+i2\pi} = e^z.$$

Therefore, the function e^z ***is periodic with period*** $i2\pi$***.***

Hence, the function e^z is periodic with period $i2\pi$ and therefore not one-to-one (not single-valued). Different complex numbers correspond to the same image under e^z.

3.1.2 Transformation under e^z

Since $z = x + iy$, we study the image of different types of lines under the transformation

$$w = e^z.$$

Definition 3.3 ***In complex analysis, the*** w***-plane is the image plane corresponding to the complex variable***

$$w = f(z),$$

where $z = x + iy$ ***is a complex variable in the*** z***-plane (the domain).***

1. The z-plane: The z-plane represents the domain of the complex function $f(z)$. Each point in this plane corresponds to a complex number $z = x + iy$, where

$$x = \Re(z), \qquad y = \Im(z).$$

2. The w-plane: The w-plane represents the image (or codomain) of the function $f(z)$. Each point corresponds to a complex number $w = u + iv$, where

$$u = \Re(w), \qquad v = \Im(w).$$

Example 3.1 ***If*** $f(z) = e^z$***, then***

$$w = e^z = e^{x+iy} = e^x(\cos y + i \sin y).$$

Hence,

$$u = e^x \cos y, \qquad v = e^x \sin y.$$

This means that each point (x, y) ***in the*** z***-plane is mapped to a point*** (u, v) ***in the*** w***-plane.***

Remark 3.3 ***Geometrically:***

- ***Vertical lines*** $x =$ ***constant in the*** z***-plane are transformed into circles in the*** w***-plane.***
- ***Horizontal lines*** $y =$ ***constant in the*** z***-plane are transformed into rays (half-lines) in the*** w***-plane.***

$$z\text{-plane} \xrightarrow{w=e^z} w\text{-plane}.$$

Case 3.1 ***Case 1:*** z ***represents a vertical line in the*** z***-plane.***
Let

$$z = a + it, \qquad a = \textbf{constant}, \quad -\pi < t \leq \pi.$$

Then

$$w = e^z = e^a e^{it}.$$

In polar form:

$$r = e^a, \qquad \theta = t + 2n\pi, \quad n \in \mathbb{Z}.$$

Remark 3.4 ***This shows that every vertical line*** $x = a$ ***in the*** z***-plane is transformed by*** e^z ***into a circle in the*** w***-plane centered at the origin, with radius***

$$r = e^a > 0.$$

Hence, periodic regions in the z***-plane are mapped to families of concentric circles in the*** w***-plane.***

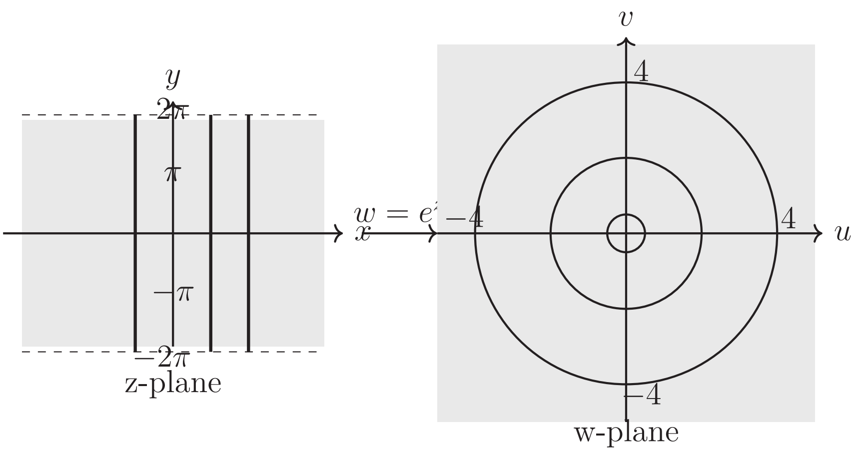


Figure 3.1: Transformation of vertical lines under $w = e^z$: the vertical strip in the z-plane maps to concentric circles in the w-plane.

Case 3.2 ***Case 2:*** z ***represents a horizontal line in the*** z***-plane.***
Let

$$z = t + ib, \qquad b = \textbf{constant}, \quad -\infty < t \leq +\infty.$$

Then

$$w = e^z = e^t e^{ib}.$$

In polar form, we have:

$$r = e^t, \qquad \theta = b + 2n\pi, \quad n \in \mathbb{Z}.$$

Remark 3.5 ***This shows that each horizontal line*** $y = b$ ***in the*** z***-plane is transformed by*** e^z ***into a half-line (ray) in the*** w***-plane starting from the origin and making an angle***

$$\theta = b$$

with the positive u***-axis. The modulus*** $r = e^t$ ***varies over*** $(0, +\infty)$***.***

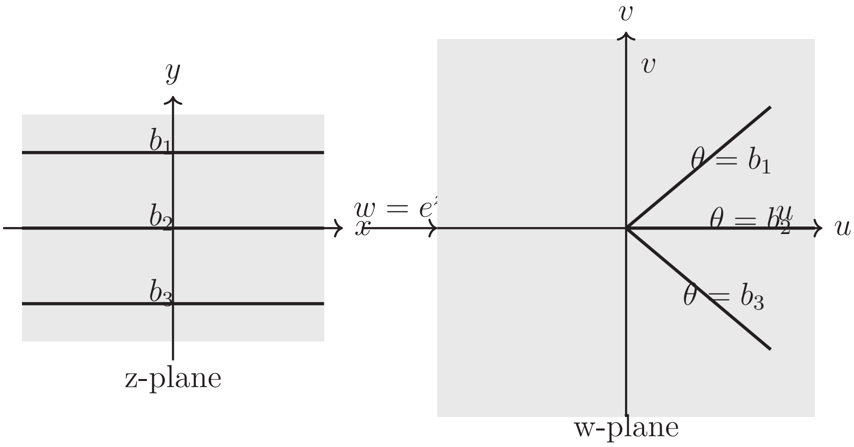


Figure 3.2: Transformation of horizontal lines under $w = e^z$: each line $y = b$ maps to a ray with argument $\theta = b$.

Proposition 3.5 ***Let*** $w = e^z$ ***with*** $z = x + iy$***. The geometric transformations of regions under the exponential mapping*** e^z ***can be summarized as follows:***

1. ***The mapping***

$$w = e^z$$

transforms the fundamental region

$$-\infty < x < \infty, \quad -\pi < y \leq \pi$$

into the set

$$|w| > 0.$$

2. ***The vertical segment***

$$x = a, \quad -\pi < y \leq \pi$$

is transformed into a circle of radius

$$|w| = e^a.$$

3. ***The horizontal line***

$$-\infty < x < \infty, \quad y = b$$

is transformed into a ray in the w***-plane making an angle***

$$\arg(w) = b.$$

3.2 Complex Logarithmic Function

As already seen in the case of the complex exponential function, the exponential mapping e^z is not single-valued as in the real case. Therefore, in the complex plane $\mathbb{C}$, for any fixed nonzero complex number $z \neq 0$, the equation

$$e^w = z$$

admits an infinite number of solutions.

In fact, because of the periodicity of the argument, the equation

$$e^w = z$$

has infinitely many solutions. This can be better understood by assuming that

$$w = u + iv$$

is a complex solution of the equation. In this case, we can write:

$$|e^w| = |z|, \qquad \arg(e^w) = \arg(z).$$

From these equalities, we obtain by identification:

$$e^u = |z|, \qquad v = \arg(z),$$

or equivalently,

$$u = \log|z|, \qquad v = \arg(z).$$

Hence, for a nonzero complex number $z \neq 0$, if $e^w = z$, then

$$w = \log|z| + i\ \arg(z).$$

Because the argument of z takes infinitely many values of the form

$$\arg(z) + 2n\pi, \qquad n \in \mathbb{Z},$$

the expression above defines a multi-valued function $w = G(z)$, as formally given by the following definition.

Definition 3.4 ***The multi-valued function***

$$\ln z = \log|z| + i\ \arg(z)$$

is called the complex logarithm of z.

3.2.1 Complex Logarithm in Polar Form

If we write a nonzero complex number in polar form

$$z = re^{i\theta}, \qquad r > 0,\ \theta \in \mathbb{R},$$

then the complex logarithm of z can be written as the multi-valued expression

$$\ln z = \log r + i(\theta + 2n\pi), \qquad n \in \mathbb{Z},$$

where $\log r$ denotes the natural (Napierian) logarithm of the positive real number r.

Remark 3.6 ***The notation $\log x$ is used here for the natural logarithm (base e) of a positive real number $x > 0$.***

Example 3.2 ***Example 2. Find all complex solutions of the following equations:***

$$\textbf{(a) } e^w = i; \qquad \textbf{(b) } e^w = 1+i; \qquad \textbf{(c) } e^w = -2.$$

Solutions.

(a) Solve $e^w = i$.
Write $i = 1 \cdot e^{i\pi/2}$. Thus

$$w = \ln i = \log|i| + i(\arg i + 2n\pi) = 0 + i\Big(\frac{\pi}{2} + 2n\pi\Big), \qquad n \in \mathbb{Z}.$$

Hence all solutions are

$$\boxed{w = i\Big(\frac{\pi}{2} + 2n\pi\Big),\ n \in \mathbb{Z}}.$$

(b) Solve $e^w = 1 + i$.
Compute $|1+i| = \sqrt{2}$ and $\arg(1+i) = \pi/4$ (principal argument). Therefore

$$w = \ln(1+i) = \log\sqrt{2} + i\Big(\frac{\pi}{4} + 2n\pi\Big) = \tfrac{1}{2}\log 2 + i\Big(\frac{\pi}{4} + 2n\pi\Big), \qquad n \in \mathbb{Z}.$$

Thus all solutions are

$$\boxed{w = \tfrac{1}{2}\log 2 + i\Big(\frac{\pi}{4} + 2n\pi\Big),\ n \in \mathbb{Z}}.$$

(c) Solve $e^w = -2$.
Write $-2 = 2e^{i\pi}$ (principal argument π). Hence

$$w = \ln(-2) = \log 2 + i(\pi + 2n\pi) = \log 2 + i\big(\pi(1+2n)\big), \qquad n \in \mathbb{Z}.$$

All solutions are

$$\boxed{w = \log 2 + i(\pi + 2n\pi),\ n \in \mathbb{Z}}.$$

3.2.2 Logarithmic Identities (multi-valued form)

The following identities are analogous to the real logarithm identities but must be understood up to integer multiples of $2\pi i$:

$$\begin{aligned}
\ln(z_1 z_2) &= \ln z_1 + \ln z_2 \quad (\textbf{mod } 2\pi i),\\
\ln\left(z^n\right) &= n \ln z \quad (\textbf{mod } 2\pi i), \quad n \in \mathbb{Z},\\
\ln\left(\frac{1}{z}\right) &= -\ln z \quad (\textbf{mod } 2\pi i),\\
\ln z &= \log|z| + i(\arg z + 2n\pi), \quad n \in \mathbb{Z}.
\end{aligned}$$

Remark 3.7 ***To obtain single-valued formulas one restricts the argument to a branch, for example the principal branch***

$$\operatorname{Ln} z = \log|z| + i \operatorname{Arg}(z), \qquad -\pi < \operatorname{Arg}(z) \leq \pi,$$

where $\operatorname{Ln} z$ ***denotes the principal value of the complex logarithm.***

3.2.3 Principal Value of the Complex Logarithm

Definition 3.5 ***The principal value of the complex logarithm, denoted by*** $\operatorname{Ln} z$***, is defined as***

$$\operatorname{Ln} z = \log|z| + i \operatorname{Arg}(z),$$

where $\operatorname{Arg}(z)$ ***is the principal argument of*** z***, satisfying***

$$-\pi < \operatorname{Arg}(z) \leq \pi.$$

Example 3.3 ***Example 3. Compute the principal values*** $\operatorname{Ln} z$ ***for the following equations:***

(a) $e^w = i$, ***(b)*** $e^w = 1+i$, ***(c)*** $e^w = -2$.

Solutions.

(a) For $e^w = i$***:***

$$|i| = 1, \quad \operatorname{Arg}(i) = \frac{\pi}{2}.$$

Thus,

$$\operatorname{Ln} i = \log 1 + i\frac{\pi}{2} = i\frac{\pi}{2}.$$

(b) For $e^w = 1+i$***:***

$$|1+i| = \sqrt{2}, \quad \operatorname{Arg}(1+i) = \frac{\pi}{4}.$$

Hence,

$$\operatorname{Ln}(1+i) = \log\sqrt{2} + i\frac{\pi}{4} = \frac{1}{2}\log 2 + i\frac{\pi}{4}.$$

(c) For $e^w = -2$***:***

$$|-2| = 2, \quad \operatorname{Arg}(-2) = \pi.$$

Then,

$$\operatorname{Ln}(-2) = \log 2 + i\pi.$$

3.2.4 Inverse Relationship with the Exponential Function

If the complex exponential function

$$f(z) = e^z$$

is considered on the fundamental region

$$-\infty < x < \infty, \qquad -\pi < y \leq \pi,$$

then $f(z)$ is one-to-one on this domain. Its inverse function is the principal value of the complex logarithm:

$$f^{-1}(z) = \operatorname{Ln} z.$$

3.2.5 Analyticity of the Complex Logarithm

Since the complex logarithm is a multi-valued function, it is not continuous or differentiable over the entire complex plane. To make it analytic, we restrict its domain by removing the negative real axis.

Thus, the principal branch of the logarithmic function is defined on

$$\mathbb{C} \setminus (-\infty, 0] = \{ z \in \mathbb{C} \mid \Re(z) > 0 \text{ or } \Im(z) \neq 0 \}.$$

On this domain, the principal branch is expressed as

$$f_1(z) = \log|z| + i \operatorname{Arg}(z) = \log r + i\theta, \qquad -\pi < \theta < \pi,$$

where $z = re^{i\theta}$ with $r > 0$. This definition excludes all points $x \in (-\infty, 0]$.

Determination of the Domain of Definition of $\ln(z)$

Let

$$z = re^{i\theta}, \quad \text{where } r > 0 \text{ and } \theta \in \mathbb{R}.$$

The complex logarithm is defined by

$$\ln(z) = \ln r + i\theta.$$

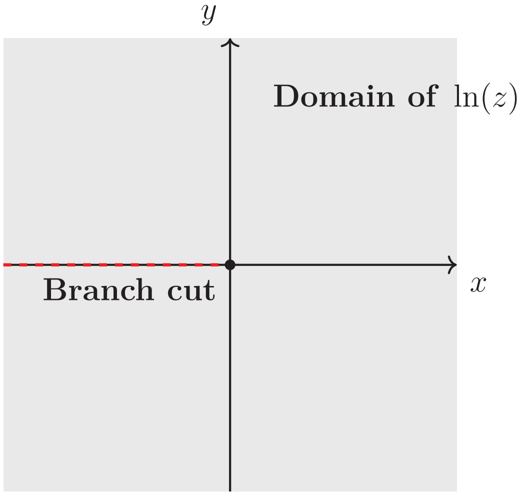
However, since θ is not unique (we can add any multiple of 2π), $\ln(z)$ is a multivalued function. To make it single-valued, we must choose a principal value for θ, usually

$$-\pi < \theta < \pi.$$

This restriction excludes the negative real axis, where $\theta = \pi$ or $\theta = -\pi$. Hence, the domain of definition of the principal branch of the logarithm is

$$D = \mathbb{C} \setminus (-\infty, 0].$$

Geometrically, this means that the branch cut (the rejected part) is the negative real axis, which we represent by a dashed line on the complex plane.



3.2.6 Analyticity of the Principal Branch of $\ln z$

Definition 3.6 ***The principal branch of the complex logarithm is the function***

$$f_1(z) = \operatorname{Ln} z = \log|z| + i\operatorname{Arg}(z),$$

defined on the domain

$$\mathbb{C} \setminus (-\infty, 0] = \{z \in \mathbb{C} \mid z \neq 0,\ -\pi < \operatorname{Arg}(z) \leq \pi\},$$

where $\operatorname{Arg}(z)$ ***denotes the principal argument satisfying*** $-\pi < \operatorname{Arg}(z) \leq \pi$***.***

Theorem 3.2 (Analyticity of the principal branch of $\ln z$) ***The principal branch*** $f_1(z) = \operatorname{Ln} z$ ***is analytic on*** $\mathbb{C} \setminus (-\infty, 0]$***. Moreover, for every*** z ***in this domain,***

$$f_1'(z) = \frac{d}{dz}\operatorname{Ln} z = \frac{1}{z}.$$

Example 3.4 ***Compute the derivatives (on the appropriate domain):***

$$\textbf{\textit{(a)}}\ F(z) = z\operatorname{Ln} z, \qquad \textbf{\textit{(b)}}\ G(z) = \operatorname{Ln}(z+1).$$

Solution 3.2 ***We work with the*** **principal branch** Ln***, defined on*** $\mathbb{C} \setminus (-\infty, 0]$ ***and satisfying***

$$\operatorname{Ln} w = \log|w| + i\operatorname{Arg}(w), \qquad -\pi < \operatorname{Arg}(w) \leq \pi.$$

(a) $F(z) = z\operatorname{Ln} z$***.*** ***Domain:*** $z \in \mathbb{C} \setminus (-\infty, 0]$ ***(so that*** $\operatorname{Ln} z$ ***is defined).***

Both z ***and*** $\operatorname{Ln} z$ ***are differentiable on this domain (indeed*** Ln ***is analytic there). Use the product rule:***

$$F'(z) = 1 \cdot \operatorname{Ln} z + z \cdot \frac{d}{dz}\big(\operatorname{Ln} z\big).$$

Since on the principal branch $\frac{d}{dz}\operatorname{Ln} z = \frac{1}{z}$***, we obtain***

$$\boxed{F'(z) = \operatorname{Ln} z + z \cdot \frac{1}{z} = \operatorname{Ln} z + 1, \qquad z \in \mathbb{C} \setminus (-\infty, 0].}$$

(b) $G(z) = \mathrm{Ln}(z+1)$. ***Domain:*** $z + 1 \in \mathbb{C} \setminus (-\infty, 0]$***, i.e.***

$$z \in \mathbb{C} \setminus \{\, t \in \mathbb{R} \mid t \leq -1 \,\}$$

(the complex plane with the ray $(-\infty, -1]$ ***removed).***

On this domain the composition is analytic and by the chain rule

$$G'(z) = \frac{d}{dz}\mathrm{Ln}(z+1) = \frac{1}{z+1}.$$

Hence

$$\boxed{G'(z) = \frac{1}{z+1}, \qquad z \in \mathbb{C} \setminus (-\infty, -1].}$$

3.2.7 Transformations under the principal logarithm

Proposition 3.6 ***Let*** $w = \mathrm{Ln}\, z = u + iv$ ***be the principal logarithm (so*** $-\pi < v \leq \pi$***). The main mapping properties are:***

1. $w = \mathrm{Ln}\, z$ ***maps the set*** $\{z \in \mathbb{C} \mid |z| > 0\}$ ***onto the horizontal strip***

$$-\infty < u < \infty, \qquad -\pi < v \leq \pi.$$

2. A circle $\{z : |z| = r\}$ ***is mapped to the vertical segment***

$$u = \log r, \qquad -\pi < v \leq \pi.$$

3. A ray $\{z : \arg z = \theta\}$ ***is mapped to the horizontal line***

$$-\infty < u < \infty, \qquad v = \theta \quad (-\pi < \theta \leq \pi).$$

Remark 3.8 ***These correspond exactly to the inverse relations of the exponential map*** $z \mapsto e^z$***: for*** $z = re^{i\theta}$***,***

$$\mathrm{Ln}\, z = \log r + i\theta \quad (\textbf{\textit{principal value}}),$$

so modulus-level sets become vertical lines in the w***-plane and argument-level sets become horizontal lines.***

Recall the principal logarithm for a nonzero complex number $z = re^{i\theta}$ **with** $r > 0$ **and principal argument** $-\pi < \theta \leq \pi$**:**

$$w = \mathrm{Ln}\, z = \log r + i\theta, \qquad w = u + iv.$$

Thus $u = \log r$ **and** $v = \theta$**. The examples below illustrate the three principal mapping properties: (1) the whole nonzero plane to the horizontal strip,**
(2) a circle to a vertical segment, and
(3) a ray to a horizontal line.

Example 3.5 ***The image of the set*** $\{z \in \mathbb{C} : |z| > 0\}$
Show that under the principal logarithm $w = \operatorname{Ln} z$ ***the set of all nonzero complex numbers***

$$\{z \in \mathbb{C} \mid |z| > 0\}$$

is mapped onto the horizontal strip

$$-\infty < u < \infty, \qquad -\pi < v \leq \pi.$$

Solution 3.3

1. ***Any nonzero complex number has a polar representation***

$$z = re^{i\theta}, \qquad r > 0, \quad -\pi < \theta \leq \pi,$$

where we choose the principal argument $\operatorname{Arg} z \in (-\pi, \pi]$***.***

2. ***Apply the principal logarithm to*** z***:***

$$w = \operatorname{Ln} z = \log r + i\theta.$$

3. ***Separate into real and imaginary parts:***

$$u = \log r, \qquad v = \theta.$$

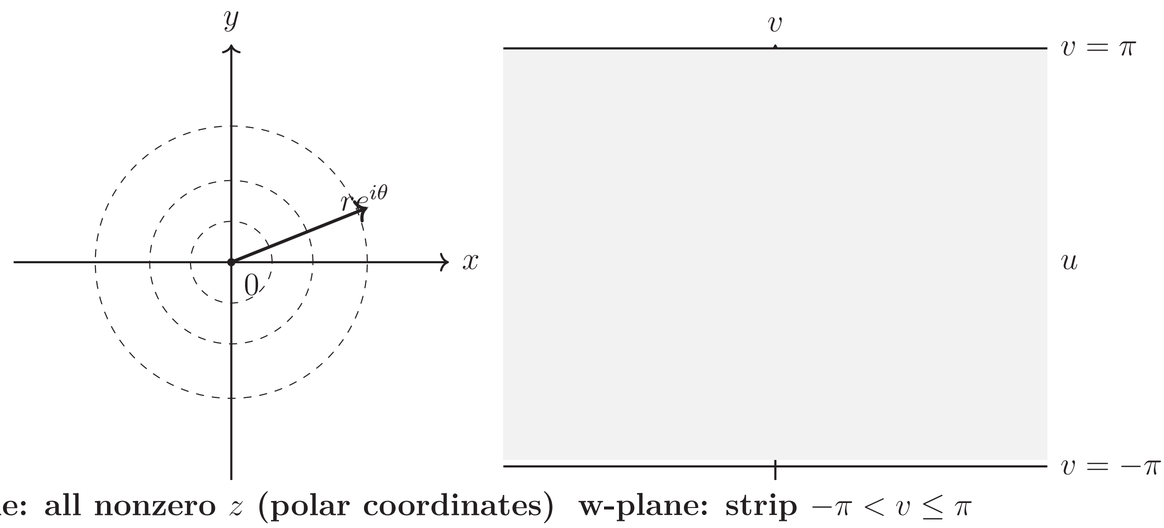
4. ***As*** r ***varies over*** $(0, \infty)$***,*** $\log r$ ***takes all real values*** $(-\infty, \infty)$***. As*** θ ***ranges over the principal interval*** $(-\pi, \pi]$***,*** v ***takes values*** $-\pi < v \leq \pi$***.***

5. ***Therefore the image is exactly the horizontal strip***

$$-\infty < u < \infty, \qquad -\pi < v \leq \pi.$$

Remark 3.9

- ***The negative real axis corresponds to*** $\theta = \pi$ ***and maps to the top boundary*** $v = \pi$***.***
- ***The origin*** $z = 0$ ***is not in the domain of*** Ln***, so it does not map to any*** w***.***
- ***The principal logarithm is single-valued only after fixing*** $\theta \in (-\pi, \pi]$***. If we allowed other branches,*** v ***would differ by multiples of*** 2π***.***



z-plane: all nonzero z (polar coordinates) w-plane: strip $-\pi < v \leq \pi$

Example 3.6 ***A circle*** $|z| = r$ ***maps to the vertical segment*** $u = \log r$
Let $|z| = 1$***. Find its image under the principal logarithm and explain why the image is a vertical segment.***

Solution 3.4 1. ***Points on the circle*** $|z| = 1$ ***can be written*** $z = e^{i\theta}$ ***with*** $-\pi < \theta \leq \pi$***.***

2. ***Compute the principal logarithm:***

$$w = \operatorname{Ln} z = \log 1 + i\theta = i\theta.$$

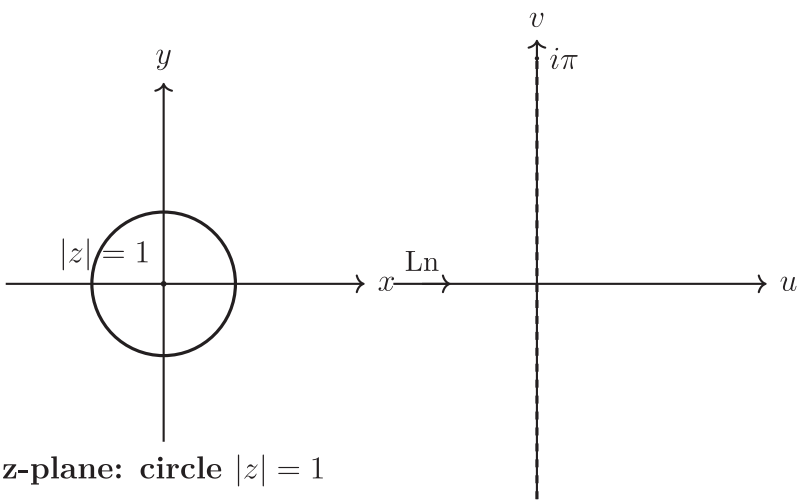
3. ***Thus***

$$u = \Re(w) = 0, \qquad v = \Im(w) = \theta, \quad -\pi < \theta \leq \pi.$$

4. ***Therefore the image is the vertical segment*** $\{w : u = 0,\ -\pi < v \leq \pi\}$***.***

5. ***For a general circle*** $|z| = r$ ***the same reasoning gives*** $u = \log r$ ***and the vertical segment*** $\{w : u = \log r,\ -\pi < v \leq \pi\}$***.***

Remark 3.10
- ***The endpoint corresponding to*** $\theta = \pi$ ***(i.e.*** $z = -1$***) maps to*** $w = i\pi$***. The value*** $\theta = -\pi$ ***is not an independent principal argument, so the parameterization interval is*** $-\pi < \theta \leq \pi$***.***
- ***The mapping is one-to-one when restricted appropriately (e.g. restrict domain to the circle with specified principal arguments).***



z-plane: circle $|z| = 1$

w-plane: vertical segment $u = 0$, $-\pi < v \leq \pi$

Example 3.7 ***A ray*** $\arg z = \theta$ ***maps to the horizontal line*** $v = \theta$ ***Let the ray*** $\arg z = \pi/4$ ***(all*** $z = re^{i\pi/4}$ ***with*** $r > 0$***). Find its image under the principal logarithm.***

Solution 3.5 1. ***Any point on the ray has the form*** $z = re^{i\pi/4}$ ***where*** $r > 0$ ***and the principal argument*** $\pi/4$ ***belongs to*** $(-\pi, \pi]$***.***

2. ***Apply the principal logarithm:***

$$w = \operatorname{Ln} z = \log r + i\frac{\pi}{4}.$$

3. ***Separate parts:***

$$u = \log r, \qquad v = \frac{\pi}{4}.$$

4. ***As*** r ***ranges over*** $(0, \infty)$***,*** $u = \log r$ ***ranges over*** $(-\infty, \infty)$***. Therefore the image is the horizontal line***

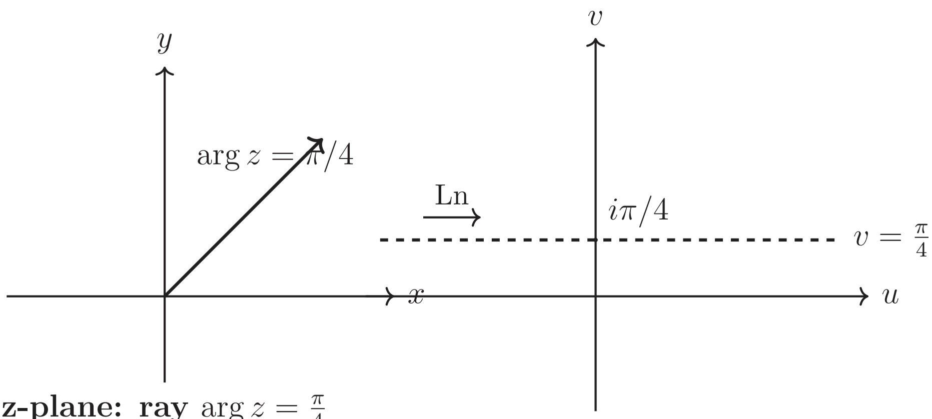
$$\{w : -\infty < u < \infty,\ v = \tfrac{\pi}{4}\}.$$

5. ***Orientation note: increasing*** r ***corresponds to increasing*** u***. For example*** $z = 1 \cdot e^{i\pi/4}$ ***maps to*** $w = i\pi/4$***, and*** $z = e \cdot e^{i\pi/4}$ ***maps to*** $w = 1 + i\pi/4$***.***

Remark 3.11 • ***The ray does not include the origin;*** Ln ***is not defined at*** 0***.***

- ***The horizontal line is full in*** u ***(unbounded both ways).***

TikZ drawing.



z-plane: ray $\arg z = \frac{\pi}{4}$

w-plane: horizontal line $v = \frac{\pi}{4}$

1. **Compute the image under** Ln **of the circle** $|z| = e^{-1}$ **and draw the result.**

2. **Compute the image under** Ln **of the ray** $\arg z = -\frac{3\pi}{4}$**.**

3. **Show that the set** $\{z : 1 < |z| < e$ **and** $0 < \arg z < \frac{\pi}{2}\}$ **is mapped to the rectangle**

$$0 < u < 1, \qquad 0 < v < \tfrac{\pi}{2}$$

in the w**-plane.**

3.3 Complex Powers

We already know how to manipulate an integer power of a complex number such as z^n. However, when the exponent is itself a complex number, the computation of an expression like $(1+i)^i$ is less straightforward. In such cases, we use the complex exponential and logarithmic functions.

Definition 3.7 (Complex Power) ***If α is a complex number and $z \neq 0$, the complex power is defined as***

$$z^\alpha = e^{\alpha \ln z}.$$

For integer exponents n, the power function z^n is single-valued, while for complex exponents α, the function z^α is multivalued because of the multivalued nature of $\ln z$.

Definition 3.8 (Principal Value) ***If α is a complex number and $z \neq 0$, the function***

$$z^\alpha = e^{\alpha \operatorname{Ln} z}$$

is called the principal value of the complex power z^α, where $\operatorname{Ln} z = \ln|z| + i \operatorname{Arg} z$ with $\operatorname{Arg} z \in (-\pi, \pi]$.

Remark 3.12 ***The function z^α inherits the multivalued character of the logarithm function, and different branches correspond to different possible values of the argument $\operatorname{Arg} z = \theta + 2k\pi$.***

—

Example 3.8 ***Compute the complex power i^{2i}.***

Solution 3.6 ***We know that***

$$i = e^{i(\pi/2 + 2k\pi)}, \qquad k \in \mathbb{Z}.$$

Hence,

$$\ln i = i\left(\frac{\pi}{2} + 2k\pi\right), \quad \textit{and therefore} \quad i^{2i} = e^{2i \ln i} = e^{-2(\frac{\pi}{2} + 2k\pi)} = e^{-\pi(1+4k)}.$$

Thus, the general values are

$$\boxed{i^{2i} = \{e^{-\pi(1+4k)} : k \in \mathbb{Z}\}.}$$

For the principal value ($k = 0$),

$$\boxed{i^{2i}\Big|_{\text{principal}} = e^{-\pi} \approx 0.0432139183.}$$

—

Example 3.9 ***Compute the complex power $(1+i)^i$.***

Solution 3.7 ***We write***

$$1 + i = \sqrt{2}\, e^{i(\pi/4 + 2k\pi)}.$$

Thus,

$$\ln(1 + i) = \tfrac{1}{2}\ln 2 + i\Big(\tfrac{\pi}{4} + 2k\pi\Big).$$

Then,

$$(1 + i)^i = e^{i\ln(1+i)} = e^{i(\frac{1}{2}\ln 2 + i(\frac{\pi}{4} + 2k\pi))} = e^{-(\frac{\pi}{4} + 2k\pi)}\, e^{i(\frac{1}{2}\ln 2)}.$$

Hence,

$$\boxed{(1 + i)^i = e^{-(\frac{\pi}{4} + 2k\pi)}\, e^{i(\frac{1}{2}\ln 2)}.}$$

For the principal value ($k = 0$),

$$(1 + i)^i\Big|_{\text{principal}} = e^{-\pi/4}\, e^{i(\frac{1}{2}\ln 2)} \approx 0.4288290063 + 0.1548717525\, i.$$

3.3.1 Analyticity of the Complex Power

We recall that, on the principal branch domain $D = \mathbb{C} \setminus (-\infty, 0]$, the principal branch

$$f_1(z) = z^\alpha = e^{\alpha\, \mathrm{Ln}\, z}, \qquad -\pi < \mathrm{Arg}\, z < \pi,$$

is analytic and satisfies

$$f_1'(z) = \frac{d}{dz}\Big(e^{\alpha \mathrm{Ln}\, z}\Big) = \alpha\, z^{\alpha - 1}.$$

Example 3.10 ***Find the derivative of the principal value of z^i at the point $z = 1 + i$.***

Solution 3.8 ***Step 1 - general derivative formula. For $\alpha = i$ we have, on the principal branch,***

$$\frac{d}{dz} z^i = i\, z^{i-1}.$$

Step 2 - principal logarithm of the point. For $z_0 = 1 + i$ the principal polar data are

$$|1 + i| = \sqrt{2}, \qquad \mathrm{Arg}(1 + i) = \frac{\pi}{4}.$$

Thus the principal logarithm is

$$\mathrm{Ln}(1 + i) = \ln|1 + i| + i\,\mathrm{Arg}(1 + i) = \tfrac{1}{2}\ln 2 + i\frac{\pi}{4}.$$

Set

$$a := \tfrac{1}{2}\ln 2, \qquad b := \frac{\pi}{4}.$$

Step 3 - compute z_0^{i-1}. Using the principal logarithm,

$$z_0^{i-1} = e^{(i-1)\mathrm{Ln}(1+i)} = e^{(i-1)(a+ib)}.$$

Expand the product in the exponent:

$$(i - 1)(a + ib) = (ia - a) + (i^2 b - ib) = (-a - b) + i(a - b).$$

Therefore

$$z_0^{i-1} = e^{-a-b}\, e^{i(a-b)}.$$

Step 4 - multiply by i to get the derivative.

$$f'(1+i) = i\, z_0^{i-1} = i\, e^{-a-b}\, e^{i(a-b)}.$$

We may write the final result in rectangular form by multiplying i with the complex exponential:

$$i\, e^{i\phi} = i(\cos\phi + i\sin\phi) = -\sin\phi + i\cos\phi, \qquad \phi := a-b.$$

Hence

$$\boxed{f'(1+i) = e^{-a-b}\Big(-\sin(a-b) + i\cos(a-b)\Big)}$$

with $a = \frac{1}{2}\ln 2$ and $b = \frac{\pi}{4}$.

Numerical approximation. Using $a \approx 0.3465735903$ and $b \approx 0.7853981634$,

$$a-b \approx -0.4388245731, \qquad -a-b \approx -1.1319717537.$$

Thus

$$z_0^{i-1} \approx 0.2918503794 - 0.1369786269\, i,$$

and

$$\boxed{f'(1+i) \approx 0.1369786269 + 0.2918503794\, i.}$$

(These digits are rounded; keep more if required.)

Remark 3.13 ***All steps used the principal branch Ln (i.e. $\mathrm{Arg} \in (-\pi, \pi]$). If a different branch were used the values of z^{i-1} and hence of $f'(1+i)$ would differ by the corresponding $2k\pi$ shifts in the argument.***

3.4 Complex Trigonometric Functions

Definition 3.9 (Complex Sine and Cosine) ***The complex sine and cosine functions are defined for every complex number z by:***

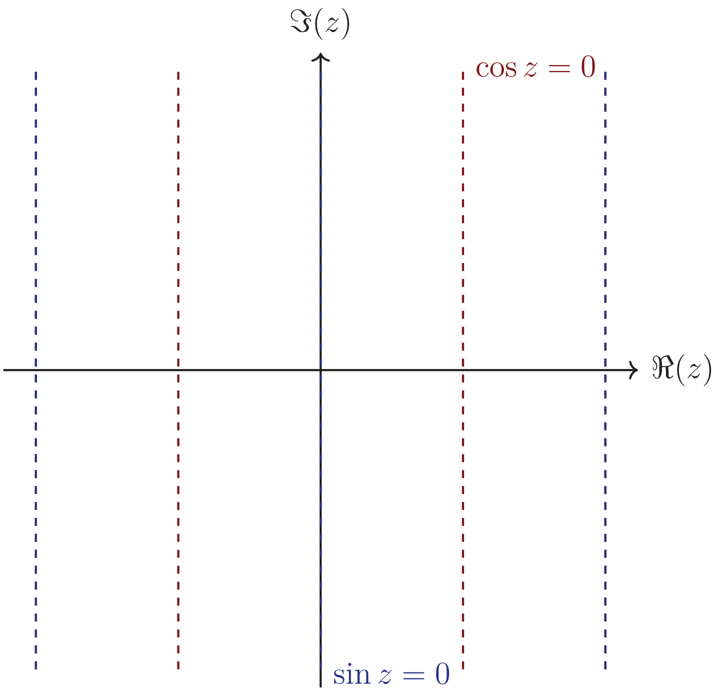
$$\sin z = \frac{e^{iz} - e^{-iz}}{2i}, \qquad \cos z = \frac{e^{iz} + e^{-iz}}{2}.$$

These definitions are direct extensions of the real trigonometric functions to the complex plane. They are entire (analytic on all of $\mathbb{C}$) since they are built from the exponential function, which is entire.

Definition 3.10 (Derived Complex Trigonometric Functions) ***Using the complex sine and cosine, we define the other trigonometric functions in the same way as in the real case:***

$$\tan z = \frac{\sin z}{\cos z}, \qquad \cot z = \frac{\cos z}{\sin z}, \qquad \sec z = \frac{1}{\cos z}, \qquad \csc z = \frac{1}{\sin z}.$$

Remark 3.14 ***Unlike in the real case, these functions are*** **meromorphic** ***on*** $\mathbb{C}$***: they are analytic everywhere except at the zeros of their denominators. For instance,*** $\tan z$ ***and*** $\sec z$ ***are undefined when*** $\cos z = 0$***, and*** $\cot z$ ***and*** $\csc z$ ***are undefined when*** $\sin z = 0$***.***



Complex z**-plane: singular lines of** $\tan z$ **and** $\cot z$

Example 3.11 ***Express each of the following complex trigonometric values in the form*** $a + ib$***:***

(a) $\cos i$***,***

(b) $\sin(2 + i)$***,***

(c) $\tan(\pi - 2i)$***.***

Solution 3.9 ***We use the standard identities (valid for complex arguments):***

$$\cos(x + iy) = \cos x \cosh y - i \sin x \sinh y, \qquad \sin(x + iy) = \sin x \cosh y + i \cos x \sinh y,$$

and for tangent the convenient formula

$$\tan(x + iy) = \frac{\sin 2x + i \sinh 2y}{\cos 2x + \cosh 2y}.$$

(a) $\cos i$***. Put*** $x = 0,\ y = 1$ ***(since*** $i = 0 + 1 \cdot i$***). Using*** $\cos(0) = 1,\ \sin(0) = 0$***,***

$$\cos i = \cos 0 \cosh 1 - i \sin 0 \sinh 1 = \cosh 1.$$

Therefore

$$\boxed{\cos i = \cosh 1\ +\ 0 \cdot i.}$$

Numerically $\cosh 1 = \frac{e+e^{-1}}{2} \approx 1.5430806348$**.**

(b) $\sin(2+i)$***. Here*** $x = 2,\ y = 1$***. Apply the identity:***

$$\sin(2+i) = \sin 2 \cosh 1 + i \cos 2 \sinh 1.$$

Thus in $a + ib$ ***form:***

$$\boxed{\sin(2+i) = \Big(\sin 2 \cosh 1\Big) \ + \ i\Big(\cos 2 \sinh 1\Big).}$$

Numerically (rounded):

$$\sin 2 \approx 0.9092974268, \qquad \cos 2 \approx -0.4161468365,$$

$$\cosh 1 \approx 1.5430806348, \qquad \sinh 1 \approx 1.1752011936.$$

So

$$\sin 2 \cosh 1 \approx 1.4031192506, \qquad \cos 2 \sinh 1 \approx -0.4890561256,$$

and

$$\boxed{\sin(2+i) \approx 1.4031192506 - 0.4890561256\, i.}$$

(c) $\tan(\pi - 2i)$***. Write*** $z = \pi - 2i$ ***so*** $x = \pi,\ y = -2$***. Use the tangent formula above:***

$$\tan(\pi - 2i) = \frac{\sin(2\pi) + i \sinh(-4)}{\cos(2\pi) + \cosh(-4)}.$$

Compute the elementary values:

$$\sin(2\pi) = 0, \qquad \cos(2\pi) = 1, \qquad \sinh(-4) = -\sinh 4, \qquad \cosh(-4) = \cosh 4.$$

Thus

$$\tan(\pi - 2i) = \frac{i(-\sinh 4)}{1 + \cosh 4} = -i\,\frac{\sinh 4}{1 + \cosh 4}.$$

Use the identities $\sinh 4 = 2 \sinh 2 \cosh 2$ ***and*** $1 + \cosh 4 = 2\cosh^2 2$ ***to simplify:***

$$\frac{\sinh 4}{1 + \cosh 4} = \frac{2 \sinh 2 \cosh 2}{2 \cosh^2 2} = \tanh 2.$$

Hence

$$\boxed{\tan(\pi - 2i) = -i\ \tanh 2 = 0 \ + \ \Big(-\tanh 2\Big)\, i.}$$

Numerically $\tanh 2 \approx 0.9640275801$***, so***

$$\boxed{\tan(\pi - 2i) \approx -0.9640275801\, i.}$$

Remark 3.15 ***Note the useful special cases:*** $\cos(iy) = \cosh y$ ***and*** $\sin(x + iy) = \sin x \cosh y + i \cos x \sinh y$***. When an identity reduces an expression to purely real or purely imaginary form, it is often possible to simplify further using hyperbolic identities (as in part (c)).***

3.4.1 Trigonometric Identities

As for real trigonometric functions, the complex trigonometric functions satisfy the following identities:

$$\sin(-z) = -\sin z, \qquad \cos(-z) = \cos z$$

$$\cos^2 z + \sin^2 z = 1$$

$$\sin(z_1 + z_2) = \sin z_1 \cos z_2 + \cos z_1 \sin z_2$$

$$\sin(z_1 - z_2) = \sin z_1 \cos z_2 - \cos z_1 \sin z_2$$

$$\cos(z_1 + z_2) = \cos z_1 \cos z_2 - \sin z_1 \sin z_2$$

$$\cos(z_1 - z_2) = \cos z_1 \cos z_2 + \sin z_1 \sin z_2$$

$$\sin(2z) = 2 \sin z \cos z, \qquad \cos(2z) = \cos^2 z - \sin^2 z$$

3.4.2 Periodicity

It follows directly that the complex trigonometric functions sine and cosine are periodic, and their period is 2π:

$$\sin(z + 2\pi) = \sin z, \qquad \cos(z + 2\pi) = \cos z$$

3.4.3 Zeros of Complex Sine and Cosine Functions

The complex sine and cosine functions have zeros, analogously to the real case, for the following values:

$$\sin z = 0 \quad \textbf{if and only if} \quad z = n\pi$$

$$\cos z = 0 \quad \textbf{if and only if} \quad z = \frac{(2n+1)\pi}{2}, \qquad n = 0, \pm 1, \pm 2, \ldots$$

3.4.4 Analyticity

The derivatives of the complex sine and cosine functions are obtained from their definitions in terms of the complex exponential function. We have:

$$\frac{d}{dz}(\sin z) = \cos z, \qquad \frac{d}{dz}(\cos z) = -\sin z$$

$$\frac{d}{dz}(\tan z) = \sec^2 z, \qquad \frac{d}{dz}(\cot z) = -\csc^2 z$$

$$\frac{d}{dz}(\sec z) = \sec z\, \tan z, \qquad \frac{d}{dz}(\csc z) = -\csc z\, \cot z$$

3.5 Complex Hyperbolic Functions

Definition 3.11 ***The complex hyperbolic functions are defined by:***

$$\sinh z = \frac{e^z - e^{-z}}{2}, \qquad \cosh z = \frac{e^z + e^{-z}}{2}.$$

From these, the other hyperbolic functions are deduced as follows:

$$\tanh z = \frac{\sinh z}{\cosh z}, \qquad \coth z = \frac{\cosh z}{\sinh z}, \qquad \operatorname{sech} z = \frac{1}{\cosh z}, \qquad \operatorname{csch} z = \frac{1}{\sinh z}.$$

It should also be noted that the hyperbolic functions $\sinh z$ and $\cosh z$ are *entire* functions, since the exponential function e^z itself is entire.

3.5.1 Derivatives of Complex Hyperbolic Functions

The derivatives of the complex hyperbolic functions can be obtained directly from their definitions in terms of the complex exponential function. We have:

$$\frac{d}{dz}(\sinh z) = \cosh z, \qquad \frac{d}{dz}(\cosh z) = \sinh z,$$

$$\frac{d}{dz}(\tanh z) = \operatorname{sech}^2 z, \qquad \frac{d}{dz}(\coth z) = -\operatorname{csch}^2 z,$$

$$\frac{d}{dz}(\operatorname{sech} z) = -\operatorname{sech} z\, \tanh z, \qquad \frac{d}{dz}(\operatorname{csch} z) = -\operatorname{csch} z\, \coth z.$$

3.5.2 Relations with Complex Trigonometric Functions

It is quite interesting to observe the relationship between hyperbolic and trigonometric complex functions, since their expressions have the same form up to a constant factor.

Let us replace z by iz in the definitions above. We obtain:

$$\sinh(iz) = \frac{e^{iz} - e^{-iz}}{2} = i\left(\frac{e^{iz} - e^{-iz}}{2i}\right) = i \sin z.$$

Equivalently,

$$\sin z = -i \sinh(iz).$$

Similarly, we can establish the correspondence between the different hyperbolic and trigonometric functions as follows:

$$\sin z = -i \sinh(iz), \qquad \cos z = \cosh(iz),$$

$$\sinh z = -i \sin(iz), \qquad \cosh z = \cos(iz),$$

$$\tan(iz) = \frac{\sin(iz)}{\cos(iz)} = \frac{i \sinh z}{\cosh z} = i \tanh z.$$

3.5.3 Identities of Complex Hyperbolic Functions

The equalities above can be used to establish the following identities for the complex hyperbolic functions:

$$\sinh(-z) = -\sinh z, \qquad \cosh(-z) = \cosh z$$

$$\cosh^2 z - \sinh^2 z = 1$$

$$\sinh(z_1 \pm z_2) = \sinh z_1 \cosh z_2 \pm \cosh z_1 \sinh z_2$$

$$\cosh(z_1 \pm z_2) = \cosh z_1 \cosh z_2 \pm \sinh z_1 \sinh z_2$$

$$\sinh(2z) = 2 \sinh z \cosh z, \qquad \cosh(2z) = \cosh^2 z + \sinh^2 z$$

3.5.4 Inverse Trigonometric and Hyperbolic Functions

Inverse Trigonometric Functions

When seeking solutions to the equation $\sin w = z$, it is necessary to define the inverse function such that $z = \sin^{-1} w$.

Starting from the definition of the complex sine function:

$$\sin w = \frac{e^{iw} - e^{-iw}}{2i} = z,$$

we obtain:

$$e^{2iw} - 2iz\, e^{iw} - 1 = 0.$$

This is a quadratic equation in e^{iw}, whose solution is:

$$e^{iw} = iz + \sqrt{1 - z^2}.$$

Taking the logarithm, we find:

$$iw = \ln\Big[iz + \sqrt{1 - z^2} \Big], \qquad \textbf{hence} \qquad w = -i \ln\Big[iz + \sqrt{1 - z^2} \Big].$$

This leads to the definition of the inverse sine function:

$$\boxed{\sin^{-1} z = -i \ln\Big[iz + \sqrt{1 - z^2} \Big]} \quad \textbf{(Inverse sine)}.$$

This function is ***multivalued***, since it involves the complex logarithm, which is itself multivalued.

By applying the same procedure, we define the other inverse trigonometric functions:

$$\boxed{\cos^{-1} z = -i \ln\Big[z + i\sqrt{1 - z^2} \Big]} \quad \textbf{(Inverse cosine)}$$

$$\boxed{\tan^{-1} z = \frac{i}{2}\ln\left(\frac{i+z}{i-z}\right)} \quad \textbf{(Inverse tangent)}.$$

The derivatives of these functions are defined only on domains where the functions are continuous and differentiable (that is, after choosing suitable branches of these multivalued functions):

$$\frac{d}{dz}\sin^{-1} z = \frac{1}{\sqrt{1-z^2}}, \qquad \frac{d}{dz}\cos^{-1} z = -\frac{1}{\sqrt{1-z^2}}, \qquad \frac{d}{dz}\tan^{-1} z = \frac{1}{1+z^2}.$$

3.6 Inverse Hyperbolic Functions

In the same way as for trigonometric functions, we can define the inverse hyperbolic functions as the solutions of equations of the form $\sinh w = z$. We then obtain:

$$\sinh^{-1} z = \ln\left[z + \sqrt{z^2+1}\right] \quad \textbf{(Inverse hyperbolic sine)}$$

$$\cosh^{-1} z = \ln\left[z + i\sqrt{z^2-1}\right] \quad \textbf{(Inverse hyperbolic cosine)}$$

$$\tanh^{-1} z = \frac{1}{2}\ln\left(\frac{1+z}{1-z}\right) \quad \textbf{(Inverse hyperbolic tangent)}$$

For suitable branches defined on a domain D where these functions are continuous and differentiable, we have:

$$\frac{d}{dz}\left(\sinh^{-1} z\right) = \frac{1}{\sqrt{z^2+1}}, \qquad \frac{d}{dz}\left(\cosh^{-1} z\right) = \frac{1}{\sqrt{z^2-1}}, \qquad \frac{d}{dz}\left(\tanh^{-1} z\right) = \frac{1}{1-z^2}.$$

Exercise 3.2 ***(1) Find all possible values of*** $\sin^{-1}(\sqrt{5})$.

(2) Find the derivative of $\sin^{-1} z$ ***at*** $z = i$***, and verify that this point belongs to the domain of definition.***

(3) Assume that $\cosh^{-1} z$ ***represents a branch of the inverse hyperbolic cosine defined using the complex square root and logarithm branches***

$$f_2(z) = \sqrt{r}\, e^{i\theta/2}, \quad 0 < \theta < 2\pi.$$

(4) Compute:

(a) $\cosh^{-1}\left(\frac{\sqrt{2}}{2}\right)$;

(b) $\left.\frac{d}{dz}\cosh^{-1} z\right|_{z=\frac{\sqrt{2}}{2}}$.

Solution 3.10 ***(1) Using the formula***

$$\sin^{-1} z = -i \ln\left[iz + \sqrt{1-z^2}\right],$$

we get for $z = \sqrt{5}$***:***

$$\sin^{-1}(\sqrt{5}) = -i\ln\left[i\sqrt{5} + \sqrt{1-5}\right] = -i\ln\left[i\sqrt{5} + 2i\right] = -i\ln\left[i(\sqrt{5}+2)\right].$$

Hence,

$$\boxed{\sin^{-1}(\sqrt{5}) = \frac{\pi}{2} + i\ln(\sqrt{5}+2) + 2k\pi, \quad k \in \mathbb{Z}.}$$

(2) The derivative is

$$\frac{d}{dz}\left(\sin^{-1} z\right) = \frac{1}{\sqrt{1-z^2}}.$$

At $z = i$***:***

$$\frac{d}{dz}\left(\sin^{-1} z\right)\Big|_{z=i} = \frac{1}{\sqrt{1-i^2}} = \frac{1}{\sqrt{2}}.$$

The point $z = i$ ***belongs to the domain since*** $1 - z^2 = 2 \neq 0$***.***

$$\boxed{\frac{d}{dz}\left(\sin^{-1} z\right)\Big|_{z=i} = \frac{1}{\sqrt{2}}.}$$

(a) Using

$$\cosh^{-1} z = \ln\left[z + i\sqrt{z^2-1}\right],$$

for $z = \frac{\sqrt{2}}{2}$***:***

$$\cosh^{-1}\left(\frac{\sqrt{2}}{2}\right) = \ln\left[\frac{\sqrt{2}}{2} + i\sqrt{\frac{1}{2}-1}\right] = \ln\left[\frac{\sqrt{2}}{2} - i\frac{i}{\sqrt{2}}\right] = \frac{1}{2}\ln(2).$$

$$\boxed{\cosh^{-1}\left(\frac{\sqrt{2}}{2}\right) = \frac{1}{2}\ln(2)}$$

(b) The derivative is

$$\frac{d}{dz}\left(\cosh^{-1} z\right) = \frac{1}{\sqrt{z^2-1}}.$$

At $z = \frac{\sqrt{2}}{2}$***:***

$$\frac{d}{dz}\left(\cosh^{-1} z\right)\Big|_{z=\frac{\sqrt{2}}{2}} = \frac{1}{\sqrt{\frac{1}{2}-1}} = \frac{1}{\sqrt{-\frac{1}{2}}} = \frac{1}{-i/\sqrt{2}} = i\sqrt{2}.$$

$$\boxed{\frac{d}{dz}\left(\cosh^{-1} z\right)\Big|_{z=\frac{\sqrt{2}}{2}} = i\sqrt{2}.}$$

3.7 Solved Exercises

Exercise 3.3 ***Compute the following complex derivatives:***

$$\textbf{(a)}\ f(z) = iz^4(z^2 - e^z), \qquad \textbf{(b)}\ g(z) = \exp\{z^2 - (1+i)z + 3\}.$$

Solution 3.11 ***(a) Let***

$$f(z) = iz^4(z^2 - e^z).$$

We use the product rule:

$$f'(z) = i\left[(z^4)'(z^2 - e^z) + z^4(z^2 - e^z)'\right].$$

Compute each derivative:

$$(z^4)' = 4z^3, \qquad (z^2 - e^z)' = 2z - e^z.$$

Hence

$$\begin{aligned} f'(z) &= i\left[4z^3(z^2 - e^z) + z^4(2z - e^z)\right] \\ &= i\left(4z^5 - 4z^3e^z + 2z^5 - z^4e^z\right) \\ &= i\left(6z^5 - e^z(4z^3 + z^4)\right). \end{aligned}$$

Therefore,

$$\boxed{f'(z) = i\left(6z^5 - e^z(4z^3 + z^4)\right)}.$$

(b) Let

$$g(z) = \exp\{z^2 - (1+i)z + 3\}.$$

We apply the chain rule:

$$g'(z) = \exp\{z^2 - (1+i)z + 3\} \cdot (2z - (1+i)).$$

Thus,

$$\boxed{g'(z) = (2z - (1+i))\, e^{z^2 - (1+i)z + 3}}.$$

Second Method

Solution 3.12 ***(a) Function*** $f(z) = iz^4(z^2 - e^z)$***.***

Let $z = x + iy$***. Then***

$$f(z) = i(x+iy)^4\left[(x+iy)^2 - e^{x+iy}\right].$$

We expand each term:

$$(x+iy)^2 = (x^2 - y^2) + 2ixy, \qquad e^{x+iy} = e^x(\cos y + i\sin y).$$

Hence

$$\begin{aligned} z^2 - e^z &= (x^2 - y^2 - e^x\cos y) + i(2xy - e^x\sin y), \\ z^4 &= (x^4 - 6x^2y^2 + y^4) + i(4x^3y - 4xy^3). \end{aligned}$$

Now, let

$$z^4(z^2 - e^z) = (A + iB), \quad \textbf{\textit{so that}}\ f(z) = i(A + iB) = iA - B.$$

Thus

$$u(x, y) = -B, \qquad v(x, y) = A.$$

Since u ***and*** v ***are differentiable functions of*** x ***and*** y***, the function*** f ***is holomorphic everywhere (product and sum of entire functions).***

Therefore, by definition of the complex derivative:

$$f'(z) = u_x + iv_x.$$

We compute symbolically (using standard differentiation rules for holomorphic functions):

$$f'(z) = i\left[4z^3(z^2 - e^z) + z^4(2z - e^z)\right]$$
$$= i\left(6z^5 - e^z(4z^3 + z^4)\right).$$

Hence

$$\boxed{f'(z) = u_x + iv_x = i\Big(6z^5 - e^z(4z^3 + z^4)\Big)}.$$

(b) Function $g(z) = e^{z^2-(1+i)z+3}$***.***

Let

$$z = x + iy \quad \textbf{\textit{and}} \quad \Phi(z) = z^2 - (1 + i)z + 3 = (x^2 - y^2 - x + y + 3) + i(2xy - y - x).$$

Then

$$e^{\Phi(z)} = e^{x^2-y^2-x+y+3}\Big[\cos(2xy - y - x) + i\sin(2xy - y - x)\Big].$$

Hence

$$u(x, y) = e^{x^2-y^2-x+y+3}\cos(2xy - y - x),$$
$$v(x, y) = e^{x^2-y^2-x+y+3}\sin(2xy - y - x).$$

We compute

$$u_x = e^{x^2-y^2-x+y+3}\Big[(2x - 1)\cos(2xy - y - x) - (2y - 1)\sin(2xy - y - x)\Big],$$

$$v_x = e^{x^2-y^2-x+y+3}\Big[(2x - 1)\sin(2xy - y - x) + (2y - 1)\cos(2xy - y - x)\Big].$$

Thus, by definition,

$$f'(z) = u_x + iv_x = e^{z^2-(1+i)z+3}\,(2z - (1 + i)).$$

$$\boxed{g'(z) = u_x + iv_x = (2z - (1 + i))\,e^{z^2-(1+i)z+3}}.$$

Exercise 3.4 ***Compute the complex power*** $(-3)^{i/\pi}$***.***

Solution 3.13 ***We write***

$$-3 = 3e^{i(\pi+2k\pi)}, \qquad k \in \mathbb{Z}.$$

Thus,

$$\ln(-3) = \ln 3 + i(\pi + 2k\pi),$$

and hence

$$(-3)^{\,i/\pi} = e^{(i/\pi)\ln(-3)} = e^{(i/\pi)(\ln 3 + i(\pi+2k\pi))} = e^{-(1+2k)}\,e^{\,i(\ln 3)/\pi}.$$

Therefore,

$$\boxed{(-3)^{\,i/\pi} = e^{-(1+2k)}\,e^{\,i(\ln 3)/\pi}.}$$

For the principal value ($k = 0$ ***),***

$$(-3)^{\,i/\pi}\Big|_{\text{principal}} = e^{-1}\,e^{\,i(\ln 3)/\pi} \approx 0.3456138434 + 0.1260410825\,i.$$

Exercise 3.5 ***Compute the complex power*** $(2i)^{\,1-i}$***.***

Solution 3.14 ***We write***

$$2i = 2e^{i(\pi/2+2k\pi)}.$$

Hence,

$$\ln(2i) = \ln 2 + i(\pi/2 + 2k\pi).$$

Then,

$$(2i)^{1-i} = e^{(1-i)\ln(2i)} = e^{(1-i)(\ln 2 + i(\pi/2+2k\pi))}.$$

Expanding the exponent:

$$(1-i)(\ln 2 + iB) = (\ln 2 + B) + i(B - \ln 2), \qquad \textbf{\textit{with}}\ B = \frac{\pi}{2} + 2k\pi.$$

Hence,

$$(2i)^{1-i} = e^{\ln 2 + B}\,e^{\,i(B-\ln 2)} = 2e^{B}\,e^{\,i(B-\ln 2)}.$$

Thus,

$$\boxed{(2i)^{1-i} = 2e^{\,(\frac{\pi}{2}+2k\pi)}\,e^{\,i(\frac{\pi}{2}+2k\pi-\ln 2)}.}$$

For the principal value ($k = 0$ ***),***

$$(2i)^{1-i}\Big|_{\text{principal}} = 2e^{\pi/2}\,e^{\,i(\pi/2-\ln 2)} \approx 6.1474175340 + 7.4008126711\,i.$$

Remark 3.16 ***The above examples illustrate how the multivalued nature of the complex logarithm leads to multiple values of complex powers. The*** principal value ***corresponds to taking*** $\operatorname{Arg} z$ ***within*** $(-\pi, \pi]$***.***

Expression	Principal Value (approx.)
i^{2i}	$e^{-\pi} \approx 0.0432139183$
$(1+i)^i$	$0.4288290063 + 0.1548717525\,i$
$(-3)^{\,i/\pi}$	$0.3456138434 + 0.1260410825\,i$
$(2i)^{\,1-i}$	$6.1474175340 + 7.4008126711\,i$

Exercise 3.6 (Analyticity and derivatives of complex trigonometric functions) ***Show that the complex sine and cosine functions are entire (analytic on*** $\mathbb{C}$***) and compute the derivatives***

$$\frac{d}{dz}\sin z, \quad \frac{d}{dz}\cos z, \quad \frac{d}{dz}\tan z, \quad \frac{d}{dz}\cot z, \quad \frac{d}{dz}\sec z, \quad \frac{d}{dz}\csc z.$$

State clearly the domains where the derivatives of $\tan z,\ \cot z,\ \sec z,\ \csc z$ ***are defined.***

Solution 3.15 ***1. Entirety of*** $\sin z$ ***and*** $\cos z$***.***

Recall the exponential definitions for all $z \in \mathbb{C}$***:***

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i}, \qquad \cos z = \frac{e^{iz} + e^{-iz}}{2}.$$

The complex exponential e^w ***is entire (holomorphic on*** $\mathbb{C}$***). Since*** $\sin z$ ***and*** $\cos z$ ***are linear combinations and compositions of entire functions, they are entire as well. In particular they are differentiable everywhere in*** $\mathbb{C}$***.***

2. Derivative of $\sin z$ ***and*** $\cos z$***.***

Differentiate the exponential expressions term-by-term (valid for entire functions):

$$\frac{d}{dz}\sin z = \frac{d}{dz}\left(\frac{e^{iz} - e^{-iz}}{2i}\right) = \frac{ie^{iz} + ie^{-iz}}{2i} = \frac{e^{iz} + e^{-iz}}{2} = \cos z.$$

$$\frac{d}{dz}\cos z = \frac{d}{dz}\left(\frac{e^{iz} + e^{-iz}}{2}\right) = \frac{ie^{iz} - ie^{-iz}}{2} = -\frac{e^{iz} - e^{-iz}}{2i} = -\sin z.$$

Thus

$$\frac{d}{dz}\sin z = \cos z, \qquad \frac{d}{dz}\cos z = -\sin z.$$

3. Derivative of $\tan z$***.***

By definition $\tan z = \dfrac{\sin z}{\cos z}$***. Use the quotient rule on the domain where*** $\cos z \neq 0$***:***

$$\frac{d}{dz}\tan z = \frac{\cos z \cdot (\cos z) - \sin z \cdot (-\sin z)}{\cos^2 z} = \frac{\cos^2 z + \sin^2 z}{\cos^2 z} = \frac{1}{\cos^2 z} = \sec^2 z.$$

So $\dfrac{d}{dz}\tan z = \sec^2 z$ ***on the domain*** $\{z \in \mathbb{C} : \cos z \neq 0\}$ ***(i.e. excluding the poles of*** $\tan z$***, which occur at*** $z = (2n+1)\pi/2,\ n \in \mathbb{Z}$***).***

4. Derivative of $\cot z$***.***

$\cot z = \dfrac{\cos z}{\sin z}$***. Use the quotient rule where*** $\sin z \neq 0$***:***

$$\frac{d}{dz}\cot z = \frac{\sin z \cdot (-\sin z) - \cos z \cdot \cos z}{\sin^2 z} = -\frac{\sin^2 z + \cos^2 z}{\sin^2 z} = -\frac{1}{\sin^2 z} = -\csc^2 z.$$

So $\dfrac{d}{dz}\cot z = -\csc^2 z$ ***on*** $\{z \in \mathbb{C} : \sin z \neq 0\}$ ***(excluding poles*** $z = n\pi,\ n \in \mathbb{Z}$***).***

5. Derivative of $\sec z$***.***

$\sec z = 1/\cos z$***. Differentiate on the domain*** $\cos z \neq 0$***:***

$$\frac{d}{dz}\sec z = \frac{d}{dz}\left(\frac{1}{\cos z}\right) = -\frac{-\sin z}{\cos^2 z} = \frac{\sin z}{\cos^2 z} = \frac{1}{\cos z}\cdot\frac{\sin z}{\cos z} = \sec z\ \tan z.$$

Thus $\frac{d}{dz}\sec z = \sec z\ \tan z$ ***for*** $\cos z \neq 0$***.***

6. Derivative of $\csc z$***.***

$\csc z = 1/\sin z$***. Differentiate on*** $\sin z \neq 0$***:***

$$\frac{d}{dz}\csc z = \frac{d}{dz}\left(\frac{1}{\sin z}\right) = -\frac{\cos z}{\sin^2 z} = -\csc z\ \cot z.$$

So $\frac{d}{dz}\csc z = -\csc z\ \cot z$ ***for*** $\sin z \neq 0$***.***

Summary. The derivatives (with their domains) are:

$$\frac{d}{dz}\sin z = \cos z \quad (\forall z \in \mathbb{C}), \qquad \frac{d}{dz}\cos z = -\sin z \quad (\forall z \in \mathbb{C}),$$

$$\frac{d}{dz}\tan z = \sec^2 z \quad (\cos z \neq 0), \qquad \frac{d}{dz}\cot z = -\csc^2 z \quad (\sin z \neq 0),$$

$$\frac{d}{dz}\sec z = \sec z\ \tan z \quad (\cos z \neq 0), \qquad \frac{d}{dz}\csc z = -\csc z\ \cot z \quad (\sin z \neq 0).$$

This completes the exercise.

Exercise 3.7 ***(1) Find all possible values of*** $\cos^{-1}(2i)$***.***

(2) Compute the derivative of $\tanh^{-1} z$ ***at*** $z = \frac{i}{2}$***, and verify that this point belongs to the domain of definition.***

(3) Assume that $\sinh^{-1} z$ ***represents a branch of the inverse hyperbolic sine defined using the complex square root and logarithm branches***

$$f_2(z) = \sqrt{r}\, e^{i\theta/2}, \quad 0 < \theta < 2\pi.$$

Compute:

(a) $\sinh^{-1}\left(\frac{3}{2}\right)$***;***

(b) $\left.\frac{d}{dz}\sinh^{-1} z\right|_{z=\frac{3}{2}}$***.***

Solution 3.16 ***(1) Values of*** $\cos^{-1}(2i)$***.***

We use the standard complex formula

$$\cos^{-1} z = -i\ln\left(z + \sqrt{z^2 - 1}\right),$$

where the square root and the logarithm are multivalued (hence the inverse cosine is multivalued).

Take $z = 2i$***. Compute***

$$z^2 - 1 = (2i)^2 - 1 = -4 - 1 = -5.$$

Thus

$$\sqrt{z^2 - 1} = \sqrt{-5} = \pm i\sqrt{5}.$$

Two possible choices give

$$z + \sqrt{z^2 - 1} = 2i \pm i\sqrt{5} = i(2 \pm \sqrt{5}).$$

Now

$$\cos^{-1}(2i) = -i \ln\Big(i(2 \pm \sqrt{5})\Big).$$

Write $i(2 \pm \sqrt{5}) = r\, e^{i\theta}$ ***with***

$$r = |2 \pm \sqrt{5}|, \qquad \theta = \arg\Big(i(2 \pm \sqrt{5})\Big).$$

Case $2 + \sqrt{5}$ ***(positive):***

$$i(2 + \sqrt{5}) = (2 + \sqrt{5})e^{i\pi/2}, \qquad \ln\Big(i(2 + \sqrt{5})\Big) = \ln(2 + \sqrt{5}) + i\Big(\frac{\pi}{2} + 2k\pi\Big).$$

Hence

$$\cos^{-1}(2i) = -i\Big(\ln(2 + \sqrt{5}) + i\Big(\frac{\pi}{2} + 2k\pi\Big)\Big) = \frac{\pi}{2} + 2k\pi \;-\; i\ln(2 + \sqrt{5}), \quad k \in \mathbb{Z}.$$

Case $2 - \sqrt{5}$ ***(negative; note*** $2 - \sqrt{5} < 0$***): let*** $a = \sqrt{5} - 2 > 0$***. Then*** $2 - \sqrt{5} = -a$ ***and***

$$i(2 - \sqrt{5}) = i(-a) = ae^{-i\pi/2}.$$

So

$$\ln\Big(i(2 - \sqrt{5})\Big) = \ln(a) + i\Big(-\frac{\pi}{2} + 2k\pi\Big),$$

and

$$\cos^{-1}(2i) = -i\Big(\ln(a) + i\Big(-\frac{\pi}{2} + 2k\pi\Big)\Big) = -\frac{\pi}{2} + 2k\pi \;-\; i\ln(\sqrt{5} - 2), \quad k \in \mathbb{Z}.$$

Compact form of the multivalued results:

$$\boxed{\cos^{-1}(2i) = \pm\frac{\pi}{2} + 2k\pi \;-\; i\ln\left|2 \pm \sqrt{5}\right|, \qquad k \in \mathbb{Z},}$$

where the two signs correspond to the two choices of the square root. (The explicit forms found above are usually written as)

$$\boxed{\cos^{-1}(2i) = \frac{\pi}{2} + 2k\pi - i\ln(2 + \sqrt{5}) \quad \textbf{\textit{or}} \quad \cos^{-1}(2i) = -\frac{\pi}{2} + 2k\pi - i\ln(\sqrt{5} - 2).}$$

(2) Derivative of $\tanh^{-1} z$ at $z = \frac{i}{2}$.

Recall

$$\tanh^{-1} z = \frac{1}{2}\ln\left(\frac{1+z}{1-z}\right)$$

and its derivative (on domains where $1 - z^2 \neq 0$) is

$$\frac{d}{dz}\tanh^{-1} z = \frac{1}{1-z^2}.$$

Evaluate at $z = \dfrac{i}{2}$:

$$1 - z^2 = 1 - \left(\frac{i}{2}\right)^2 = 1 - \left(-\frac{1}{4}\right) = 1 + \frac{1}{4} = \frac{5}{4}.$$

Thus

$$\frac{d}{dz}\tanh^{-1} z\bigg|_{z=\frac{i}{2}} = \frac{1}{5/4} = \frac{4}{5}.$$

Since $1-z^2 \neq 0$ at $z = i/2$, the derivative is defined there. Also the principal formula for $\tanh^{-1}$ is analytic except at points where $1 \pm z = 0$ or equivalently where $z = \pm 1$ (and more generally where the logarithm branch cut is chosen). The point $z = i/2$ is not a singularity, so it belongs to the domain for the standard branches.

$$\boxed{\frac{d}{dz}\tanh^{-1} z\bigg|_{z=\frac{i}{2}} = \frac{4}{5}.}$$

(3) Inverse hyperbolic sine at $\frac{3}{2}$ and its derivative.

We use

$$\sinh^{-1} z = \ln\left(z + \sqrt{z^2+1}\right),$$

with the principal (positive) square root for real positive arguments.

(a) For $z = \dfrac{3}{2}$ (real, positive),

$$z^2 + 1 = \frac{9}{4} + 1 = \frac{13}{4}, \qquad \sqrt{z^2+1} = \frac{\sqrt{13}}{2}.$$

Therefore

$$\sinh^{-1}\left(\tfrac{3}{2}\right) = \ln\left(\frac{3}{2} + \frac{\sqrt{13}}{2}\right) = \ln\left(\frac{3+\sqrt{13}}{2}\right).$$

$$\boxed{\sinh^{-1}\left(\tfrac{3}{2}\right) = \ln\left(\frac{3+\sqrt{13}}{2}\right).}$$

(b) The derivative formula (on domains where the chosen square root is nonzero) is

$$\frac{d}{dz}\sinh^{-1} z = \frac{1}{\sqrt{z^2+1}}.$$

Hence at $z = \dfrac{3}{2}$***,***

$$\left.\frac{d}{dz}\sinh^{-1} z\right|_{z=\frac{3}{2}} = \frac{1}{\sqrt{\frac{13}{4}}} = \frac{1}{\frac{\sqrt{13}}{2}} = \frac{2}{\sqrt{13}}.$$

$$\boxed{\left.\frac{d}{dz}\sinh^{-1} z\right|_{z=\frac{3}{2}} = \frac{2}{\sqrt{13}}.}$$

Chapter 4

Fundamental Theorems on Holomorphic Functions

Part 1: Integration

4.1 Integration

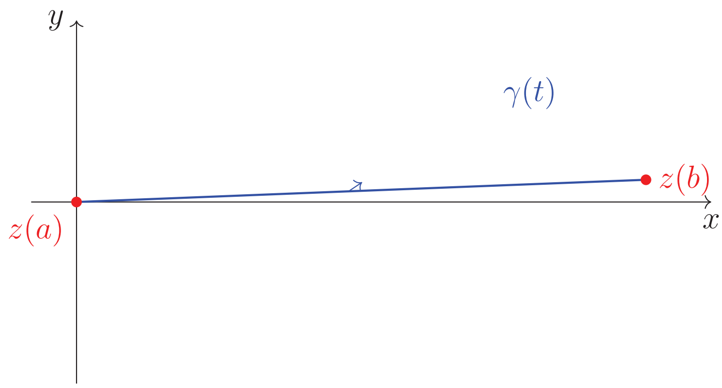
4.1.1 Paths and in the Complex Plane

Definition 4.1 (Path or Curve) ***A path (or curve) in the complex plane is a continuous function***

$$\gamma : [a, b] \to \mathbb{C}, \quad t \mapsto z(t) = x(t) + i\,y(t),$$

where $x(t)$ ***and*** $y(t)$ ***are real-valued functions of class*** $\mathcal{C}^1$ ***on*** $[a, b]$***. The set of points described by*** $\gamma(t)$ ***as*** t ***varies from*** a ***to*** b ***is called an arc of curve.***

The points $z(a)$ ***and*** $z(b)$ ***are called respectively the initial point and the terminal point of the path.***

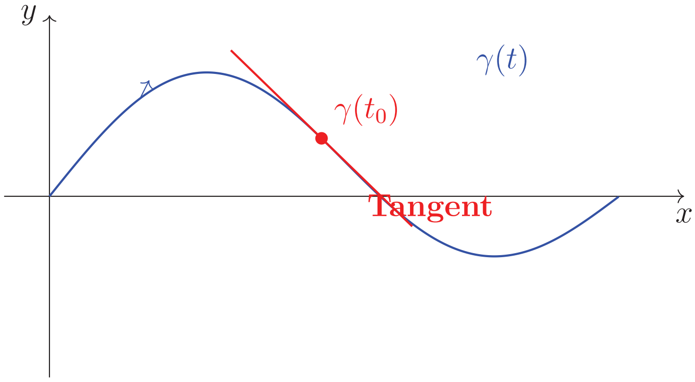


Definition 4.2 (Smooth Path) ***A path*** $\gamma(t)$ ***is said to be smooth if its derivative***

$$\gamma'(t) = x'(t) + i\,y'(t)$$

exists and is continuous on $[a, b]$***, and*** $\gamma'(t) \neq 0$ ***for all*** $t \in (a, b)$***.***

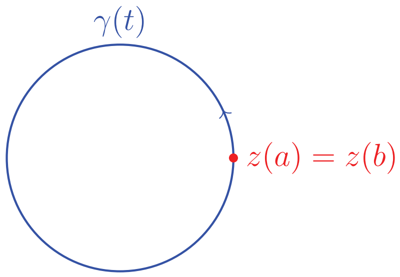
Geometrically, this means that the curve has a smoothly varying tangent direction at every point, and the tangent vector is never zero, so the curve never stops or forms a sharp corner.



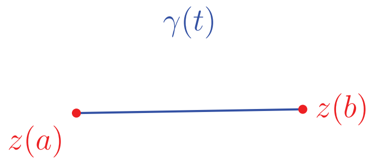
Definition 4.3 (Closed Path) ***A path is called closed (or a loop) if its initial and terminal points coincide:***

$$\gamma(a) = \gamma(b).$$

In this case, the path forms a closed curve in the complex plane.



Definition 4.4 (Simple Path) ***A path*** γ ***is called simple if it does not intersect itself, except possibly at the endpoints in the case of a closed path. That is, if*** $\gamma(t_1) = \gamma(t_2)$ ***implies either*** $t_1 = t_2$ ***or*** $t_1, t_2 \in \{a, b\}$***.***



4.2 Line Integrals

Line integrals are a fundamental tool in physics and complex analysis. They allow us to "sum" a function or a vector field along a curve in the plane or in space.

4.2.1 Line Integral of a of a Scalar Function

Definition 4.5 (Line integral of a scalar function) ***Let*** γ ***be a curve in the plane, parameterized by***

$$\gamma(t) = (x(t), y(t)), \quad t \in [a, b].$$

If $f(x,y)$ ***is a continuous function along*** γ***, the line integral of*** f ***along*** γ ***is defined by***

$$\int_\gamma f(x,y)\,ds = \int_a^b f(x(t),y(t))\sqrt{\left(x'(t)\right)^2 + \left(y'(t)\right)^2}\,dt.$$

Here, $ds = \sqrt{(dx)^2+(dy)^2}$ ***represents an infinitesimal segment of the curve.***

Remark 4.1 ***This integral "weights" the function*** f ***by the length of the path.***

4.2.2 Line Integral of a Vector Field

Definition 4.6 (Line integral of a vector field) ***Let*** $\mathbf{F}(x,y) = (P(x,y), Q(x,y))$ ***be a continuous vector field, and*** $\gamma(t) = (x(t),y(t)), t \in [a,b]$ ***an oriented curve. The line integral of*** $\mathbf{F}$ ***along*** γ ***is defined by***

$$\int_\gamma \mathbf{F}\cdot d\mathbf{r} = \int_a^b \Big[P(x(t),y(t))\,x'(t) + Q(x(t),y(t))\,y'(t)\Big]\,dt.$$

Remark 4.2 (Physical interpretation:) ***- If*** $\mathbf{F}$ ***represents a force, this integral corresponds to the work done by the force along the path*** γ***. - The projection of the vector*** $\mathbf{F}$ ***onto the tangent vector*** $d\mathbf{r}$ ***of the curve is used.***

Example 4.1 ***Let*** $f(x,y) = x+y$ ***and*** γ ***the line from*** $(0,0)$ ***to*** $(1,1)$ ***parameterized by*** $\gamma(t) = (t,t), t \in [0,1]$***.***

$$x'(t) = 1, \quad y'(t) = 1, \quad ds = \sqrt{1^2+1^2}\,dt = \sqrt{2}\,dt.$$

The line integral is then:

$$\int_\gamma f(x,y)\,ds = \int_0^1 (t+t)\sqrt{2}\,dt = \sqrt{2}\int_0^1 2t\,dt = \sqrt{2}.$$

Example 4.2 ***Let*** $\mathbf{F}(x,y) = (y,-x)$ ***and*** $\gamma(t) = (\cos t, \sin t), t \in [0,\pi/2]$ ***(a quarter circle).***

$$x'(t) = -\sin t, \quad y'(t) = \cos t$$

$$\int_\gamma \mathbf{F}\cdot d\mathbf{r} = \int_0^{\pi/2} \Big[y(t)x'(t) + (-x(t))y'(t)\Big]dt = \int_0^{\pi/2} \Big[\sin t(-\sin t) + (-\cos t)(\cos t)\Big]dt$$

$$= \int_0^{\pi/2} -(\sin^2 t + \cos^2 t)\,dt = \int_0^{\pi/2} -1\,dt = -\frac{\pi}{2}.$$

Remark 4.3

- ***The line integral depends on the orientation of the curve.***
- ***If*** $\mathbf{F}$ ***is a conservative field, the integral depends only on the start and end points.***
- ***In complex analysis, line integrals are essential for computing integrals of holomorphic functions along paths in the complex plane.***

4.3 Complex Line Integrals

Line integrals are fundamental in complex analysis and physics. They allow us to integrate a complex function along a curve in the complex plane.

Definition 4.7 (Complex Line Integral) ***Let*** γ ***be a smooth curve in the complex plane, parameterized by***

$$\gamma(t) = x(t) + iy(t), \quad t \in [a, b].$$

If $f : \mathbb{C} \to \mathbb{C}$ ***is continuous along*** γ***, the line integral of*** f ***along*** γ ***is defined by***

$$\int_\gamma f(z)\, dz = \int_a^b f(\gamma(t))\, \gamma'(t)\, dt,$$

where $\gamma'(t) = x'(t) + iy'(t)$***.***

Remark 4.4 ***This is the complex analogue of a vector line integral. The integral combines the function values with the infinitesimal displacements along the curve.***

4.3.1 Decomposition into Real Components

Definition 4.8 (Real Component Form) ***If*** $f(z) = u(x, y) + iv(x, y)$ ***and*** $\gamma(t) = x(t) + iy(t),\ \ t \in [a, b]$, ***then***

$$\int_\gamma f(z)\, dz = \int_a^b \Big[u(x(t), y(t)) + iv(x(t), y(t))\Big]\Big[x'(t) + iy'(t)\Big]dt.$$

Expanding, we get

$$\int_\gamma f(z)\, dz = \int_a^b \Big(ux' - vy'\Big)dt + i\int_a^b \Big(vx' + uy'\Big)dt,$$

which expresses the integral in terms of real integrals.

Example 4.3 (Simple Complex Line Integral) ***Let*** $f(z) = z$ ***and*** $\gamma(t) = t + it$***,*** $t \in [0, 1]$***. Then***

$$\gamma'(t) = 1 + i, \quad f(\gamma(t)) = t + it.$$

The line integral is

$$\int_\gamma f(z)\, dz = \int_0^1 (t + it)(1 + i)\, dt = \int_0^1 \Big[(t - t) + i(t + t)\Big]dt = \int_0^1 2it\, dt = i.$$

Example 4.4 (Vector Field Analogy) ***Let*** $f(z) = f(x+iy) = u(x, y)+iv(x, y)$***, where*** $u(x, y) = x - y$ ***and*** $v(x, y) = x + y$***, and let*** $\gamma(t) = t + it$***,*** $t \in [0, 1]$***.***

$$\textbf{\textit{Compute}} \ \int_\gamma f(z)\, dz.$$

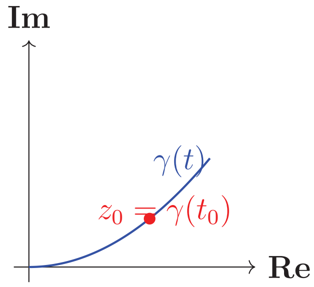
Solution:

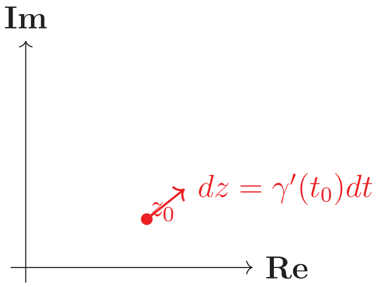
$$\gamma'(t) = 1 + i, \quad f(\gamma(t))f(t + it) = u(t, t) + iv(t, t) = 0 + i(2t) = 2it.$$

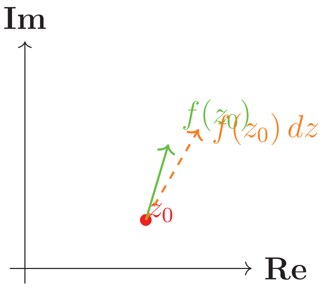
Hence,

$$\int_\gamma f(z)\, dz = \int_0^1 2it\, dt = i.$$

Remarks 4.1 *1- The line integral depends on the orientation of the curve. Reversing the path changes the sign of the integral.*







Remark 4.5

- *The quantity $\gamma'(t) = x'(t) + i\,y'(t)$ is the complex derivative of the path.*
- *The integral depends on the orientation of the curve. If the path is traversed in the opposite direction, then*

$$\int_{-\gamma} f(z)\,dz = -\int_{\gamma} f(z)\,dz.$$

Definition 4.9 (Integral along a Path) *A path Γ may be composed of a finite number of smooth curves:*

$$\Gamma = \gamma_1 + \gamma_2 + \cdots + \gamma_n,$$

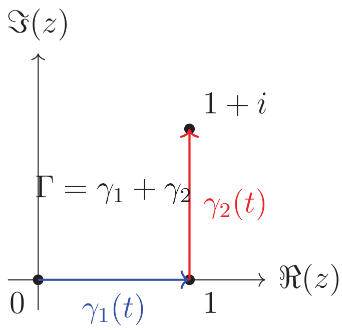
where each $\gamma_k : [a_k, b_k] \to \mathbb{C}$ is of class $\mathcal{C}^1$, and $\gamma_k(b_k) = \gamma_{k+1}(a_{k+1})$. If f is continuous on a domain containing Γ, we define

$$\boxed{\int_{\Gamma} f(z)\,dz = \sum_{k=1}^{n} \int_{\gamma_k} f(z)\,dz.}$$

Example 4.5 (Integral along a Path) *Let the path* $\Gamma = \gamma_1 + \gamma_2$ *be composed of two line segments:*

$$\gamma_1(t) = t, \quad t \in [0,1], \qquad \gamma_2(t) = 1 + it, \quad t \in [0,1].$$

Hence, the path goes from 0 *to* 1*, then from* 1 *to* $1+i$*. Consider* $f(z) = z^2$*.*



By definition, the integral of f *along the path* Γ *is*

$$\int_\Gamma f(z)\,dz = \int_{\gamma_1} f(z)\,dz + \int_{\gamma_2} f(z)\,dz,$$

where, for each curve γ_k*,*

$$\int_{\gamma_k} f(z)\,dz = \int_{a_k}^{b_k} f(\gamma_k(t))\,\gamma_k'(t)\,dt.$$

For the first segment:

$$\gamma_1(t) = t, \quad \gamma_1'(t) = 1.$$

Thus,

$$\int_{\gamma_1} f(z)\,dz = \int_0^1 f(\gamma_1(t))\,\gamma_1'(t)\,dt = \int_0^1 t^2\,dt = \frac{1}{3}.$$

For the second segment:

$$\gamma_2(t) = 1 + it, \quad \gamma_2'(t) = i.$$

Hence,

$$\begin{aligned}
\int_{\gamma_2} f(z)\,dz &= \int_0^1 f(\gamma_2(t))\,\gamma_2'(t)\,dt \\
&= \int_0^1 (1+it)^2 \cdot i\,dt \\
&= i\int_0^1 (1 + 2it - t^2)\,dt \\
&= i\left[t + it^2 - \frac{t^3}{3}\right]_0^1 \\
&= i\left(1 + i - \frac{1}{3}\right) = i\left(\frac{2}{3} + i\right) = -1 + \frac{2i}{3}.
\end{aligned}$$

Therefore, the total integral is:

$$\int_\Gamma f(z)\,dz = \int_{\gamma_1} f(z)\,dz + \int_{\gamma_2} f(z)\,dz$$
$$= \frac{1}{3} + \left(-1 + \frac{2i}{3}\right)$$
$$= -\frac{2}{3} + \frac{2i}{3}.$$

$$\boxed{\int_\Gamma z^2\,dz = \frac{2}{3}(i-1).}$$

Remark 4.6 ***Choice of the Integration Limits on Each Curve***

To compute a complex integral along a path composed of several curves, we first need to parameterize each part of the path and determine its corresponding limits of integration.

Let the path be

$$\Gamma = \gamma_1 + \gamma_2,$$

where

$$\gamma_1(t) = t, \quad t \in [0,1], \qquad \gamma_2(t) = 1 + it, \quad t \in [0,1].$$

For the first curve γ_1***: It goes from the point*** $z = 0$ ***to*** $z = 1$ ***along the real axis. When*** $t = 0$***, we have*** $\gamma_1(0) = 0$***; when*** $t = 1$***, we have*** $\gamma_1(1) = 1$***. Hence, the integration limits are*** $t \in [0,1]$***.***

For the second curve γ_2***: It goes from the point*** $z = 1$ ***to*** $z = 1+i$ ***vertically along the imaginary direction. When*** $t = 0$***, we have*** $\gamma_2(0) = 1$***; when*** $t = 1$***, we have*** $\gamma_2(1) = 1 + i$***. Therefore, the integration limits are also*** $t \in [0,1]$***.***

In general, for a parameterized curve $\gamma : [a,b] \to \mathbb{C}$***, the complex integral is defined by***

$$\int_\gamma f(z)\,dz = \int_a^b f(\gamma(t))\,\gamma'(t)\,dt.$$

Thus, the integration limits correspond to the parameter values a ***and*** b ***such that*** $\gamma(a)$ ***is the starting point and*** $\gamma(b)$ ***is the endpoint of the curve.***

4.3.2 Properties of Complex Line Integrals

Definition 4.10 ***Properties of Complex Line Integrals***

Let $f(z)$ ***and*** $g(z)$ ***be continuous complex functions defined on a path*** Γ***, and let*** $\alpha, \beta \in \mathbb{C}$ ***be complex constants. The following properties hold:***

1. ***Linearity:***

$$\int_\Gamma \big(\alpha f(z) + \beta g(z)\big)\,dz = \alpha \int_\Gamma f(z)\,dz + \beta \int_\Gamma g(z)\,dz.$$

2. ***Constant factor:***

$$\int_C k\,f(z)\,dz = k \int_C f(z)\,dz, \qquad k \in \mathbb{C}.$$

3. ***Additivity (linearity) with respect to functions:***

$$\int_C \Big(f(z) + g(z)\Big)\, dz = \int_C f(z)\, dz + \int_C g(z)\, dz.$$

4. ***Additivity with respect to the path: If the path Γ is composed of two consecutive curves,***

$$\Gamma = \Gamma_1 + \Gamma_2,$$

then

$$\int_\Gamma f(z)\, dz = \int_{\Gamma_1} f(z)\, dz + \int_{\Gamma_2} f(z)\, dz.$$

5. ***Reversal of the path: If $-\Gamma$ denotes the same path traversed in the opposite direction, then***

$$\int_{-\Gamma} f(z)\, dz = -\int_\Gamma f(z)\, dz.$$

6. ***Integral of the conjugate: The complex conjugate satisfies***

$$\overline{\int_\Gamma f(z)\, dz} = \int_\Gamma \overline{f(z)}\, \overline{dz} = \int_\Gamma \overline{f(z)}\, d\overline{z}.$$

Remark 4.7 ***These properties allow us to manipulate complex line integrals in a manner similar to real integrals, but with careful attention to the orientation and parametrization of the path. The additivity property is particularly useful when a path is composed of several segments.***

Theorem 4.1 (Upper Bound Theorem) ***Let f be a continuous complex function defined on a smooth curve C parameterized by***

$$z = z(t) = x(t) + iy(t), \quad t \in [a, b].$$

If

$$|f(z)| \leq M, \quad \forall z \in C,$$

then the following inequality holds:

$$\left| \int_C f(z)\, dz \right| \leq ML,$$

where

$$L = \int_a^b |z'(t)|\, dt = \int_a^b \sqrt{(x'(t))^2 + (y'(t))^2}\, dt$$

is the length of the curve C.

Example 4.6 ***Upper bound for the integral***

$$\int_C \frac{e^z}{z^2 + 1}\, dz, \qquad C : |z| = 4.$$

Solution.

We apply the Upper Bound Theorem***:***

$$\left|\int_C f(z)\,dz\right| \le ML,$$

where

$$M = \max_{z\in C} |f(z)|, \qquad L = \textbf{\textit{length of }} C.$$

1. The function is

$$f(z) = \frac{e^z}{z^2+1}.$$

2. On the circle $|z| = 4$:

$$|e^z| = e^{\Re(z)} \le e^{|z|} = e^4.$$

Also,

$$|z^2+1| \ge ||z|^2 - 1| = |16-1| = 15.$$

Hence,

$$|f(z)| = \left|\frac{e^z}{z^2+1}\right| \le \frac{|e^z|}{|z^2+1|} \le \frac{e^4}{15}.$$

Thus $M = \dfrac{e^4}{15}$.

3. The length of the circle C is

$$L = 2\pi r = 2\pi(4) = 8\pi.$$

4. By the upper bound theorem:

$$\left|\int_C \frac{e^z}{z^2+1}\,dz\right| \le ML = \frac{e^4}{15}\cdot 8\pi = \frac{8\pi e^4}{15}.$$

$$\boxed{\left|\int_C \frac{e^z}{z^2+1}\,dz\right| \le \frac{8\pi e^4}{15}.}$$

Remark 4.8 ***Theorem 4.2 (Reverse Triangle Inequality) For any complex numbers $a, b \in \mathbb{C}$, we have:***

$$|\,|a| - |b|\,| \le |a+b|.$$

Equivalently,

$$|a+b| \ge \Big|\,|a| - |b|\,\Big|.$$

Proof. Starting from the standard triangle inequality:

$$|a| = |(a+b) - b| \le |a+b| + |b|.$$

Hence,

$$|a| - |b| \le |a+b|.$$

By interchanging the roles of a and b, we also get

$$|b| - |a| \le |a+b|.$$

Combining both inequalities gives

$$-|a+b| \le |a| - |b| \le |a+b|.$$

Therefore,

$$\Big| |a| - |b| \Big| \le |a+b|.$$

■

4.3.3 Integral over a closed contour

Definition 4.11 ***Let*** $\gamma : [a,b] \to \mathbb{C}$ ***be a piecewise smooth and continuous curve given by*** $\gamma(t) = x(t) + i\,y(t)$***, such that*** $\gamma(a) = \gamma(b)$***. The curve*** γ ***is called a closed contour in the complex plane.***

If $f : D \to \mathbb{C}$ ***is a continuous function on a domain*** D ***containing*** γ***, the integral of*** f ***over the closed contour*** γ ***is defined by:***

$$\oint_\gamma f(z)\,dz = \int_a^b f(\gamma(t))\,\gamma'(t)\,dt.$$

Remark 4.9

1. ***If the curve is not closed (that is,*** $\gamma(a) \ne \gamma(b)$***), the usual notation is used:***

$$\int_\gamma f(z)\,dz,$$

and the integral is called an integral along an open path.

2. ***The notation*** $\oint_\gamma f(z)\,dz$ ***is reserved for the case where the path*** γ ***is closed:***

$$\gamma(a) = \gamma(b).$$

This type of integral naturally appears in the Cauchy Integral Theorem and the Cauchy Integral Formula.

Example 4.7 ***Example: Integral over a closed contour***
Let γ ***be the unit circle centered at the origin, defined by the parametrization:***

$$\gamma(t) = e^{it}, \quad t \in [0, 2\pi].$$

Then, γ ***is a closed contour since*** $\gamma(0) = \gamma(2\pi) = 1$***.***

(a) Case 1: $f(z) = z$

$$\oint_\gamma z\,dz = \int_0^{2\pi} \gamma(t)\,\gamma'(t)\,dt = \int_0^{2\pi} e^{it} \cdot (ie^{it})\,dt = i\int_0^{2\pi} e^{2it}\,dt.$$

Since

$$\int_0^{2\pi} e^{2it}\,dt = \left[\frac{1}{2i}e^{2it}\right]_0^{2\pi} = 0,$$

we obtain:

$$\boxed{\oint_\gamma z\,dz = 0.}$$

(b) Case 2: $f(z) = \dfrac{1}{z}$

$$\oint_\gamma \frac{1}{z}\,dz = \int_0^{2\pi} \frac{1}{\gamma(t)}\gamma'(t)\,dt = \int_0^{2\pi} e^{-it}\cdot(ie^{it})\,dt = i\int_0^{2\pi} dt = 2\pi i.$$

Thus,

$$\boxed{\oint_\gamma \frac{1}{z}\,dz = 2\pi i.}$$

Conclusion:

- ***For*** $f(z) = z$***, the integral over the closed contour is zero.***
- ***For*** $f(z) = \dfrac{1}{z}$***, the integral over the closed contour is nonzero and equals*** $2\pi i$***.***

This second case will play a fundamental role in the Cauchy Integral Formula.

4.4 Cauchy Integral Theorem and Cauchy Integral Formula

4.4.1 Simply Connected Domain

In physics, a system is said to be *connected* if all its parts are linked together, that is, if one can move from one point to another without leaving the system. This idea directly applies to complex analysis: a region in the complex plane is *connected* if it forms a continuous whole, with no separated pieces. Similarly, a *connected curve* is one that can be traced without any jump or interruption.

For instance, imagine a continuous metallic wire (fil métallique) forming a circle or a spiral. This wire represents a connected curve, since it can be followed entirely without lifting the pen. If the wire is cut into two separate pieces, we obtain two disjoint curves, which are no longer connected.

Mathematically, a curve

$$\gamma : [a, b] \to \mathbb{C}$$

is connected because its image is the set of points $\gamma(t)$ obtained as t varies continuously from a to b. In other words, it is a continuous curve without any break.

More generally, a subset $D \subset \mathbb{C}$ is said to be *connected* if, for any two points $z_1, z_2 \in D$, there exists at least one continuous path joining z_1 to z_2 while remaining entirely within D.

As a physical illustration, consider the electric field inside a solid conductor: it is defined on a connected region (the conductor is a single piece). If the conductor is split into two isolated parts, the region is no longer connected, since one cannot move from one point to another without leaving the domain.

4.4.2 Cauchy Integral Theorem

Theorem 4.3 (Cauchy Integral Theorem) ***Let $D \subset \mathbb{C}$ be a simply connected domain, and let $f : D \to \mathbb{C}$ be a holomorphic function on D.***

If γ is a closed contour lying entirely in D, then the integral of f around γ is zero:

$$\oint_\gamma f(z)\, dz = 0.$$

Remark 4.10

1. ***The domain D must be simply connected, meaning that it contains no holes (every closed curve in D can be continuously deformed to a point within D).***
2. ***The theorem expresses that the integral of a holomorphic function around any closed path in D is zero. This implies that the value of the integral between two points depends only on the endpoints, not on the path chosen.***
3. ***Geometrically, it means that the complex vector field defined by f is conservative.***

Example 4.8 ***For the function $f(z) = z^2$, and for the closed contour $\gamma(t) = e^{it}$, $t \in [0, 2\pi]$, we have:***

$$\oint_\gamma f(z)\, dz = \oint_\gamma z^2\, dz = 0,$$

which is consistent with Cauchy's Theorem.
Explicit calculation: Let $f(z) = z^2$ and let γ be the unit circle parametrized by $\gamma(t) = e^{it}$, for $t \in [0, 2\pi]$. We will compute the integral

$$\oint_\gamma z^2\, dz.$$

Using the parametrization $\gamma(t) = e^{it}$ and $\gamma'(t) = ie^{it}$, we have:

$$\oint_\gamma z^2\, dz = \int_0^{2\pi} \big(\gamma(t)\big)^2\, \gamma'(t)\, dt = \int_0^{2\pi} e^{2it}\, (ie^{it})\, dt = i \int_0^{2\pi} e^{3it}\, dt.$$

Now integrate:

$$i \int_0^{2\pi} e^{3it}\, dt = i \left[\frac{1}{3i} e^{3it}\right]_0^{2\pi} = \frac{i}{3i}\Big(e^{6\pi i} - 1\Big) = \frac{1}{3}(1 - 1) = 0.$$

Therefore,

$$\boxed{\oint_\gamma z^2\, dz = 0,}$$

which explicitly confirms Cauchy's Theorem: the integral of a holomorphic function over any closed contour in a simply connected region is zero.

4.4.3 Cauchy Integral Formula

Theorem 4.4 (Cauchy Integral Formula) ***Let $D \subset \mathbb{C}$ be a simply connected domain, and let $f : D \to \mathbb{C}$ be holomorphic. Let γ be a positively oriented (counterclockwise: dans le sens contraire des aiguilles d?une montre) closed contour contained in D, and let z_0 be any point inside γ.***

Then,

$$f(z_0) = \frac{1}{2\pi i} \oint_\gamma \frac{f(z)}{z - z_0}\, dz.$$

Remark 4.11

1. ***The formula gives the value of f inside the contour in terms of the values of f on the contour.***
2. ***The Cauchy Integral Formula is one of the most powerful tools in complex analysis; it leads to many fundamental results such as:***
   - ***the infinite differentiability of holomorphic functions,***
   - ***the Cauchy inequalities,***
   - ***the Liouville theorem,***
   - ***and the Maximum Modulus Principle.***

Example 4.9 ***Let $f(z) = \frac{1}{z+1}$, and let γ be the circle $|z| = 2$ oriented counterclockwise. We compute $f(z_0)$ for $z_0 = 1$, which lies inside γ:***

$$f(z_0) = \frac{1}{2\pi i} \oint_\gamma \frac{f(z)}{z - z_0}\, dz.$$

Substituting $f(z) = \frac{1}{z+1}$ and $z_0 = 1$:

$$f(1) = \frac{1}{2\pi i} \oint_\gamma \frac{1}{(z+1)(z-1)}\, dz.$$

We parametrize the contour γ by $z = 2e^{it}$ with $t \in [0, 2\pi]$, so that $dz = 2ie^{it}\, dt$. Then:

$$\oint_\gamma \frac{1}{(z+1)(z-1)}\, dz = \int_0^{2\pi} \frac{2ie^{it}}{(2e^{it}+1)(2e^{it}-1)}\, dt.$$

Although the integral looks complicated, note that Cauchy's Integral Formula tells us directly that:

$$\frac{1}{2\pi i} \oint_\gamma \frac{f(z)}{z-1}\, dz = f(1),$$

and by simple substitution $f(1) = \frac{1}{2}$.

Remark 4.12 ***The Cauchy Integral Formula is not primarily used to compute $f(z_0)$ directly, since this value can be easily obtained by simple substitution in $f(z)$. Its real importance lies in the fact that it allows us to determine the numerical value of the contour integral without performing the integration.***

In practice, once we identify the function $f(z)$ and verify that the point z_0 lies inside the domain (enclosed by the contour γ), we can immediately conclude that

$$f(z_0) = \frac{1}{2\pi i} \oint_\gamma \frac{f(z)}{z - z_0}\, dz,$$

or equivalently,

$$\oint_\gamma \frac{f(z)}{z - z_0}\, dz = 2\pi i\, f(z_0),$$

without calculating the integral explicitly.

Corollary 4.1 ***If f is holomorphic in D, then for any integer $n \geq 1$,***

$$\boxed{f^{(n)}(z_0) = \frac{n!}{2\pi i} \int_\Gamma \frac{f(z)}{(z - z_0)^{n+1}}\, dz.}$$

Interpretation. Cauchy's Integral Formula expresses the value of a holomorphic function inside a closed contour in terms of the values of the function on the contour itself. It is a cornerstone of complex analysis, from which many fundamental results follow (such as Cauchy's inequalities, Liouville's theorem, and Morera's theorem).

Theorem 4.5 (Generalized Cauchy Integral Formula) ***Let f be a holomorphic function on a simply connected domain D, and let γ be a positively oriented, closed, and piecewise smooth contour lying entirely in D. If z_0 is any point inside γ, then for every integer $n \geq 0$, we have:***

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \oint_\gamma \frac{f(z)}{(z - z_0)^{n+1}}\, dz.$$

Remark 4.13 ***This formula shows that all derivatives of a holomorphic function can be expressed as contour integrals. It is a direct consequence of the Cauchy Integral Formula and implies that a holomorphic function is infinitely differentiable inside its domain.***

Example 4.10 ***Let $f(z) = e^z$ and consider the contour $\gamma(t) = 2e^{it}$, with $t \in [0, 2\pi]$, which is a circle of radius 2 centered at the origin. We want to compute the following integral for $z_0 = 1$:***

$$\oint_\gamma \frac{e^z}{(z - 1)^3}\, dz.$$

According to the Cauchy Integral Formula for derivatives:

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \oint_\gamma \frac{f(z)}{(z - z_0)^{n+1}}\, dz.$$

For $n = 2$***, we have***

$$\oint_\gamma \frac{f(z)}{(z-z_0)^3}\,dz = \frac{2\pi i}{2!}\,f^{(2)}(z_0).$$

Considering the integrale

$$\oint_\gamma \frac{e^z}{(z-1)^3}\,dz.$$

we have $f(z) = e^z$***, then*** $f^{(2)}(z) = e^z$***. Therefore,***

$$\oint_\gamma \frac{e^z}{(z-1)^3}\,dz = \frac{2\pi i}{2}\,e^1 = \pi i\,e.$$

This confirms the Cauchy Integral Formula for higher-order derivatives.

4.4.4 Cauchy's Inequalities

Proposition 4.1 ***(Cauchy's Inequalities]***

Statement. Let f ***be a holomorphic function in a simply connected domain*** $D \subset \mathbb{C}$***. Let***

$$\Gamma = \{z \in \mathbb{C} : |z - z_0| = r\}$$

be a circle of center z_0 ***and radius*** r***, entirely contained in*** D***. Assume that***

$$|f(z)| \le M, \quad \forall z \in \Gamma.$$

Then, for every nonnegative integer $n \in \mathbb{N}$***,***

$$\boxed{|f^{(n)}(z_0)| \le \frac{n!\,M}{r^n}.}$$

Proof. From Cauchy?s integral formula for derivatives:

$$f^{(n)}(z_0) = \frac{n!}{2\pi i}\int_{|z-z_0|=r} \frac{f(z)}{(z-z_0)^{n+1}}\,dz.$$

Taking the modulus and using $|f(z)| \le M$**, we obtain**

$$|f^{(n)}(z_0)| \le \frac{n!}{2\pi}\int_{|z-z_0|=r} \frac{M}{r^{n+1}}\,|dz| = \frac{n!M}{r^n}.$$

Interpretation. Cauchy's inequalities show that the size of the derivatives of a holomorphic function is controlled by its maximum modulus on a surrounding circle. This is a fundamental tool in analytic estimates and proofs of Liouville's theorem.

4.4.5 Integral of $\frac{1}{z-z_0}$ over a closed contour

Theorem 4.6 ***Let γ be a positively oriented closed contour and $z_0 \in \mathbb{C}$. Then***

$$\oint_\gamma \frac{1}{z-z_0}\,dz = \begin{cases} 2\pi i, & \textit{if } z_0 \textit{ is inside } \gamma, \\ 0, & \textit{if } z_0 \textit{ is outside } \gamma. \end{cases}$$

Proof. We parametrize the contour γ as $z = z(t)$, $t \in [0, 2\pi]$.

Case 1: z_0 inside γ. Assume γ is a circle of radius r around z_0:

$$z(t) = z_0 + re^{it}, \quad dz = ire^{it}dt.$$

Then

$$\oint_\gamma \frac{1}{z-z_0}\,dz = \int_0^{2\pi} \frac{1}{re^{it}} \cdot ire^{it}dt = \int_0^{2\pi} i\,dt = 2\pi i.$$

Case 2: z_0 outside γ. In this case, $\frac{1}{z-z_0}$ is holomorphic on and inside the contour. By Cauchy's theorem,

$$\oint_\gamma \frac{1}{z-z_0}\,dz = 0.$$

■

Remark 4.14 ***This fundamental result allows us to evaluate integrals of the form $\oint_\gamma \frac{f(z)}{z-z_0}\,dz$ when f is holomorphic: we only need to check whether z_0 is inside or outside the contour. It is also the key step in proving the Cauchy integral formula. Integral is zero if z_0 is outside the contour.Indeed,***

if the point z_0 lies outside the closed contour γ, the function

$$f(z) = \frac{1}{z-z_0}$$

is holomorphic on the entire domain containing γ and its interior.

According to Cauchy's theorem, the integral of a holomorphic function over a closed contour is zero:

$$\oint_\gamma f(z)\,dz = 0.$$

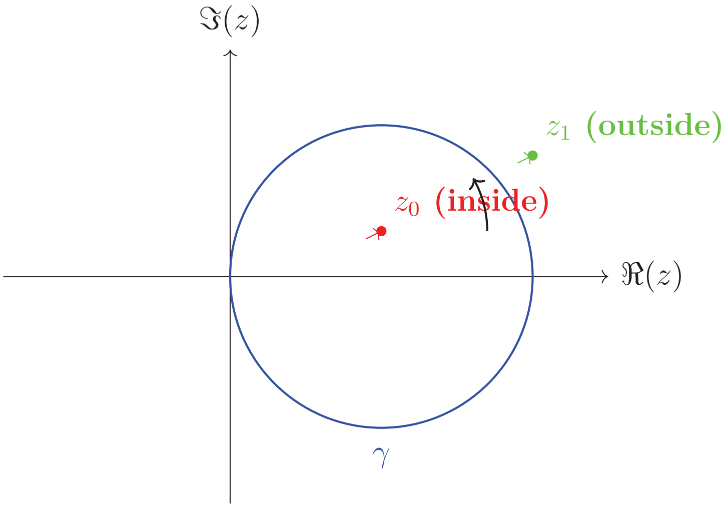
Therefore, when z_0 is outside the contour, the integral $\oint_\gamma \frac{1}{z-z_0}\,dz = 0$.

Example 4.11 ***Let $\gamma : |z| = 1$ and $z_0 = 0.5$. Then***

$$\oint_\gamma \frac{1}{z-0.5}\,dz = 2\pi i.$$

If $z_0 = 2$ (outside the circle), then

$$\oint_\gamma \frac{1}{z-2}\,dz = 0.$$



4.4.6 Cauchy-Goursat Theorem for Domains with Holes

Definition 4.12 (Multiply Connected Domain) ***A domain $D \subset \mathbb{C}$ is called multiply connected if it contains one or more "holes", i.e., subregions removed from the domain. Let $\gamma_0, \gamma_1, \ldots, \gamma_n$ be positively oriented closed contours such that:***

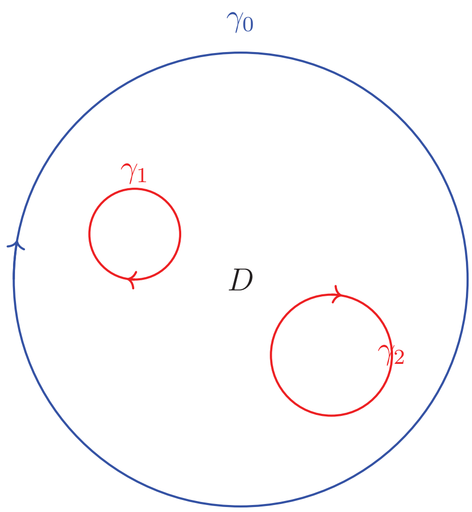
- ***γ_0 is the outer boundary of D,***
- ***$\gamma_1, \ldots, \gamma_n$ are boundaries of holes in D.***

Example 4.12 ***Consider the annulus $D = \{z \in \mathbb{C} \mid 1 < |z| < 2\}$, which has one hole (Exclusion region:the disc $|z| < 1$). Let $f(z) = z^2$, $\gamma_0 : |z| = 2$ (outer boundary) and $\gamma_1 : |z| = 1$ (inner boundary), both counterclockwise. Then***

$$\oint_{\gamma_0} z^2\, dz - \oint_{\gamma_1} z^2\, dz = 0.$$

Example 4.13 ***Let D be the domain with two holes as in the figure below. Let $f(z) = e^z$. Then f is holomorphic on D, and***

$$\oint_{\gamma_0} e^z\, dz - \oint_{\gamma_1} e^z\, dz - \oint_{\gamma_2} e^z\, dz = 0.$$

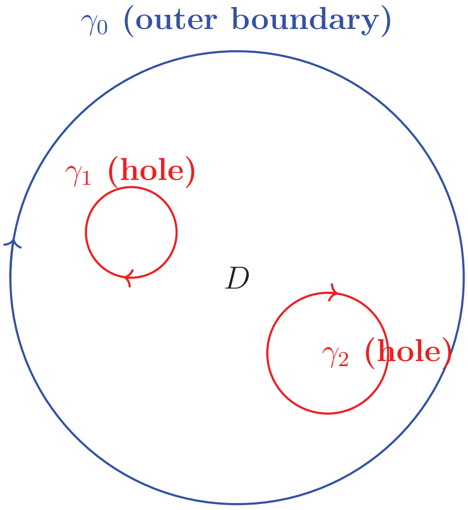


Theorem 4.7 (Cauchy-Goursat for Multiply Connected Domains) ***Let*** $D \subset \mathbb{C}$ ***be a multiply connected domain and let*** $\gamma_0, \gamma_1, \ldots, \gamma_n$ ***be positively oriented closed contours such that:***

- γ_0 ***is the outer boundary of*** D***,***
- $\gamma_1, \ldots, \gamma_n$ ***are boundaries of holes in*** D***,***

and all contours are oriented positively with respect to the domain. If f ***is holomorphic on*** D ***and continuous on its closure, then***

$$\oint_{\gamma_0} f(z)\, dz = \sum_{k=1}^{n} \oint_{\gamma_k} f(z)\, dz.$$

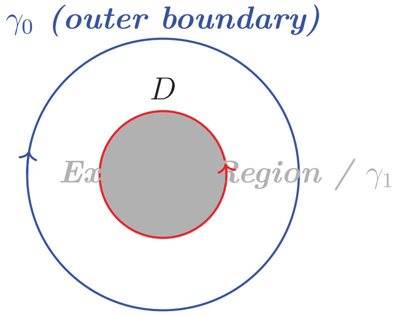


Remark 4.15 ***This theorem generalizes the classical Cauchy-Goursat theorem to domains with holes. It states that the integral of a holomorphic function over the outer boundary is equal to the sum of integrals over the inner boundaries (holes), taking orientation into account.***

Example 4.14 (Annulus with one hole) ***Consider the annulus***

$$D = \{z \in \mathbb{C} \mid 1 < |z| < 2\},$$

which has one inner excluded region (the disc $|z| < 1$***).***



Let $f(z) = z^2$***, and let*** $\gamma_0 : |z| = 2$ ***be the outer boundary, and*** $\gamma_1 : |z| = 1$ ***be the inner boundary, both oriented counterclockwise.***

1. $f(z) = z^2$ ***is holomorphic on the entire domain*** D ***because it is a polynomial (no singularities).***

2. ***According to the Cauchy-Goursat theorem for multiply connected domains, the integral along the outer boundary minus the integral along inner boundaries is zero:***
$$\oint_{\gamma_0} z^2\,dz - \oint_{\gamma_1} z^2\,dz = 0.$$

3. ***Intuitively, the "effect" of the inner hole*** γ_1 ***cancels part of the integral around*** γ_0***, making the sum zero. This is why the presence of holes changes the classical Cauchy theorem.***

4. ***If you compute explicitly using parameterization:***
$$\gamma_0(t) = 2e^{it}, \quad t \in [0, 2\pi] \quad \Rightarrow \quad dz = 2ie^{it}dt,$$
$$\oint_{\gamma_0} z^2 dz = \int_0^{2\pi} (2e^{it})^2 \cdot 2ie^{it}dt = 8i \int_0^{2\pi} e^{3it}dt = 0$$
and similarly
$$\oint_{\gamma_1} z^2 dz = \int_0^{2\pi} (1e^{it})^2 \cdot ie^{it}dt = i \int_0^{2\pi} e^{3it}dt = 0.$$
So indeed the formula is verified.

4.5 Primitives and Morera's Theorem

4.5.1 Primitives

Definition 4.13 ***Let*** f ***be a function holomorphic on a domain*** $D \subset \mathbb{C}$***. A function*** $F : D \to \mathbb{C}$ ***is called a primitive (or antiderivative) of*** f ***on*** D ***if***
$$F'(z) = f(z), \quad \forall z \in D.$$

Remark 4.16 ***If*** F ***is a primitive of*** f ***in*** D***, then for any contour*** γ ***in*** D ***from*** z_0 ***to*** z_1***, we have***
$$\int_\gamma f(z)\,dz = F(z_1) - F(z_0).$$
In particular, if γ ***is a closed contour, the integral vanishes:***
$$\oint_\gamma f(z)\,dz = 0.$$

Example 4.15 ***Let*** $f(z) = e^z$***. Then a primitive of*** f ***is*** $F(z) = e^z$***, since*** $F'(z) = e^z = f(z)$***. Hence, for any contour*** γ ***from*** z_0 ***to*** z_1***,***
$$\int_\gamma e^z dz = e^{z_1} - e^{z_0}.$$

4.5.2 Morera's Theorem

Theorem 4.8 (Morera) ***Let f be a continuous function on a domain $D \subset \mathbb{C}$. If for every closed contour γ in D we have***

$$\oint_\gamma f(z)\,dz = 0,$$

then f is holomorphic on D.

Remark 4.17 ***Morera's theorem is essentially the converse of Cauchy's theorem: - Cauchy's theorem says: If f is holomorphic, then its integral around any closed contour is zero. - Morera's theorem says: If a continuous function has zero integral around every closed contour, then it is holomorphic.***

Example 4.16 ***Let $f(z) = \overline{z}$ on $D = \mathbb{C}$. For the closed contour $\gamma(t) = e^{it}, t \in [0, 2\pi]$, we have***

$$\oint_\gamma f(z)dz = \oint_0^{2\pi} \overline{e^{it}} ie^{it}dt = i\int_0^{2\pi} e^{-it}e^{it}dt = i\int_0^{2\pi} 1dt = 2\pi i \neq 0.$$

Hence, by Morera's theorem, f is not holomorphic on $\mathbb{C}$.

Interpretation. Morera's theorem provides a converse to Cauchy's theorem: while Cauchy's theorem states that the integral of a holomorphic function over a closed path is zero, Morera's theorem asserts that if a continuous function has zero integral over every closed contour, then it must be holomorphic.

Remark 4.18 ***Morera's theorem is particularly useful for proving that limits of sequences of holomorphic functions are holomorphic, provided the convergence is uniform on compact subsets.***

Example 4.17 ***Using Morera's Theorem show that $e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$ is holomorphic.***

Solution 4.1 ***Let $f_N(z) = \sum_{n=0}^{N} \frac{z^n}{n!}$ denote the N-th partial sum. Each finite sum f_N is a polynomial, hence entire (holomorphic on $\mathbb{C}$). We will show that for every closed, piecewise-smooth contour Γ contained in $\mathbb{C}$,***

$$\int_\Gamma f(z)\,dz = 0,$$

and then invoke Morera's theorem.

Fix a closed contour Γ. For each N,

$$\int_\Gamma f_N(z)\,dz = \int_\Gamma \sum_{n=0}^{N} \frac{z^n}{n!}\,dz = \sum_{n=0}^{N} \frac{1}{n!}\int_\Gamma z^n\,dz.$$

But $\int_\Gamma z^n\,dz = 0$ *for every integer* $n \geq 0$ *because* z^n *is entire (Cauchy's theorem). Hence*

$$\int_\Gamma f_N(z)\,dz = 0 \quad \textit{for all } N.$$

Next we pass to the limit $N \to \infty$*. The power series for* f *has infinite radius of convergence, so it converges uniformly on every compact subset of* $\mathbb{C}$*. In particular, the sequence* (f_N) *converges uniformly to* f *on the compact set given by the image of* Γ*. Therefore we may interchange limit and integral:*

$$\int_\Gamma f(z)\,dz = \lim_{N\to\infty} \int_\Gamma f_N(z)\,dz = \lim_{N\to\infty} 0 = 0.$$

By Morera's theorem (continuous function whose integral over every closed contour is zero is holomorphic), f *is holomorphic on* $\mathbb{C}$*. That is, the power series defines an entire function.* □

4.6 Identity Principle, Mean Value, Maximum Principle, and Classical Theorems

4.6.1 Identity Principle

Definition 4.14 (Accumulation Point) *Let* $S \subset D \subset \mathbb{C}$*. A point* $z_0 \in D$ *is called an accumulation point of* S *if, for every* $\varepsilon > 0$*, the disk*

$$B(z_0, \varepsilon) = \{z \in \mathbb{C} \mid |z - z_0| < \varepsilon\}$$

contains at least one point of S *different from* z_0*.*

In simple terms: no matter how small a circle you draw around z_0*, there is always at least one point of* S *inside that circle.*

Example 4.18 *Let*

$$S = \left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots\right\} \subset \mathbb{C}.$$

The point 0 *is an accumulation point of* S *because the elements* $1/n$ *get arbitrarily close to* 0*. For any small circle around* 0*, there is always a point* $1/n \in S$ *inside it.*

On the other hand, the point 1 *is not an accumulation point of* S*, because we can find a small circle around* 1 *that contains no other points of* S *besides* 1 *itself.*

Theorem 4.9 (Identity Principle) *Let* f *and* g *be holomorphic on a connected domain* $D \subset \mathbb{C}$*. If* $f(z) = g(z)$ *on a set* $S \subset D$ *that has an accumulation point in* D*, then* $f \equiv g$ *on* D*.*

Example 4.19 *Consider the functions*

$$f(z) = e^z, \quad g(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!} \quad \textit{on } \mathbb{C}.$$

Define the set of points

$$S = \{0, 0.1, 0.2, 0.3, \dots\}.$$

- $f(z) = g(z)$ ***for all*** $z \in S$***, since the power series for*** $g(z)$ ***converges to*** e^z***.***
- S ***has an accumulation point at*** 0***.***

By the Identity Principle, since both f ***and*** g ***are holomorphic and agree on a set with an accumulation point, we conclude that***

$$f(z) \equiv g(z) \quad \textbf{\textit{on}}\ \mathbb{C}.$$

Remark 4.19 ***This principle shows the*** **rigidity** ***of holomorphic functions: knowing the values on even a tiny set with an accumulation point determines the function everywhere.***

4.6.2 Mean Value Property for Holomorphic Functions

Theorem 4.10 (Mean Value Property) ***Let*** f ***be holomorphic on a domain*** $D \subset \mathbb{C}$***. For any closed disk*** $\overline{B(z_0, r)} \subset D$***:***

$$f(z_0) = \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + re^{i\theta})\, d\theta$$

where z_0 ***is the center and*** r ***the radius.***

Remarks 4.2 ***1. If*** f ***is holomorphic (complex-differentiable) on a domain*** D***, then the value of*** f ***at the center of any disk entirely contained in*** D ***is equal to the*** **average of** f **on the boundary of the disk.**

2.
- z_0 ***is the center of the disk.***
- $r > 0$ ***is the radius.***
- $\overline{B(z_0, r)} = \{z \in \mathbb{C} \mid |z - z_0| \leq r\}$ ***is the closed disk.***
- ***The integral***

$$\frac{1}{2\pi} \int_0^{2\pi} f(z_0 + re^{i\theta})\, d\theta$$

is the average of f ***along the circle of radius*** r ***centered at*** z_0***.***

3. The property follows directly from Cauchy's Integral Formula:

$$f(z_0) = \frac{1}{2\pi i} \oint_{|z - z_0| = r} \frac{f(z)}{z - z_0}\, dz.$$

Parametrize the circle by $z = z_0 + re^{i\theta}$***, so that*** $dz = ire^{i\theta}\, d\theta$***. Substituting gives:***

$$f(z_0) = \frac{1}{2\pi i} \int_0^{2\pi} \frac{f(z_0 + re^{i\theta})}{re^{i\theta}} \cdot ire^{i\theta}\, d\theta = \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + re^{i\theta})\, d\theta,$$

which is exactly the Mean Value Property.

4. ***Holomorphic functions are very* rigid*: the value at the center of a disk is completely determined by the average along any surrounding circle. This property is special to holomorphic functions and does not generally hold for real differentiable functions.***

5. - ***If two holomorphic functions agree on the boundary of a disk, they must agree at the center.***

Example 4.20 Let $f(z) = z^2$ ***and consider the disk centered at*** 0 ***with radius*** r***:***

$$\frac{1}{2\pi}\int_0^{2\pi}(re^{i\theta})^2\,d\theta = \frac{r^2}{2\pi}\int_0^{2\pi} e^{2i\theta}\,d\theta = 0,$$

which equals $f(0) = 0$ ***as expected.***

Theorem 4.11 (Gauss Mean Value Theorem) ***Let*** f ***be holomorphic in a domain*** $U \subset \mathbb{C}$***, and let*** $z_0 \in U$***. Then*** $f(z_0)$ ***is equal to the average of*** f ***on the boundary of any disk centered at*** z_0 ***and contained in*** U***. That is, for any disk*** $D(z_0, r) \subset U$***:***

$$f(z_0) = \frac{1}{2\pi}\int_0^{2\pi} f\Big(z_0 + re^{i\theta}\Big)\,d\theta.$$

Example 4.21 ***Let***

$$f(z) = z^2$$

which is holomorphic on $\mathbb{C}$***. Let*** $z_0 = 1 + i$ ***and consider a disk of radius*** $r > 0$ ***centered at*** z_0***,*** $D(z_0, r)$***.***

According to the Gauss mean value theorem, we have

$$f(z_0) = \frac{1}{2\pi}\int_0^{2\pi} f\Big(z_0 + re^{i\theta}\Big)\,d\theta.$$

Computation:

$$f(z_0 + re^{i\theta}) = (z_0 + re^{i\theta})^2 = z_0^2 + 2z_0re^{i\theta} + r^2e^{2i\theta}.$$

Now integrate over $\theta \in [0, 2\pi]$***:***

$$\frac{1}{2\pi}\int_0^{2\pi}\Big(z_0^2 + 2z_0re^{i\theta} + r^2e^{2i\theta}\Big)d\theta = z_0^2 + \frac{2z_0r}{2\pi}\int_0^{2\pi} e^{i\theta}d\theta + \frac{r^2}{2\pi}\int_0^{2\pi} e^{2i\theta}d\theta.$$

Since

$$\int_0^{2\pi} e^{i\theta}d\theta = 0 \quad \textbf{\textit{and}} \quad \int_0^{2\pi} e^{2i\theta}d\theta = 0,$$

we get

$$\frac{1}{2\pi}\int_0^{2\pi} f(z_0 + re^{i\theta})\,d\theta = z_0^2 = f(z_0).$$

Thus, the value at the center equals the average on the circle, as stated by the theorem.

4.6.3 Maximum Principle

Theorem 4.12 (Maximum Principle) ***Let f be holomorphic and non-constant on a connected domain $D \subset \mathbb{C}$. Then $|f(z)|$ cannot attain a maximum value inside D. Any maximum occurs on the boundary of D.***

Remarks 4.3

1. ***If f is holomorphic and not constant on D, the largest value of $|f(z)|$ occurs on the boundary, not inside the domain.***

2. ***Suppose $|f(z)|$ attains a maximum at some $z_0 \in D$. Consider a small disk centered at z_0, that is to suppose, that $|f(z)|$ attains a maximum at some interior point $z_0 \in D$. Consider a small disk centered at z_0 entirely contained in D. By the Mean Value Property:***

   $$f(z_0) = \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + re^{i\theta})\, d\theta.$$

   Since $|f(z)| \leq |f(z_0)|$ on the circle of radius r, the average of f on the circle cannot exceed $|f(z_0)|$. This forces f to be constant on the circle.

   Repeating this argument for overlapping disks covering the connected domain D shows that f must be constant on D.

   Contradiction: This is impossible because we assumed at the start that f is non-constant.

   Therefore, the assumption that $|f(z)|$ attains a maximum inside D must be false, proving that any maximum occurs on the boundary of D.

3. ***Holomorphic functions are "rigid"; the modulus cannot have a peak in the interior without the function being constant. The maximum occurs at the boundary.***

4.

5. ***The minimum of $|f|$ for non-zero functions occurs on the boundary.***

   Example 4.22 Let $f(z) = z$ and $D = \{z \in \mathbb{C} : |z| < 1\}$. Then:

   $$|f(z)| = |z| < 1 \text{ for all } z \in D,$$

   and the maximum $|f(z)| = 1$ occurs only on the boundary $|z| = 1$.

4.6.4 Maximum Modulus Theorem

Definition 4.15 (Local Maximum of a Complex Function) ***Let f be a complex-valued function. We say that $|f|$ has a*** local maximum ***at $z = z_0$ if there exists a neighborhood U of z_0 such that***

$$\forall z \in U : \quad |f(z)| \leq |f(z_0)|.$$

If the inequality is strict, i.e.,

$$\forall z \in U \setminus \{z_0\} : \quad |f(z)| < |f(z_0)|,$$

then the local maximum is said to be strict.

Theorem 4.13 (Maximum Modulus Theorem) ***Let f be a non-constant holomorphic function on a domain $U \subset \mathbb{C}$. Then the modulus $|f(z)|$ cannot attain a local maximum at any point inside U. In other words, if $z_0 \in U$ and there exists a neighborhood V of z_0 such that***

$$|f(z)| \leq |f(z_0)| \quad \textbf{\textit{for all }} z \in V,$$

then f must be constant.

4.6.5 Liouville's Theorem

Theorem 4.14 ***Let $f : \mathbb{C} \to \mathbb{C}$ be an entire function (that is, holomorphic everywhere on) $\mathbb{C}$. If there exists a constant $M > 0$ such that***

$$|f(z)| \leq M, \quad \forall z \in \mathbb{C},$$

then f is constant.

Proof. By Cauchy's integral formula, for any $R > 0$ and any z such that $|z| < R$,

$$f'(z) = \frac{1}{2\pi i} \int_{|w|=R} \frac{f(w)}{(w-z)^2}\, dw.$$

Taking the modulus, we get

$$|f'(z)| \leq \frac{1}{2\pi} \int_{|w|=R} \frac{|f(w)|}{|w-z|^2}\, |dw| \leq \frac{M}{R-|z|}.$$

Letting $R \to \infty$, we obtain $f'(z) = 0$ for all $z \in \mathbb{C}$. Hence f is constant.

Corollary 4.2 (Fundamental result) ***Every bounded entire function is constant.***

Intuitive interpretation. Liouville's theorem shows that holomorphic functions cannot remain bounded over the entire complex plane unless they are constant. This result plays a key role in proving the Fundamental Theorem of Algebra.

Remarks 4.4

1. ***If a function f is holomorphic everywhere on $\mathbb{C}$ (entire) and there exists $M > 0$ such that $|f(z)| \leq M$ for all $z \in \mathbb{C}$, then f must be constant.***

2. ***Using Cauchy's estimate for the derivative on a disk of radius R around z_0:***

   $$f'(z_0) = \frac{1}{2\pi i} \oint_{|z-z_0|=R} \frac{f(z)}{(z-z_0)^2} dz.$$

   Taking absolute values and using $|f(z)| \leq M$:

   $$|f'(z_0)| \leq \frac{1}{2\pi} \cdot 2\pi R \cdot \frac{M}{R^2} = \frac{M}{R}.$$

   Letting $R \to \infty$ gives $|f'(z_0)| \to 0$, so $f'(z_0) = 0$. Since z_0 is arbitrary, $f'(z) \equiv 0$, hence f is constant.

3. ***Holomorphic functions are rigid. If bounded everywhere, they cannot oscillate or grow, so they must be flat (constant).***

4. ***Any non-constant entire function must grow unbounded somewhere in $\mathbb{C}$.***

***Example 4.23** Let $f(z) = e^{iz}$. This function is entire but not bounded, since*

$$|e^{iz}| = |e^{i(x+iy)}| = |e^{ix}e^{-y}| = e^{-y},$$

which grows unbounded as $y \to -\infty$. Therefore, Liouville's theorem does not apply. If f were bounded, it would have to be constant.

4.6.6 D'Alembert's Theorem (Fundamental Theorem of Algebra)

Theorem 4.15 (D'Alembert) ***Every non-constant polynomial $P(z)$ with complex coefficients has at least one complex root.***

Remarks 4.5

1. ***Any polynomial of degree $n \geq 1$ with complex coefficients has at least one root in $\mathbb{C}$.***

2. ***Polynomials grow large as $|z| \to \infty$. If a polynomial had no root, then $1/P(z)$ would be defined and entire. Boundedness of $1/P(z)$ at infinity would then force it to be constant, contradicting the fact that $P(z)$ is non-constant.***

3. ***Proof***

   - ***Suppose $P(z)$ has no roots in $\mathbb{C}$, i.e., $P(z) \neq 0$ for all $z \in \mathbb{C}$.***
   - ***Define $f(z) = 1/P(z)$. Then f is entire.***
   - ***For large $|z|$, $|P(z)| \sim |a_n||z|^n$, so***

     $$|f(z)| = \frac{1}{|P(z)|} \leq \frac{C}{|z|^n} \to 0 \quad \textbf{\textit{as}}\ |z| \to \infty,$$

     for some constant $C > 0$.
   - ***Therefore, f is bounded on $\mathbb{C}$.***
   - ***By Liouville's Theorem, f must be constant.***
   - ***Contradiction, since $P(z)$ is non-constant.***
   - ***Hence, $P(z)$ has at least one root in $\mathbb{C}$.***

4. ***Consequences: Repeating the argument allows factoring any polynomial completely into linear factors over $\mathbb{C}$:***

   $$P(z) = a_n(z - z_1)(z - z_2)\ldots(z - z_n).$$

5. ***This theorem is a direct consequence of Liouville's Theorem: if a polynomial had no roots, $1/P(z)$ would be entire and bounded, leading to a contradiction.***

4.6.7 Rouché's Theorem

Theorem 4.16 ***Let f and g be holomorphic on a domain containing a simple closed contour C and its interior. If***

$$|g(z)| < |f(z)| \quad \textbf{\textit{for all}}\ z \in C,$$

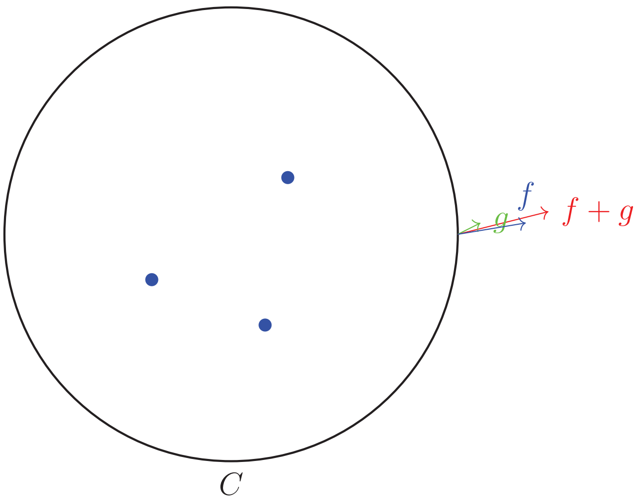
then f and $f+g$ have the same number of zeros inside C, counting multiplicities.

- f is the dominant function on the contour C.
- g is a small addition compared to f.
- The zeros of $f+g$ cannot leave the contour C, so the number of zeros inside remains the same as for f.

Example 4.24 ***Soit***

$$f(z) = z^3, \quad g(z) = 1, \quad C : |z| = 2.$$

- ***On the circle:*** $|f(z)| = |z|^3 = 8$, $|g(z)| = 1 < 8$.
- ***Therefore, f dominates g on the contour.***
- ***Conclusion:*** $f+g = z^3+1$ ***has the same number of zeros inside C as $f = z^3$, that is, 3 zeros.***



Explanation of the figure: The circle represents the contour C. The blue points are the zeros inside. The arrows show the dominant function f and the small addition g. Since f dominates g, the zeros remain inside.

Example 4.25 ***Let***

$$f(z) = z^5 + 3z^2 + 2.$$

We want to determine the number of zeros of $f(z)$ inside the unit circle

$$C : |z| = 1.$$

Step 1. Split $f(z)$ into two parts

We write

$$f(z) = g(z) + h(z),$$

where

$$g(z) = 3z^2, \qquad h(z) = z^5 + 2.$$

Step 2. Check Rouché's condition on $C : |z| = 1$

On the circle $|z| = 1$:

$$|g(z)| = |3z^2| = 3|z|^2 = 3.$$

and

$$|h(z)| = |z^5 + 2| \leq |z^5| + |2| = 1 + 2 = 3.$$

We need a strict inequality $|h(z)| < |g(z)|$. However, this is not true on all of $|z| = 1$, so let us choose another dominant part.

Step 3. Try a different splitting

Take instead:

$$g(z) = z^5, \qquad h(z) = 3z^2 + 2.$$

Then on $|z| = 1$:

$$|g(z)| = |z^5| = 1, \quad |h(z)| = |3z^2 + 2| \leq 3 + 2 = 5.$$

Here $|h(z)| > |g(z)|$, so we swap the roles: set $g(z) = 3z^2 + 2$ and $h(z) = z^5$.

Now:

$$|g(z)| \geq |3z^2| - |2| = 3 - 2 = 1, \quad |h(z)| = |z^5| = 1.$$

The inequality $|h(z)| < |g(z)|$ is true except possibly at a few points, so by Rouché's theorem, $f(z)$ and $g(z)$ have the same number of zeros inside $|z| = 1$.

Step 4. Count zeros of $g(z) = 3z^2 + 2$

We solve $3z^2 + 2 = 0$:

$$z^2 = -\frac{2}{3}, \quad z = \pm i\sqrt{\frac{2}{3}}.$$

Each of these satisfies

$$|z| = \sqrt{\frac{2}{3}} < 1.$$

Hence $g(z)$ has two zeros inside the unit circle.

Step 5. Conclusion

By Rouché's theorem, $f(z) = z^5 + 3z^2 + 2$ also has exactly two zeros inside $|z| = 1$.

$$\boxed{\textbf{Number of zeros of } f(z) \textbf{ inside } |z| = 1 \textbf{ is } 2.}$$

4.7 Solved Exercises

Exercise 4.1 ***Calculate***

$$\oint_C \frac{5z+7}{z^2-2z-3}\,dz, \qquad C:|z-2|=2,$$

using the result for $\oint 1/(z-a)\,dz$***.***

Step 1: Factor the denominator

$$z^2-2z-3=(z-3)(z+1).$$

Step 2: Partial fraction decomposition

$$\frac{5z+7}{(z-3)(z+1)}=\frac{A}{z-3}+\frac{B}{z+1}.$$

Solving for A ***and*** B***:***

$$A=\frac{11}{2}, \qquad B=-\frac{1}{2}.$$

So

$$\frac{5z+7}{(z-3)(z+1)}=\frac{11/2}{z-3}-\frac{1/2}{z+1}.$$

Step 3: Split the integral

$$\oint_C \frac{5z+7}{(z-3)(z+1)}\,dz=\frac{11}{2}\oint_C \frac{1}{z-3}\,dz-\frac{1}{2}\oint_C \frac{1}{z+1}\,dz.$$

Step 4: Determine which poles are inside C

- $z=3$***:*** $|3-2|=1<2$. ***(inside)***

Then

$$\oint_C \frac{1}{z-3}\,dz=2\pi i$$

- $z=-1$***:*** $|-1-2|=3>2$. ***(outside)***

Then

$$\oint_C \frac{1}{z+1}\,dz=0.$$

Step 5: Compute the integral

$$\oint_C \frac{5z+7}{z^2-2z-3}\,dz=\frac{11}{2}\cdot 2\pi i-\frac{1}{2}\cdot 0=11\pi i.$$

$$\boxed{\oint_C \frac{5z+7}{z^2-2z-3}\,dz=11\pi i}.$$

Exercise 4.2 (Complex integral using Cauchy-Goursat theorem) ***Evaluate***

$$\oint_C \frac{dz}{z^2+1}, \qquad C:|z|=4$$

using the Cauchy-Goursat theorem for multiply connected domains.

Step 1: Identify singularities.
The function is

$$f(z) = \frac{1}{z^2+1} = \frac{1}{(z-i)(z+i)}.$$

The singularities (poles) are

$$z = i \quad \textbf{\textit{and}} \quad z = -i.$$

Step 2: Define contours as sets.
- Outer contour (outer boundary):

$$\gamma_0 = \{z \in \mathbb{C} \mid |z| = 4\}$$

- Inner contour around $z = i$ (first singularity):

$$\gamma_1 = \{z \in \mathbb{C} \mid |z-i| = r\}, \quad r > 0 \textbf{\textit{ small enough}}$$

- Inner contour around $z = -i$ (second singularity):

$$\gamma_2 = \{z \in \mathbb{C} \mid |z+i| = r\}, \quad r > 0 \textbf{\textit{ small enough}}$$

Remark: Choosing the radius r for inner contours.
We can take any small positive r for the inner contours. The only requirements are:
1. $r > 0$ so that the contour actually surrounds the singularity.
2. r is small enough that the inner contour does not enclose any other singularities (here, each contour should enclose only one singularity).
So for our example:

$$\gamma_1 = \{z \in \mathbb{C} \mid |z-i| = r\}, \quad \gamma_2 = \{z \in \mathbb{C} \mid |z+i| = r\}, \quad r > 0 \textbf{\textit{ small enough.}}$$

Any r satisfying $0 < r < 2$ works (since the distance between the singularities is 2). The exact value of r does not affect the integral, because of Cauchy's theorem: the integral around a simple pole depends only on whether the pole is inside, not on the radius of the contour.
Step 3: Parameterize the contours.

$$\gamma_0(t) = 4e^{it}, \quad \gamma_1(t) = i + re^{it}, \quad \gamma_2(t) = -i + re^{it}, \quad t \in [0, 2\pi]$$

$$dz = iRe^{it}dt \quad \textbf{\textit{for each contour with radius }} R$$

Step 4: Partial fraction decomposition.

$$\frac{1}{z^2+1} = -\frac{i}{2}\frac{1}{z-i} + \frac{i}{2}\frac{1}{z+i}$$

Step 5: Integrate around the inner contours explicitly.

Contour γ_1 around $z = i$:

$$\oint_{\gamma_1} f(z)dz = \oint_{\gamma_1} -\frac{i}{2}\frac{1}{z-i}dz + \oint_{\gamma_1} \frac{i}{2}\frac{1}{z+i}dz$$

- *First term:*

$$\oint_{\gamma_1} -\frac{i}{2}\frac{1}{z-i}dz = -\frac{i}{2}\int_0^{2\pi} \frac{1}{re^{it}} ire^{it}dt = -\frac{i}{2} \cdot i \cdot 2\pi = \pi$$

- *Second term (holomorphic inside γ_1):*

$$\oint_{\gamma_1} \frac{i}{2}\frac{1}{z+i}dz \approx 0$$

Total:

$$\oint_{\gamma_1} f(z)dz = \pi$$

Contour γ_2 around $z = -i$:

$$\oint_{\gamma_2} f(z)dz = \oint_{\gamma_2} -\frac{i}{2}\frac{1}{z-i}dz + \oint_{\gamma_2} \frac{i}{2}\frac{1}{z+i}dz$$

- *First term (holomorphic inside γ_2):*

$$\oint_{\gamma_2} -\frac{i}{2}\frac{1}{z-i}dz \approx 0$$

- *Second term:*

$$\oint_{\gamma_2} \frac{i}{2}\frac{1}{z+i}dz = \frac{i}{2}\int_0^{2\pi} \frac{1}{re^{it}} ire^{it}dt = -\pi$$

Total:

$$\oint_{\gamma_2} f(z)dz = -\pi$$

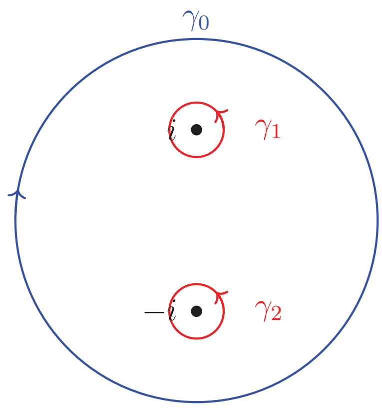
Step 6: Apply the Cauchy-Goursat theorem for multiply connected domains.

$$\oint_{\gamma_0} f(z)dz = \oint_{\gamma_1} f(z)dz + \oint_{\gamma_2} f(z)dz = \pi + (-\pi) = 0$$

Conclusion:

$$\boxed{\oint_C \frac{dz}{z^2+1} = 0}$$

This confirms that the integral along the outer contour equals the sum of integrals around the inner contours, as predicted by the Cauchy-Goursat theorem for multiply connected domains.



Exercise 4.3 ***Let $f : \mathbb{C} \to \mathbb{C}$ be defined by***

$$f(z) = \begin{cases} \dfrac{z^2}{|z|^2}, & z \neq 0, \\ 0, & z = 0. \end{cases}$$

1. ***Show that f is continuous at every point $z \neq 0$.***
2. ***Consider the circle $\gamma(t) = e^{it}$, $t \in [0, 2\pi]$. Compute***

$$\oint_\gamma f(z)\, dz.$$

3. ***Using Morera's theorem, determine whether f is holomorphic at $z = 0$.***

Solution 4.2 Step 1: Continuity for $z \neq 0$
For $z \neq 0$,

$$f(z) = \frac{z^2}{|z|^2} = \frac{z^2}{z\overline{z}} = \frac{z}{\overline{z}}.$$

Both z and $\overline{z}$ are continuous for $z \neq 0$, so $f(z)$ is continuous on $\mathbb{C} \setminus \{0\}$.
Step 2: Compute the integral along γ
Take $\gamma(t) = e^{it}$, $t \in [0, 2\pi]$. Then $dz = ie^{it}dt$ and

$$f(\gamma(t)) = \frac{\gamma(t)^2}{|\gamma(t)|^2} = \frac{e^{2it}}{1} = e^{2it}.$$

Thus, the integral becomes:

$$\oint_\gamma f(z)\, dz = \int_0^{2\pi} f(\gamma(t))\, dz = \int_0^{2\pi} e^{2it} ie^{it} dt = i \int_0^{2\pi} e^{3it} dt.$$

But

$$\int_0^{2\pi} e^{3it} dt = \frac{1}{3}\left[e^{3it}\right]_0^{2\pi} = \frac{1}{3}(e^{6\pi i} - 1) = 0.$$

Hence,

$$\oint_\gamma f(z)\, dz = 0.$$

Step 3: Apply Morera's theorem

Morera's theorem states that if f ***is continuous on a domain*** D ***and*** $\oint_\gamma f(z)\,dz = 0$ ***for every closed contour*** $\gamma \subset D$***, then*** f ***is holomorphic on*** D***.***

- f ***is continuous on*** $\mathbb{C} \setminus \{0\}$***, and the integral around any small circle not enclosing other singularities is zero. - Therefore,*** f ***is holomorphic on*** $\mathbb{C} \setminus \{0\}$***.***

Step 4: Check holomorphy at $z = 0$

We examine the derivative at $z = 0$***:***

$$\lim_{z \to 0} \frac{f(z) - f(0)}{z - 0} = \lim_{z \to 0} \frac{f(z)}{z} = \lim_{z \to 0} \frac{z/\overline{z}}{z} = \lim_{z \to 0} \frac{1}{\overline{z}}.$$

The limit depends on the path:

- Along $z = x$ ***(real axis):*** $\frac{1}{\overline{z}} = \frac{1}{x} \to \infty$***.***

- Along $z = iy$ ***(imaginary axis):*** $\frac{1}{\overline{z}} = \frac{1}{-iy} = \frac{i}{y} \to \infty$***.***

Hence, the limit does not exist, and f ***is not holomorphic at*** 0***.***

Example 4.26 ***Using Cauchy's theorem, calculate***

$$\oint_C \frac{z}{z^2 + 9}\, dz, \quad C : |z - 2i| = 4$$

Step 1: Factor the denominator and find singularities.

We have

$$z^2 + 9 = (z + 3i)(z - 3i).$$

Hence, the singularities (poles) are

$$z_1 = 3i, \quad z_2 = -3i.$$

Step 2: Determine which singularities are inside C***.***

The contour is centered at $2i$ ***with radius*** 4***:***

$$C = \{z \in \mathbb{C} \mid |z - 2i| = 4\}.$$

The distances are:

$$|3i - 2i| = 1 < 4 \quad \textbf{\textit{inside,}} \qquad |-3i - 2i| = 5 > 4 \quad \textbf{\textit{outside.}}$$

So only $z_1 = 3i$ ***is inside*** C***.***

Step 3: Partial fraction decomposition.

$$\frac{z}{z^2 + 9} = \frac{z}{(z - 3i)(z + 3i)} = \frac{A}{z - 3i} + \frac{B}{z + 3i}.$$

Multiply both sides by $(z - 3i)(z + 3i)$***:***

$$z = A(z + 3i) + B(z - 3i)$$

Set $z = 3i$***:***

$$3i = A(6i) \implies A = \frac{1}{2}$$

Set $z = -3i$***:***

$$-3i = B(-6i) \implies B = \frac{1}{2}$$

Hence:

$$\frac{z}{z^2+9} = \frac{1/2}{z-3i} + \frac{1/2}{z+3i}.$$

Step 4: Integrate along C***.***
Only the singularity inside C ***contributes:***

$$\oint_C \frac{z}{z^2+9}dz = \oint_C \frac{1/2}{z-3i}dz + \oint_C \frac{1/2}{z+3i}dz = \oint_C \frac{1/2}{z-3i}dz + 0.$$

Step 5: Apply Cauchy's integral formula.
For a simple pole z_0 ***inside*** C***:***

$$\oint_C \frac{dz}{z-z_0} = 2\pi i.$$

Hence:

$$\oint_C \frac{1/2}{z-3i}dz = \frac{1}{2} \cdot 2\pi i = \pi i.$$

Step 6: Final answer.

$$\boxed{\oint_C \frac{z}{z^2+9}dz = \pi i}$$

Second Method: Using Cauchy's Integral Formula

Consider the integral

$$\oint_C \frac{z}{z^2+9}\,dz, \quad C : |z-2i| = 4$$

where C ***is oriented counterclockwise.***
Step 1: Factor the denominator and identify singularities.

$$z^2 + 9 = (z-3i)(z+3i)$$

The singularities are

$$z_1 = 3i \quad \textit{(inside } C\textit{)}, \qquad z_2 = -3i \quad \textit{(outside } C\textit{)}.$$

Step 2: Rewrite the integrand in the form $\frac{f(z)}{z-z_0}$***.***

$$\frac{z}{z^2+9} = \frac{z}{(z-3i)(z+3i)} = \frac{\frac{z}{z+3i}}{z-3i}.$$

Let $z_0 = 3i$ ***(the singularity inside*** C***) and define***

$$f(z) = \frac{z}{z+3i}.$$

Notice that $f(z)$ ***is holomorphic inside and on*** C ***because its only singularity*** $z = -3i$ ***lies outside*** C***.***

Step 3: Apply Cauchy's integral formula.

Cauchy's integral formula states:

$$\oint_C \frac{f(z)}{z - z_0} dz = 2\pi i\, f(z_0)$$

Step 4: Evaluate $f(z_0)$***.***

$$f(3i) = \frac{3i}{3i + 3i} = \frac{3i}{6i} = \frac{1}{2}$$

Step 5: Compute the integral.

$$\oint_C \frac{z}{z^2 + 9} dz = 2\pi i \cdot \frac{1}{2} = \pi i$$

Conclusion:

Using this second method with Cauchy's integral formula and the holomorphic function

$$f(z) = \frac{z}{z + 3i},$$

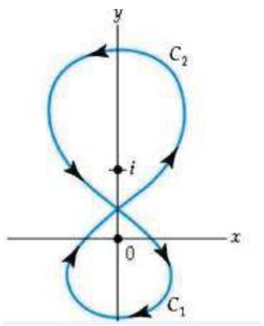
we find

$$\boxed{\oint_C \frac{z}{z^2 + 9} dz = \pi i}.$$

Exercise 4.4 ***Evaluate the integral***

$$\oint_C \frac{z^3 + 3}{z(z - i)^2} dz$$

where $C = C_1 + C_2$.



Part 2: Power Series and Laurent Series

4.8 Series in the Complex Plane

4.8.1 Sequence of Complex Numbers

Definition 4.16 (Sequence of Complex Numbers) ***A complex sequence*** $\{z_n\}$ ***is a correspondence that associates to each positive integer*** $n \in \mathbb{N} = \{1, 2, 3, \ldots\}$ ***a unique complex value*** $z_n \in \mathbb{C}$***. In other words, a sequence is simply a list of complex numbers ordered by their indices.***

Example. Consider the sequence defined by

$$z_n = 1 + in.$$

Its first few terms are:

$$z_1 = 1 + i, \quad z_2 = 1 + 2i, \quad z_3 = 1 + 3i, \quad z_4 = 1 + 4i, \ \ldots$$

Each term lies on a vertical line parallel to the imaginary axis in the complex plane.

We say that a sequence $\{z_n\}$ ***converges to a complex number*** L ***if the limit***

$$\lim_{n \to \infty} z_n = L$$

exists. In that case, L ***is called the limit of the sequence.***

Proposition 4.2 (Convergence of a Complex Sequence) ***A sequence of complex numbers*** $\{z_n\}$ ***converges to a complex limit***

$$L = a + ib$$

if and only if the real parts and imaginary parts converge separately, that is,

$$\Re(z_n) \to a \quad \textbf{\textit{and}} \quad \Im(z_n) \to b \quad \textbf{\textit{as}}\ n \to \infty.$$

Example 4.27 (Convergence of a Complex Sequence) ***Determine whether the sequence***

$$z_n = \frac{3 + ni}{n + 2ni}, \quad n \in \mathbb{N},$$

converges as $n \to \infty$. If it does, find its limit.

Solution: We multiply the numerator and the denominator by the conjugate of the denominator:

$$z_n = \frac{(3+ni)(n-2ni)}{(n+2ni)(n-2ni)} = \frac{(3+ni)(n-2ni)}{n^2+4n^2}.$$

Expanding the numerator gives

$$(3+ni)(n-2ni) = 3n - 6ni + n^2 i - 2n^2 i^2 = (2n^2+3n) + i(n^2-6n).$$

Hence,

$$z_n = \frac{2n^2+3n}{5n^2} + i\,\frac{n^2-6n}{5n^2}.$$

Now, we identify the real and imaginary parts:

$$\Re(z_n) = \frac{2n^2+3n}{5n^2} = \frac{2}{5} + \frac{3}{5n} \longrightarrow \frac{2}{5},$$

$$\Im(z_n) = \frac{n^2-6n}{5n^2} = \frac{1}{5} - \frac{6}{5n} \longrightarrow \frac{1}{5},$$

as $n \to \infty$.

Conclusion: From Theorem 6.1, since both the real and imaginary parts converge, the sequence (z_n) converges in $\mathbb{C}$ to

$$a + ib = \frac{2}{5} + \frac{i}{5}.$$

$$\boxed{\lim_{n\to\infty} z_n = \frac{2}{5} + \frac{i}{5}.}$$

4.8.2 Series of Complex Numbers

Definition 4.17 *Let $\{z_k\}_{k=1}^{\infty}$ be a sequence of complex numbers. The expression*

$$\sum_{k=1}^{\infty} z_k = z_1 + z_2 + z_3 + \cdots + z_n + \cdots$$

is called an infinite series of complex terms. We define its partial sums by

$$S_n = \sum_{k=1}^{n} z_k = z_1 + z_2 + z_3 + \cdots + z_n.$$

The series is said to converge if the sequence (S_n) approaches a finite limit L as $n \to \infty$; in that case, we write

$$\sum_{k=1}^{\infty} z_k = L,$$

and we say that the series converges to L, or equivalently, that L is the sum of the series.

Geometric Series

Definition 4.18 ***A geometric series is an infinite series of the form***

$$\sum_{k=1}^{\infty} az^{k-1} = a + az + az^2 + \cdots + az^{n-1} + \cdots \tag{4.1}$$

where a ***and*** z ***are complex numbers.***

The n***th partial sum of this series is given by***

$$S_n = a + az + az^2 + \cdots + az^{n-1} = a\,\frac{1-z^n}{1-z}.$$

If $|z| < 1$***, then*** $z^n \to 0$ ***as*** $n \to \infty$***, and consequently***

$$S_n \longrightarrow \frac{a}{1-z}.$$

Hence, for all complex numbers z ***satisfying*** $|z| < 1$***, the geometric series*** (4.32) ***converges and its sum is***

$$\frac{a}{1-z} = a + az + az^2 + \cdots + az^{n-1} + \cdots\,. \tag{4.2}$$

On the other hand, when $|z| \geq 1$***, the sequence of partial sums*** (S_n) ***does not converge, and therefore the geometric series*** (4.32) ***diverges.***

Remark 4.20 ***Two important cases of geometric series arise for specific values of the parameters*** a ***and*** z***.***

1. ***When*** $a = 1$ ***and*** z ***is any complex number such that*** $|z| < 1$***, the geometric series becomes***

$$\frac{1}{1-z} = 1 + z + z^2 + z^3 + \cdots\,. \tag{4.3}$$

2. ***Similarly, by replacing*** z ***with*** $-z$ ***in*** (4.3)***, we obtain***

$$\frac{1}{1+z} = 1 - z + z^2 - z^3 + \cdots\,, \tag{4.4}$$

which is valid whenever $|z| < 1$***.***

These two identities are special forms of the geometric series (4.32) ***and are often used in analytic computations and power series expansions.***

Remark 4.21 ***For any complex number*** $z \neq 1$***, we have the finite geometric identity***

$$\frac{1-z^n}{1-z} = 1 + z + z^2 + z^3 + \cdots + z^{n-1}. \tag{4.5}$$

If we decompose the left-hand side of (4.5) ***as***

$$\frac{1-z^n}{1-z} = \frac{1}{1-z} - \frac{z^n}{1-z},$$

we can isolate $\frac{1}{1-z}$ ***and obtain the alternative expression***

$$\frac{1}{1-z} = 1 + z + z^2 + z^3 + \cdots + z^{n-1} + \frac{z^n}{1-z}. \tag{4.6}$$

Equation (4.6) ***provides an explicit formula for the remainder term*** $\frac{z^n}{1-z}$***, which tends to zero as*** $n \to \infty$ ***whenever*** $|z| < 1$***.***

Example 4.28 ***Consider the infinite series***

$$\sum_{k=1}^{\infty} \frac{(1+2i)^k}{5^k} = \frac{1+2i}{5} + \frac{(1+2i)^2}{5^2} + \frac{(1+2i)^3}{5^3} + \cdots \tag{4.7}$$

which is a geometric series.

It can be written in the standard form (4.32) ***with***

$$a = \frac{1+2i}{5} \quad \textit{and} \quad z = \frac{1+2i}{5}.$$

To determine whether the series converges, we compute the modulus of z***:***

$$|z| = \frac{|1+2i|}{5} = \frac{\sqrt{1^2+2^2}}{5} = \frac{\sqrt{5}}{5} < 1.$$

Since $|z| < 1$***, the series*** (4.7) ***converges.***

Using the general formula for the sum of a convergent geometric series,

$$\sum_{k=1}^{\infty} a z^{k-1} = \frac{a}{1-z},$$

we obtain

$$\sum_{k=1}^{\infty} \frac{(1+2i)^k}{5^k} = \frac{\frac{1+2i}{5}}{1 - \frac{1+2i}{5}} = \frac{1+2i}{5-(1+2i)} = \frac{1+2i}{4-2i}.$$

To simplify this expression, multiply numerator and denominator by the conjugate $4+2i$***:***

$$\frac{1+2i}{4-2i} \cdot \frac{4+2i}{4+2i} = \frac{(1+2i)(4+2i)}{(4-2i)(4+2i)} = \frac{4+2i+8i+4i^2}{16+4} = \frac{4-4+10i}{20} = \frac{i}{2}.$$

Therefore, the geometric series (4.7) ***is convergent and its sum is***

$$\sum_{k=1}^{\infty} \frac{(1+2i)^k}{5^k} = \frac{i}{2}. \tag{4.8}$$

4.8.3 Convergence Criteria

Theorem 4.17 (Necessary Condition for Convergence) ***Let*** $\sum_{k=1}^{\infty} z_k$ ***be an infinite series of complex numbers. If this series converges, then the sequence of its terms tends to zero, that is,***

$$\lim_{k\to\infty} z_k = 0.$$

Remark 4.22 ***This condition is*** **necessary** ***but not*** **sufficient.** ***In other words, even if*** $\lim_{k\to\infty} z_k = 0$***, the series*** $\sum_{k=1}^{\infty} z_k$ ***may still diverge. For example, the harmonic series***

$$\sum_{k=1}^{\infty} \frac{1}{k}$$

diverges, although its general term tends to zero.

Theorem 4.18 (The n**th-Term Test for Divergence)** ***If***

$$\lim_{n\to\infty} z_n \neq 0,$$

then the series

$$\sum_{k=1}^{\infty} z_k$$

diverges.

Remark 4.23 ***This test provides a simple way to detect divergence: if the individual terms of a series do not approach zero, the sequence of partial sums cannot converge. However, the converse is not true- the condition*** $\lim_{n\to\infty} z_n = 0$ ***does*** **not** ***guarantee convergence.***

Theorem 4.19 (Ratio Test for Complex Series) ***Let***

$$\sum_{k=1}^{\infty} z_k$$

be a series whose terms z_k ***are nonzero complex numbers, and suppose that***

$$\lim_{n\to\infty} \left| \frac{z_{n+1}}{z_n} \right| = L.$$

Then:

(i) If $L < 1$***, the series converges absolutely.***

(ii) If $L > 1$ ***or*** $L = \infty$***, the series diverges.***

(iii) If $L = 1$***, the test gives no information (the result is inconclusive).***

Theorem 4.20 (Root Test for Complex Series) ***Let***

$$\sum_{k=1}^{\infty} z_k$$

be a series of complex terms. Assume that the following limit exists:

$$\lim_{n\to\infty} \sqrt[n]{|z_n|} = L.$$

Then:

(i) If $L < 1$***, the series converges absolutely.***

(ii) If $L > 1$ ***or*** $L = \infty$***, the series diverges.***

(iii) If $L = 1$***, the test is inconclusive.***

4.8.4 Complex Polynomials

Definition 4.19 ***A complex polynomial function is a function of the form***

$$P(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n,$$

where $a_0, a_1, \ldots, a_n \in \mathbb{C}$ ***and*** n ***is a non-negative integer.***
The degree of the polynomial is n ***if*** $a_n \neq 0$***.***

Example 4.29

$$P(z) = 3z^2 + 2z + 1 \quad \textit{and} \quad Q(z) = z^3 - iz + 2$$

are complex polynomials of degree 2 ***and*** 3***, respectively.***

Remark 4.24 ***Polynomials are the simplest examples of entire functions, since they are differentiable (and hence holomorphic) everywhere in*** $\mathbb{C}$***.***

4.8.5 From Polynomials to Power Series

Polynomials involve a finite number of terms. It is natural to ask what happens if we allow an infinite number of terms of the form $a_n(z - z_0)^n$.

This leads us to the notion of a power series, which can be viewed as an infinite-degree polynomial.

Definition 4.20 (Complex Power Series) ***A complex power series centered at*** $z_0 \in \mathbb{C}$ ***is an infinite series of the form***

$$\sum_{n=0}^{\infty} a_n (z - z_0)^n,$$

where (a_n) ***is a sequence of complex numbers.***

For each fixed z, the series may converge or diverge. The set of all $z \in \mathbb{C}$ for which the series converges is called its domain of convergence.

4.8.6 Radius and Disk of Convergence

Definition 4.21 (Radius of Convergence) ***There exists a real number*** $R \geq 0$***, possibly*** $R = +\infty$***, such that:***

$$\begin{cases} \textit{The series converges absolutely for all } z \textit{ such that } |z - z_0| < R, \\ \textit{The series diverges for all } z \textit{ such that } |z - z_0| > R. \end{cases}$$

The number R ***is called the radius of convergence, and the disk***

$$D(z_0, R) = \{\, z \in \mathbb{C} : |z - z_0| < R \,\}$$

is called the disk of convergence.

Theorem 4.21 (Radius of Convergence) ***For a general power series***

$$\sum_{k=0}^{\infty} a_k (z - z_0)^k,$$

the limit involved in the ratio test depends only on the coefficients a_k***. Specifically, if***

1. $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = L \neq 0$***, then the radius of convergence is***

$$R = \frac{1}{L}.$$

2. $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = 0$***, then the radius of convergence is*** $R = \infty$***.***

3. $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \infty$***, then the radius of convergence is*** $R = 0$***.***

A similar results follows from the root test using

$$\lim_{n\to\infty} \sqrt[n]{|a_n|}.$$

Theorem 4.22 (Root Test and Radius of Convergence) ***If a power series***

$$\sum_{k=0}^{\infty} a_k (z - z_0)^k$$

satisfies

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = L \neq 0,$$

then the radius of convergence is given by

$$R = \frac{1}{L}.$$

If the limit is $L = 0$***, then*** $R = \infty$***; if*** $L = \infty$***, then*** $R = 0$***.***

Example 4.30 *For the series*

$$\sum_{n=0}^{\infty} \frac{z^n}{n!},$$

we can check that

$$\frac{|a_{n+1}|}{|a_n|} = \frac{|z|}{n+1} \to 0 \quad \textit{when } n \to \infty.$$

This means that the series converges for every value of z in the complex plane.

Therefore, the radius of convergence is infinite:

$$R = +\infty.$$

Because the series converges everywhere, the function

$$f(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!} = e^z$$

is well defined for all complex numbers z.

Each term $\frac{z^n}{n!}$ is a polynomial, so it is easy to see that $f(z)$ is smooth and differentiable everywhere. That is why we say that e^z is an entire function, which means "holomorphic on the whole complex plane."

Example 4.31 *Consider the power series*

$$\sum_{k=1}^{\infty} \frac{z^{k+1}}{k}.$$

By applying the ratio test, we examine the limit

$$\lim_{n\to\infty} \left| \frac{\frac{z^{n+2}}{n+1}}{\frac{z^{n+1}}{n}} \right| = \lim_{n\to\infty} \frac{n}{n+1} |z| = |z|. \tag{4.9}$$

Hence, the series converges absolutely for $|z| < 1$.

The circle of convergence is therefore defined by $|z| = 1$, and the corresponding radius of convergence is $R = 1$.

On the circle $|z| = 1$, the series does not converge absolutely, since

$$\sum_{k=1}^{\infty} \frac{1}{k}$$

is the classical divergent harmonic series. However, the series may converge at some points on the circle. For instance, at $z = -1$,

$$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}$$

is the convergent alternating harmonic series. In fact, this particular series converges for every point on the unit circle except $z = 1$.

4.8.7 Taylor Series

Power Series and Their Associated Functions

There exists a one-to-one correspondence between any complex number z located within the circle of convergence and the value to which the series

$$\sum_{k=0}^{\infty} a_k(z-z_0)^k$$

converges. In this sense, a power series *defines* or *represents* a function f. For each specific z inside the circle of convergence, the number L to which the power series converges is defined as the value of f at z; that is,

$$f(z) = L.$$

In this section, we present some important properties concerning the nature of this function f.

In the previous section, we established that every power series possesses a radius of convergence R. Throughout this discussion, we will assume that the power series

$$\sum_{k=0}^{\infty} a_k(z-z_0)^k$$

has either a positive or an infinite radius of convergence R.

Proposition 4.3 ***Suppose that f has a power series expansion***

$$f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n, \quad z \in D(z_0, R),$$

and let $0 < r < R$. Then we have

$$\sum_{n=0}^{\infty} |a_n| r^n = \frac{1}{2\pi}\int_0^{2\pi} \left| f(z_0 + re^{i\theta}) \right| d\theta.$$

Differentiation and Integration of Power Series

The following three theorems show that a function f, defined by a power series, is continuous, differentiable, and integrable within its circle of convergence.

Theorem 4.23 (Continuity) ***A power series***

$$\sum_{k=0}^{\infty} a_k(z-z_0)^k$$

represents a continuous function f within its circle of convergence $|z-z_0| = R$.

Theorem 4.24 (Term-by-Term Differentiation) ***A power series***

$$\sum_{k=0}^{\infty} a_k(z-z_0)^k$$

can be differentiated term by term within its circle of convergence $|z-z_0|=R$.

Differentiating the series term by term yields:

$$\frac{d}{dz}\left(\sum_{k=0}^{\infty} a_k(z-z_0)^k\right)=\sum_{k=0}^{\infty} a_k\frac{d}{dz}\left((z-z_0)^k\right)=\sum_{k=1}^{\infty} k a_k(z-z_0)^{k-1}.$$

Theorem 4.25 (Term-by-Term Integration) ***A power series***

$$\sum_{k=0}^{\infty} a_k(z-z_0)^k$$

can be integrated term by term within its circle of convergence $|z-z_0|=R$, ***for every contour*** C ***lying entirely inside the circle of convergence.***

The theorem states that

$$\int_C\left(\sum_{k=0}^{\infty} a_k(z-z_0)^k\right)dz=\sum_{k=0}^{\infty} a_k\int_C(z-z_0)^k\,dz=\sum_{k=0}^{\infty}\frac{a_k}{k+1}(z-z_0)^{k+1}+\text{constant},$$

whenever the contour C lies in the interior of the circle of convergence $|z-z_0|=R$. Suppose a power series represents a function f within the circle of convergence $|z-z_0|=R$; that is,

$$f(z)=\sum_{k=0}^{\infty} a_k(z-z_0)^k=a_0+a_1(z-z_0)+a_2(z-z_0)^2+a_3(z-z_0)^3+\cdots. \tag{4.10}$$

Taylor Series

It follows from Theorem 4.24 that the derivatives of f are given by the following series:

$$f'(z)=\sum_{k=1}^{\infty} k a_k(z-z_0)^{k-1}=a_1+2a_2(z-z_0)+3a_3(z-z_0)^2+\cdots. \tag{4.11}$$

$$f''(z)=\sum_{k=2}^{\infty} k(k-1)a_k(z-z_0)^{k-2}=2\cdot 1\,a_2+3\cdot 2\,a_3(z-z_0)+\cdots. \tag{4.12}$$

$$f^{(3)}(z)=\sum_{k=3}^{\infty} k(k-1)(k-2)a_k(z-z_0)^{k-3}=3\cdot 2\cdot 1\,a_3+\cdots. \tag{4.13}$$

There is a relationship between the coefficients a_k in (4.10) and the derivatives of f. Evaluating (4.10), (4.11), (4.12), and (4.13) at $z=z_0$ gives:

$$f(z_0)=a_0,\quad f'(z_0)=1!\,a_1,\quad f''(z_0)=2!\,a_2,\quad f^{(3)}(z_0)=3!\,a_3.$$

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$$f(z_0) = a_0, \quad f'(z_0) = 1!\,a_1, \quad f''(z_0) = 2!\,a_2, \quad f^{(3)}(z_0) = 3!\,a_3,$$

respectively.
In general,

$$f^{(n)}(z_0) = n!\,a_n, \quad \textbf{or equivalently,} \quad a_n = \frac{f^{(n)}(z_0)}{n!}, \quad n \geq 0. \tag{4.14}$$

When $n = 0$ in (4.14), we interpret the zero-order derivative as $f(z_0)$ and note that $0! = 1$, so that the formula gives $a_0 = f(z_0)$.
Substituting (4.14) into (4.10), we obtain

$$f(z) = \sum_{k=0}^{\infty} \frac{f^{(k)}(z_0)}{k!}(z - z_0)^k. \tag{4.15}$$

The series given in (4.15) is called the Taylor series for f centered at z_0. A Taylor series with center $z_0 = 0$ is referred to as a Maclaurin series, and is given by

$$f(z) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!}\, z^k. \tag{4.16}$$

If we are given a function f that is analytic in some domain D, can we represent it by a power series of the form (4.15) or (4.16)?

Theorem 4.26 (Taylor's Theorem) ***Let f be analytic within a domain D, and let z_0 be a point in D. Then f has the series representation***

$$f(z) = \sum_{k=0}^{\infty} \frac{f^{(k)}(z_0)}{k!}(z - z_0)^k, \tag{4.17}$$

which is valid for the largest circle C with center at z_0 and radius R that lies entirely within D.

Example 4.32 (Some Important Maclaurin Series) ***The following are some of the most common Maclaurin series expansions:***

$$e^z = 1 + \frac{z}{1!} + \frac{z^2}{2!} + \cdots = \sum_{k=0}^{\infty} \frac{z^k}{k!}. \tag{4.18}$$

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots = \sum_{k=0}^{\infty} (-1)^k \frac{z^{2k+1}}{(2k+1)!}. \tag{4.19}$$

$$\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \cdots = \sum_{k=0}^{\infty} (-1)^k \frac{z^{2k}}{(2k)!}. \tag{4.20}$$

Example 4.33 ***We aim to find the Maclaurin expansion of*** $f(z) = \dfrac{1}{(1-z)^2}$.

1. Find the Maclaurin expansion of $g(z) = \dfrac{1}{1-z}$,

2. Find the Maclaurin expansion of $\dfrac{d}{dz}\dfrac{1}{1-z}$ ***3- Deduce the Maclaurin expansion of*** $f(z) = \dfrac{1}{(1-z)^2}$.

Solution 4.3 ***For*** $|z| < 1$,

$$\frac{1}{1-z} = 1 + z + z^2 + z^3 + \cdots ,$$

we can differentiate both sides of this expression with respect to z***. Thus,***

$$\frac{d}{dz}\left(\frac{1}{1-z}\right) = \frac{d}{dz}(1) + \frac{d}{dz}(z) + \frac{d}{dz}(z^2) + \frac{d}{dz}(z^3) + \cdots ,$$

which simplifies to

$$\frac{1}{(1-z)^2} = 0 + 1 + 2z + 3z^2 + \cdots = \sum_{k=1}^{\infty} kz^{k-1}.$$

Motivating Question

We know that if a complex function $f(z)$ is analytic in a neighborhood of a point z_0, it can be expanded into a Taylor series about z_0.

But what happens if $f(z)$ is not analytic at z_0, even though it is analytic in some region around this point?

Can we still represent $f(z)$ by a series expansion similar to the Taylor series?

The answer is yes. In this situation, we use the ***Laurent series***, which generalizes the Taylor expansion by allowing both positive and negative powers of $(z - z_0)$.

4.9 Laurent Series

4.9.1 Singular Point

Definition 4.22 ***If a complex function*** f ***fails to be analytic at a point*** $z = z_0$***, then this point is called a singularity or singular point of the function. For instance, the complex numbers*** $z = 2i$ ***and*** $z = -2i$ ***are singularities of the function***

$$f(z) = \frac{z}{z^2 + 4},$$

because f ***is discontinuous at each of these points.***

In this section, we introduce a new type of "power series" expansion of f about an ***isolated singularity*** z_0. This new expansion, called the ***Laurent series***, involves both negative and nonnegative integer powers of $(z - z_0)$.

4.9.2 Laurent Series Expansion

Theorem 4.27 ***Let f be a complex function that is analytic in an annular region***

$$R_1 < |z - z_0| < R_2,$$

where R_1 and R_2 are real numbers such that $0 \leq R_1 < R_2 \leq \infty$. Then f can be represented by a Laurent series about the point z_0:

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n,$$

where the coefficients are given by

$$a_n = \frac{1}{2\pi i} \oint_C \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} \, d\zeta,$$

and C is a positively oriented simple closed contour contained in the annulus.

Remark 4.25

- ***The terms with $n \geq 0$ form the analytic part of the series.***
- ***The terms with $n < 0$ form the principal part.***
- ***If the principal part contains only finitely many terms, the singularity is called a pole.***
- ***If infinitely many negative powers appear, the singularity is called an essential singularity.***

Example 4.34 ***Consider the complex function***

$$f(z) = \frac{8z + 1}{z(z - 1)}$$

and the region $0 < |z| < 1$.

1. ***Find the Laurent series of $f(z)$ centered at $z_0 = 0$.***
2. ***Identify the principal part and the analytic part.***
3. ***Illustrate the region of convergence and indicate the principal and analytic parts on a diagram.***

Solution 4.4 ***Step 1: Partial fraction decomposition.***

$$f(z) = \frac{8z + 1}{z(z - 1)} = \frac{A}{z} + \frac{B}{z - 1}.$$

Multiplying both sides by $z(z - 1)$:

$$8z + 1 = A(z - 1) + Bz.$$

Compare coefficients:

$$A = -1, \quad B = 9$$

Hence:

$$f(z) = -\frac{1}{z} + \frac{9}{z-1}.$$

Step 2: Expand $\frac{9}{z-1}$ ***for*** $|z| < 1$ ***using a geometric series.***

$$\frac{1}{z-1} = -\frac{1}{1-z} = -\sum_{n=0}^{\infty} z^n \quad \Rightarrow \quad \frac{9}{z-1} = -9\sum_{n=0}^{\infty} z^n.$$

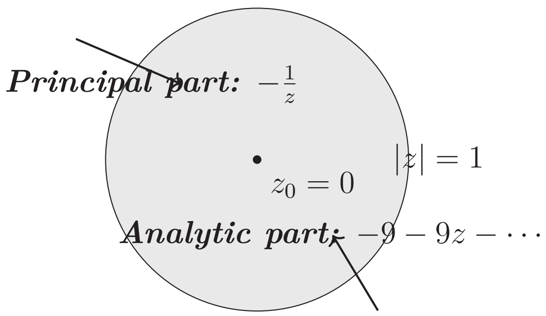
Step 3: Combine terms to obtain the Laurent series.

$$f(z) = -\frac{1}{z} - 9 - 9z - 9z^2 - \cdots = -\frac{1}{z} - \sum_{n=0}^{\infty} 9z^n$$

Step 4: Identify principal and analytic parts.

- ***Principal part (negative powers):*** $-\frac{1}{z}$
- ***Analytic part (non-negative powers):*** $-9 - 9z - 9z^2 - \cdots = -\sum_{n=0}^{\infty} 9z^n$

Step 5: Diagram of the region and series parts.



Conclusion:
The Laurent series of $f(z)$ ***around*** $z_0 = 0$***, valid for*** $0 < |z| < 1$***, is***

$$\boxed{f(z) = -\frac{1}{z} - 9 - 9z - 9z^2 - \cdots = -\frac{1}{z} - \sum_{n=0}^{\infty} 9z^n}$$

with the principal part consisting of $-\frac{1}{z}$ ***and the analytic part*** $-9-9z-9z^2-\cdots$***.***

4.9.3 Isolated Singular Points

Definition 4.23 ***Let $f(z)$ be a complex function. A point $z_0 \in \mathbb{C}$ is called a*** **singular point** ***or*** **singularity** ***of f if f is not analytic at z_0, but is analytic at some point arbitrarily close to z_0.***

A singularity z_0 is said to be an **isolated singularity** ***if there exists a small radius $r > 0$ such that $f(z)$ is analytic for all z in the punctured disk***

$$0 < |z - z_0| < r.$$

Example 4.35 ***1. The function***

$$f(z) = \frac{1}{z}$$

has a singularity at $z_0 = 0$. For any small disk around 0, $f(z)$ is analytic for all $z \neq 0$. Hence $z_0 = 0$ is an isolated singularity.

2. The function

$$f(z) = \frac{1}{z^2 + 1} = \frac{1}{(z - i)(z + i)}$$

has singularities at $z = i$ and $z = -i$. Each of these points is isolated because the function is analytic in a neighborhood around each singularity, excluding the singular point itself.

3. The logarithm function

$$f(z) = \ln z$$

has a branch point at $z_0 = 0$, which is also considered a singularity. It is isolated if we consider a domain that avoids the branch cut along the negative real axis.

Classification of Isolated Singular Points

- An isolated singularity allows the function to be expanded in a Laurent series in a punctured disk around the singular point.
- If singular points accumulate (i.e., there is no punctured disk where the function is analytic), the singularity is not isolated.
- An isolated singular point $z = z_0$ of a complex function f is classified according to the principal part of its Laurent expansion. Let the Laurent expansion of f around z_0 be

$$f(z) = \sum_{k=-\infty}^{\infty} a_k (z - z_0)^k,$$

and let the principal part be

$$\sum_{k=1}^{\infty} \frac{a_{-k}}{(z - z_0)^k}.$$

. Isolated singularities are classified into three types:
1. *Removable singularities*
2. *Poles*

3. ***Essential singularities***

This classification is based on the behavior of the function near the singular point and the terms of the Laurent series.

1. **Removable singularity: If the principal part is zero, that is, all coefficients $a_{-k} = 0$, then $z = z_0$ is called a *removable singularity.* In this case, f can be redefined at z_0 to become analytic.**
2. **Pole: If the principal part contains a finite number of nonzero terms, then $z = z_0$ is called a *pole.* If the last nonzero term in the principal part is $a_{-n}/(z - z_0)^n$ with $n \geq 1$, then $z = z_0$ is a *pole of order* n. In particular, if $n = 1$, the principal part contains exactly one term with coefficient a_{-1}, and the pole is called a *simple pole.***
3. **Essential singularity: If the principal part contains infinitely many nonzero terms, then $z = z_0$ is called an *essential singularity.***

All these cases can be summarized in the following table.

$z = z_0$	Laurent Series for $0 < \lvert z - z_0\rvert < R$
Removable singularity	$a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + \cdots$
Pole of order n	$\frac{a_{-n}}{(z - z_0)^n} + \frac{a_{-(n-1)}}{(z - z_0)^{n-1}} + \cdots + \frac{a_{-1}}{z - z_0} + a_0 + a_1(z - z_0) + \cdots$
Simple pole	$\frac{a_{-1}}{z - z_0} + a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + \cdots$
Essential singularity	$\cdots + \frac{a_{-2}}{(z - z_0)^2} + \frac{a_{-1}}{z - z_0} + a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + \cdots$

Example 4.36 (Removable Singularity) ***Dividing the Maclaurin series for*** $\sin z$ ***by*** z***, we obtain***

$$\frac{\sin z}{z} = 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \cdots \tag{4.21}$$

From equation (4.40)***, we see that all the coefficients in the principal part of the Laurent series are zero. Hence,***

$$z = 0 \textit{ is a removable singularity of the function } f(z) = \frac{\sin z}{z}. \tag{4.22}$$

Example 4.37 (Poles and Essential Singularity) ***Dividing the terms of the Maclaurin series***

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots \tag{4.23}$$

by z^2 ***gives***

$$\frac{\sin z}{z^2} = \frac{1}{z} - \frac{z}{3!} + \frac{z^3}{5!} - \cdots, \quad 0 < |z| < \infty \tag{4.24}$$

From equation (4.24)***, we see that the coefficient*** $a_{-1} \neq 0$***, and hence***

$$z = 0 \textit{ is a simple pole of the function } f(z) = \frac{\sin z}{z^2}. \tag{4.25}$$

4.9.4 Zeros of a Function

Recall, a number z_0 is a *zero* of a function f if

$$f(z_0) = 0.$$

Definition 4.24 ***We say that an analytic function f has a*** **zero of order n** ***at $z = z_0$ if z_0 is a zero of f and of its first $n-1$ derivatives, that is,***

$$f(z_0) = 0, \quad f'(z_0) = 0, \quad f''(z_0) = 0, \ \ldots, \quad f^{(n-1)}(z_0) = 0, \quad \textbf{\textit{but}} \quad f^{(n)}(z_0) \neq 0. \quad (4.26)$$

A zero of order n is also referred to as a zero of **multiplicity n.**

Example:
For

$$f(z) = (z-5)^3,$$

we have $f(5) = 0, f'(5) = 0, f''(5) = 0, f^{(3)}(5) = 6 \neq 0$.
Thus f has a zero of order (or multiplicity) 3 at $z_0 = 5$.
A zero of order 1 is called a *simple zero*.

Theorem 4.28 (Zero of Order n) ***A function f that is analytic in some disk $|z - z_0| < R$ has a zero of order n at $z = z_0$ if and only if it can be written as***

$$f(z) = (z - z_0)^n \, \phi(z), \quad (4.27)$$

where $\phi(z)$ is analytic at z_0 and $\phi(z_0) \neq 0$.

Theorem 4.29 (Pole of Order n) ***A function f analytic in a punctured disk $0 < |z - z_0| < R$ has a pole of order n at $z = z_0$ if and only if it can be written as***

$$f(z) = \frac{\phi(z)}{(z - z_0)^n}, \quad (4.28)$$

where $\phi(z)$ is analytic at z_0 and $\phi(z_0) \neq 0$.

Definition 4.25 (Zeros Again) ***A zero $z = z_0$ of an analytic function f is*** **isolated** ***in the sense that there exists some neighborhood of z_0 in which***

$$f(z) = 0$$

at every point z except at $z = z_0$ itself.

As a consequence, if z_0 is a zero of a nontrivial analytic function f, then the function

$$\frac{1}{f(z)}$$

has an isolated singularity at the point $z = z_0$.

Theorem 4.30 (Pole of Order n) ***Let g and h be functions analytic at $z = z_0$. If h has a zero of order n at $z = z_0$ and $g(z_0) \neq 0$, then the function***

$$f(z) = \frac{g(z)}{h(z)} \tag{4.29}$$

has a pole of order n at $z = z_0$.

Example 4.38 (Poles of a Rational Function) ***Consider the rational function***

$$f(z) = \frac{2z + 5}{(z - 1)(z + 5)(z - 2)^4}. \tag{4.30}$$

Let us denote the numerator by

$$N(z) = 2z + 5. \tag{4.31}$$

We observe that

$$N(1) = 7 \neq 0, \quad N(-5) = -5 \neq 0, \quad N(2) = 9 \neq 0.$$

Hence, the zeros of the denominator determine the poles of $f(z)$:

- $z = 1$ ***and*** $z = -5$ ***are*** simple poles, ***because the corresponding factors in the denominator are of order 1.***
- $z = 2$ ***is a*** pole of order 4, ***because the factor $(z - 2)^4$ appears in the denominator.***

Definition 4.26 (Holomorphic function with isolated poles) ***A function f defined on a domain $D \subseteq \mathbb{C}$ is said to be holomorphic with isolated poles if***

- ***f is holomorphic on D except at a finite or countable set of points $\{z_1, z_2, \dots\} \subset D$,***
- ***and at each such point z_k, f can be written as***

$$f(z) = \frac{g(z)}{(z - z_k)^{n_k}},$$

where $g(z)$ is holomorphic at z_k and $g(z_k) \neq 0$.

In other words, f is holomorphic everywhere in D except at isolated singularities, and each singularity is a pole of finite order.

Example 4.39 ***The function***

$$f(z) = \frac{1}{z^2 - 1} = \frac{1}{(z - 1)(z + 1)}$$

is holomorphic on $\mathbb{C} \setminus \{-1, 1\}$, and has two isolated poles at $z = 1$ and $z = -1$, with $g(z) = 1$.

Definition 4.27 (Meromorphic Function) ***Let $D \subseteq \mathbb{C}$ be an open domain. A function $f : D \to \mathbb{C}$ is said to be meromorphic on D if f is holomorphic on D except at a discrete set of points $\{z_1, z_2, \dots\} \subset D$, where each z_k is an isolated pole of f.***

Equivalently, around each such point z_k, there exists an integer $m \geq 1$ and a function g holomorphic at z_k with $g(z_k) \neq 0$ such that

$$f(z) = \frac{g(z)}{(z - z_k)^m}.$$

4.9.5 Argument Principle

Theorem 4.31 (Argument Principle) ***Let f be a meromorphic function in a simply connected domain Ω. Let C be a closed contour contained in Ω, enclosing all zeros and poles of f within Ω. Then we have:***

$$\frac{1}{2\pi i} \int_C \frac{f'(z)}{f(z)}\, dz = N(f) - P(f),$$

where $N(f)$ and $P(f)$ denote, respectively, the number of zeros and poles of f inside C (each counted with their multiplicities).

Example 4.40 ***Let***

$$f(z) = z^3 - 1.$$

We will use the Argument Principle to determine the number of zeros of $f(z)$ inside the unit circle

$$C : |z| = 1.$$

Step 1. Compute $\dfrac{f'(z)}{f(z)}$

$$f'(z) = 3z^2, \qquad \frac{f'(z)}{f(z)} = \frac{3z^2}{z^3 - 1}.$$

Step 2. Apply the Argument Principle
According to the theorem,

$$\frac{1}{2\pi i} \int_C \frac{f'(z)}{f(z)}\, dz = N - P,$$

where:

- **N is the number of zeros of $f(z)$ inside C,**
- **P is the number of poles of $f(z)$ inside C.**

Since $f(z)$ is a polynomial, it has no poles ($P = 0$).
Step 3. Evaluate the change of argument
For $|z| = 1$, set $z = e^{i\theta}$ with $0 \leq \theta \leq 2\pi$. Then,

$$f(z) = e^{3i\theta} - 1.$$

As θ goes from 0 to 2π, the term $e^{3i\theta}$ describes three full turns around the origin. Subtracting 1 shifts the circle but does not change the number of turns around the origin. Hence, the image $f(C)$ winds three times around the origin.

$$\Rightarrow \frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)}\,dz = 3.$$

Step 4. Conclusion
Therefore,

$$N - P = 3 \quad \Rightarrow \quad N = 3.$$

$$\boxed{f(z) = z^3 - 1 \text{ has three zeros inside the unit circle } |z| = 1.}$$

Corollary 4.3 ***If f is analytic inside and on a positively oriented simple closed contour C, and if f has no zeros on C, then***

$$\frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)}\,dz = N(f),$$

where $N(f)$ denotes the number of zeros of f inside C (counted with multiplicities).

Remark 4.26 ***The definition of the Argument Principle can also be expressed in terms of the total variation of the argument of $f(z)$ along the contour C:***

$$\frac{1}{2\pi}\Delta_C \arg f(z) = N(f) - P(f),$$

where:

- $\arg f(z)$ ***denotes the argument of $f(z)$, i.e.***

$$f(z) = |f(z)|e^{i\arg f(z)}.$$

- $\Delta_C \arg f(z)$ ***represents the total change in the argument of $f(z)$ as z traverses the contour C.***

Example 4.41 ***Let the function***

$$f(z) = \frac{2z+1}{z}.$$

Determine the number of times the variable $w = f(z)$ winds around the origin when z traverses the unit circle $|z| = 1$ once in the positive (counterclockwise) direction.

Solution 4.5 ***Step 1: Identify zeros and poles.***
- Zeros: $f(z) = 0 \implies 2z+1 = 0 \implies z = -\frac{1}{2}$. Multiplicity: 1 Inside the unit circle: $|-1/2| = 1/2 < 1$.
- Poles: $f(z)$ has a pole at $z = 0$. Multiplicity: 1 Inside the unit circle: $|0| = 0 < 1$.

Step 2: Apply the argument principle.
The argument principle states:

$$\Delta_\Gamma \arg f(z) = 2\pi(N - P),$$

where N ***is the number of zeros inside*** Γ ***and*** P ***is the number of poles inside*** Γ***.***
Here: $N = 1$***,*** $P = 1$***, so***

$$\Delta_\Gamma \arg f(z) = 2\pi(1 - 1) = 0.$$

Step 3: Conclusion.
The number of times $w = f(z)$ ***winds around the origin is***

$$\boxed{0}.$$

Example 4.42 *Let*

$$f(z) = \frac{2z + 1}{z(3z + 1)}.$$

Solution 4.6 ***Step 1: Identify zeros and poles.***
- Zeros: $2z + 1 = 0 \implies z = -\frac{1}{2}$ ***(multiplicity 1) - Poles:*** $z = 0$ ***and*** $3z + 1 = 0 \implies z = -\frac{1}{3}$ ***(multiplicity 1 each)***
Step 2: Check which are inside $|z| < 1$***.***
- Zero: $z = -1/2$***: inside - Poles:*** $z = 0$ ***and*** $z = -1/3$ ***: both inside***
So $N = 1$***,*** $P = 2$***.***
Step 3: Apply the argument principle.

$$\Delta_\Gamma \arg f(z) = 2\pi(N - P) = 2\pi(1 - 2) = -2\pi.$$

Step 4: Conclusion.
The negative sign indicates a clockwise rotation around the origin. Hence, the number of windings of $w = f(z)$ ***around the origin is***

$$\boxed{\frac{\Delta_\Gamma \arg f(z)}{2\pi} = -1}.$$

4.10 Solved Exercises

Exercise 4.5 *Consider the power series*

$$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k!}(z - 1 - i)^k.$$

1. ***Determine the radius of convergence*** R ***of this series using the ratio test.***

2. ***Find the domain of absolute convergence in the complex plane.***

3. ***Identify the analytic function $f(z)$ represented by this power series.***

Solution 4.7 ***Solution. Set $w = z - 1 - i$. The series can be written as***

$$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k!}(z-1-i)^k = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k!} w^k.$$

(1) **Radius of convergence via the ratio test.**
Denote the general coefficient by $a_k = \frac{(-1)^{k+1}}{k!}$. Apply the ratio test:

$$\left|\frac{a_{k+1}}{a_k}\right| = \left|\frac{(-1)^{k+2}/(k+1)!}{(-1)^{k+1}/k!}\right| = \frac{1}{k+1}.$$

Taking the limit as $k \to \infty$ gives

$$\lim_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right| = 0.$$

By the ratio test (or the standard consequence for power series), if this limit is 0 then the radius of convergence is $R = \infty$.

(2) **Domain of absolute convergence.**
Since $R = \infty$, the series converges absolutely for every complex z. In other words, the domain of absolute convergence is the whole complex plane $\mathbb{C}$.

(3) **Identification of the analytic function $f(z)$.**
Recall the Taylor expansion of the exponential:

$$e^t = \sum_{k=0}^{\infty} \frac{t^k}{k!} \qquad (\textbf{\textit{valid for all }} t \in \mathbb{C}).$$

Replace t by $-w = -(z-1-i)$:

$$e^{-w} = \sum_{k=0}^{\infty} \frac{(-w)^k}{k!} = 1 + \sum_{k=1}^{\infty} \frac{(-1)^k}{k!} w^k.$$

Therefore

$$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k!} w^k = -\sum_{k=1}^{\infty} \frac{(-1)^k}{k!} w^k = -(e^{-w} - 1) = 1 - e^{-w}.$$

Returning to z (with $w = z - 1 - i$), we obtain the closed form

$$f(z) = 1 - \exp\Big(-(z-1-i)\Big).$$

Because the exponential is entire, f is entire as well (consistent with the radius $R = \infty$). The equality above holds for all $z \in \mathbb{C}$ by uniqueness of power-series expansion about the center $z_0 = 1 + i$.

***Conclusion.* The radius of convergence is $R = \infty$; the series converges absolutely for every $z \in \mathbb{C}$; and it represents the entire function**

$$\boxed{f(z) = 1 - \exp\Big(-(z-1-i)\Big)}.$$

Exercise 4.6 ***Find the Taylor (Maclaurin) series expansion of the function***

$$f(z) = \frac{1}{1-z}$$

about $z_0 = 0$. Determine the radius of convergence.

Solution 4.8 ***We compute the Maclaurin expansion by observing that for $|z| < 1$ the geometric series converges:***

$$\frac{1}{1-z} = 1 + z + z^2 + z^3 + \cdots = \sum_{k=0}^{\infty} z^k. \tag{4.32}$$

One may also obtain the same result by differentiating the partial sums or by using the Taylor coefficient formula

$$a_k = \frac{f^{(k)}(0)}{k!}.$$

Indeed, $f^{(k)}(z) = k!(1-z)^{-k-1}$, hence $f^{(k)}(0) = k!$ and

$$a_k = \frac{f^{(k)}(0)}{k!} = \frac{k!}{k!} = 1, \qquad k \geq 0, \tag{4.33}$$

which gives the series in (4.32).

Finally, since the geometric series converges for $|z| < 1$, the radius of convergence is

$$R = 1. \tag{4.34}$$

Thus the Maclaurin series of $f(z) = \dfrac{1}{1-z}$ is $\sum_{k=0}^{\infty} z^k$ with radius of convergence $R = 1$.

Exercise 4.7 ***Find the Taylor series of***

$$f(z) = \frac{1}{1+z^2}$$

about a general center z_0 (with $z_0 \neq \pm i$). Determine the coefficients of the expansion and the radius of convergence.

Solution 4.9 ***Write f by partial fractions using the factorization $1 + z^2 = (z - i)(z+i)$:***

$$\frac{1}{1+z^2} = \frac{1}{2i}\left(\frac{1}{z-i} - \frac{1}{z+i}\right).$$

Fix a center z_0 ***with*** $z_0 \neq \pm i$***. For each simple pole*** $a \in \{i, -i\}$ ***we use the geometric-type expansion***

$$\frac{1}{z-a} = \frac{1}{z_0 - a} \cdot \frac{1}{1 - \dfrac{z - z_0}{z_0 - a}} = \sum_{k=0}^{\infty} \frac{(z - z_0)^k}{(z_0 - a)^{k+1}},$$

which is valid when $\left|\dfrac{z - z_0}{z_0 - a}\right| < 1$***, i.e.*** $|z - z_0| < |z_0 - a|$***.***

Applying this to $a = i$ ***and*** $a = -i$ ***and combining gives the Taylor expansion about*** z_0***:***

$$\boxed{\frac{1}{1+z^2} = \sum_{k=0}^{\infty} a_k (z - z_0)^k} \tag{4.35}$$

with the coefficients

$$a_k \;=\; \frac{1}{2i}\left(\frac{1}{(z_0 - i)^{k+1}} - \frac{1}{(z_0 + i)^{k+1}}\right), \qquad k \geq 0. \tag{4.36}$$

The region of validity is determined by the nearest singularity to z_0***. The series*** (4.35) ***converges for***

$$|z - z_0| < R, \qquad R = \min\{|z_0 - i|,\ |z_0 + i|\}. \tag{4.37}$$

Special case. ***For the Maclaurin expansion (***$z_0 = 0$***) the formulas simplify. From*** (4.36) ***with*** $z_0 = 0$ ***we obtain***

$$a_k = \frac{1}{2i}\left(\frac{1}{(-i)^{k+1}} - \frac{1}{(i)^{k+1}}\right).$$

After simplification one gets the well-known Maclaurin series (valid for $|z| < 1$***):***

$$\frac{1}{1+z^2} = \sum_{m=0}^{\infty} (-1)^m z^{2m}. \tag{4.38}$$

(Only even powers appear; coefficients of odd powers are zero.)

Partant de la formule générale obtenue précédemment,

$$a_k = \frac{1}{2i}\left(\frac{1}{(z_0 - i)^{k+1}} - \frac{1}{(z_0 + i)^{k+1}}\right),$$

prenons $z_0 = 1$**. Alors** $1 - i = \sqrt{2}\, e^{-i\pi/4}$ **et** $1 + i = \sqrt{2}\, e^{i\pi/4}$**. D'où**

$$\frac{1}{(1-i)^{k+1}} - \frac{1}{(1+i)^{k+1}} = (\sqrt{2})^{-(k+1)}\Big(e^{i(k+1)\pi/4} - e^{-i(k+1)\pi/4}\Big) = 2i\,(\sqrt{2})^{-(k+1)} \sin\Big((k+1)\tfrac{\pi}{4}\Big).$$

En multipliant par $1/(2i)$ **on obtient la forme réelle et simple des coefficients :**

$$\boxed{a_k \;=\; 2^{-\frac{k+1}{2}} \sin\Big((k+1)\tfrac{\pi}{4}\Big), \qquad k \geq 0.}$$

Ainsi la série de Taylor de $\dfrac{1}{1+z^2}$ autour de $z_0 = 1$ s'écrit

$$\frac{1}{1+z^2} = \sum_{k=0}^{\infty} a_k (z-1)^k, \qquad a_k = 2^{-\frac{k+1}{2}} \sin\Big((k+1)\tfrac{\pi}{4}\Big).$$

Les premiers coefficients (et premiers termes) sont :

$$\begin{aligned} a_0 &= 2^{-1/2} \sin\tfrac{\pi}{4} = \tfrac{1}{2}, \\ a_1 &= 2^{-1} \sin\tfrac{\pi}{2} = \tfrac{1}{2}, \\ a_2 &= 2^{-3/2} \sin\tfrac{3\pi}{4} = \tfrac{1}{4}, \\ a_3 &= 2^{-2} \sin\pi = 0, \\ a_4 &= 2^{-5/2} \sin\tfrac{5\pi}{4} = -\tfrac{1}{8}, \end{aligned}$$

donc

$$\frac{1}{1+z^2} = \tfrac{1}{2} + \tfrac{1}{2}(z-1) + \tfrac{1}{4}(z-1)^2 + 0 \cdot (z-1)^3 - \tfrac{1}{8}(z-1)^4 + \cdots .$$

Le rayon de convergence est la distance du centre à la singularité la plus proche,

$$R = \min\{|1-i|,\ |1+i|\} = \sqrt{2}.$$

Remarque : la formule compacte $a_k = 2^{-(k+1)/2} \sin\big((k+1)\pi/4\big)$ permet de générer tous les coefficients (on voit la périodicité des signes/coefs via la valeur du sinus).

Exercise 4.8 (Laurent Series Expansion) ***Consider the complex function***

$$f(z) = \frac{1}{1+z}.$$

1. ***Find the Laurent series of $f(z)$ centered at $z_0 = 0$ for the region $|z| > 1$.***
2. ***Identify the principal part and the analytic part of the series.***

Solution 4.10 ***Step 1: Rewrite $f(z)$ in a suitable form for expansion.***
For $|z| > 1$, we can factor out $1/z$:

$$f(z) = \frac{1}{1+z} = \frac{1}{z} \cdot \frac{1}{1+1/z} = \frac{1}{z} \cdot \frac{1}{1-(-1/z)}.$$

Step 2: Expand using the geometric series.
For $|1/z| < 1$ (i.e., $|z| > 1$), we can write

$$\frac{1}{1-(-1/z)} = \sum_{n=0}^{\infty} \left(-\frac{1}{z}\right)^n = \sum_{n=0}^{\infty} (-1)^n z^{-n}.$$

Step 3: Multiply by $1/z$ to obtain the Laurent series.

$$f(z) = \frac{1}{z} \sum_{n=0}^{\infty} (-1)^n z^{-n} = \sum_{n=0}^{\infty} (-1)^n z^{-n-1}.$$

Step 4: Identify the principal part and the analytic part.
- The series contains only negative powers of z***, so the principal part is***

$$\sum_{n=0}^{\infty}(-1)^n z^{-n-1} = \frac{1}{z} - \frac{1}{z^2} + \frac{1}{z^3} - \cdots$$

- There is no part with non-negative powers of z***, so the analytic part is*** 0***.***

Conclusion:
The Laurent series of $f(z)$ ***around*** $z_0 = 0$ ***valid for*** $|z| > 1$ ***is***

$$\boxed{f(z) = \sum_{n=0}^{\infty}(-1)^n z^{-n-1} = \frac{1}{z} - \frac{1}{z^2} + \frac{1}{z^3} - \cdots}$$

with the principal part consisting of all terms and the analytic part equal to zero.

Exercise 4.9 (Classification of Isolated Singularities) ***Classify the singularities of the following complex functions and determine their type (removable, pole, or essential):***

1. $f_1(z) = \frac{\sin z}{z}$
2. $f_2(z) = \frac{1}{z^3}$
3. $f_3(z) = e^{1/z}$

Solution 4.11 ***1. Function*** $f_1(z) = \frac{\sin z}{z}$
The Maclaurin series for $\sin z$ ***is***

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots \tag{4.39}$$

Dividing by z ***gives the Laurent series around*** $z = 0$***:***

$$\frac{\sin z}{z} = 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \cdots \tag{4.40}$$

All coefficients in the principal part are zero. Therefore, $z = 0$ ***is a*** **removable singularity*****.***

2. Function $f_2(z) = \frac{1}{z^3}$
The function has a singularity at $z = 0$***, and its Laurent series is***

$$f_2(z) = \frac{1}{z^3}. \tag{4.41}$$

The principal part contains a finite number of terms (here, only $1/z^3$***), hence*** $z = 0$ ***is a*** **pole of order 3*****.***

3. Function $f_3(z) = e^{1/z}$
Expanding around $z = 0$ ***gives the Laurent series***

$$e^{1/z} = 1 + \frac{1}{z} + \frac{1}{2!z^2} + \frac{1}{3!z^3} + \cdots \tag{4.42}$$

The principal part contains infinitely many terms, so $z = 0$ ***is an*** **essential singularity*****.***

Exercise 4.10 (Poles and Zeros of Order n) *Consider the following complex functions:*

1. $f_1(z) = \frac{(z-1)^3}{(z+2)^2}$
2. $f_2(z) = z^4(z-3)$
3. $f_3(z) = \frac{z^2+1}{(z-2)^5}$

For each function, identify all zeros and poles, and determine their order.

Solution 4.12 *1. Function* $f_1(z) = \frac{(z-1)^3}{(z+2)^2}$

1. *Zeros:*

   - The zeros of a function occur where the numerator vanishes. - The numerator is $(z-1)^3$*. Solve* $(z-1)^3 = 0$*:*

   $$z - 1 = 0 \implies z = 1.$$

   - Since the factor is raised to the power 3*,* $z = 1$ *is a zero of* **order 3.**

2. *Poles:*

   - Poles occur where the denominator vanishes. - The denominator is $(z+2)^2$*. Solve* $(z+2)^2 = 0$*:*

   $$z + 2 = 0 \implies z = -2.$$

   - The exponent 2 *indicates a* **pole of order 2** *at* $z = -2$*.*

3. *Remark:*

   - Near $z = -2$*, the function behaves as* $f_1(z) \sim \frac{\textit{constant}}{(z+2)^2}$*, which confirms the pole of order* 2*.*

2. Function $f_2(z) = z^4(z-3)$

1. *Zeros:*

   - Solve $z^4(z-3) = 0$*. Factorization shows two contributions:*

   $$z^4 = 0 \implies z = 0 \text{ with multiplicity } 4,$$
   $$z - 3 = 0 \implies z = 3 \text{ with multiplicity } 1.$$

   - Therefore, $z = 0$ *is a zero of order* 4*, and* $z = 3$ *is a zero of order* 1*.*

2. *Poles:*

   - The function is a polynomial; it has no denominator. - Hence, there are **no poles.**

3. Function $f_3(z) = \frac{z^2+1}{(z-2)^5}$

1. ***Zeros:***

   - ***Zeros occur where*** $z^2 + 1 = 0$***:***

   $$z^2 = -1 \implies z = i,\, z = -i.$$

   - ***Each factor*** $(z - i)$ ***and*** $(z + i)$ ***appears only once, so both zeros are of*** **order 1*.***

2. ***Poles:***

   - ***Poles occur where*** $(z - 2)^5 = 0$***:***

   $$z - 2 = 0 \implies z = 2.$$

   - ***The factor is raised to the 5th power, so*** $z = 2$ ***is a*** **pole of order 5*.***

3. ***Remark:***

   - ***Near*** $z = 2$***, the function behaves as***

   $$f_3(z) \sim \frac{\textit{\textbf{constant}}}{(z-2)^5},$$

   confirming the pole of order 5.

Chapter 5

Residue Theorem and Applications

5.1 Residues and Residue Theorem

In the previous chapter, we saw that if a complex function f has an isolated singularity at a point z_0, then it can be expressed as a Laurent series of the form

$$f(z) = \sum_{k=-\infty}^{\infty} a_k (z - z_0)^k = \cdots + \frac{a_{-2}}{(z - z_0)^2} + \frac{a_{-1}}{z - z_0} + a_0 + a_1(z - z_0) + \cdots ,$$

which converges in a neighborhood of z_0. More precisely, this expansion is valid in a punctured disk

$$0 < |z - z_0| < R.$$

In this section, we will focus exclusively on the coefficient a_{-1}, known as the ***residue***, and its central role in evaluating contour integrals.

The coefficient a_{-1} of the term $\frac{1}{z-z_0}$ in the Laurent series of a function f at an isolated singularity z_0 is called the ***residue*** of f at z_0. We denote it by

$$a_{-1} = \mathrm{Res}(f(z), z_0).$$

Definition 5.1 (Residue of a Function) ***Let f be a complex function that is analytic in a domain D except at an isolated singularity z_0. Then f admits a Laurent series expansion around z_0 of the form***

$$f(z) = \sum_{n=-\infty}^{+\infty} a_n (z - z_0)^n, \quad 0 < |z - z_0| < R.$$

The residue of f at z_0, denoted by $\mathrm{Res}(f, z_0)$, is defined as the coefficient a_{-1} of the term $(z - z_0)^{-1}$ in this expansion:

$$\boxed{\mathrm{Res}(f, z_0) = a_{-1}.}$$

Recall that if the principal part of the Laurent series, valid for $0 < |z - z_0| < R$, contains a finite number of terms, with a_{-n} being the last nonzero coefficient, then z_0 is a ***pole of order*** n. On the other hand, if the principal part contains infinitely many nonzero terms, then z_0 is an ***essential singularity.***

Theorem 5.1 (Residue at a Simple Pole) ***If a function f has a*** simple pole ***at $z = z_0$, then the residue of f at z_0 is given by***

$$\text{Res}(f(z), z_0) = \lim_{z \to z_0} (z - z_0) f(z).$$

Example 5.1 (Residue at a Simple Pole) ***Find the residue of the function***

$$f(z) = \frac{e^z}{z - 2}$$

at its simple pole $z_0 = 2$.

Solution 5.1 ***Step 1: Identify the simple pole.***
- The function $f(z) = \frac{e^z}{z-2}$ has a singularity at $z = 2$.
- The singularity is a simple pole because the denominator is linear and nonzero derivative at $z = 2$.

Step 2: Apply the formula for a simple pole.
For a simple pole at z_0, the residue is

$$\text{Res}(f(z), z_0) = \lim_{z \to z_0} (z - z_0) f(z).$$

Step 3: Compute the limit.

$$\text{Res}(f(z), 2) = \lim_{z \to 2} (z - 2) \frac{e^z}{z - 2} = \lim_{z \to 2} e^z = e^2$$

Answer:

$$\boxed{e^2}$$

Theorem 5.2 (Residue at a Pole of Order n) ***If a function f has a pole of order n at $z = z_0$, then the residue of f at z_0 is given by***

$$\text{Res}(f(z), z_0) = \frac{1}{(n-1)!} \lim_{z \to z_0} \frac{d^{n-1}}{dz^{n-1}} \left[(z - z_0)^n f(z) \right].$$

Example 5.2 ***Consider the function***

$$f(z) = \frac{1}{(z-1)^2 (z-3)}.$$

The function f has two isolated singularities: a double pole at $z = 1$ and a simple pole at $z = 3$.

Residue at $z = 3$: Since $z = 3$ is a simple pole, we use the formula

$$\text{Res}(f, 3) = \lim_{z \to 3} (z - 3) f(z).$$

Thus,

$$\text{Res}(f, 3) = \lim_{z \to 3} \frac{1}{(z-1)^2} = \frac{1}{(3-1)^2} = \frac{1}{4}.$$

Residue at $z=1$: Since $z=1$ is a pole of order 2, we use

$$\operatorname{Res}(f,1)=\lim_{z\to 1}\frac{d}{dz}\Big[(z-1)^2 f(z)\Big].$$

Then,

$$(z-1)^2 f(z)=\frac{1}{z-3},\quad \textit{so}\quad \frac{d}{dz}\left(\frac{1}{z-3}\right)=-\frac{1}{(z-3)^2}.$$

Hence,

$$\operatorname{Res}(f,1)=\lim_{z\to 1}\left(-\frac{1}{(z-3)^2}\right)=-\frac{1}{(1-3)^2}=-\frac{1}{4}.$$

Theorem 5.3 (Residue Theorem) ***Let f be a complex function that is analytic on and inside a simple closed positively oriented contour Γ, except for finitely many isolated singularities $z_1, z_2, \ldots, z_n$ inside Γ.***

Then the integral of f around Γ is given by

$$\oint_\Gamma f(z)\,dz=2\pi i\sum_{k=1}^{n}\operatorname{Res}(f,z_k).$$

Example 5.3 ***Evaluate the contour integral***

$$\oint_C \frac{2z+6}{z^2+4}\,dz,$$

where C is the positively oriented circle $|z-i|=2$.

Step 1. Singularities. Factor the denominator:

$$z^2+4=(z-2i)(z+2i),$$

so the integrand has simple poles at $z=2i$ and $z=-2i$.

Step 2. Determine which poles lie inside C. The circle is centered at i with radius 2.

$$|2i-i|=|i|=1<2\quad \textit{(so } 2i \textit{ is inside } C\textit{)},$$
$$|-2i-i|=|-3i|=3>2\quad \textit{(so } -2i \textit{ is outside } C\textit{)}.$$

Therefore only the pole $z=2i$ lies inside C.

Step 3. Compute the residue at $z=2i$. Since $z=2i$ is a simple pole,

$$\operatorname{Res}\left(\frac{2z+6}{z^2+4},2i\right)=\lim_{z\to 2i}(z-2i)\frac{2z+6}{(z-2i)(z+2i)}=\frac{2(2i)+6}{2i+2i}=\frac{4i+6}{4i}=1-\frac{3}{2}i.$$

Step 4. Apply the Residue Theorem. Only the residue at $2i$ contributes:

$$\oint_C \frac{2z+6}{z^2+4}\,dz=2\pi i\ \operatorname{Res}\left(\frac{2z+6}{z^2+4},2i\right)=2\pi i\left(1-\frac{3}{2}i\right).$$

Simplifying,

$$2\pi i\left(1-\frac{3}{2}i\right)=2\pi i-3\pi i^2=2\pi i+3\pi.$$

Final answer:

$$\boxed{\oint_C \frac{2z+6}{z^2+4}\,dz=3\pi+2\pi i.}$$

5.1.1 Jordan's Lemmas

In the evaluation of contour integrals, it is sometimes necessary to use Jordan's Lemmas. These lemmas provide essential estimates for integrals taken over large circular arcs in the complex plane.

In order to perform such calculations correctly, it is indispensable to know the asymptotic behaviour of integrals of the form

$$\int_{\Gamma_R} f(z)\, dz,$$

where Γ_R denotes an arc of a circle of fixed opening angle and radius $R \to \infty$.

Jordan's Lemmas establish sufficient conditions under which these integrals tend to zero when the radius of the arc becomes infinite. They play a fundamental role in the application of the ***Residue Theorem*** to the evaluation of real and trigonometric integrals by contour integration.

Lemma 5.1 (Jordan's Lemma 1) ***Let a be a simple pole and let Γ_R be an arc of a circle of radius R, centered at a, with an opening angle $\Delta\theta = \theta_2 - \theta_1$. Then***

$$\lim_{R\to 0} \int_{\Gamma_R} f(z)\, dz = i\, \Delta\theta\ \mathrm{Res}(f; a).$$

Geometric Context

Let us consider the semicircular contour in the complex plane defined by

$$\Gamma_R = \{\, z = Re^{i\theta} \mid \theta_1 \le \theta \le \theta_2 \,\},$$

where $R > 0$ and the angles θ_1, θ_2 determine the opening of the arc.

Geometrically, this contour corresponds to an arc of a circle of radius R centered at the origin. The parameter θ measures the angular position of a point z on the arc, and as $R \to \infty$, the arc expands outward, covering a larger portion of the complex plane.

When evaluating complex integrals, particularly those of the form

$$\int_{\Gamma_R} e^{iaz} f(z)\, dz,$$

Jordan's lemma provides conditions under which the contribution of the integral over the arc Γ_R vanishes as $R \to \infty$. This is crucial when closing a contour in the upper or lower half-plane, depending on the sign of a, to apply the residue theorem effectively.

Geometric Interpretation. As the radius R increases, the exponential term e^{iaz} tends to zero along the arc in the appropriate half-plane (upper or lower), because the imaginary part of z makes e^{iaz} exponentially decreasing. Consequently, the contribution of the arc integral becomes negligible, and only the integral along the real axis (or another finite path) remains significant.

This geometric idea explains why, under certain conditions on $f(z)$, the integral over the arc of infinite radius can be ignored.

Lemma 5.2 (Jordan's Lemma 2) ***Let Γ_R be an arc of a circle of radius R centered at a finite point a. If***

$$\lim_{R\to\infty} |z|\,|f(z)| = 0,$$

then

$$\lim_{R\to\infty} \int_{\Gamma_R} f(z)\,dz = 0.$$

Lemma 5.3 (Jordan's Lemma 3) ***Let $f(z)$ be a complex function such that $|f(z)| \leq M_R$ on a semicircle Γ_R of radius R centered at the origin, lying in one of the half-planes, and assume that $M_R \to 0$ as $R \to \infty$. Then, for any $a > 0$, we have:***

1. ***If Γ_R lies in the upper half-plane $(\Im z > 0)$, then***

$$\lim_{R\to\infty} \int_{\Gamma_R} f(z)e^{iaz}\,dz = 0.$$

2. ***If Γ_R lies in the lower half-plane $(\Im z < 0)$, then***

$$\lim_{R\to\infty} \int_{\Gamma_R} f(z)e^{-iaz}\,dz = 0.$$

3. ***If Γ_R lies in the right half-plane $(\Re z > 0)$, then***

$$\lim_{R\to\infty} \int_{\Gamma_R} f(z)e^{az}\,dz = 0.$$

4. ***If Γ_R lies in the left half-plane $(\Re z < 0)$, then***

$$\lim_{R\to\infty} \int_{\Gamma_R} f(z)e^{-az}\,dz = 0.$$

Remark 5.1 ***Jordan's Lemmas are particularly useful when applying the Residue Theorem to evaluate real or trigonometric integrals. They justify neglecting the integral over the circular arc when the radius tends to infinity, allowing one to express the real integral as $2\pi i$ times the sum of the residues of the enclosed singularities.***

Example 5.4 (First Form - Upper Half-Plane) ***Consider the integral***

$$I = \int_{-\infty}^{+\infty} \frac{e^{iaz}}{z^2+1}\,dz, \qquad a > 0.$$

We integrate along the real axis and close the contour by a semicircle Γ_R in the upper half-plane.

On $\Gamma_R : z = Re^{i\theta},\ 0 \leq \theta \leq \pi$.

Then,

$$|e^{iaz}| = |e^{iaR(\cos\theta + i\sin\theta)}| = |e^{iaR\cos\theta}e^{iaR\sin\theta}| = |e^{-aR\sin\theta}| \times |e^{iaR(\cos\theta}| = |e^{-aR\sin\theta}| \times (1)$$

Hence, as $R \to \infty$***, the exponential term tends to zero, and Jordan's lemma ensures that***

$$\int_{\Gamma_R} \frac{e^{iaz}}{z^2+1}\, dz \to 0.$$

The integral over the real line is then equal to $2\pi i$ ***times the sum of the residues of*** $\frac{e^{iaz}}{z^2+1}$ ***inside the contour (at*** $z = i$***).***

$$I = 2\pi i \cdot \text{Res}\left(\frac{e^{iaz}}{z^2+1}, \ z = i\right) = 2\pi i \cdot \frac{e^{-a}}{2i} = \pi e^{-a}.$$

Example 5.5 (Second Form - Lower Half-Plane) ***Consider the integral***

$$I = \int_{-\infty}^{+\infty} \frac{e^{-iaz}}{z^2+1}\, dz, \qquad a > 0.$$

Now we close the contour by a semicircle in the lower half-plane. On this contour,

$$|e^{-iaz}| = e^{-aR|\sin\theta|},$$

so again Jordan's lemma implies that the integral over the arc tends to zero as $R \to \infty$***.***

The pole inside the lower half-plane is at $z = -i$***, and thus***

$$I = 2\pi i \cdot \text{Res}\left(\frac{e^{-iaz}}{z^2+1}, \ z = -i\right) = 2\pi i \cdot \frac{e^{-a}}{-2i} = \pi e^{-a}.$$

Example 5.6 (Third Form - Oscillatory Function) ***Let***

$$I = \int_{-\infty}^{+\infty} \frac{\cos(ax)}{x^2+1}\, dx, \qquad a > 0.$$

We express the cosine as $\cos(ax) = \Re(e^{iax})$***, and consider the contour integral***

$$\int_{C_R} \frac{e^{iaz}}{z^2+1}\, dz,$$

where C_R ***is the contour in the upper half-plane.***

Using Jordan's lemma, the integral over the semicircle vanishes. Thus, by the residue theorem,

$$\int_{-\infty}^{+\infty} \frac{e^{iaz}}{z^2+1}\, dz = 2\pi i \cdot \frac{e^{-a}}{2i} = \pi e^{-a}.$$

Taking the real part gives:

$$I = \int_{-\infty}^{+\infty} \frac{\cos(ax)}{x^2+1}\, dx = \pi e^{-a}.$$

5.1.2 Evaluation of integrals of the form $\int_0^{2\pi} F(\cos\theta, \sin\theta)\, d\theta$ by residues

General method. Let F be such that the integral below is meaningful. Use the substitution

$$z = e^{i\theta}, \qquad \cos\theta = \frac{z + z^{-1}}{2}, \qquad \sin\theta = \frac{z - z^{-1}}{2i}, \qquad d\theta = \frac{dz}{iz},$$

so that the integral over $\theta \in [0, 2\pi]$ becomes a contour integral over the unit circle $|z| = 1$ (positively oriented):

$$\int_0^{2\pi} F(\cos\theta, \sin\theta)\, d\theta = \oint_{|z|=1} F\Big(\frac{z + z^{-1}}{2}, \frac{z - z^{-1}}{2i}\Big)\frac{dz}{iz}.$$

After simplification, the integrand is typically a rational function $R(z)$ (possibly multiplied by e^{az} etc.). Then:

1. **Express the integrand as a meromorphic function $R(z)$ on and inside $|z| = 1$.**
2. **Identify all poles of $R(z)$ and determine which lie strictly inside the unit circle.**
3. **Compute the residues of R at those interior poles.**
4. **Apply the Residue Theorem:**

$$\oint_{|z|=1} R(z)\, dz = 2\pi i \sum_{\text{poles } z_k \text{ inside } |z|<1} \operatorname{Res}(R, z_k),$$

and hence recover the desired real integral (remember the factor $1/i$ from $d\theta = dz/(iz)$ if present).

Remarks 5.1
- ***Many standard real integrals reduce to rational integrals in z because $\cos\theta, \sin\theta$ become Laurent polynomials in z.***
- ***Quadratic denominators in $\cos\theta$ produce quadratic polynomials in z whose two roots are reciprocals ? exactly one of them lies inside $|z| = 1$ (if the roots are distinct and not on unit circle).***
- ***When F is even/odd in $\sin\theta$ or $\cos\theta$ this may simplify residue calculations (symmetries).***

Solution 5.2 ***Worked example. Evaluate***

$$I = \int_0^{2\pi} \frac{d\theta}{a + b\cos\theta},$$

where $a, b \in \mathbb{R}$ and $|a| > |b|$ (so denominator does not vanish on $[0, 2\pi]$).

Step 1. Substitution. ***Put*** $z = e^{i\theta}$***. Then***

$$\cos\theta = \frac{z+z^{-1}}{2}, \qquad d\theta = \frac{dz}{iz}.$$

Thus

$$a + b\cos\theta = a + \frac{b}{2}\left(z+z^{-1}\right) = \frac{bz^2+2az+b}{2z},$$

and

$$\frac{d\theta}{a+b\cos\theta} = \frac{dz}{iz}\cdot\frac{2z}{bz^2+2az+b} = \frac{2}{i}\cdot\frac{dz}{bz^2+2az+b}.$$

Therefore

$$I = \oint_{|z|=1} \frac{2}{i}\cdot\frac{dz}{bz^2+2az+b}.$$

Step 2. Poles. ***The denominator polynomial is***

$$P(z) = bz^2 + 2az + b.$$

Its roots are

$$z_\pm = \frac{-a \pm \sqrt{a^2-b^2}}{b},$$

and satisfy $z_+z_- = 1$***. Under the hypothesis*** $|a| > |b|$ ***the discriminant*** $a^2 - b^2 > 0$ ***and exactly one root, say*** z_0***, lies inside the unit circle. Choose***

$$z_0 = \frac{-a+\sqrt{a^2-b^2}}{b},$$

which indeed satisfies $|z_0| < 1$ ***when*** $a > |b|$ ***(one can check the reciprocal relation).***

Step 3. Residue at the simple pole z_0**.** ***Since*** z_0 ***is a simple root of*** P***,***

$$\operatorname{Res}\left(\frac{2}{i}\cdot\frac{1}{P(z)}, z_0\right) = \frac{2}{i}\cdot\frac{1}{P'(z_0)} = \frac{2}{i}\cdot\frac{1}{2bz_0+2a} = \frac{1}{i}\cdot\frac{1}{bz_0+a}.$$

Step 4. Evaluate the contour integral. ***By the Residue Theorem,***

$$I = 2\pi i\cdot\operatorname{Res}\left(\frac{2}{i}\cdot\frac{1}{P(z)}, z_0\right) = 2\pi i\cdot\frac{1}{i}\cdot\frac{1}{bz_0+a} = 2\pi\cdot\frac{1}{bz_0+a}.$$

Now substitute the value of z_0***. A short algebraic simplification (using*** $bz_0 = -a+\sqrt{a^2-b^2}$***) gives***

$$bz_0 + a = \sqrt{a^2-b^2}.$$

Indeed, we have

$$z_0 = \frac{-a+\sqrt{a^2-b^2}}{b}.$$

Multiplying both sides by b ***and adding*** a***, we obtain***

$$bz_0 + a = b\cdot\frac{-a+\sqrt{a^2-b^2}}{b} + a.$$

Simplifying the first term,

$$bz_0 + a = (-a + \sqrt{a^2 - b^2}) + a.$$

The terms $-a$ ***and*** $+a$ ***cancel, giving***

$$\boxed{bz_0 + a = \sqrt{a^2 - b^2}.}$$

Hence

$$\boxed{\int_0^{2\pi} \frac{d\theta}{a + b\cos\theta} = \frac{2\pi}{\sqrt{a^2 - b^2}}, \qquad \textbf{\textit{for }} |a| > |b|.}$$

This is the standard result. (One must choose the principal positive square root so that the right-hand side is positive when $a > |b|$***.)***

5.1.3 Complement Formula

Definition 5.2 (Euler Gamma Function) ***The*** **Euler Gamma function** $\Gamma(\alpha)$ ***is defined by the improper integral:***

$$\Gamma(\alpha) = \int_0^{+\infty} e^{-t} t^{\alpha-1}\, dt,$$

where $\Re(\alpha) > 0$ ***and*** $t \in (0, +\infty)$***.***

Proposition 5.1 (Complement Formula) ***For all*** $\alpha \in \mathbb{C} \setminus \mathbb{Z}$***, the Gamma function satisfies the identity:***

$$\Gamma(\alpha)\,\Gamma(1-\alpha) = \frac{\pi}{\sin(\pi\alpha)}.$$

This relation is known as the **reflection formula** ***or the*** **Euler complement formula*****.***

Remark 5.2 ***The Gamma function extends the factorial to the complex plane since***

$$\Gamma(n) = (n-1)! \quad \textbf{\textit{for all }} n \in \mathbb{N}^*.$$

The complement formula reveals a fundamental symmetry of $\Gamma(\alpha)$ ***with respect to*** $\alpha = \frac{1}{2}$***.***

Lemma 5.4 ***For every*** $x \in (0, 1)$***, the following equality holds:***

$$\int_0^{+\infty} \frac{t^{x-1}}{1+t}\, dt = \frac{\pi}{\sin(\pi x)}.$$

Remark 5.3 ***There exist several ways to introduce the Euler Gamma function, and it possesses many remarkable properties. This lemma is often used in the derivation of the Euler complement (or reflection) formula for the Gamma function.***

Proposition 5.2 (Main Properties of the Euler Gamma Function) ***The Euler Gamma function*** $\Gamma(z)$ ***satisfies the following properties:***

1. ***Domain and Analyticity.*** $\Gamma(z)$ ***is defined and analytic in the region*** $\Re(z) > 0$.

2. ***Functional Equation. The function satisfies the recurrence relation***

$$\Gamma(z+1) = z\,\Gamma(z), \quad \Re(z) > 0.$$

In particular, for any integer $n \geq 0$,

$$\Gamma(n+1) = n!.$$

3. ***Derivatives. For all*** $n \in \mathbb{N}$ ***and*** $z \in \mathbb{C}$ ***with*** $\Re(z) > 0$,

$$\Gamma^{(n)}(z) = \int_0^{+\infty} e^{-t} t^{z-1} (\ln t)^n \, dt.$$

4. ***Analytic Continuation. Using the functional equation*** $\Gamma(z+1) = z\Gamma(z)$, ***the function can be extended to a meromorphic function on*** $\mathbb{C} \setminus \{-\mathbb{N}\}$.

5. ***Limit Representation. For*** $\Re(z) > 0$,

$$\Gamma(z) = \int_0^{+\infty} e^{-t} t^{z-1} \, dt = \lim_{n \to +\infty} \int_0^n \left(1 - \frac{t}{n}\right)^n t^{z-1} \, dt.$$

6. ***Weierstrass Product Formula. For*** $\Re(z) > 0$,

$$\frac{1}{\Gamma(z)} = z e^{\gamma z} \prod_{n=1}^{\infty} \left(1 + \frac{z}{n}\right) e^{-z/n},$$

where $\gamma \approx 0.57721$ ***is the*** Euler-Mascheroni constant, ***defined by***

$$\gamma = \lim_{n \to +\infty} \left(\sum_{k=1}^{n} \frac{1}{k} - \ln n \right).$$

7. ***Stirling's Formula. For*** $|z|$ ***large and*** $\Re(z) > 0$,

$$\Gamma(z) \sim \sqrt{2\pi}\, z^{z-\frac{1}{2}} e^{-z}.$$

In particular,

$$n! \sim \sqrt{2\pi n} \left(\frac{n}{e}\right)^n.$$

8. ***Legendre's Duplication Formula.***

$$2^{2z-1} \pi^{-1/2} \Gamma(z)\, \Gamma\!\left(z + \frac{1}{2}\right) = \Gamma(2z).$$

5.1.4 Residue at Infinity

Let $f(z)$ be a complex function that is analytic everywhere on the extended complex plane $\mathbb{C} \cup \{\infty\}$ except possibly at $z = \infty$.

Definition 5.3 ***The residue of f at infinity is defined by:***

$$\operatorname{Res}(f, \infty) = -\operatorname{Res}\left(\frac{1}{z^2} f\left(\frac{1}{z}\right), 0\right)$$

That is, the residue at infinity is the negative of the residue at zero of the function $\frac{f(1/z)}{z^2}$.

Remark 5.4 ***Studying the behavior of $f(z)$ near infinity is equivalent to studying the behavior of $f(1/z)$ near $z = 0$.***

Theorem 5.4 (Sum of Residues) ***If f is a rational function that is analytic on the extended complex plane except at finitely many poles $z_1, z_2, \ldots, z_n$ (and possibly at ∞), then:***

$$\operatorname{Res}(f, \infty) + \sum_{k=1}^{n} \operatorname{Res}(f, z_k) = 0$$

Proof. Consider a large positively oriented circle C_R centered at the origin with radius R. Let Γ denote the contour formed by C_R enclosing all finite poles of f. By the residue theorem:

$$\int_\Gamma f(z)\, dz = 2\pi i \sum_{k=1}^{n} \operatorname{Res}(f, z_k)$$

However, if we traverse the same contour in the opposite (clockwise) direction around the point at infinity, we have:

$$\int_\Gamma f(z)\, dz = -2\pi i \operatorname{Res}(f, \infty)$$

Comparing the two expressions yields:

$$\sum_{k=1}^{n} \operatorname{Res}(f, z_k) + \operatorname{Res}(f, \infty) = 0$$

■

Example 5.7 ***Compute the residue of***

$$f(z) = \frac{1}{z^2 + 1}$$

at infinity.

Solution 5.3 ***We use the definition:***

$$\operatorname{Res}(f,\infty) = -\operatorname{Res}\left(\frac{1}{z^2}f\left(\frac{1}{z}\right),0\right)$$

Compute:

$$f\left(\frac{1}{z}\right) = \frac{1}{\frac{1}{z^2}+1} = \frac{z^2}{1+z^2}$$

Hence,

$$\frac{1}{z^2}f\left(\frac{1}{z}\right) = \frac{1}{1+z^2}$$

Therefore,

$$\operatorname{Res}\left(\frac{1}{z^2}f\left(\frac{1}{z}\right),0\right) = \operatorname{Res}\left(\frac{1}{1+z^2},0\right) = 0$$

Thus,

$$\boxed{\operatorname{Res}(f,\infty) = 0}$$

Example 5.8 ***Compute the residue at infinity of***

$$f(z) = z^3 + 2z.$$

Solution 5.4 ***We use:***

$$\operatorname{Res}(f,\infty) = -\operatorname{Res}\left(\frac{1}{z^2}f\left(\frac{1}{z}\right),0\right)$$

Compute:

$$f\left(\frac{1}{z}\right) = \frac{1}{z^3} + \frac{2}{z}$$

Then,

$$\frac{1}{z^2}f\left(\frac{1}{z}\right) = \frac{1}{z^5} + \frac{2}{z^3}$$

There is no term in $\frac{1}{z}$***, so the residue at*** $z = 0$ ***is*** 0***. Therefore,***

$$\boxed{\operatorname{Res}(f,\infty) = 0}$$

Example 5.9 ***Compute the residue at infinity of***

$$f(z) = \frac{1}{z}.$$

Solution 5.5 ***We have:***

$$f\left(\frac{1}{z}\right) = z \quad \Rightarrow \quad \frac{1}{z^2}f\left(\frac{1}{z}\right) = \frac{1}{z}$$

The residue of $\frac{1}{z}$ ***at*** $z = 0$ ***is*** 1***, therefore:***

$$\boxed{\operatorname{Res}(f,\infty) = -1}$$

Meaning and Role of the Residue at Infinity

Remark 5.5 ***The residue at infinity measures the behavior of a function*** $f(z)$ ***when*** $|z| \to \infty$***. It plays a crucial role in completing the theory of residues by treating the point at infinity as a*** possible singularity, ***just like any finite point.***

1. The Geometric Idea

In complex analysis, we often work on the ***extended complex plane*** (also called the Riemann sphere):

$$\mathbb{C}_\infty = \mathbb{C} \cup \{\infty\}.$$

This means that ∞ is considered as an additional point where functions can also have singularities.

Just as we study the behavior of $f(z)$ near $z = z_0$, we can study its behavior near $z = \infty$ by considering the new variable $w = \dfrac{1}{z}$.

Then, the neighborhood of $z = \infty$ corresponds to the neighborhood of $w = 0$.

2. Why We Define a Residue at Infinity?

When we integrate a function $f(z)$ around a large contour enclosing ***all*** finite singularities, the residue theorem gives:

$$\int_\Gamma f(z)\, dz = 2\pi i \sum_{k=1}^{n} \operatorname{Res}(f, z_k).$$

However, if we look at the same contour as enclosing the point at infinity (in the opposite orientation), the same integral is also equal to:

$$\int_\Gamma f(z)\, dz = -2\pi i \operatorname{Res}(f, \infty).$$

Combining these two expressions leads to the fundamental relation:

$$\boxed{\operatorname{Res}(f, \infty) + \sum_{k=1}^{n} \operatorname{Res}(f, z_k) = 0.}$$

Thus, the residue at infinity acts as a balance term that completes the sum of all residues on the Riemann sphere.

3. Practical Uses

- (a) Simplifying calculations: Sometimes, it is easier to compute $\operatorname{Res}(f, \infty)$ than all other residues, or vice versa. The relation

  $$\operatorname{Res}(f, \infty) = -\sum_{k=1}^{n} \operatorname{Res}(f, z_k)$$

  can then give the value of one term directly.

- (b) Integrals over large contours: When evaluating integrals over a large circle $|z| = R \to \infty$, the residue at infinity determines the limit of the contour integral:

  $$\int_{|z|=R} f(z)\, dz \xrightarrow[R\to\infty]{} -2\pi i \operatorname{Res}(f, \infty).$$

  This is particularly useful for rational functions that vanish at infinity.

- **(c) Theoretical completeness: In the Riemann sphere, every meromorphic function has a finite number of singularities, including possibly one at infinity. The residue theorem then holds globally:**

  $$\text{The sum of all residues on } \mathbb{C}_\infty \text{ is always } 0.$$

  This property is central in complex geometry and in the study of analytic continuation.

The residue at infinity measures the contribution of the function $f(z)$ near $z = \infty$ to complex contour integrals. It ensures that the sum of all residues (finite and infinite) on the Riemann sphere equals zero.

$$\boxed{\sum_{\text{all poles (including } \infty)} \operatorname{Res}(f, z) = 0}$$

Application of the Residue at Infinity to the Evaluation of Real Integrals

1. General Idea

When evaluating improper real integrals of rational functions, one common approach is to extend the real integral to a *contour integral* in the complex plane. The residue at infinity can then be used to compute the integral over a large circle when the function decays sufficiently fast as $|z| \to \infty$.

Remark 5.6 ***The main idea is to interpret an integral of a rational function over the real line as part of a closed contour integral in the complex plane. The contribution of the large semicircular arc often tends to zero, and the remaining part of the contour integral can be computed using the residues of the function-including possibly the one at infinity.***

2. Typical Integral Form

We consider integrals of the form:

$$I = \int_{-\infty}^{\infty} R(x)\, dx,$$

where $R(x) = \dfrac{P(x)}{Q(x)}$ is a rational function such that:

$$\deg(Q) \geq \deg(P) + 2.$$

- **The degree condition ensures that $R(z) \to 0$ as $|z| \to \infty$.**
- **This guarantees that the integral over the large semicircle in the upper (or lower) half-plane vanishes as its radius tends to infinity.**

3. Using the Residue Theorem

Let C_R denote the contour composed of:

- **the real axis from $-R$ to R,**

- **and the upper semicircular arc Γ_R of radius R centered at the origin.**

Then,

$$\int_{C_R} R(z)\,dz = 2\pi i \sum_{\substack{\text{poles of } R \\ \text{in the upper half-plane}}} \operatorname{Res}(R, z_k).$$

If $\deg(Q) \geq \deg(P) + 2$, **then as** $R \to \infty$,

$$\int_{\Gamma_R} R(z)\,dz \to 0.$$

Thus,

$$\boxed{\int_{-\infty}^{\infty} R(x)\,dx = 2\pi i \sum_{\substack{\text{poles of } R \\ \text{in the upper half-plane}}} \operatorname{Res}(R, z_k).}$$

4. Role of the Residue at Infinity

When $\deg(Q) = \deg(P) + 1$, **the integral over the large circle *does not vanish.* In that case, the contribution of the arc at infinity can be computed using the residue at infinity.**

Indeed, for any closed contour enclosing all finite poles:

$$\int_{C_R} R(z)\,dz = -2\pi i \operatorname{Res}(R, \infty).$$

Hence,

$$\int_{-\infty}^{\infty} R(x)\,dx = -2\pi i \operatorname{Res}(R, \infty) \quad \text{if } R \text{ has no finite poles in the upper half-plane.}$$

Remark 5.7 ***The residue at infinity is therefore useful to:***

- ***account for the contribution of the arc when it does not vanish,***
- ***or replace the sum of residues at finite poles when the function is entire except at infinity.***

5. Example 1

Example 5.10 ***Compute the integral***

$$I = \int_{-\infty}^{\infty} \frac{dx}{x^2 + 1}.$$

Solution 5.6 ***Let***

$$f(z) = \frac{1}{z^2 + 1}.$$

It has simple poles at $z = i$ ***and*** $z = -i$.

We close the contour in the upper half-plane. Only the pole at $z = i$ ***lies inside.***

$$\operatorname{Res}(f, i) = \lim_{z \to i} (z - i) f(z) = \frac{1}{2i}.$$

Hence,

$$\int_{-\infty}^{\infty} \frac{dx}{x^2 + 1} = 2\pi i \left(\frac{1}{2i}\right) = \pi.$$

Since $f(z) \to 0$ ***as*** $z \to \infty$, ***we do not need*** $\operatorname{Res}(f, \infty)$ ***here.***

6. Example 2 (Using the Residue at Infinity)

Example 5.11 ***Compute***

$$I = \int_{-\infty}^{\infty} \frac{x^2 - 1}{x^2 + 1}\, dx.$$

Solution 5.7 ***Let***

$$f(z) = \frac{z^2 - 1}{z^2 + 1}.$$

This function is meromorphic with poles at $z = i$ ***and*** $z = -i$***. However, note that***

$$f(z) \xrightarrow[|z|\to\infty]{} 1.$$

So the integral along the large arc does not vanish, we must use the residue at infinity.

We have:

$$\operatorname{Res}(f, \infty) = -\operatorname{Res}\left(\frac{1}{z^2} f\left(\frac{1}{z}\right), 0\right).$$

Compute:

$$f\left(\frac{1}{z}\right) = \frac{\frac{1}{z^2} - 1}{\frac{1}{z^2} + 1} = \frac{1 - z^2}{1 + z^2}.$$

Then:

$$\frac{1}{z^2} f\left(\frac{1}{z}\right) = \frac{1 - z^2}{z^2(1 + z^2)} = \frac{1}{z^2} - \frac{1}{1 + z^2}.$$

The residue at $z = 0$ ***is the coefficient of*** $\frac{1}{z}$ ***in this expansion, which is*** 0.

Hence:

$$\operatorname{Res}(f, \infty) = 0.$$

Therefore, using the global residue theorem:

$$\operatorname{Res}(f, i) + \operatorname{Res}(f, -i) + \operatorname{Res}(f, \infty) = 0.$$

We can compute one residue and deduce the other, or use symmetry to conclude that the integral of the odd part is zero.

In fact, since the integrand tends to 1 ***as*** $x \to \infty$***, the integral diverges,confirming that the residue at infinity correctly indicates that the integral does not converge.***

7. Summary For rational functions:

- **If** $f(z) \to 0$ **as** $|z| \to \infty$**, the integral along the large circle vanishes. The integral over** $\mathbb{R}$ **equals** $2\pi i$ **times the sum of residues in the upper half-plane.**
- **If** $f(z)$ **does not vanish at infinity, the contribution of the large circle is expressed through the residue at infinity:**

$$\int_{-\infty}^{\infty} f(x)\, dx = -2\pi i \operatorname{Res}(f, \infty).$$

5.1.5 Integrals of Multivalued Functions

Example 5.12 ***Consider the function***

$$f(z) = \sqrt{z}.$$

This function is multivalued because for every nonzero complex number $z = re^{i\theta}$***, we have two possible values:***

$$\sqrt{z} = \sqrt{r}e^{i\theta/2} \quad \textbf{\textit{and}} \quad -\sqrt{r}e^{i\theta/2}.$$

Remark 5.8 ***This happens because when*** z ***completes one full turn around the origin (***$\theta \to \theta + 2\pi$***), the value of*** $\sqrt{z}$ ***changes sign:***

$$\sqrt{re^{i(\theta+2\pi)}} = \sqrt{r}e^{i(\theta/2+\pi)} = -\sqrt{r}e^{i\theta/2}.$$

Hence, $\sqrt{z}$ ***is two-valued.***

To make $\sqrt{z}$ single-valued, we restrict its domain by introducing a branch cut, usually along the negative real axis. On this restricted domain, the function behaves normally and we can integrate it.

Integral Around the Origin

Example 5.13 ***Let us compute***

$$I = \oint_C \sqrt{z}\, dz,$$

where C ***is the unit circle*** $z = e^{i\theta}$***,*** $0 \leq \theta \leq 2\pi$***.***

Solution 5.8 ***Parameterize the contour:***

$$z = e^{i\theta}, \quad dz = ie^{i\theta} d\theta.$$

Then

$$I = \int_0^{2\pi} \sqrt{e^{i\theta}}\, ie^{i\theta}\, d\theta.$$

Using the principal branch $\sqrt{e^{i\theta}} = e^{i\theta/2}$***, we have***

$$I = i\int_0^{2\pi} e^{i(3\theta/2)}\, d\theta = i\left[\frac{2}{3i}e^{i(3\theta/2)}\right]_0^{2\pi} = \frac{2}{3}i\left(e^{i3\pi} - 1\right) = \frac{2}{3}i(-1-1) = -\frac{4}{3}i.$$

However, this result is not valid because $\sqrt{z}$ ***is not single-valued on the full circle.***

Indeed, it is not possible to integrate the function $\sqrt{z}$ around a closed contour that encloses the branch point (here $z = 0$) if we require the function to remain single-valued on the entire contour. There exists no single-valued branch of $\sqrt{z}$ on a contour that makes a full turn around the point $z = 0$ once.

If we choose one specific branch (for example, the principal branch), we can integrate only along the corresponding arc $\theta \in (-\pi, \pi)$. However, this path is not closed on the same sheet of the Riemann surface.

To obtain a closed integral that returns to the same value of the function, the contour must go around the origin on both sheets, in other words, we must make two complete turns for $\sqrt{z}$.

The computed values are:

$$\int_{\theta \in (-\pi,\pi)} \sqrt{z}\, dz = -\frac{4i}{3}, \qquad \text{and for the closed path over two turns:} \quad \oint \sqrt{z}\, dz = -\frac{8i}{3}.$$

Conclusion

$$\boxed{\int_{-\pi}^{\pi} \sqrt{z}\, dz = -\frac{4i}{3} \quad \text{(one turn on principal branch)}}$$

$$\boxed{\oint_{\text{closed over two sheets}} \sqrt{z}\, dz = -\frac{8i}{3} \quad \text{(true closed contour)}}$$

$$\boxed{\text{and if the contour does not encircle } 0\text{, the integral is } 0.}$$

Remark 5.9 ***When the contour C encircles the origin completely, the function $\sqrt{z}$ changes its sign after one full turn, meaning the integral depends on the path. This is why the integral of a multivalued function around its branch point is generally not zero and sometimes even undefined.***

How to Handle Integrals of Multivalued Functions

When dealing with a function that is not single-valued, such as $z^{1/n}$ or $\log z$, the integration process requires special care.

1. Identify multivalued behavior: Check whether the function involves expressions like $z^{1/n}$ or $\log z$, which take several values depending on the argument of z.

2. Choose a branch cut: Select a curve (usually a half-line or segment) that connects the branch point to infinity (for example, the negative real axis). This cut ensures the function becomes discontinuous only along that curve.

3. Define a single branch: Restrict the argument of z (e.g. $\theta \in (-\pi, \pi)$) so that the function becomes single-valued on the chosen domain.

4. Select a contour that avoids the branch cut: Integrate only along paths that do not cross the branch cut. If a closed contour is needed, it may have to run on both sides of the cut or cover several sheets of the Riemann surface.

Remark. This procedure allows one to define integrals of multivalued functions consistently, without ambiguity in the value of the integrand. In many cases (such as $\sqrt{z}$ or $\log z$), the contour must be taken over two sheets of the Riemann surface to return to the initial value of the function.

Example 5.14 ***For instance, if we take the branch cut along the negative real axis and integrate***

$$I = \int_0^{\infty} \sqrt{x}\, e^{-x}\, dx,$$

then $\sqrt{x}$ ***is well-defined and real, and we can evaluate the integral normally using standard real methods.***

5.2 Solved Exercises

Exercise 5.1 (Evaluation by the Residue Theorem) ***Evaluate the contour integral***

$$\oint_C \frac{1}{(z-1)^2(z-3)}\, dz$$

for the following contours:

(a) C ***is the rectangle with sides*** $x = 0, x = 4, y = -1, y = 1$ ***(positively oriented).***

(b) C ***is the circle*** $|z| = 2$ ***(positively oriented).***

Solution 5.9 *Let*

$$f(z) = \frac{1}{(z-1)^2(z-3)}.$$

The function f ***has a pole of order*** 2 ***at*** $z = 1$ ***and a simple pole at*** $z = 3$***.***

Residues.

$$\mathrm{Res}(f, 3) = \lim_{z\to 3}(z-3)f(z) = \lim_{z\to 3} \frac{1}{(z-1)^2} = \frac{1}{(3-1)^2} = \frac{1}{4}.$$

For the double pole at $z = 1$ ***we use***

$$\mathrm{Res}(f, 1) = \lim_{z\to 1} \frac{d}{dz}\Big[(z-1)^2 f(z)\Big] = \lim_{z\to 1} \frac{d}{dz}\Big(\frac{1}{z-3}\Big) = \lim_{z\to 1}\left(-\frac{1}{(z-3)^2}\right) = -\frac{1}{(1-3)^2} = -\frac{1}{4}.$$

(a) Rectangle $0 \le x \le 4,\ -1 \le y \le 1$***. Both points*** $z = 1$ ***and*** $z = 3$ ***lie inside this rectangle. Hence by the Residue Theorem***

$$\oint_C f(z)\, dz = 2\pi i\Big(\mathrm{Res}(f, 1) + \mathrm{Res}(f, 3)\Big) = 2\pi i\left(-\frac{1}{4} + \frac{1}{4}\right) = 0.$$

$$\boxed{\oint_C \frac{1}{(z-1)^2(z-3)}\, dz = 0.}$$

(b) Circle $|z| = 2$***. Here*** $|1| = 1 < 2$ ***so*** $z = 1$ ***lies inside the circle, while*** $|3| = 3 > 2$ ***so*** $z = 3$ ***lies outside. Only the residue at*** $z = 1$ ***contributes:***

$$\oint_C f(z)\, dz = 2\pi i\ \mathrm{Res}(f, 1) = 2\pi i\left(-\frac{1}{4}\right) = -\frac{\pi i}{2}.$$

$$\boxed{\oint_C \frac{1}{(z-1)^2(z-3)}\, dz = -\frac{\pi i}{2}.}$$

Exercise 5.2 (Evaluation by the Residue Theorem) ***Evaluate the contour integral***

$$\oint_C \frac{e^z}{z^4 + 5z^3}\, dz,$$

where C ***is the positively oriented circle*** $|z| = 2$***.***

Solution 5.10 ***Write the integrand as***

$$\frac{e^z}{z^4 + 5z^3} = \frac{e^z}{z^3(z+5)}.$$

Thus the integrand has a pole of order 3 ***at*** $z = 0$ ***and a simple pole at*** $z = -5$***. On the circle*** $|z| = 2$ ***the point*** $z = 0$ ***lies inside while*** $z = -5$ ***lies outside. Hence only the pole at*** $z = 0$ ***contributes.***

Let

$$g(z) = \frac{e^z}{z+5}.$$

Since the pole at $z = 0$ ***is of order*** $m = 3$***, the residue at*** 0 ***is given by***

$$\operatorname{Res}\left(\frac{e^z}{z^3(z+5)}, 0\right) = \frac{1}{2!} \lim_{z\to 0} \frac{d^2}{dz^2}\left[z^3 \cdot \tfrac{e^z}{z^3(z+5)}\right] = \frac{1}{2} \lim_{z\to 0} \frac{d^2}{dz^2}\left(\frac{e^z}{z+5}\right) = \frac{1}{2} g''(0).$$

We compute $g''(z)$***. First***

$$g'(z) = \frac{d}{dz}\left(\frac{e^z}{z+5}\right) = \frac{e^z(z+5) - e^z}{(z+5)^2} = \frac{e^z(z+4)}{(z+5)^2}.$$

Differentiating again,

$$g''(z) = \frac{d}{dz}\left(\frac{e^z(z+4)}{(z+5)^2}\right) = \frac{e^z(z+5)^2 - 2e^z(z+4)}{(z+5)^3} = \frac{e^z\left(z^2 + 8z + 17\right)}{(z+5)^3}.$$

Hence

$$g''(0) = \frac{e^0 \cdot 17}{5^3} = \frac{17}{125},$$

and therefore

$$\operatorname{Res}\left(\frac{e^z}{z^4 + 5z^3}, 0\right) = \frac{1}{2} \cdot \frac{17}{125} = \frac{17}{250}.$$

By the Residue Theorem (only the residue at 0 ***contributes),***

$$\oint_C \frac{e^z}{z^4 + 5z^3}\, dz = 2\pi i \cdot \frac{17}{250} = \frac{17\pi i}{125}.$$

Final answer:

$$\boxed{\oint_C \frac{e^z}{z^4 + 5z^3}\, dz = \frac{17\pi i}{125}.}$$

Exercise 5.3 ***Use the properties of the Gamma function to show that:***

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi} \quad \textit{and} \quad \Gamma\left(\frac{3}{2}\right) = \frac{\sqrt{\pi}}{2}.$$

Solution 5.11 *Start from the definition*

$$\Gamma\left(\tfrac{1}{2}\right) = \int_0^\infty e^{-t}\, t^{-1/2}\, dt.$$

Perform the substitution $t = u^2$*. Then* $dt = 2u\, du$ *and*

$$t^{-1/2} = (u^2)^{-1/2} = \frac{1}{u},$$

so

$$\Gamma\left(\tfrac{1}{2}\right) = \int_0^\infty e^{-u^2} \frac{1}{u}\,(2u\, du) = 2\int_0^\infty e^{-u^2}\, du.$$

Thus it remains to evaluate the Gaussian integral $\int_0^\infty e^{-u^2}\, du$*.*

Let $I = \int_0^\infty e^{-u^2}\, du$*. Then*

$$I^2 = \left(\int_0^\infty e^{-x^2}\, dx\right)\left(\int_0^\infty e^{-y^2}\, dy\right) = \int_0^\infty \int_0^\infty e^{-(x^2+y^2)}\, dx\, dy.$$

Switching to polar coordinates $x = r\cos\theta$, $y = r\sin\theta$ *on the first quadrant:*

$$I^2 = \int_0^{\pi/2} \int_0^\infty e^{-r^2}\, r\, dr\, d\theta = \left(\int_0^{\pi/2} d\theta\right)\left(\int_0^\infty re^{-r^2}\, dr\right).$$

Compute the radial integral with $s = r^2$*,* $ds = 2r\, dr$*:*

$$\int_0^\infty re^{-r^2}\, dr = \frac{1}{2}\int_0^\infty e^{-s}\, ds = \frac{1}{2}.$$

Hence

$$I^2 = \frac{\pi}{2}\cdot\frac{1}{2} = \frac{\pi}{4},$$

so $I = \sqrt{\pi}/2$*. Therefore*

$$\Gamma\left(\tfrac{1}{2}\right) = 2I = 2\cdot\frac{\sqrt{\pi}}{2} = \sqrt{\pi}.$$

Finally use the functional equation $\Gamma(z+1) = z\Gamma(z)$*. With* $z = \frac{1}{2}$ *we get*

$$\Gamma\left(\tfrac{3}{2}\right) = \tfrac{1}{2}\,\Gamma\left(\tfrac{1}{2}\right) = \tfrac{1}{2}\sqrt{\pi}.$$

This completes the proof.

Exercise 5.4 (Residue at Infinity) *Compute the integral*

$$I = \int_{-\infty}^\infty \frac{x^2}{x^4+1}\, dx$$

using complex residues. In particular, compute the residues of the integrand's poles, evaluate the residue at infinity (as a check), and deduce the value of I*.*

Solution 5.12 *Let*

$$f(z) = \frac{z^2}{z^4+1}.$$

We proceed in several steps.

Step 1: Locate the poles.
The poles are the roots of $z^4+1=0$***. Solving gives***

$$z^4 = -1 = e^{i\pi} \quad \Longrightarrow \quad z_k = e^{i\left(\frac{\pi}{4}+\frac{k\pi}{2}\right)}, \qquad k = 0,1,2,3.$$

Explicitly,

$$z_1 = e^{i\pi/4}, \quad z_2 = e^{i3\pi/4}, \quad z_3 = e^{i5\pi/4}, \quad z_4 = e^{i7\pi/4}.$$

The two poles in the upper half-plane are $z_1 = e^{i\pi/4}$ ***and*** $z_2 = e^{i3\pi/4}$***.***

Step 2: Determine the order of the poles and compute residues at finite poles.

Each root of z^4+1 ***is simple (the derivative*** $4z^3$ ***does not vanish at the roots), so the poles are simple. For a simple pole*** z_0 ***of*** f***, the residue is***

$$\operatorname{Res}(f,z_0) = \lim_{z\to z_0}(z-z_0)\frac{z^2}{z^4+1} = \frac{z_0^2}{(z^4+1)'\big|_{z=z_0}} = \frac{z_0^2}{4z_0^3} = \frac{1}{4z_0}.$$

Hence for each pole z_k***,***

$$\operatorname{Res}(f,z_k) = \frac{1}{4z_k}.$$

In particular, for the two poles in the upper half-plane,

$$\operatorname{Res}(f,z_1) = \frac{1}{4e^{i\pi/4}}, \qquad \operatorname{Res}(f,z_2) = \frac{1}{4e^{i3\pi/4}}.$$

Step 3: Sum the residues in the upper half-plane.
Compute

$$S_{upper} = \operatorname{Res}(f,z_1) + \operatorname{Res}(f,z_2) = \frac{1}{4}\left(e^{-i\pi/4} + e^{-i3\pi/4}\right).$$

We simplify the bracket:

$$e^{-i\pi/4} + e^{-i3\pi/4} = \cos\left(\tfrac{\pi}{4}\right) - i\sin\left(\tfrac{\pi}{4}\right) + \cos\left(\tfrac{3\pi}{4}\right) - i\sin\left(\tfrac{3\pi}{4}\right).$$

Using $\cos(\pi/4) = \frac{\sqrt{2}}{2}$***,*** $\cos(3\pi/4) = -\frac{\sqrt{2}}{2}$***,*** $\sin(\pi/4) = \frac{\sqrt{2}}{2}$***,*** $\sin(3\pi/4) = \frac{\sqrt{2}}{2}$***, we get***

$$e^{-i\pi/4} + e^{-i3\pi/4} = 0 - i\left(\tfrac{\sqrt{2}}{2} + \tfrac{\sqrt{2}}{2}\right) = -i\sqrt{2}.$$

Therefore

$$S_{upper} = \frac{1}{4}\Big(-i\sqrt{2}\Big) = -\frac{i}{2\sqrt{2}}.$$

Step 4: Evaluate the real integral via residues.

The integrand $f(z)$ is even and behaves like $1/z^2$ as $|z| \to \infty$, so the integral over the large semicircle in the upper half-plane vanishes as the radius $R \to \infty$. Hence, closing the contour in the upper half-plane and applying the residue theorem gives

$$\int_{-\infty}^{\infty} \frac{x^2}{x^4+1}\,dx = 2\pi i \sum_{\substack{\textit{poles}\\ \textit{in upper half-plane}}} \operatorname{Res}(f, z_k) = 2\pi i \cdot S_{upper}.$$

Substituting $S_{upper} = -\dfrac{i}{2\sqrt{2}}$ ***yields***

$$I = 2\pi i \left(-\frac{i}{2\sqrt{2}}\right) = 2\pi \cdot \frac{1}{2\sqrt{2}} = \frac{\pi}{\sqrt{2}}.$$

Step 5: Compute the residue at infinity as a check.
By definition,

$$\operatorname{Res}(f, \infty) = -\operatorname{Res}\left(\frac{1}{z^2} f\left(\frac{1}{z}\right), 0\right).$$

Compute

$$f\left(\frac{1}{z}\right) = \frac{1/z^2}{1/z^4+1} = \frac{z^2}{1+z^2}, \quad \textit{so} \quad \frac{1}{z^2} f\left(\frac{1}{z}\right) = \frac{1}{1+z^2}.$$

The function $\dfrac{1}{1+z^2}$ ***has no $1/z$-term in its Laurent expansion at 0, hence its residue at 0 is 0. Therefore***

$$\operatorname{Res}(f, \infty) = 0.$$

This is consistent with the fact that the sum of all finite residues is 0 (which can be verified by summing $\frac{1}{4z_k}$ over the four roots z_k).

Conclusion. The value of the integral is

$$\boxed{I = \int_{-\infty}^{\infty} \frac{x^2}{x^4+1}\,dx = \frac{\pi}{\sqrt{2}}}.$$

Chapter 6

Applications

6.1 Equivalence Between Holomorphy and Analyticity

Definition 6.1 (Holomorphic Function) ***A complex function*** $f : \Omega \subset \mathbb{C} \to \mathbb{C}$ ***is called*** **holomorphic** ***at a point*** $z_0 \in \Omega$ ***if the complex derivative***

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

exists. If f ***is holomorphic at every point in*** Ω***, we say that*** f ***is holomorphic on*** Ω***.***

Definition 6.2 (Analytic Function) ***A function*** $f : \Omega \to \mathbb{C}$ ***is said to be*** **analytic** ***at*** $z_0 \in \Omega$ ***if there exists a power series***

$$\sum_{n=0}^{\infty} a_n (z - z_0)^n$$

which converges to $f(z)$ ***in some neighborhood of*** z_0***:***

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n, \quad |z - z_0| < R.$$

Theorem 6.1 (Equivalence of Holomorphy and Analyticity) ***Let*** $f : \Omega \subset \mathbb{C} \to \mathbb{C}$ ***be a complex function. Then:***

$$f \text{ is holomorphic on } \Omega \iff f \text{ is analytic on } \Omega.$$

Sketch of Proof.

- **Holomorphic $\implies$ Analytic: If f is holomorphic in an open set Ω, then by Cauchy's integral formula, for any $z_0 \in \Omega$:**

$$f(z) = \frac{1}{2\pi i} \int_{|w - z_0| = r} \frac{f(w)}{w - z} \, dw, \quad |z - z_0| < r.$$

Expanding $1/(w - z)$ as a geometric series gives a convergent power series around z_0, showing that f is analytic.

- **Analytic $\implies$ Holomorphic: If f can be expressed as a convergent power series around z_0, then term-by-term differentiation of the series is valid, giving**

$$f'(z) = \sum_{n=1}^{\infty} n a_n (z - z_0)^{n-1},$$

which exists in the same neighborhood. Hence f is holomorphic.

■

Remark 6.1 ***This equivalence is a fundamental property of complex analysis and distinguishes it from real analysis: in the real case, differentiability does not necessarily imply analyticity, but in the complex case, holomorphy automatically implies that the function is locally representable by a convergent power series.***

Definition 6.3 (Radius of Convergence) ***Let***

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

be a power series centered at $z_0 \in \mathbb{C}$.

The radius of convergence R ***is the largest number such that the series converges for every z satisfying***

$$|z - z_0| < R.$$

The open disk

$$D(z_0, R) = \{z \in \mathbb{C} : |z - z_0| < R\}$$

is called the disk of convergence***.***

Corollary 6.1 ***If***

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

has radius of convergence R, then f is holomorphic at every point z in the open disk

$$|z - z_0| < R.$$

Example 6.1 ***Let***

$$f(z) = \begin{cases} \dfrac{\sin z}{z}, & z \neq 0, \\ 1, & z = 0. \end{cases}$$

(a) Show that f is an entire function.

(b) Compute $\lim_{z \to 0} f(z)$.

Solution 6.1 ***Step (a): Show that f is entire.***

For $z \neq 0$, recall the power series expansion of $\sin z$:

$$\sin z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!}.$$

Dividing by z for $z \neq 0$, we get

$$f(z) = \frac{\sin z}{z} = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n+1)!}.$$

Step (a1): Analyticity at $z = 0$.

To prove f is analytic at 0, observe that the series above is of the form

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \quad \textbf{\textit{with}}\ a_n = \begin{cases} \frac{(-1)^{n/2}}{(2n+1)!}, & n\ \textbf{\textit{even}}, \\ 0, & n\ \textbf{\textit{odd}}. \end{cases}$$

- This series converges for all $z \in \mathbb{C}$ (infinite radius of convergence). - The constant term $a_0 = 1$, which matches $f(0) = 1$.

Hence, f can be expressed as a convergent power series around 0, which proves that f is analytic at $z = 0$.

Step (a2): Analyticity elsewhere.

For $z \neq 0$, $f(z) = \sin z/z$ is the ratio of two analytic functions with a removable singularity at $z = 0$. Therefore, f is analytic on all of $\mathbb{C}$, i.e., f is **entire.**

Step (b): Compute the limit at 0.

From the series or the standard limit, we have:

$$\lim_{z\to 0} f(z) = \lim_{z\to 0} \frac{\sin z}{z} = 1.$$

Thus, $f(0) = 1 = \lim_{z\to 0} f(z)$, confirming continuity at 0.

Theorem 6.2 (Gauss Mean Value Theorem) ***Let f be holomorphic in a domain $U \subset \mathbb{C}$, and let $z_0 \in U$. Then $f(z_0)$ is equal to the average of f on the boundary of any disk centered at z_0 and contained in U. That is, for any disk $D(z_0, r) \subset U$:***

$$f(z_0) = \frac{1}{2\pi} \int_0^{2\pi} f\left(z_0 + re^{i\theta}\right) d\theta.$$

Example 6.2 ***Let***

$$f(z) = z^2$$

which is holomorphic on $\mathbb{C}$. Let $z_0 = 1 + i$ and consider a disk of radius $r > 0$ centered at z_0, $D(z_0, r)$.

According to the Gauss mean value theorem, we have

$$f(z_0) = \frac{1}{2\pi} \int_0^{2\pi} f\left(z_0 + re^{i\theta}\right) d\theta.$$

Computation:

$$f(z_0 + re^{i\theta}) = (z_0 + re^{i\theta})^2 = z_0^2 + 2z_0 re^{i\theta} + r^2 e^{2i\theta}.$$

Now integrate over $\theta \in [0, 2\pi]$***:***

$$\frac{1}{2\pi}\int_0^{2\pi}\left(z_0^2 + 2z_0 re^{i\theta} + r^2 e^{2i\theta}\right)d\theta = z_0^2 + \frac{2z_0 r}{2\pi}\int_0^{2\pi} e^{i\theta}d\theta + \frac{r^2}{2\pi}\int_0^{2\pi} e^{2i\theta}d\theta.$$

Since

$$\int_0^{2\pi} e^{i\theta}d\theta = 0 \quad \textbf{\textit{and}} \quad \int_0^{2\pi} e^{2i\theta}d\theta = 0,$$

we get

$$\frac{1}{2\pi}\int_0^{2\pi} f(z_0 + re^{i\theta})\,d\theta = z_0^2 = f(z_0).$$

Thus, the value at the center equals the average on the circle, as stated by the theorem.

6.2 Calculation of Integrals by the Residue Method

6.2.1 Integrals over Special Contours

The residue theorem is a powerful tool to evaluate complex integrals, especially when the integrals are difficult or impossible to compute by elementary methods. Let us recall:

Some common special contours are:

1. **Circle around the origin:** $C_R : |z| = R$ **If** $f(z)$ **has isolated singularities inside** C_R**, the integral over the circle can be computed using**

$$\int_{C_R} f(z)\,dz = 2\pi i \sum_{\textbf{inside } C_R} \text{Res}(f, z_k)$$

   and it is often useful for computing integrals of the type

$$\int_0^{2\pi} F(e^{i\theta})\,d\theta.$$

2. **Semi-circular contour: Often used to compute integrals on the real axis by closing the contour in the upper or lower half-plane. Let** $f(z)$ **be analytic except at poles in the upper half-plane. Then for** $R \to \infty$**:**

$$\int_{-\infty}^{\infty} f(x)\,dx = 2\pi i \sum_{\textbf{upper poles}} \text{Res}(f, z_k),$$

   provided the integral over the semicircular arc vanishes as $R \to \infty$**.**

3. **Keyhole contour: Useful for integrals involving branch cuts, for example**

$$\int_0^{\infty} \frac{x^{\alpha-1}}{x+1}\,dx, \quad 0 < \alpha < 1.$$

4. Rectangular or other contours: Sometimes used to evaluate sums or integrals using periodicity or decay conditions of $f(z)$.

Example 6.3 *Compute*

$$\int_0^{2\pi} \frac{d\theta}{5+4\cos\theta}.$$

Solution 6.2 *1. Use the substitution* $z = e^{i\theta}$*,* $d\theta = \frac{dz}{iz}$*,* $\cos\theta = \frac{z+z^{-1}}{2}$*. Then*

$$\int_0^{2\pi} \frac{d\theta}{5+4\cos\theta} = \oint_{|z|=1} \frac{dz}{iz(5+2(z+z^{-1}))} = \oint_{|z|=1} \frac{dz}{i(2z^2+5z+2)}.$$

2. Factorize the denominator:

$$2z^2 + 5z + 2 = (2z+1)(z+2)$$

3. Identify the pole inside the unit circle: $2z+1=0 \implies z=-\frac{1}{2}$

4. Compute the residue:

$$\text{Res}\left(\frac{1}{i(2z^2+5z+2)}, z=-\frac{1}{2}\right) = \frac{1}{i(5-2)} = \frac{1}{3i}$$

5. Multiply by $2\pi i$*:*

$$\int_0^{2\pi} \frac{d\theta}{5+4\cos\theta} = 2\pi i \cdot \frac{1}{3i} = \frac{2\pi}{3}.$$

Remark 6.2 *The key steps are: substitution* $z = e^{i\theta}$*, identify the poles inside the contour, compute residues, and multiply by* $2\pi i$*. This method is extremely efficient for integrals over* $[0, 2\pi]$ *or* $(-\infty, \infty)$*.*

6.2.2 Improper Integrals Involving an Exponential

Integrals of the type

$$\int_{-\infty}^{\infty} \frac{P(x)}{Q(x)} e^{i\alpha x}\, dx, \quad \alpha > 0$$

where $P(x)$ and $Q(x)$ are polynomials, can often be evaluated using the residue theorem by extending the integral to the complex plane.

Method

1. Consider the complex function

$$f(z) = \frac{P(z)}{Q(z)} e^{i\alpha z}.$$

2. Choose a contour in the complex plane:

   - If $\alpha > 0$, close the contour in the "upper half-plane" to ensure $e^{i\alpha z} \to 0$ on the large semicircle as $R \to \infty$.
   - If $\alpha < 0$, close in the "lower half-plane"

3. **Identify the poles of $f(z)$ inside the chosen contour.**

4. **Apply the residue theorem:**

$$\int_{-\infty}^{\infty} f(x)\,dx = 2\pi i \sum_{\text{poles inside contour}} \text{Res}(f, z_k),$$

provided that the integral over the semicircle vanishes as $R \to \infty$ (Jordan's lemma).

Example 6.4 ***Compute***

$$\int_{-\infty}^{\infty} \frac{e^{ix}}{x^2+1}\,dx.$$

Solution 6.3 1. Consider

$$f(z) = \frac{e^{iz}}{z^2+1}.$$

2. Poles: $z = i$ and $z = -i$ (simple poles).

3. Close the contour in the upper half-plane ($\alpha = 1 > 0$), so only $z = i$ is inside.

4. Compute the residue at $z = i$:

$$\text{Res}\left(\frac{e^{iz}}{z^2+1}, z = i\right) = \lim_{z\to i}(z-i)\frac{e^{iz}}{(z-i)(z+i)} = \frac{e^{ii}}{2i} = \frac{e^{-1}}{2i}.$$

5. Apply the residue theorem:

$$\int_{-\infty}^{\infty} \frac{e^{ix}}{x^2+1}\,dx = 2\pi i \cdot \frac{e^{-1}}{2i} = \pi e^{-1}.$$

Remark 6.3 ***- The choice of the semicircle direction depends on the sign of the exponent in $e^{i\alpha x}$.***
- Jordan's lemma ensures that the integral over the large semicircle vanishes as $R \to \infty$.
- This method works for integrals of the form $\int_{-\infty}^{\infty} \frac{P(x)}{Q(x)} e^{i\alpha x} dx$ with proper decay at infinity.

Remark 6.4 ***In the context of integrals of the form***

$$\int_{-\infty}^{\infty} \frac{P(x)}{Q(x)} e^{i\alpha x}\,dx,$$

the phrase proper decay at infinity means that the integrand goes to zero sufficiently fast as $|x| \to \infty$ so that the integral converges and contour integration can be applied.

More precisely:

- ***When closing the contour in the upper or lower half-plane, the exponential term $e^{i\alpha z}$ must decay:***

- ***If*** $\alpha > 0$***, we close the contour in the upper half-plane because*** $e^{i\alpha z} = e^{i\alpha x - \alpha y}$ ***decays as*** $y \to \infty$***.***
- ***If*** $\alpha < 0$***, we close the contour in the lower half-plane.***

- ***The rational factor*** $P(z)/Q(z)$ ***must satisfy*** $\deg(Q) \geq \deg(P) + 1$ ***so that*** $|P(z)/Q(z)| \to 0$ ***as*** $|z| \to \infty$***. This ensures that the integral over the semi-circular arc vanishes (by Jordan's lemma).***

In short, proper decayensures that

$$\lim_{|z|\to\infty,\, z \text{ on contour}} \frac{P(z)}{Q(z)} e^{i\alpha z} = 0,$$

allowing the residue theorem to be applied safely.

Exercise 6.1 ***Evaluate the integral***

$$I = \int_0^\infty \frac{e^{-x}}{x^2 + 4x + 5}\, dx$$

using the residue theorem.

Solution 6.4 ***Step 1: Identify the poles.***
We factor the denominator:

$$x^2 + 4x + 5 = (x + 2 - i)(x + 2 + i),$$

so the function

$$f(z) = \frac{e^{-z}}{(z + 2 - i)(z + 2 + i)}$$

has simple poles at

$$z_1 = -2 + i, \quad z_2 = -2 - i.$$

Step 2: Choose the contour.
We consider a contour in the upper half-plane consisting of:

- ***The segment along the real axis from*** 0 ***to*** R***.***
- ***A semicircular arc of radius*** R ***in the upper half-plane connecting*** R ***back to*** 0***.***

This contour is chosen because:

- ***The integral is over*** $[0, \infty)$***.***
- ***The factor*** e^{-z} ***decays in the upper half-plane, so the integral over the arc vanishes as*** $R \to \infty$ ***(Jordan's lemma).***

Step 3: Determine which poles lie inside the contour.

- Only the pole $z_1 = -2 + i$ is in the upper half-plane.
- The pole $z_2 = -2 - i$ lies below the real axis and is outside the contour.

Step 4: Compute the residue at z_1.
For a simple pole:

$$\boldsymbol{Res}(f, z_1) = \lim_{z \to z_1} (z - z_1) f(z) = \frac{e^{-z_1}}{z_1 - z_2} = \frac{e^{-(-2+i)}}{(-2+i) - (-2-i)} = \frac{e^{2-i}}{2i}.$$

Step 5: Apply the residue theorem.
The integral over the full contour is

$$\oint f(z)\, dz = 2\pi i\, \boldsymbol{Res}(f, z_1) = 2\pi i \cdot \frac{e^{2-i}}{2i} = \pi e^{2-i}.$$

Step 6: Justify the arc contribution.
On the semicircular arc $z = Re^{i\theta}, \theta \in [0, \pi]$:

$$\left| \int_{\boldsymbol{arc}} f(z) dz \right| \leq \pi R \cdot \frac{e^{-\Re(z)}}{|z + 2 - i||z + 2 + i|} \to 0 \quad \boldsymbol{as}\ R \to \infty.$$

Thus, the integral over the arc vanishes in the limit.
Step 7: Conclude.

$$\int_0^\infty \frac{e^{-x}}{x^2 + 4x + 5}\, dx = \pi e^{2-i} = \pi e^2 (\cos 1 - i \sin 1).$$

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