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Ministère de l'Enseignement Supérieur et de la Recherche Scientifique

Université Djilali Bounaama, Khemis Miliana
Faculté des Sciences de la Matière et d'informatique
Conseil Scientifique de la Faculté

جامعة جيلالى بونعامة خميس مليانة
كليّة علوم المادة والإعلام الآلي
المجلس العلمي للكلية

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EXTRAIT DU PV
DE LA REUNION ORDINAIRE DU CONSEIL SCIENTIFIQUE
Du 02/05/2026

Objet : : Expertise de polycopié pédagogique

En l'an deux mille vingt-six (2026), le deux (02) mai 09 h 30, une réunion ordinaire du Conseil Scientifique de la Faculté des Sciences de la Matière et de l'Informatique s'est tenue dans la salle de réunion de la faculté (Bloc B).

Suite aux rapports favorables reçus de la part des experts cités ci-après concernant l'expertise du polycopié pédagogique, le CSF a prononcé favorablement pour la conformité du polycopié pédagogique en vue de préparer son professorat.

- **Auteur du polycopié** : Dr. AYADI Souad (MCA)
- **Intitulé du polycopié** : Series and Differential Equations : Course and Exercises
- **Destiné aux étudiants de** : L2 Physique
- **Experts du polycopié** :

  - BOUCENNA Amina MCA ENS - Kouba
  - CHITA Fouzia MCA UDB - Khemis Miliana

Président du
Conseil Scientifique de la Faculté SMI
Dr. BOUDERBALA Mihoub

رئيس المجلس العلمي
الامضاء: بودربالة ميهوب

University of Khemis Miliana - Djilali Bounaama
Faculty of Science Materials and Computer Science
Department of Physics

Series & Differential Equations

Course and Exercises

2nd Year Licence (LMD System)

Physics Track – Material Sciences

First Semester

Author: **AYADI SOUAD**

Academic Year: **2025–2026**

Contents

Introduction

The present pedagogical handout "**Series and Differential Equations**" is intended for second-year undergraduate students (LMD system), Physics track, within the field of Material Sciences. It is part of the fundamental unit **UEF-3** of the first semester.

This support aims to provide students with a comprehensive and structured resource in order to facilitate autonomous learning and the progressive assimilation of fundamental concepts. Through a balance of theory, detailed examples, and solved exercises, it is designed to strengthen students' analytical skills and to enable them to apply the studied methods in various contexts of physics and mathematics.

The main objective of this module is to enable students to master the techniques of differential and integral calculus essential in physics. More specifically, the targeted skills are:

- To master the different methods of computation and evaluation of single, multiple, definite, and improper integrals.
- To acquire analytical methods for solving differential equations.
- To apply Laplace and Fourier transforms in solving ordinary differential equations as well as certain partial differential equations.
- To discover numerical series, study their properties, and analyze some convergence criteria.

The manuscript is divided into six chapters corresponding to the program of this fundamental unit, while strictly following the content and the sequence of chapters as defined by the official framework provided by the Ministry. The document begins with a study of single and multiple integrals, aiming to provide a thorough understanding of integration methods, their applications, and the techniques used to solve various types of problems. Improper integrals are also discussed, with particular attention to cases where the limits of integration are infinite. A substantial part of this handout is devoted to differential equations, covering various aspects, including first-order linear differential equations. In addition, second-order linear differential equations are studied in detail, with special emphasis on their practical applications. This manuscript also includes a section devoted to numerical series, power series and Fourier series, presenting their properties, convergence criteria, and applications in various mathematical fields. Moreover, an in-depth study of the Fourier transform is provided, detailing its definition, properties, and applications in the analysis of signals and periodic functions. The Laplace transform is also discussed extensively, highlighting its applications in solving differential equations and other mathematical and scientific problems.

Finally, each chapter is enriched with a collection of solved exercises, offering readers the opportunity to consolidate their understanding of the presented concepts and to improve their problem-solving skills. I hope that this document will effectively support students in their learning and meet their expectations, helping them to gain a solid mastery of this course.

Finally, I would be very grateful to all readers-both students and instructors-who wish to share their feedback, comments, or suggestions regarding the content or the form of this manuscript. You may contact me at: `souad.ayadi@univ-dbkm.dz`

Chapter 1

Simple and Multiple Integrals

The goal of this chapter is to review fundamental notions about integrals and primitives(antiderivativ which are essential tools for differential equations and applications in physics

1.1 Review of the Riemann Integral and the Calculation of Primitives

The concept of the integral arises naturally in physics and mathematics whenever we want to compute a continuous quantity as the limit of discrete approximations. For instance, one can obtain the total mass from a density distribution, the total charge from a charge density, or the work from a force applied along a trajectory. The Riemann integral provides the rigorous framework for this transition from discrete sums to continuous accumulation.

Example: The work W done by a force $F(x)$ along a straight line:

$$W = \int_{x_0}^{x_1} F(x)\,dx$$

We can approximate this work by dividing the interval $[x_0, x_1]$ into small subintervals of width Δx. For each subinterval, we approximate the work by $F(x_i)\,\Delta x$. Taking the limit as $\Delta x \to 0$ gives the integral:

$$W = \lim_{\Delta x \to 0} \sum_i F(x_i)\,\Delta x$$

This provides an intuitive definition of the integral as the limit of the sum of small local contributions.

1.1.1 Function Riemann integrable

Let us consider some continuous positive function f defined on a bounded interval $[a\ b]$ where a and b are real numbers. A **partition** of $[a\ b]$ is a finite sequence

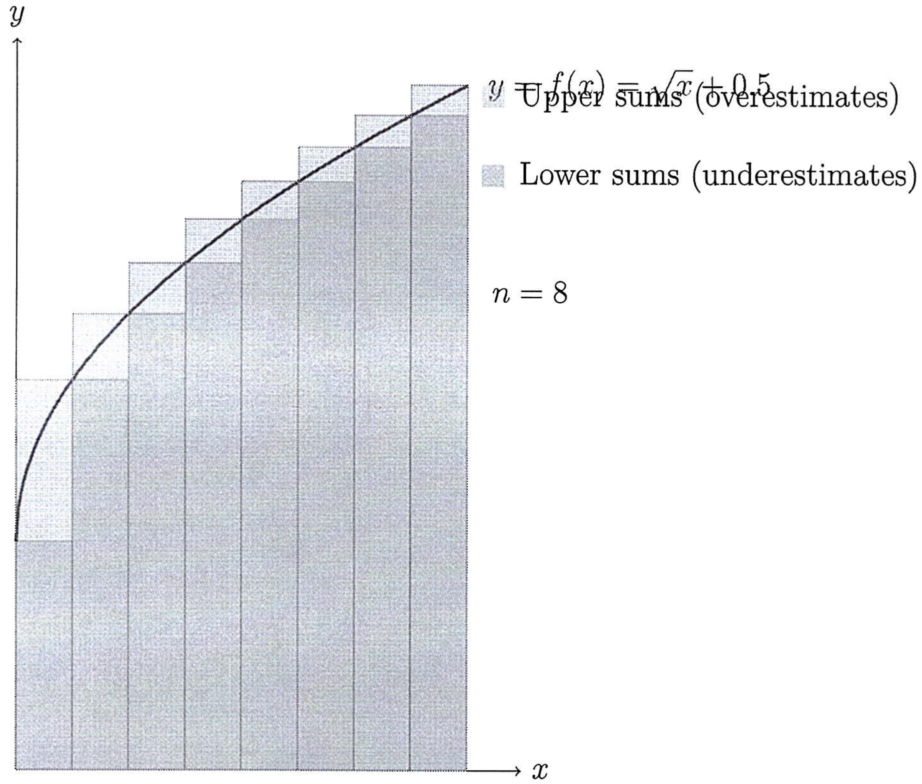
$$P = \{a = x_0 < x_1 < x_2 < \cdots < x_n = b\}.$$

For each subinterval $[x_{i-1}\ x_i]$, we define

$$m_i = \inf_{x\in[x_{i-1}\ x_i]} |f(x)|, \qquad M_i = \sup_{x\in[x_{i-1}\ x_i]} |f(x)|.$$

We then define the lower sum and upper sum

$$L(f, P) = \sum_{i=1}^{n} (x_i - x_{i-1})\, m_i, \quad U(f, P) = \sum_{i=1}^{n} (x_i - x_{i-1})\, M_i,$$



Definition 1.1 *The positive function f is Riemann integrable if*

$$\sup_P L(f, P) = \inf_P U(f, P),$$

and we call this common value the Riemann integral of f over $[a\ b]$ and ti is denoted by $\int_a^b f(x)dx$. Geometrically, the Riemann integral represents the area limited by the abscise axis and the curve with equation $y = f(x)$ and the straight lines with equations $x = a$ and $x = b$.

Definition 1.2 *A function $f : [a, b] \to \mathbb{R}$ is called* ***Riemann-integrable*** *if it is bounded and if the following limit exists:*

$$\lim_{\|P\| \to 0} \sum_{i=1}^{n} f(\xi_i)\, \Delta x_i,$$

where:

- $P = \{x_0, x_1, \ldots, x_n\}$ *is a partition of* $[a, b]$,
- $\Delta x_i = x_i - x_{i-1}$,
- $\xi_i \in [x_{i-1}, x_i]$ *is any sample point in the subinterval, and*

- $\|P\| = \max_i \Delta x_i$ *is the norm of the partition (the length of the largest subinterval).*

If this limit exists and is independent of the choice of points ξ_i*, it is called the* ***Riemann integral*** *of* f *over* $[a, b]$*, denoted by*

$$\int_a^b f(x)\,dx.$$

Remark 1.1 *Functions Do Not Need to Be Positive to Be Riemann-Integrable.In fact A function* $f : [a, b] \to \mathbb{R}$ *does* ***not need to be positive*** *to have a Riemann integral. What matters is that* f *is* ***bounded*** *and that the limit of its Riemann sums exists as the partition is refined. In other words, Riemann-integrability depends on* ***boundedness*** *and the existence of the limit of Riemann sums,* ***not on the sign*** *of the function.*

Examples:

- $f(x) = x$ *on* $[-1, 1]$ *changes sign (negative for* $x < 0$*, positive for* $x > 0$*), but it is Riemann-integrable:*

$$\int_{-1}^{1} x\,dx = 0$$

- $f(x) = \sin(x)$ *on* $[0, 2\pi]$ *also changes sign, yet it is Riemann-integrable:*

$$\int_0^{2\pi} \sin(x)\,dx = 0$$

- *The Riemann integral always produces a number, which represents a cumulative quantity of the function over the interval.*

- *If the function* $f(x)$ *is* ***non-negative*** *on* $[a, b]$*, the integral corresponds to the* ***geometric area*** *under the curve.*

- *If the function* $f(x)$ ***changes sign****, the integral represents the* ***net area*** *(algebraic sum of positive and negative contributions), and not the total geometric area.*

- *Therefore, the Riemann integral does* ***not always measure the literal surface****, but it always gives a meaningful accumulated value of the function over the interval.*

Examples of Geometric Area, Net Area, and Total Geometric Area

Example 1: Geometric Area (Positive Function)

Consider the function

$$f(x) = x^2 \quad \text{on } [0, 1].$$

Since $f(x) \geq 0$ *for all* $x \in [0, 1]$*, the Riemann integral represents the actual geometric area under the curve:*

$$\int_0^1 x^2\,dx = \frac{1}{3}.$$

Example 2: Net Area (Function Changing Sign)

Consider the function

$$f(x) = \begin{cases} 1, & 0 \leq x \leq 1, \\ -1, & 1 < x \leq 2 \end{cases}$$

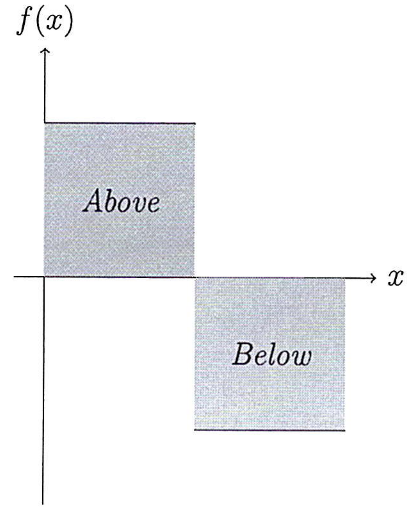
The integral gives the **net area**, *taking positive contributions above the x-axis and negative contributions below the x-axis: Although the geometric area of each rectangle is 1, the integral sums them algebraically, giving a net area of 0. Indeed:*

The geometric areas are:

- *Area above x-axis:* $1 \times 1 = 1$
- *Area below x-axis:* $1 \times 1 = 1$

The **net area** *(Riemann integral) is:*

$$\int_0^2 f(x)\,dx = 1 + (-1) = 0$$



Example 3: Total Geometric Area (Ignoring Sign)

The **total geometric area** *is the actual area under the curve,* **ignoring the sign** *of* $f(x)$. *We sum all parts, whether the function is above or below the x-axis.*

Consider the same function as in Example 2:

$$f(x) = \begin{cases} 1, & 0 \leq x \leq 1, \\ -1, & 1 < x \leq 2 \end{cases}$$

The total geometric area is

$$\int_0^1 |f(x)|\,dx + \int_1^2 |f(x)|\,dx = 1 + 1 = 2.$$

Here, we take the absolute value of the function to compute the **real area under the curve.**

Proposition 1.1 *Every continuous function on a closed interval* $[a, b]$ *is Riemann integrable.*

Remark 1.2 *This is a very useful property because it guarantees that most functions encountered in physics and engineering are automatically Riemann integrable.*

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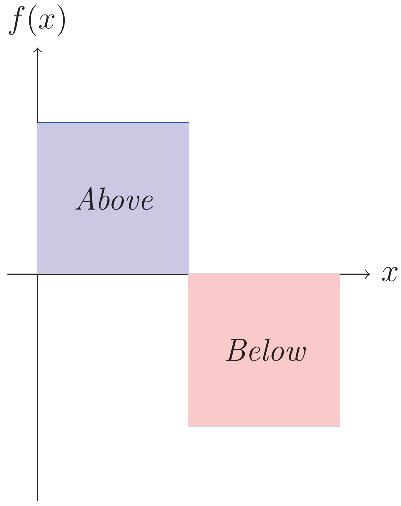

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Properties of the Riemann Integral

Let f and g be Riemann-integrable functions on $[a, b]$, and let $\alpha, \beta \in \mathbb{R}$. The main properties are:

1. **Linearity:**
$$\int_a^b \Big(\alpha f(x) + \beta g(x)\Big)\,dx = \alpha \int_a^b f(x)\,dx + \beta \int_a^b g(x)\,dx$$
Interpretation: Constants can be factored out, and the integral of a sum is the sum of the integrals.

2. **Positivity:** If $f(x) \geq 0$ for all $x \in [a, b]$, then
$$\int_a^b f(x)\,dx \geq 0$$
Interpretation: The area under the curve is non-negative when the function is above the x-axis.

3. **Order (Monotonicity):** If $f(x) \leq g(x)$ for all $x \in [a, b]$, then
$$\int_a^b f(x)\,dx \leq \int_a^b g(x)\,dx$$
Interpretation: A function that is larger everywhere has a larger integral.

4. **Additivity over intervals:** For $a < c < b$,
$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$
Interpretation: The integral over an interval can be split into the sum of integrals over subintervals.

5. **Inequality with absolute value:**
$$\left|\int_a^b f(x)\,dx\right| \leq \int_a^b |f(x)|\,dx$$
Interpretation: The integral of the absolute value is always greater than or equal to the absolute value of the integral.

6.

Mean Value of a Riemann Integrable Function

Let f be a Riemann integrable function on the interval $[a, b]$. The **mean value or average value** of f over $[a, b]$ is defined by:

$$f_{\text{avg}} = m = \frac{1}{b-a} \int_a^b f(x)\,dx.$$

Explanation: This quantity represents the constant value that the function would need to take on the entire interval $[a, b]$ to produce the same total integral as $f(x)$ does. It is widely used in physics to compute mean quantities, such as mean velocity, mean force, or mean energy.

Remark 1.3 *If $f(x)$ is non-negative, f_{avg} can also be interpreted as the height of a rectangle over $[a,b]$ whose area equals the area under the curve of f.*

Root Mean Square (RMS) of a Riemann Integrable Function

Let f be a Riemann integrable function on the interval $[a,b]$. The **root mean square (RMS) value** of f over $[a,b]$ is defined by:

$$f_{\text{RMS}} = \sqrt{\frac{1}{b-a}\int_a^b [f(x)]^2\,dx}.$$

Explanation: The RMS value is a measure of the *effective magnitude* of a function. It is widely used in physics and engineering, for example to compute the effective voltage, current, or other quantities that vary in time.

Remark 1.4 *For a constant function $f(x) = C$, we have $f_{RMS} = |C|$. This generalizes the idea of magnitude for varying functions.*

1.1.2 Indefinite Integrals

Primitives

Example 1.1 (Introductory Example) *Consider the functions f and F defined on the interval $]0,+\infty[$ by*

$$F(x) = \ln(x) + \frac{1}{2}x^2 \quad \textit{and} \quad f(x) = \frac{1}{x} + x.$$

We can easily see that both functions are continuous. Moreover, F is differentiable on $]0,+\infty[$ and

$$\forall x \in]0,+\infty[\,, \quad F'(x) = f(x).$$

Such a function F is called an ***antiderivative*** *(or* ***primitive****) of f on I.*

Definition 1.3 *Let $f(x)$ be a continuous function defined on an interval I. A function $F(x)$ is called an* ***indefinite integral*** *or* ***primitive*** *of f if*

$$F'(x) = f(x) \quad \textit{for all } x \in I.$$

Example 1.2 *The function $F : x \longmapsto F(x) = \frac{1}{2}e^{2x+1}$ is a primitive of the function $f : x \longmapsto f(x) = e^{2x+1}$ because $F'(x) = e^{2x+1}$.*

Notation: The indefinite integral of f is denoted by

$$\int f(x)\,dx = F(x) + k,$$

where k is an arbitrary constant of integration.

Remark 1.5 *The indefinite integral represents the* family of all antiderivatives *of the function f. That is if F is a primitive of the continuous function f then $\int f(x)\,dx = F(x) + k$.*

Example 1.3 *Let* $F(x) = x\ln(x) - x$*, defined for* $x > 0$*.*
1. Prove that F *is a primitive of the function* $f(x) = \ln(x)$*.*
2. Find the set of all primitives of $f(x)$*.*

Solution 1.1 1. *By the product rule,*

$$F'(x) = \frac{d}{dx}[x\ln(x) - x] = 1 \cdot \ln(x) + x \cdot \frac{1}{x} - 1 = \ln(x) = f(x).$$

Thus, F *is a primitive of* $f(x) = \ln(x)$ *on* $(0, +\infty)$*.*

2. *Using the indefinite integral notation, we write*

$$\int \ln(x)\,dx = x\ln(x) - x + C, \quad C \in \mathbb{R}.$$

This expression represents the set of all primitives of $\ln(x)$ *on* $(0, +\infty)$*.*

Proposition 1.2 *Let* f *be a continuous function on* $I \subset \mathbb{R}$*.*

1. *If* F *is an antiderivative of* f*, then* $F + k$ *is also an antiderivative of* f*, where* k *is a constant function on* I*.*

2. *If* F *and* G *are two antiderivatives of* f*, then* $F - G$ *is constant. In other words, if* F *is an antiderivative of* f*, any other antiderivative* G *of* f *is of the form* $G = F + k$*, where* k *is constant.*

3. *If* F *is an antiderivative of* f *and* $x_0 \in I$*, then there exists a unique antiderivative* G *of* f *such that* $G(x_0) = k$*.*

Example 1.4 *Let* $F(x) = (x-1)e^x$ *defined* $\mathbb{R}$ *be a primitive of* $f(x) = xe^x$*.*
Find the primitive of f *which is equal to* 3 *at* 0

Solution 1.2 *Since* $F(x) = (x-1)e^x$ *is a primitive of* $f(x) = xe^x$*, then any other primitive* G *of* f *is of the form* $G(x) = F(x) + C$ *for all* $x \in \mathbb{R}$ *and where* C *is an arbitrary real number. Hence,* $G(0) = F(0) + C = -1 + C$*.*
Therefore,

$$G(0) = 3 \Longleftrightarrow -1 + C = 3,$$

that is $C = 4$*. and the required primitive is* $G(x) = (x-1)e^x + 4$*.*

Properties

1. Linearity: $\int \Big(af(x) + bg(x)\Big)\,dx = a\int f(x)\,dx + b\int g(x)\,dx$, with $a, b \in \mathbb{R}$.

2. Integration of a derivative: $\int F'(x)\,dx = F(x) + k$.

Example 1.5 $\int(-5x^2)\,dx = -5\int x^2\,dx$

Example 1.6 $\int(-5x^2 + \frac{1}{x})\,dx = \int -5x^2\,dx + \int \frac{1}{x}\,dx$

Some Immediate Indefinite Integrals

$\int 0\,dx = k, \quad k \in \mathbb{R}$	$\int \frac{1}{\cos^2(x)}\,dx = \tan(x) + k$
$\int a\,dx = ax + k, \quad k \in \mathbb{R}$	$\int \frac{1}{\sin^2(x)}\,dx = -\cot(x) + k$
$\int x^m\,dx = \frac{x^{m+1}}{m+1} + k, \quad k \in \mathbb{R}, m \neq -1$	$\int \frac{1}{\sqrt{1-x^2}}\,dx = \arcsin(x) + k$
$\int \frac{1}{x}\,dx = \ln(\lvert x \rvert) + k, \quad k \in \mathbb{R}$	$\int \frac{-1}{\sqrt{1-x^2}}\,dx = \arccos(x) + k$
$\int e^x\,dx = e^x + k, \quad k \in \mathbb{R}$	$\int \frac{1}{1+x^2}\,dx = \arctan(x) + k$
$\int \sin(x)\,dx = -\cos(x) + k, \quad k \in \mathbb{R}$	$\int \frac{-1}{1+x^2}\,dx = \operatorname{arccotan}(x) + k$
$\int \cos(x)\,dx = \sin(x) + k, \quad k \in \mathbb{R}$	$k \in \mathbb{R}$

Remark 1.6 *In the general case,*

$$\int (f(x) \times g(x))\,dx \neq \int f(x)\,dx \times \int g(x)\,dx.$$

The question we ask is: how can we compute an antiderivative of a product of two functions?

Integration by Parts

Integration by parts is a method used to find the integral of a product of two functions.

Theorem 1.1 (Integration by Parts) *Let $u(x)$ and $v(x)$ be differentiable functions. Then,*

$$\int u(x)\,v'(x)\,dx = u(x)v(x) - \int u'(x)\,v(x)\,dx.$$

Remark 1.7 *The formula is derived from the product rule for differentiation:*

$$(u(x)v(x))' = u'(x)v(x) + u(x)v'(x).$$

Example 1.7 *Compute $\int xe^x\,dx$.*

Solution: *Take $u = x$ and $dv = e^x dx$. Then $du = dx$ and $v = e^x$. Applying the formula:*

$$\int xe^x\,dx = xe^x - \int 1 \cdot e^x\,dx = xe^x - e^x + k.$$

Although integration by parts is an important tool for computing the antiderivatives of products of functions, it does not allow us to calculate all types of function products. Another widely used integration technique is integration by substitution (change of the integration variable).

Integration by Substitution (Change of Variable)

The following integration formulas follow directly from the rules of differentiation by using a simple change of variable.
We set $y = f(x)$, which implies

$$\frac{dy}{dx} = f'(x),$$

hence

$$f'(x)\,dx = dy.$$

The following table is obtained by using the technic of variable change

$$\int (f(x))^{n+1} f^{'}(x)\,dx = \frac{(f(x))^{n+1}}{n+1} + k, \quad k \in \mathbb{R}.$$

$$\int f^{'}(x) e^{f(x)} dx = e^{f(x)} + k, \quad k \in \mathbb{R}.$$

$$\int \frac{f^{'}(x)}{f(x)} = \ln|f(x)| + k, \quad k \in \mathbb{R}.$$

$$\int f^{'}(x)\sin(f(x))\,dx = -\cos(f(x)) + k, \quad k \in \mathbb{R}.$$

$$\int f^{'}(x)\cos(f(x))\,dx = -\sin(f(x)) + k, \quad k \in \mathbb{R}.$$

$$\int \frac{f^{'}(x)}{1+f^2(x)} = \arctan(f(x)) + k, \quad k \in \mathbb{R}.$$

$$\int \frac{f^{'}(x)}{\sqrt{1-f^2(x)}} = \arcsin(f(x)) + k, \quad k \in \mathbb{R}.$$

$$\int \frac{-f^{'}(x)}{\sqrt{1-f^2(x)}} = \arccos(f(x)) + k, \quad k \in \mathbb{R}.$$

Exercise 1.1 *Prove the results of the table above.*

Examples 1.1 *1)* $\int (2x+5)^3\,dx = \frac{1}{2}\int 2(2x+5)^3\,dx = \frac{1}{8}(2x+5)^4 + k, \quad k \in \mathbb{R}.$

2) $\int e^{3x+1}\,dx = \frac{1}{3}\int 3e^{3x+1}\,dx = \frac{1}{3}e^{3x+1} + k, \quad k \in \mathbb{R}.$

3) $\int \frac{2x}{1+(x)^2} = \arctan(x^2) + k, \quad k \in \mathbb{R}.$

4) $\int \frac{2x}{\sqrt{1-(x)^2}} = \arcsin(x^2) + k, \quad k \in \mathbb{R}.$

We can generalize this result as follows:

Proposition 1.3 *Let g be a continuous function on $I \subset \mathbb{R}$ and G its antiderivative on I, and let f be a differentiable function on I with derivative f'. Then*

$$\int f'(x) g(f(x))\,dx = G(f(x)) + k, \quad k \in \mathbb{R}.$$

Proof. Set $u = f(x)$, hence $\frac{du}{dx} = f'(x)$, which gives $du = f'(x)\,dx$.
It follows that

$$\int f'(x) g(f(x))\,dx = \int g(u)\,du = G(u) + k = G(f(x)) + k, \quad k \in \mathbb{R}.$$

■

Example 1.8 *Compute*

$$\int x(3x+4)^5\,dx.$$

Set $u = 3x+4$, *so that* $\dfrac{du}{dx} = 3$, *hence* $dx = \dfrac{1}{3}du$.

Moreover, $x = \dfrac{u-4}{3} = \dfrac{1}{3}u - \dfrac{4}{3}$.
Then we obtain

$$\begin{aligned}\int x(3x+4)^5\,dx &= \int\left(\frac{1}{3}u - \frac{4}{3}\right)u^5\cdot\frac{1}{3}\,du\\ &= \frac{1}{9}u^6 - \frac{4}{9}u^5 + k, \quad k\in\mathbb{R}\\ &= \frac{1}{9}(3x+4)^6 - \frac{4}{9}(3x+4)^5 + k, \quad k\in\mathbb{R}.\end{aligned}$$

Unfortunately, the previous rules are not always directly applicable. In some cases, we need to simplify the function before calculating its antiderivative. This is, for example, the case for rational functions.

Integration of the function $x \longmapsto \dfrac{1}{ax^2+bx+c}$

We aim to calculate the antiderivatives of the function $x \mapsto \dfrac{1}{ax^2+bx+c}$, $a, b, c \in \mathbb{R}$.
First case: If $a = 0, b \neq 0, c \in \mathbb{R}$.

$$\int \frac{1}{bx+c}\,dx = \frac{1}{b}\int\frac{b}{bx+c}\,dx = \frac{1}{b}\ln|bx+c| + k, \quad k\in\mathbb{R}.$$

Example 1.9 $\displaystyle\int\frac{1}{2x+1}\,dx = \frac{1}{2}\ln|2x+1| + k, \quad k\in\mathbb{R}.$

Second case: If $a \neq 0, b, c \in \mathbb{R}$
We have

$$ax^2+bx+c = a\left[\left(x+\frac{b}{2a}\right)^2 - \frac{\Delta}{4a^2}\right].$$

The integral to calculate depends on Δ, and we distinguish three situations.

1. If $\Delta < 0$ we set $\Delta = -\rho,\ \rho > 0$. Then we have:

$$ax^2+bx+c = \frac{\rho}{4a}\left[\left(\frac{2a}{\sqrt{\rho}}x + \frac{b}{\sqrt{\rho}}\right)^2 + 1\right],$$

hence

$$\frac{1}{ax^2+bx+c} = \frac{4a}{\rho}\frac{1}{\left(\frac{2a}{\sqrt{\rho}}x + \frac{b}{\sqrt{\rho}}\right)^2 + 1}.$$

Using the change of variable $u = \dfrac{2a}{\sqrt{\rho}}x + \dfrac{b}{\sqrt{\rho}}$,

we have $dx = \dfrac{\sqrt{\rho}}{2a}\,du$. Therefore,

$$\begin{aligned}
\int \frac{1}{ax^2+bx+c}\,dx &= \frac{2}{\sqrt{\rho}}\int \frac{1}{u^2+1}\,du \\
&= \frac{2}{\sqrt{\rho}}\arctan(u) + k, \quad k \in \mathbb{R}, \\
&= \frac{2}{\sqrt{\rho}}\arctan\left(\frac{2a}{\sqrt{\rho}}x + \frac{b}{\sqrt{\rho}}\right) + k, \quad k \in \mathbb{R}.
\end{aligned}$$

Example 1.10 *Calculate* $\displaystyle\int \frac{1}{2x^2+x+5}\,dx.$

$\Delta = -39, \rho = 39, a = 2, u = \dfrac{4}{\sqrt{39}}x + \dfrac{1}{\sqrt{39}}$

$$\begin{aligned}
\int \frac{1}{2x^2+x+5}\,dx &= \frac{2}{\sqrt{39}}\int \frac{1}{u^2+1}\,du, \\
&= \frac{2}{\sqrt{39}}\arctan(u) + k, \quad k \in \mathbb{R}, \\
&= \frac{2}{\sqrt{39}}\arctan\left(\frac{4}{\sqrt{39}}x + \frac{1}{\sqrt{39}}\right) + k, \quad k \in \mathbb{R}.
\end{aligned}$$

2. If $\Delta = 0$, then $ax^2+bx+c = a\left(x + \dfrac{b}{2a}\right)^2$.

   Using the change of variable $u = x + \dfrac{b}{2a}$, $du = dx$, we have

$$\begin{aligned}
\int \frac{1}{ax^2+bx+c}\,dx &= \int \frac{1}{a\left(x+\frac{b}{2a}\right)^2}\,dx, \\
&= \frac{1}{a}\int \frac{1}{u^2}\,du, \\
&= \frac{1}{a}\int u^{-2}\,du, \\
&= \frac{1}{a}\frac{u^{-1}}{-1} + k, \quad k \in \mathbb{R}, \\
&= -\frac{1}{a}\frac{1}{u} + k, \quad k \in \mathbb{R}, \\
&= -\frac{1}{a}\frac{1}{x+\frac{b}{2a}} + k, \quad k \in \mathbb{R}.
\end{aligned}$$

Example 1.11

$$\begin{aligned}
\int \frac{1}{3x^2+6x+3}\,dx &= \int \frac{1}{3\,(x-1)^2}\,dx,\\
&= \frac{1}{3}\int \frac{1}{u^2}\,du,\\
&= \frac{1}{3}\left(\frac{u^{-1}}{-1}\right)+k, \quad k\in\mathbb{R},\\
&= \frac{-1}{3\,(x-1)}+k, \quad k\in\mathbb{R}.
\end{aligned}$$

3. If $\Delta > 0$, then $ax^2+bx+c = a\,(x-x_1)\,(x-x_2)\,.$

   We decompose the fraction $\dfrac{1}{ax^2+bx+c}$ as

$$\frac{1}{ax^2+bx+c} = \frac{1}{a}\left[\frac{A}{x-x_1}+\frac{B}{x-x_2}\right],$$

   where A, B are real numbers determined after reducing to a common denominator and identification.

$$\begin{aligned}
\int \frac{1}{ax^2+bx+c}\,dx &= \int \frac{1}{a}\left[\frac{A}{x-x_1}+\frac{B}{x-x_2}\right]dx,\\
&= \frac{1}{a}\left[\int \frac{A}{x-x_1}\,dx+\int \frac{B}{x-x_2}\,dx\right],\\
&= \frac{A}{a}\ln|x-x_1|+\frac{B}{a}\ln|x-x_2|+k, \quad k\in\mathbb{R}.
\end{aligned}$$

Example 1.12 *Calculate*

$$\int \frac{1}{x^2-5x+6}\,dx$$

We have

$$x^2-5x+6=(x-2)(x-3).$$

It follows that

$$\int \frac{1}{x^2-5x+6}\,dx = A\ln|x-2|+B\ln|x-3|+k,$$

where A *and* B *satisfy*

$$\begin{aligned}
\frac{1}{(x-2)(x-3)} &= \frac{A}{x-2}+\frac{B}{x-3},\\
&= \frac{Ax-3A+Bx-2B}{x^2-5x+6},\\
&= \frac{(A+B)x-3A-2B}{x^2-5x+6},\\
&\Longrightarrow [(A+B)=0]\wedge[-3A-2B=1]\,,\\
&\Longrightarrow [A=-B]\wedge[-3A-2B=1]\,,\\
&\Longrightarrow [A=-1\wedge B=1]\,.
\end{aligned}$$

Hence,

$$\int \frac{1}{x^2-5x+6}\,dx = -\ln|x-2|+\ln|x-3|+k = \ln\left|\frac{x-3}{x-2}\right|+k, \quad k\in\mathbb{R}.$$

Integration of the function $x \longmapsto \dfrac{Ax+B}{ax^2+bx+c}$, $a \neq 0, A \neq 0$

We start by simplifying the fraction $\dfrac{Ax+B}{ax^2+bx+c}$ by making the derivative of the denominator appear. We have:

$$\begin{aligned} Ax+B &= Ax+B+\frac{Ab}{2a}-\frac{Ab}{2a} \\ &= \left(Ax+\frac{Ab}{2a}\right)+\left(B-\frac{Ab}{2a}\right) \\ &= \frac{A}{2a}(2ax+b)+\left(B-\frac{Ab}{2a}\right), \end{aligned}$$

hence

$$\begin{aligned} \frac{Ax+B}{ax^2+bx+c} &= \frac{\frac{A}{2a}(2ax+b)+\left(B-\frac{Ab}{2a}\right)}{ax^2+bx+c} \\ &= \frac{A}{2a}\frac{2ax+b}{ax^2+bx+c}+\left(B-\frac{Ab}{2a}\right)\frac{1}{ax^2+bx+c}. \end{aligned}$$

Therefore,

$$\int \frac{Ax+B}{ax^2+bx+c}\,dx = \frac{A}{2a}\int \frac{2ax+b}{ax^2+bx+c}\,dx + \left(B-\frac{Ab}{2a}\right)\int \frac{1}{ax^2+bx+c}\,dx.$$

By using direct integration, and letting

$$I_1 = \int \frac{1}{ax^2+bx+c}\,dx,$$

we get

$$\int \frac{Ax+B}{ax^2+bx+c}\,dx = \frac{A}{2a}\ln\left|ax^2+bx+c\right| + \left(B-\frac{Ab}{2a}\right)I_1 + k, \quad k \in \mathbb{R}.$$

Remark 1.8 *We recall that I_1 can be computed using one of the methods described in the previous subsection.*

Example 1.13

$$\int \frac{3x+4}{2x^2-5x+6} = \frac{3}{4}\ln\left|2x^2-5x+6\right| + \frac{31}{4}\int \frac{1}{2x^2-5x+6}\,dx + k, \quad k \in \mathbb{R}.$$

Integration of the function $x \longmapsto \dfrac{1}{\sqrt{ax^2+bx+c}}$, $a \neq 0$

1. If $a > 0$ and $\Delta = 0$, then

$$\sqrt{ax^2+bx+c} = \sqrt{a}\left|x+\frac{b}{2a}\right|,$$

   and we have:

$$\int \frac{1}{\sqrt{ax^2+bx+c}}\,dx = \frac{1}{\sqrt{a}}\int \frac{1}{\left|x+\frac{b}{2a}\right|}\,dx.$$

2. If $a > 0$ and $\Delta < 0$, then

$$\sqrt{ax^2+bx+c} = \sqrt{a}\sqrt{\left(x+\frac{b}{2a}\right)^2+m^2}, \quad \text{where } m^2 = \frac{-\Delta}{4a^2}.$$

$$\begin{aligned}
\int \frac{1}{\sqrt{ax^2+bx+c}}\,dx &= \int \frac{1}{\sqrt{a}\sqrt{\left(x+\frac{b}{2a}\right)^2+m^2}}\,dx \\
&= \frac{1}{\sqrt{a}}\int \frac{1}{\sqrt{u^2+m^2}}\,du \\
&= \frac{1}{\sqrt{a}}\ln\left|u+\sqrt{u^2+m^2}\right| + k, \quad k \in \mathbb{R} \\
&= \frac{1}{\sqrt{a}}\ln\left|\left(x+\frac{b}{2a}\right)+\sqrt{\left(x+\frac{b}{2a}\right)^2-\frac{\Delta}{4a^2}}\right| + k, \quad k \in \mathbb{R}.
\end{aligned}$$

A simple calculation shows that the derivative of the function $u \longmapsto \ln\left|u+\sqrt{u^2+m^2}\right|$ is $u \longmapsto \dfrac{1}{\sqrt{u^2+m^2}}$.

Example 1.14

$$\int \frac{1}{\sqrt{5x^2+1}}\,dx = \frac{1}{\sqrt{5}}\ln\left|x+\sqrt{x^2+\frac{1}{5}}\right| + k, \quad k \in \mathbb{R}.$$

3. If $a > 0$ and $\Delta > 0$, then $ax^2+bx+c > 0$ on $]-\infty, x_1[\,\cup\,]x_2, +\infty[$, and we have

$$\sqrt{ax^2+bx+c} = \sqrt{a}\sqrt{\left(x+\frac{b}{2a}\right)^2-m^2}, \quad \text{where } m^2 = \frac{\Delta}{4a^2}.$$

$$\begin{aligned}
\int \frac{1}{\sqrt{ax^2+bx+c}}\,dx &= \int \frac{1}{\sqrt{a}\sqrt{\left(x+\frac{b}{2a}\right)^2-m^2}}\,dx \\
&= \frac{1}{\sqrt{a}}\int \frac{1}{\sqrt{u^2-m^2}}\,du \\
&= \frac{1}{\sqrt{a}}\ln\left|u+\sqrt{u^2-m^2}\right| + k, \quad k \in \mathbb{R} \\
&= \frac{1}{\sqrt{a}}\ln\left|\left(x+\frac{b}{2a}\right)+\sqrt{\left(x+\frac{b}{2a}\right)^2-\frac{\Delta}{4a^2}}\right| + k, \quad k \in \mathbb{R}.
\end{aligned}$$

A simple calculation shows that the derivative of the function

$$u \longmapsto \ln\left|u+\sqrt{u^2-m^2}\right|$$

is the function

$$u \longmapsto \frac{1}{\sqrt{u^2-m^2}}.$$

Example 1.15

$$\int \frac{1}{\sqrt{4x^2-1}}\,dx = \frac{1}{2}\ln\left|x+\sqrt{x^2-\frac{1}{4}}\right| + k, \quad k \in \mathbb{R}.$$

4. If $a < 0$ and $\Delta < 0$ then $\sqrt{ax^2+bx+c}$ is not defined.
5. If $a < 0$ and $\Delta = 0$ then $\sqrt{ax^2+bx+c}$ is not defined on $\mathbb{R} \setminus \left\{-\frac{b}{2a}\right\}$.
6. If $a < 0$ and $\Delta > 0$ then $\sqrt{ax^2+bx+c}$ is defined on the interval $]x_1, x_2[$. Moreover,

$$ax^2+bx+c = a\left[\left(x+\frac{b}{2a}\right)^2 - \frac{\Delta}{4a^2}\right].$$

In this case, let $a = -a'$, with $a' > 0$, and we get

$$\begin{aligned} ax^2+bx+c &= -a'\left[\left(x+\frac{b}{2a}\right)^2 - \frac{\Delta}{4a^2}\right] \\ &= a'\left[-\left(x+\frac{b}{2a}\right)^2 + \frac{\Delta}{4a^2}\right] \\ &= \frac{\Delta}{4a'}\left[1-\left(\frac{2a'}{\sqrt{\Delta}}x - \frac{b}{\sqrt{\Delta}}\right)^2\right]. \end{aligned}$$

Setting

$$u = \frac{2a'}{\sqrt{\Delta}}x - \frac{b}{\sqrt{\Delta}}, \quad dx = \frac{\sqrt{\Delta}}{2a'}\,du,$$

we then have

$$\begin{aligned} \int \frac{1}{\sqrt{ax^2+bx+c}}\,dx &= \frac{1}{\sqrt{a'}}\int \frac{1}{\sqrt{1-u^2}}\,du \\ &= \frac{1}{\sqrt{a'}}\arcsin(u) + k, \quad k \in \mathbb{R} \\ &= \frac{1}{\sqrt{a'}}\arcsin\left(\frac{2a'}{\sqrt{\Delta}}x - \frac{b}{\sqrt{\Delta}}\right) + k, \quad k \in \mathbb{R}. \end{aligned}$$

Example 1.16

$$\int \frac{1}{\sqrt{1-4x^2}}\,dx = \frac{1}{2}\arcsin(2x) + k, \quad k \in \mathbb{R}.$$

Techniques for Decomposing a Fraction into Partial Fractions

In the following, we aim to integrate a fraction of the form

$$\frac{P(x)}{Q(x)},$$

where $P(x)$ and $Q(x)$ are polynomials of respectively degrees n and m ($n, m \in \mathbb{N}$), with $Q(x)$ factorable.

We start by simplifying the fraction $\frac{P(x)}{Q(x)}$ depending on the values of n and m. We distinguish three cases: $n < m$, $n = m$, and $n > m$.

1. If $n < m$:

   - If
   $$Q(x) = (a_1x + b_1)(a_2x + b_2)\cdots(a_mx + b_m),$$
   then
   $$\frac{P(x)}{Q(x)} = \frac{A_1}{a_1x + b_1} + \frac{A_2}{a_2x + b_2} + \cdots + \frac{A_m}{a_mx + b_m},$$
   where $A_1, A_2, \ldots, A_m$ are real numbers determined by identification.

   Example 1.17 *Simplify the fraction* $\frac{x}{x^2-1}$.
   We have $\deg(x) = 1 < \deg(x^2 - 1) = 2$, *and* $x^2 - 1 = (x - 1)(x + 1)$. *Therefore,*
   $$\begin{aligned}\frac{x}{x^2-1} &= \frac{A_1}{x-1} + \frac{A_2}{x+1}\\ &= \frac{A_1(x+1)}{x^2-1} + \frac{A_2(x-1)}{x^2-1}\\ &= \frac{(A_1 + A_2)x + (A_1 - A_2)}{x^2-1}.\end{aligned}$$
   By identification:
   $$\begin{cases} A_1 + A_2 = 1 \\ A_1 - A_2 = 0 \end{cases} \Rightarrow A_1 = A_2 = \frac{1}{2}.$$
   Hence,
   $$\frac{x}{x^2-1} = \frac{\frac{1}{2}}{x-1} + \frac{\frac{1}{2}}{x+1}.$$
   Application:
   $$\begin{aligned}\int \frac{x}{x^2-1}\,dx &= \int \frac{\frac{1}{2}}{x-1}\,dx + \int \frac{\frac{1}{2}}{x+1}\,dx\\ &= \frac{1}{2}\int \frac{1}{x-1}\,dx + \frac{1}{2}\int \frac{1}{x+1}\,dx\\ &= \frac{1}{2}\ln|x-1| + \frac{1}{2}\ln|x+1| + k, \quad k \in \mathbb{R}.\end{aligned}$$

   - If $Q(x)$ contains at least one factor with a power:

   Example 1.18
   $$\frac{x+1}{x(x-2)^2(x-1)^3} = \frac{A_1}{x} + \frac{A_2}{x-2} + \frac{A_3}{(x-2)^2} + \frac{A_4}{x-1} + \frac{A_5}{(x-1)^2} + \frac{A_6}{(x-1)^3}$$

   - If $Q(x)$ contains a quadratic factor $ax^2 + bx + c$ with $\Delta < 0$:

Example 1.19

$$\frac{x^3+x^2+1}{x(x-2)^2(x^2+1)^3(x^2+x+2)} = \frac{A_1}{x} + \frac{A_2}{x-2} + \frac{A_3}{(x-2)^2} + \frac{A_4x+A_5}{x^2+1} + \frac{A_6x+A_7}{(x^2+1)^2} + \frac{A_8x+A_9}{(x^2+1)^3} + \frac{B_1x+B_2}{x^2+x+2}$$

2. If $n = m$, then
$$\frac{P(x)}{Q(x)} = A + \frac{R(x)}{Q(x)},$$
where $R(x)$ is a polynomial of degree less than $\deg(Q(x))$, and $A \in \mathbb{R}$ is determined by Euclidean division or identification. The fraction $\frac{R(x)}{Q(x)}$ is then decomposed as in case 1.

Example 1.20
$$\frac{x^2+4}{3x^2+x+1} = \frac{1}{3} + \frac{\frac{1}{3}(-x+11)}{3x^2+x+1}.$$

3. If $n > m$, then
$$\frac{P(x)}{Q(x)} = K(x) + \frac{R(x)}{Q(x)},$$
where $K(x)$ and $R(x)$ are the quotient and remainder of the Euclidean division of $P(x)$ by $Q(x)$. Clearly, $\deg(R(x)) < \deg(Q(x))$, so we reduce $\frac{R(x)}{Q(x)}$ to case 1.

Bioche's Rules

In this section, we focus on the computation of integrals of rational functions in sine and cosine. Bioche's rules allow us to simplify fractions of the form

$$\frac{P(\sin(x), \cos(x))}{Q(\sin(x), \cos(x))}$$

by reducing them to integrals of simpler rational fractions. We aim to calculate the following integral:
$$I = \int \frac{P(\sin(x), \cos(x))}{Q(\sin(x), \cos(x))}\, dx. \tag{1.1}$$

We set
$$f(x) = \frac{P(\sin(x), \cos(x))}{Q(\sin(x), \cos(x))}\, dx.$$

The following Bioche's rules allow us to determine the appropriate substitution to compute the integral of type (1.1):

1. If $f(-x) = f(x)$, we set $t = \cos(x)$.
2. If $f(\pi - x) = f(x)$, we set $t = \sin(x)$.
3. If $f(\pi + x) = f(x)$, we set $t = \tan(x)$.

4. If none of the previous three conditions is satisfied, then we set $t = \tan\left(\frac{x}{2}\right)$. In this case, we have:

$$\sin(x) = \frac{2t}{1+t^2}, \quad \cos(x) = \frac{1-t^2}{1+t^2}, \quad \tan(x) = \frac{2t}{1-t^2}, \quad dx = \frac{2}{1+t^2}\,dt.$$

Example 1.21 *Calculate*

$$I = \int \frac{\sin^3(x)}{1+\cos^2(x)}\,dx$$

Solution 1.3 *Let*

$$f(x) = \frac{\sin^3(x)}{1+\cos^2(x)}\,dx.$$

We have

$$f(-x) = \frac{-\sin^3(x)}{1+\cos^2(x)}\,d(-x) = f(x),$$

so we use the substitution

$$\begin{cases} t = \cos(x), \\ \dfrac{dt}{dx} = -\sin(x), \\ dt = -\sin(x)\,dx, \end{cases}$$

to compute the integral I:

$$\begin{aligned} I &= \int \frac{\sin^3(x)}{1+\cos^2(x)}\,dx \\ &= \int \frac{\sin(x)\sin^2(x)}{1+\cos^2(x)}\,dx \\ &= -\int \frac{1-\cos^2(x)}{1+\cos^2(x)}\,dt \\ &= \int \frac{t^2-1}{1+t^2}\,dt \\ &= \int dt - 2\int \frac{1}{1+t^2}\,dt \\ &= t - 2\arctan(t) + k, \quad k \in \mathbb{R} \\ &= \cos(x) - 2\arctan(\cos(x)) + k, \quad k \in \mathbb{R}. \end{aligned}$$

1.1.3 Definite Integrals

Definition 1.4 *Let f be a continuous function on $[a, b]$, and let F be an antiderivative of f on this interval. The definite integral of f from a to b is the real number defined by*

$$\int_a^b f(x)\,dx = [F(x)]_a^b = F(b) - F(a).$$

Remarks 1.1 1. *One must distinguish between $\int_a^b f(x)\,dx$, which gives a number, and $\int f(x)\,dx$, which gives all functions whose derivative is $f(x)$.*

2. $\int_a^x f(t)\,dt$*: this is a function of* x *that equals zero when* $x = a$*.*

Examples 1.2 *1.* $\int \frac{1}{x}\,dx = \ln|x| + k, \quad k \in \mathbb{R}.$

2. $\int_1^2 \frac{1}{x}\,dx = [\ln|x|]_1^2 = \ln 2 - \ln 1 = \ln 2.$

3. $\int_1^x \frac{1}{t}\,dt = [\ln|t|]_1^x = \ln x - \ln 1 = \ln x.$

Properties

Let f and g be two functions defined and continuous on $[a,b]$ and $\lambda \in \mathbb{R}$. All the properties of indefinite integrals remain valid for definite integrals. In addition, we have:

1. $\int_a^a f(x)\,dx = 0$
2. $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$
3. $\int_a^\lambda f(x)\,dx + \int_\lambda^b f(x)\,dx = \int_a^b f(x)\,dx$
4. $\int_a^b dx = b - a$
5. If $f(x) = 0$ on $[a,b]$, then $\int_a^b f(x)\,dx = 0$ (Note: the converse is not always true).
6. If $f(x)$ is positive on $[a,b]$, then $\int_a^b f(x)\,dx \geq 0$.
7. If $f(x) - g(x) \geq 0$ on $[a,b]$, then $\int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx$.
8. If there exist two real numbers α and β such that for all $x \in [a,b]$, $\alpha \leq f(x) \leq \beta$, then
$$\alpha(b-a) \leq \int_a^b f(x)\,dx \leq \beta(b-a)$$
9. If f is a continuous and even function on $[-a,a]$, then
$$\int_{-a}^a f(x)\,dx = 2\int_0^a f(x)\,dx$$
10. If f is a continuous and odd function on $[-a,a]$, then
$$\int_{-a}^a f(x)\,dx = 0$$
11. If f is a periodic function with period T, then
$$\int_a^{a+T} f(x)\,dx = \int_0^T f(x)\,dx$$

1.2 Generalization: Multiple Integrals (Several Dimensions)

1.2.1 Double Integrals

1. Double integral over a rectangle

Theorem 1.2 (Fubini) *Let $f : [a,b]\times[c,d] \to \mathbb{R}$ be a continuous function. Then the double integral of f over the rectangle $[a,b]\times[c,d]$ exists, and we have*

$$\iint_{[a,b]\times[c,d]} f(x,y)\,dx\,dy = \int_a^b \left(\int_c^d f(x,y)\,dy \right) dx = \int_c^d \left(\int_a^b f(x,y)\,dx \right) dy.$$

Moreover, if $f(x,y) = g(x)h(y)$, then

$$\iint_{[a,b]\times[c,d]} f(x,y)\,dx\,dy = \left(\int_a^b g(x)\,dx \right) \left(\int_c^d h(y)\,dy \right).$$

Example 1.22 *Compute the double integral*

$$I = \int_0^1 \int_{-2}^{-1} xy\,dy\,dx.$$

Step 1: Integration with respect to y.

$$\int_{-2}^{-1} xy\,dy = x\int_{-2}^{-1} y\,dy = x\left[\frac{y^2}{2}\right]_{-2}^{-1} = x\left(\frac{1}{2} - 2\right) = -\tfrac{3}{2}x.$$

Step 2: Integration with respect to x.

$$I = \int_0^1 \left(-\tfrac{3}{2}x\right) dx = -\tfrac{3}{2}\int_0^1 x\,dx = -\tfrac{3}{2}\left[\tfrac{x^2}{2}\right]_0^1 = -\tfrac{3}{4}.$$

Final answer $\boxed{I = -\tfrac{3}{4}}$.

2. Double integral of a function with separated variables

Let $f : \mathbb{R}^2 \to \mathbb{R}$ be continuous on $[a,b]\times[c,d]$, such that $f(x,y) = h(x)g(y)$. Then

$$\int_a^b \int_c^d f(x,y)\,dy\,dx = \left(\int_a^b h(x)\,dx \right) \left(\int_c^d g(y)\,dy \right).$$

Example 1.23

$$\int_a^b \int_c^d xe^y\,dx\,dy = \left(\int_a^b x\,dx \right)\left(\int_c^d e^y\,dy \right) = \left[\frac{x^2}{2}\right]_a^b . \left[e^y\right]_c^d = \left(\frac{b^2}{2} - \frac{a^2}{2}\right)\left(e^d - e^c\right).$$

3. Double integral over a bounded domain

Let $D_1 = \{(x,y) \in \mathbb{R}^2 \ / \ a \leq x \leq b, \ \ \psi_1(x) \leq y \leq \psi_2(x) \ \}$ and $D_2 = \{(x,y) \in \mathbb{R}^2 \ / \ a \leq y \leq b, \ \ \psi_1(y) \leq$
the integrals of $f(x,y)$ on D_1 and respectively on D_1, are given by:

$$\iint_{D_1} f(x,y)\,dx\,dy = \int_a^b \left[\int_{\psi_1(x)}^{\psi_2(x)} f(x,y)\,dy\right] dx.$$

$$\iint_{D_2} f(x,y)\,dx\,dy = \int_a^b \left[\int_{\psi_1(y)}^{\psi_2(y)} f(x,y)\,dx\right] dy.$$

Example 1.24 *Calculate*

$$\int\int_D dx\,dy \ \ \textit{tel-que} \ \ D = \left\{(x,y) \in \mathbb{R}^2 \ / \ 0 \leq x \leq 1, \ \ x^2 \leq y \leq \sqrt{x} \ \right\}.$$

$$\int\int_D dx\,dy = \int_0^1 \left(\int_{x^2}^{\sqrt{x}} dy\right) dx = \int_0^1 \left(\sqrt{x} - x^2\right) dx = \left[\frac{2}{3}x^{\frac{3}{2}} - \frac{1}{3}x^3\right]_0^1 = \frac{1}{3}.$$

4. Compute a double integral by change of variables

$$\int\int_D f\,(x,y)\ dx\,dy = \int\int_S g\,(u,v)\ detJ\,du\,dv,$$

where

$$\det J = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix}$$

Application: Compute a double integral by using Polar coordinates

$$x = r\cos(\theta), \ y = r\sin(\theta), \ 0 \leq \theta \leq 2\pi, \ \ r > 0.$$

$$J = \begin{pmatrix} \dfrac{\partial x}{\partial r} & \dfrac{\partial x}{\partial \theta} \\ \dfrac{\partial y}{\partial r} & \dfrac{\partial y}{\partial \theta} \end{pmatrix}$$

$$\frac{\partial x}{\partial r} = \cos(\theta), \frac{\partial y}{\partial r} = \sin(\theta), \frac{\partial x}{\partial \theta} = -r\sin(\theta), \frac{\partial y}{\partial \theta} = r\cos(\theta),$$

$$\det J = \begin{vmatrix} \cos(\theta) & -r\sin(\theta) \\ \sin(\theta) & r\cos(\theta) \end{vmatrix} = r\cos^2(\theta) + r\sin^2(\theta) = r\left(\cos^2(\theta) + \sin^2(\theta)\right) = r,$$

that is,

$$\int\int_D f\,(x,y)\ dx\,dy = \int\int_S g\,(r,\theta)\ r\,dr\,d\theta.$$

Example 1.25

$$\int\int_D \frac{1}{1+x^2+y^2} dx\,dy \ \ \textit{tel-que} \ \ D = \left\{(x,y) \in \mathbb{R}^2 \ / x^2 + y^2 \leq 1\right\}.$$

Step 1: Identify the region.
The condition $x^2 + y^2 \leq 1$ describes the unit disk centered at the origin. In polar coordinates we have

$$x = r\cos\theta, \qquad y = r\sin\theta, \qquad r \geq 0,$$

and

$$x^2 + y^2 = r^2 \leq 1 \quad \Rightarrow \quad 0 \leq r \leq 1.$$

Step 2: Range of θ.
To cover the whole disk, the angle θ must make a full revolution around the origin:

$$0 \leq \theta < 2\pi.$$

Then

$$\int\int_D \frac{1}{1+x^2+y^2}\,dx\,dy = \int_0^{2\pi}\int_0^1 \frac{1}{1+r^2}\,r\,dr\,d\theta = \frac{1}{2}\int_0^{2\pi}\int_0^1 \frac{2r}{1+r^2}\,dr\,d\theta,$$

$$\int\int_D \frac{1}{1+x^2+y^2}\,dx\,dy = \frac{1}{2}\int_0^{2\pi}\left(\int_0^1 \frac{2r}{1+r^2}\,dr\right)d\theta = \frac{1}{2}\int_0^{2\pi}\left[\ln\left(r^2+1\right)\right]_0^1 d\theta,$$

$$\int\int_D \frac{1}{1+x^2+y^2}\,dx\,dy = \frac{1}{2}\int_0^{2\pi}\ln(2)d\theta = \frac{\ln(2)}{2}\,[\theta]_0^{2\pi} = \frac{\ln(2)}{2}\,(2\pi) = \pi\ln(2).$$

1.2.2 Triple integrals

1. Integration by iterated integrals

According to **Fubini's theorem,** a triple integral can be computed as an iterated integral, that is, by integrating successively with respect to each variable.

For example, if f is continuous on the rectangular box $[a,b]\times[c,d]\times[e,f]$, then

$$\iiint_{[a,b]\times[c,d]\times[e,f]} f(x,y,z)\,dx\,dy\,dz = \int_a^b\int_c^d\int_e^f f(x,y,z)\,dz\,dy\,dx.$$

The order of integration may be changed:

$$\iiint_{[a,b]\times[c,d]\times[e,f]} f(x,y,z)\,dx\,dy\,dz = \int_c^d\int_e^f\int_a^b f(x,y,z)\,dx\,dz\,dy,$$

and similarly for the other possible orders of integration.

Example 1.26

$$\iiint_{[0,1]^3} (x+y+z)\,dz\,dy\,dx = \int_0^1\int_0^1\int_0^1 (x+y+z)\,dz\,dy\,dx.$$

a. inner integration (integration with respect to z) :

$$\int_0^1 (x+y+z)\,dz = \left[(x+y)z + \frac{z^2}{2}\right]_{z=0}^{z=1} = (x+y)\cdot 1 + \frac{1^2}{2} - 0 = x+y+\frac{1}{2}.$$

b. intermediate integration (with respect to one variable in the middle y) :

$$\int_0^1 \left(x+y+\frac{1}{2}\right) dy = \left[xy + \frac{y^2}{2} + \frac{y}{2}\right]_{y=0}^{y=1} = x\cdot 1 + \frac{1}{2} + \frac{1}{2} = x+1.$$

c. outer integration (with respect to the last variable x) :

$$\int_0^1 (x+1)\, dx = \left[\frac{x^2}{2} + x\right]_0^1 = \frac{1}{2} + 1 = \frac{3}{2}.$$

$$\boxed{\iiint_{[0,1]^3} (x+y+z)\, dz\, dy\, dx = \frac{3}{2}.}$$

2. Triple Integrals in Cylindrical Coordinates

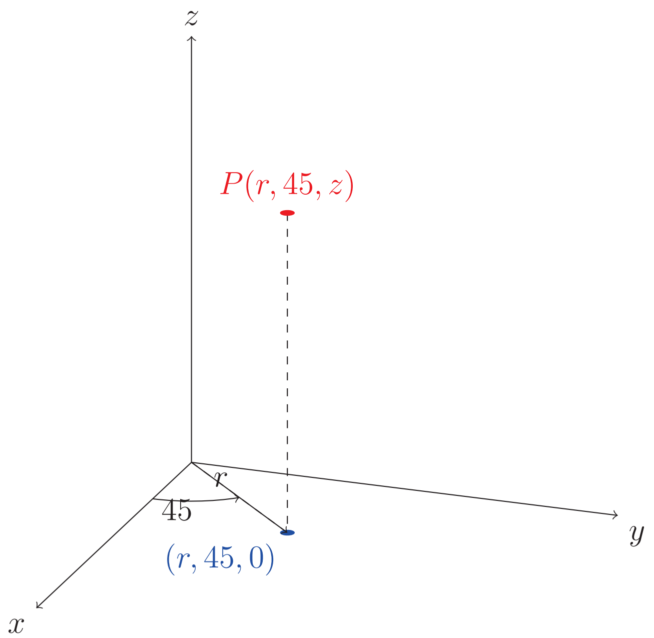
a. Cylindrical Coordinates

In cylindrical coordinates, a point (x, y, z) in $\mathbb{R}^3$ is represented as:

$$x = r\cos\theta, \quad y = r\sin\theta, \quad z = z$$

where:

- $r \geq 0$ is the radial distance from the z-axis,
- $0 \leq \theta < 2\pi$ is the azimuthal angle,
- z is the height as in Cartesian coordinates.



Explanation of Cylindrical Coordinates:

In cylindrical coordinates, a point P in space is represented by (r, θ, z), where:

- r is the distance from the z-axis to the projection of the point onto the xy-plane.
- θ is the angle between the positive x-axis and the line connecting the origin to the projection of the point on the xy-plane.
- z is the height of the point along the z-axis.

b. Jacobian of the Transformation
When changing variables from (x, y, z) to (r, θ, z), we need the Jacobian determinant:

$$J = \begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial z} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial z} \end{pmatrix} = \begin{pmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

The determinant of J is:

$$\det(J) = r$$

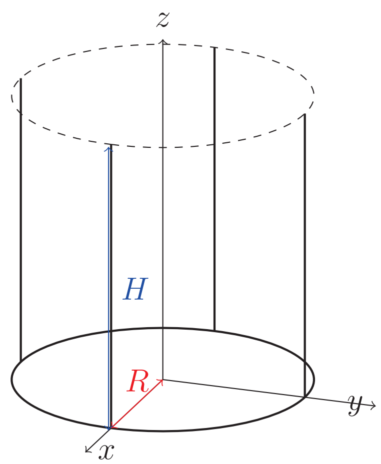
c. Triple Integral in Cylindrical Coordinates
A triple integral over a region V becomes:

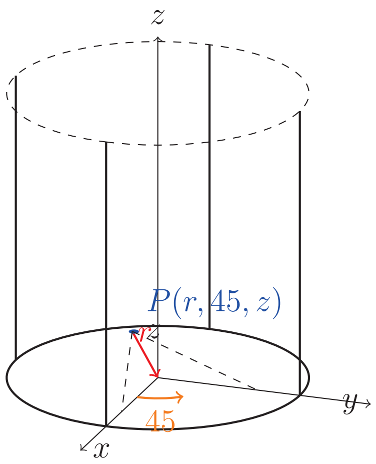
$$\iiint_V f(x, y, z)\, dx\, dy\, dz = \iiint_V f(r\cos\theta, r\sin\theta, z)\, r\, dr\, d\theta\, dz$$

Example 1.27 ***Calculate the Integral*** $\iiint_V z\, dx\, dy\, dz$ ***over a Cylinder***

a. Representation of the Cylinder
Consider a vertical cylinder with radius R and height H. In Cartesian coordinates, the cylinder is defined by:

$$x^2 + y^2 \leq R^2, \quad 0 \leq z \leq H$$





b. Transformation to Cylindrical Coordinates
Recall the cylindrical coordinates transformation:

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases} \quad \textbf{with } r \geq 0,\ 0 \leq \theta \leq 2\pi$$

c. Determining the Bounds

1. **Radius r: The distance from the center of the cylinder in the xy-plane ranges from 0 (the center) to R (the edge of the cylinder):**

$$0 \leq r \leq R$$

2. **Angle θ: To go around the full circle in the xy-plane, the angle varies from 0 to 2π:**

$$0 \leq \theta \leq 2\pi$$

3. **Height z: The vertical position ranges from the bottom of the cylinder to the top:**

$$0 \leq z \leq H$$

Solution:

$$
\begin{aligned}
\iiint_D z\,dx\,dy\,dz &= \int_0^H \int_0^{2\pi} \int_0^R z \cdot r\,dr\,d\theta\,dz \qquad \textbf{(Cylindrical coordinates)} \\
&= \int_0^H \int_0^{2\pi} \left[z\frac{r^2}{2} \right]_0^R d\theta\,dz \\
&= \int_0^H \int_0^{2\pi} z\frac{R^2}{2}\,d\theta\,dz \\
&= \int_0^H \left[z\frac{R^2}{2}\theta \right]_0^{2\pi} dz \\
&= \int_0^H z\frac{R^2}{2} \cdot 2\pi\,dz \\
&= \int_0^H \pi R^2 z\,dz \\
&= \left[\pi R^2 \frac{z^2}{2} \right]_0^H \\
&= \frac{\pi R^2 H^2}{2}
\end{aligned}
$$

Remarks:

- **We used the Jacobian r in the integrand.**
- **The calculation applied Fubini's theorem to separate the integral into r, θ, and z integrals.**
- **All steps are written in a single equation for clarity.**

Example 1.28 ***Calculate***

$$I = \int\int\int_D dx\,dy\,dz, \quad D = \left\{ (x, y, z) \in \mathbb{R}^3 \ /x^2 + y^2 + z^2 \le r^2 \right\}.$$

$$I = \int\int_D \left(\int_{-\sqrt{r^2-x^2-y^2}}^{\sqrt{r^2-x^2-y^2}} dz \right) dx\,dy = 2\int\int_{D'} \sqrt{r^2 - x^2 - y^2}\,dx\,dy,$$

D' ***the disk of center*** $O = (0, 0)$ ***and radius*** r***. Using the change of variables to polar coordinates***

$$x = \rho\cos(\theta),\ y = \rho\sin(\theta),$$

we have,

$$x^2 + y^2 = \rho^2,$$

and

$$I = 2\int\int_{D'} \sqrt{r^2 - \rho^2}\rho\,d\rho\,d\theta = -\frac{4\pi}{3}\left[\left(r^2 - \rho^2 \right)^{\frac{3}{2}} \right]_0^r = \frac{4\pi r^3}{3}.$$

3. Triple Integrals in Spherical Coordinates

a. In spherical coordinates, a point P in space is represented by three numbers:

$$(r, \theta, \phi)$$

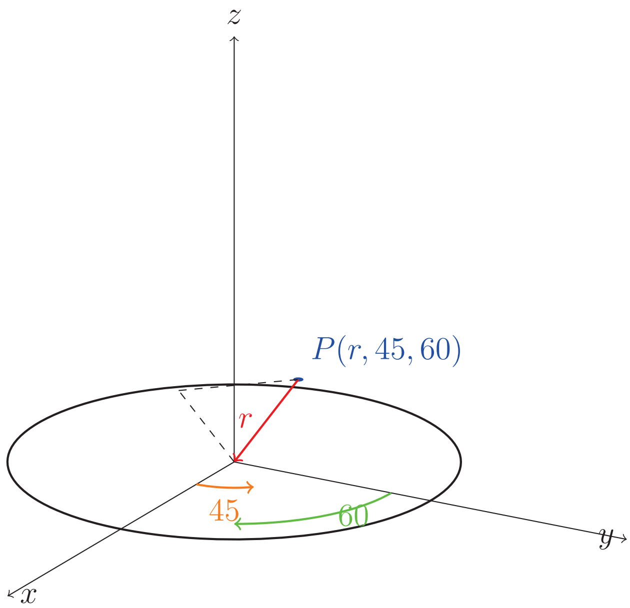
- $r \geq 0$ **is the distance from the origin to the point (radius),**
- $\theta \in [0, 2\pi)$ **is the angle in the xy-plane measured from the positive x-axis (azimuthal angle),**
- $\phi \in [0, \pi]$ **is the angle between the positive z-axis and the line connecting the origin to the point (polar angle or colatitude).**

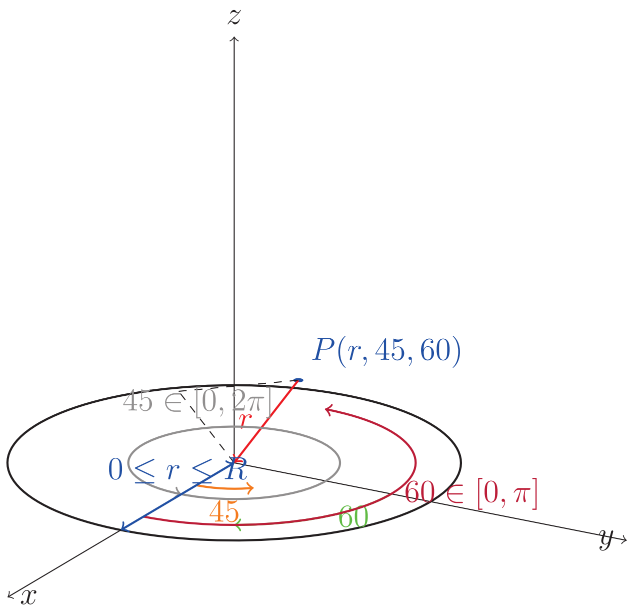
b. The relation between Cartesian coordinates (x, y, z) and spherical coordinates (r, θ, ϕ) is:

$$\begin{cases} x = r\,\sin\phi\,\cos\theta, \\ y = r\,\sin\phi\,\sin\theta, \\ z = r\,\cos\phi. \end{cases}$$

Spherical Coordinates Representation
Consider a sphere of radius R centered at the origin:

$$x^2 + y^2 + z^2 \leq R^2$$





c. Spherical Coordinates Transformation and Jacobian

The transformation from Cartesian coordinates (x, y, z) to spherical coordinates (r, θ, ϕ) is:

$$\begin{cases} x = r\sin\phi\cos\theta, \\ y = r\sin\phi\sin\theta, \\ z = r\cos\phi, \end{cases} \qquad r \geq 0,\ \ \theta \in [0, 2\pi),\ \ \phi \in [0, \pi].$$

The Jacobian matrix J is

$$J = \frac{\partial(x, y, z)}{\partial(r, \theta, \phi)} = \begin{bmatrix} \sin\phi\cos\theta & -r\sin\phi\sin\theta & r\cos\phi\cos\theta \\ \sin\phi\sin\theta & r\sin\phi\cos\theta & r\cos\phi\sin\theta \\ \cos\phi & 0 & -r\sin\phi \end{bmatrix}.$$

We compute the partial derivatives:

$$\begin{aligned} \frac{\partial x}{\partial r} &= \sin\phi\cos\theta, & \frac{\partial x}{\partial \theta} &= -r\sin\phi\sin\theta, & \frac{\partial x}{\partial \phi} &= r\cos\phi\cos\theta, \\ \frac{\partial y}{\partial r} &= \sin\phi\sin\theta, & \frac{\partial y}{\partial \theta} &= r\sin\phi\cos\theta, & \frac{\partial y}{\partial \phi} &= r\cos\phi\sin\theta, \\ \frac{\partial z}{\partial r} &= \cos\phi, & \frac{\partial z}{\partial \theta} &= 0, & \frac{\partial z}{\partial \phi} &= -r\sin\phi. \end{aligned}$$

Hence, the Jacobian matrix is:

$$J = \begin{bmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \phi} \end{bmatrix} = \begin{bmatrix} \sin\phi\cos\theta & -r\sin\phi\sin\theta & r\cos\phi\cos\theta \\ \sin\phi\sin\theta & r\sin\phi\cos\theta & r\cos\phi\sin\theta \\ \cos\phi & 0 & -r\sin\phi \end{bmatrix}.$$

d. The Jacobian determinant of this transformation is

$$\det J(\rho, \varphi, \theta) = r^2\sin\phi$$

Triple Integral in Spherical Coordinates
Let $f(x, y, z)$ be a function defined in a region $\mathcal{D} \subset \mathbb{R}^3$. In spherical coordinates:

$$\begin{cases} x = r \sin \phi \cos \theta, \\ y = r \sin \phi \sin \theta, \\ z = r \cos \phi, \end{cases}$$

where $r \geq 0$, $0 \leq \phi \leq \pi$ (polar angle), $0 \leq \theta < 2\pi$ (azimuthal angle).
e. The triple integral of f over $\mathcal{D}$ becomes:

$$\iiint_{\mathcal{D}} f(x, y, z)\, dx\, dy\, dz = \iiint_{\mathcal{D}'} f(r \sin \phi \cos \theta,\ r \sin \phi \sin \theta,\ r \cos \phi)\ r^2 \sin \phi\, dr\, d\phi\, d\theta$$

where $r^2 \sin \phi$ is the Jacobian determinant of the transformation:

Example 1.29 ***Let*** $B = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 \leq R^2\}$***. Taking*** $f(x, y, z) = 1$***, we obtain***

$$\mathrm{I} = \iiint_B 1\, dx\, dy\, dz = \int_0^{2\pi} \int_0^{\pi} \int_0^{R} r^2 \sin \varphi\, dr\, d\varphi\, d\theta.$$

First, integrate with respect to ρ***:***

$$\int_0^R \rho^2\, d\rho = \left[\tfrac{\rho^3}{3}\right]_0^R = \tfrac{R^3}{3}.$$

Next, integrate with respect to φ***:***

$$\int_0^{\pi} \sin \varphi\, d\varphi = [-\cos \varphi]_0^{\pi} = 2.$$

Finally, integrate with respect to θ***:***

$$\int_0^{2\pi} d\theta = 2\pi.$$

Thus,

$$\mathrm{I} = \frac{R^3}{3} \cdot 2 \cdot 2\pi = \frac{4}{3}\pi R^3.$$

Example 1.30 ***Let*** $B = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 \leq R^2\}$***. Taking*** $f(x, y, z) = 1$***,and***

$$x = \rho \cos(\theta) \cos(\alpha), \quad y = \rho \sin(\theta) \cos(\alpha), \quad z = \rho \sin(\alpha),$$

with,

$$0 \leq \rho \leq r, \quad \theta \in [0 \;\; 2\pi], \quad \alpha \left[-\frac{\pi}{2} \;\; \frac{\pi}{2}\right].$$

Calculate $I = \int \int \int_B f(x, y, z)\ dx\, dy\, dz.$
Solution:
We have

$$J = \begin{pmatrix} \frac{\partial x}{\partial \rho} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \alpha} \\ \frac{\partial y}{\partial \rho} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \alpha} \\ \frac{\partial z}{\partial \rho} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \alpha} \end{pmatrix} = \begin{pmatrix} \cos(\alpha)\cos(\theta) & -\rho\sin(\theta)\cos(\alpha) & -\rho\sin(\alpha)\cos(\theta) \\ \cos(\alpha)\sin(\theta) & \rho\cos(\alpha)\cos(\theta) & -\rho\sin(\alpha)\sin(\theta) \\ \sin(\alpha) & 0 & \rho\cos(\theta) \end{pmatrix}$$

$$det J = \rho^2 \cos(\alpha).$$

Then

$$\begin{aligned} I &= \int\int\int_B f(x, y, z)\, dx\, dy\, dz \\ &= \int\int\int_B \rho^2 \cos(\alpha)\, d\rho\, d\theta\, d\alpha \\ &= \left(\int_0^r \rho^2\, d\rho\right)\left(\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos(\alpha)\, d\alpha\right)\left(\int_0^{2\pi} d\theta\right) \\ &= \frac{4\pi r^3}{3}. \end{aligned}$$

Coordinate System	**Coordinates**	**Integration Element**	**Use Case**
Cartesian (2D)	(x, y)	$dx\, dy$	Double integrals over general regions
Polar (2D)	(r, θ)	$r\, dr\, d\theta$	Double integrals with circular symmetry
Cylindrical (3D)	(r, θ, z)	$r\, dr\, d\theta\, dz$	Triple integrals with axial symmetry
Spherical (3D)	(ρ, ϕ, θ)	$\rho^2 \sin\phi\, d\rho\, d\phi\, d\theta$	Triple integrals with point symmetry

Table 1.1: Summary of coordinate systems and their integration use.

1.3 Area and Volume Calculation

1.3.1 Area Calculation Using Double Integrals

The area of a surface projected onto the xy-plane, bounded by a region D, can be calculated using a double integral.

Definition 1.5 ***Let D be a region in the xy-plane and let $f(x, y)$ be a non-negative function defined on D. Then the area A under the surface $z = f(x, y)$ over the region D is given by:***

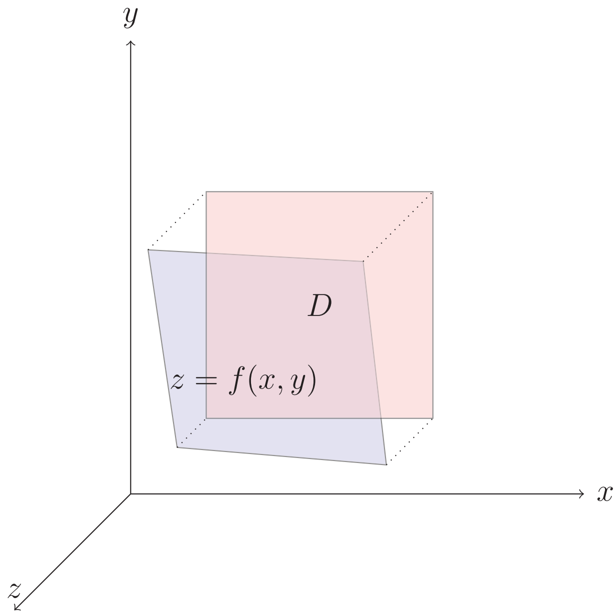
$$A = \iint_D f(x, y)\, dx\, dy$$

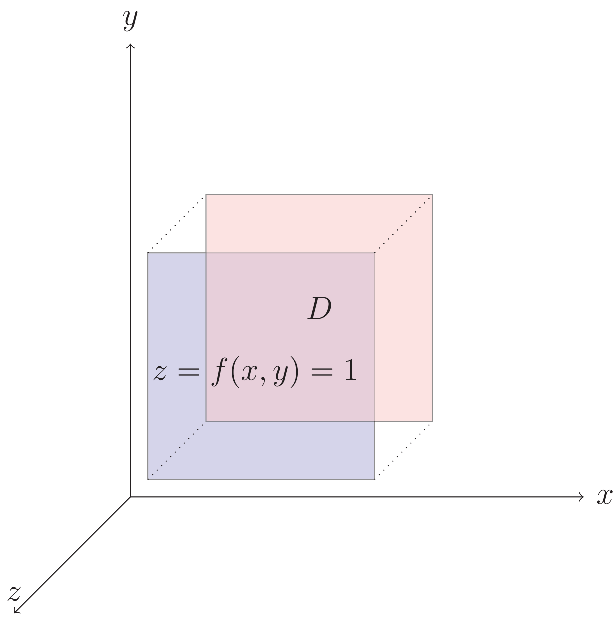
Here:

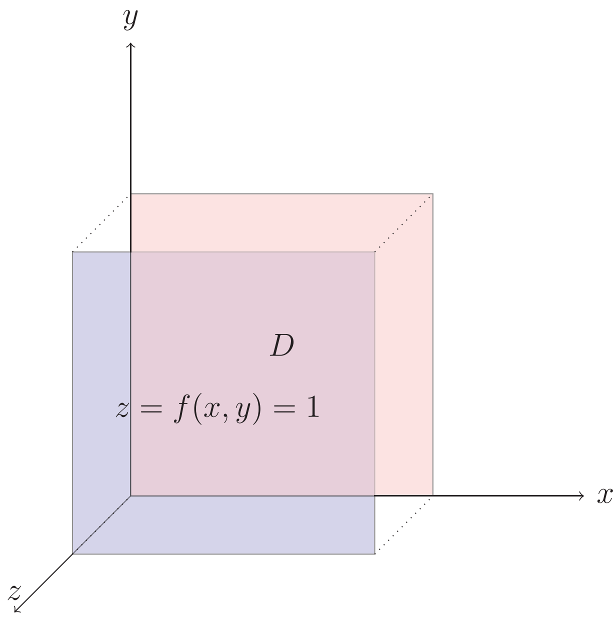
- ***D is the projection of the surface onto the xy-plane.***
- ***$f(x, y)$ represents the height of the surface above the point (x, y) in D.***

Remark: If $f(x, y) = 1$, the double integral simply gives the area of the region D itself:

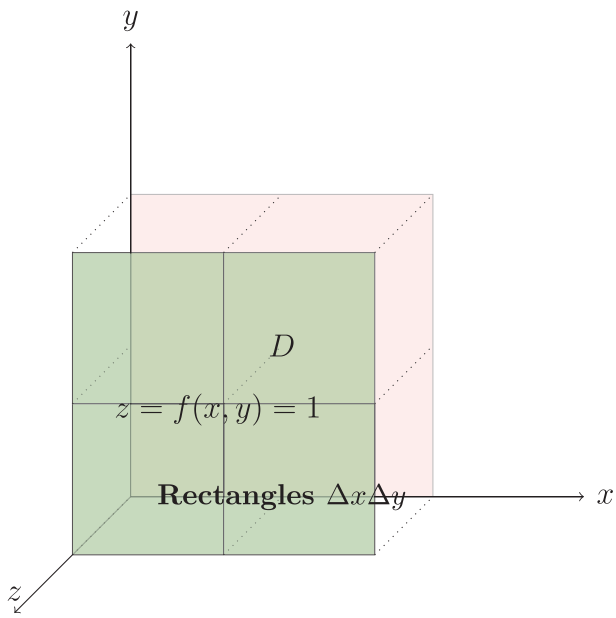
$$A = \iint_D 1 \, dx \, dy$$



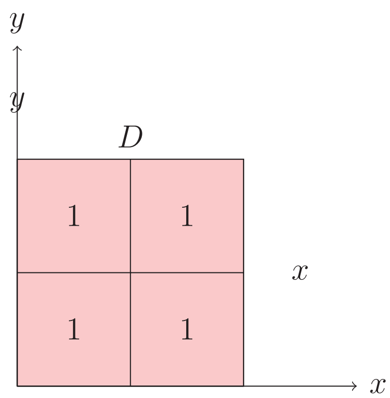




Visualisation of a Double Integral for $f(x, y) = 1$

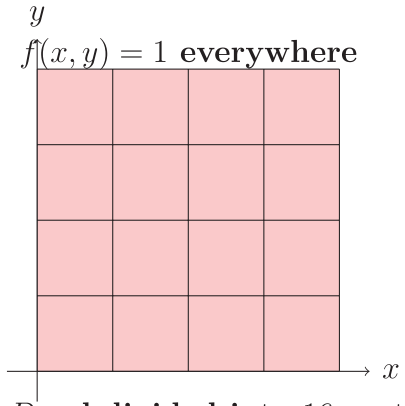


Visualisation of a Double Integral for $f(x, y) = 1$



Visualisation of a Double Integral for $f(x, y) = 1$

The domain $D = [0, 2] \times [0, 2]$ is subdivided into 16 small rectangles to illustrate the calculation of a double integral.



Domain D subdivided into 16 rectangles

1. Each small rectangle has an area $\Delta x \Delta y = 0.5 \times 0.5 = 0.25$. - The sum of all rectangle areas $= 16 \times 0.25 = 4$, which corresponds to the double integral:

$$\iint_D f(x, y)\, dx\, dy = 4.$$

2. Calculating the Area of Domain D Using a Double Integral

We want to compute the area of the domain

$$D = \{(x, y) \mid 0 \leq x \leq 2,\ 0 \leq y \leq 2\}.$$

Since the function is constant $f(x, y) = 1$, the double integral over D gives the area:

$$\textbf{Area}(D) = \iint_D 1\, dx\, dy.$$

We can write this as an iterated integral:

$$\textbf{Area}(D) = \int_0^2 \int_0^2 1\, dx\, dy.$$

First, integrate with respect to x:

$$\int_0^2 1\, dx = [x]_0^2 = 2 - 0 = 2.$$

Then, integrate with respect to y:

$$\int_0^2 2\, dy = [2y]_0^2 = 2 \times 2 - 0 = 4.$$

Hence, the exact area of the domain D is:

$$\boxed{4}.$$

Example 1.31 (Calculating Surface Using a Double Integral) ***Let us compute the surface under the function***

$$f(x, y) = x + y$$

over the domain

$$D = \{(x, y) \mid 0 \leq x \leq 1,\ 0 \leq y \leq 1\}.$$

The surface is given by the double integral:

$$\textbf{\textit{Surface}} = \iint_D f(x, y)\, dx\, dy = \int_0^1 \int_0^1 (x + y)\, dx\, dy.$$

Step 1: Integrate with respect to x***:***

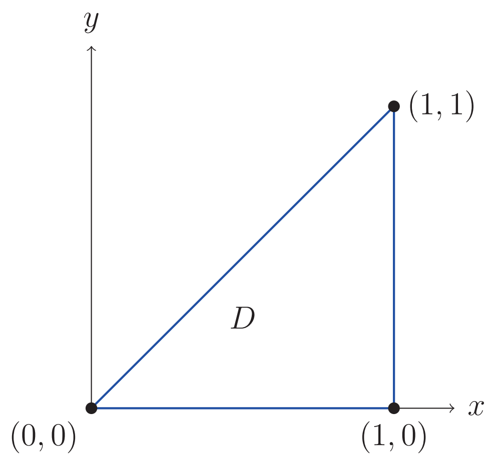
$$\int_0^1 (x + y)\, dx = \int_0^1 x\, dx + \int_0^1 y\, dx = \left[\frac{x^2}{2}\right]_0^1 + y[x]_0^1 = \frac{1}{2} + y.$$

Step 2: Integrate with respect to y***:***

$$\int_0^1 \left(\frac{1}{2} + y\right) dy = \int_0^1 \frac{1}{2}\, dy + \int_0^1 y\, dy = \left[\frac{y}{2}\right]_0^1 + \left[\frac{y^2}{2}\right]_0^1 = \frac{1}{2} + \frac{1}{2} = 1.$$

Hence, the exact surface under the function $f(x, y) = x + y$ ***over the domain*** D ***is:*** $\boxed{1}$.

Example 1.32 ***Let's calculate the area of the triangle*** D ***defined by the points*** $(0, 0), (1, 0), (1, 1)$***. 1. Drawing the domain:***



2. Define the domain: The triangle can be described as:

$$D = \{(x, y) \mid 0 \leq x \leq 1,\ 0 \leq y \leq x\}.$$

3. Write the double integral:

$$\textbf{\textit{Area}}(D) = \iint_D 1\, dA = \int_{x=0}^{1} \int_{y=0}^{x} 1\, dy\, dx.$$

4. Compute the integral:

$$\int_{x=0}^{1} \int_{y=0}^{x} 1\, dy\, dx = \int_0^1 [y]_{y=0}^{y=x}\, dx = \int_0^1 x\, dx = \left[\frac{x^2}{2}\right]_0^1 = \frac{1}{2}.$$

Conclusion: The area of the triangle is therefore $\frac{1}{2}$***.***

1.3.2 Volume Calculation Using Triple Integrals

Definition 1.6 ***The volume of a three-dimensional solid V can be calculated using a triple integral if the solid occupies a domain D in $\mathbb{R}^3$.***

Formally, if the solid is described by a continuous function $f(x, y, z)$ over a domain D, the volume is given by:

$$\textbf{\textit{Volume}} = \iiint\limits_D f(x, y, z)\, dV$$

where $dV = dx\, dy\, dz$ in Cartesian coordinates.

There are two situations:

1. **Pure volume of a solid: If we want only the geometrical volume of a solid occupying a region $D \subset \mathbb{R}^3$, then the integral is**

$$V = \iiint\limits_D 1\, dV.$$

   Here, the integrand is $f(x, y, z) = 1$ because we only sum up infinitesimal volume elements.

2. **Physical quantities (mass, charge, etc.): If the solid has a density function $\rho(x, y, z)$, then the mass is given by**

$$M = \iiint\limits_D \rho(x, y, z)\, dV.$$

   In this case, the function $\rho(x, y, z)$ replaces the 1 because we are summing up weighted infinitesimal volumes.

Remark 1.9 ***The choice of coordinates (Cartesian, cylindrical, spherical) depends on the symmetry of the region D. Depending on the symmetry of the solid, other coordinate systems can be used, such as cylindrical coordinates (r, θ, z) or spherical coordinates (r, ϕ, θ):***

$$dV = r\, dr\, d\theta\, dz \quad \textbf{\textit{(cylindrical)}}, \quad dV = r^2 \sin\phi\, dr\, d\phi\, d\theta \quad \textbf{\textit{(spherical)}}.$$

Example 1.33 ***1.*** **Volume of a Sphere (using spherical coordinates):**

Let $D \subset \mathbb{R}^3$ be the ball of radius $R > 0$ centered at the origin:

$$D = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 \le R^2\}.$$

In spherical coordinates we use the variables (r, θ, ϕ) with the convention:

$$\begin{aligned} x &= r \sin\phi \cos\theta, \\ y &= r \sin\phi \sin\theta, \qquad r \ge 0,\ \theta \in [0, 2\pi),\ \phi \in [0, \pi]. \\ z &= r \cos\phi, \end{aligned}$$

The volume element (Jacobian) is

$$dV = r^2 \sin\phi \, dr \, d\phi \, d\theta.$$

2. **Geometric volume:** ***The geometric volume*** V ***of the ball is***

$$V = \iiint_D 1 \, dV = \int_{\theta=0}^{2\pi} \int_{\phi=0}^{\pi} \int_{r=0}^{R} r^2 \sin\phi \, dr \, d\phi \, d\theta.$$

Evaluate the innermost integral in r***:***

$$\int_0^R r^2 \, dr = \frac{R^3}{3}.$$

Thus

$$V = \int_0^{2\pi} \int_0^{\pi} \frac{R^3}{3} \sin\phi \, d\phi \, d\theta = \frac{R^3}{3} \left(\int_0^{2\pi} d\theta \right) \left(\int_0^{\pi} \sin\phi \, d\phi \right).$$

Compute the angular integrals:

$$\int_0^{2\pi} d\theta = 2\pi, \qquad \int_0^{\pi} \sin\phi \, d\phi = \Big[-\cos\phi \Big]_0^{\pi} = 2.$$

Therefore

$$\boxed{V = \frac{R^3}{3} \cdot 2\pi \cdot 2 = \frac{4}{3}\pi R^3}.$$

3. **Mass with a density:** ***If the body has a density function*** $\delta(x, y, z)$ ***(note: we use*** δ ***to avoid conflict with the radial variable*** r***), then the mass is***

$$M = \iiint_D \delta(x, y, z) \, dV.$$

In spherical coordinates this becomes

$$M = \int_0^{2\pi} \int_0^{\pi} \int_0^{R} \delta\Big(r \sin\phi \cos\theta, r \sin\phi \sin\theta, r \cos\phi\Big) \, r^2 \sin\phi \, dr \, d\phi \, d\theta,$$

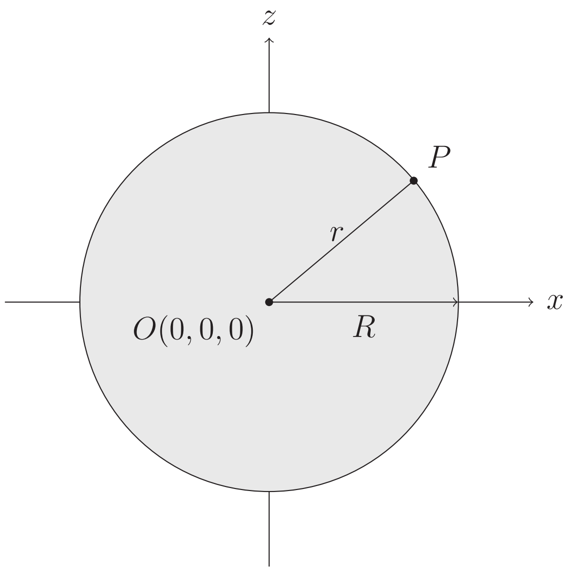
then

$$M = \int_0^{2\pi} \int_0^{\pi} \int_0^{R} \tilde{\delta}(r, \phi, \theta) \, r^2 \sin\phi \, dr \, d\phi \, d\theta.$$

where $\delta(r, \theta, \phi)$ ***denotes the same density expressed in spherical coordinates. If the density is constant,*** $\delta(x, y, z) = \delta_0$ ***(a constant), then***

$$M = \delta_0 \, V = \delta_0 \cdot \frac{4}{3}\pi R^3.$$

4. **Illustration (2D cross-section):**



This 2D figure is the vertical cross-section of the sphere by the xz-plane (it shows a circle of radius R centered at the origin). In full 3D the sphere is obtained by rotating this disk around the z-axis.

Example 1.34 ***Compute the volume of the rectangular box***

$$D = \{(x, y, z) \in \mathbb{R}^3 \mid 0 \le x \le 2,\ 0 \le y \le 1,\ 0 \le z \le 3\}$$

by a triple integral.

Solution 1.4 ***The geometric volume of the region D is obtained by integrating the function 1 over D:***

$$V = \iiint_D 1\, dV.$$

Step 1: Set up the triple integral.
Since the bounds are constant and independent, we may write the integral in Cartesian coordinates as

$$V = \int_{x=0}^{2} \int_{y=0}^{1} \int_{z=0}^{3} 1\, dz\, dy\, dx.$$

Step 2: Integrate with respect to z.

$$\int_{z=0}^{3} 1\, dz = z\Big|_0^3 = 3.$$

Hence the integral reduces to

$$V = \int_{x=0}^{2} \int_{y=0}^{1} 3\, dy\, dx.$$

Step 3: Integrate with respect to y.

$$\int_{y=0}^{1} 3\, dy = 3y\Big|_0^1 = 3.$$

Thus

$$V = \int_{x=0}^{2} 3\,dx.$$

Step 4: Integrate with respect to x***.***

$$\int_{x=0}^{2} 3\,dx = 3x\Big|_0^2 = 6.$$

Answer:

$$\boxed{V = 6}$$

The volume of the rectangular box is 6 ***cubic units.***

Example 1.35 ***Compute the volume of the solid bounded by the cylinder*** $x^2+y^2 \le 4$ ***and the planes*** $z = 0$ ***and*** $z = 1 + x^2 + y^2$***. Use cylindrical coordinates.***

Solution 1.5 ***The solid*** D ***is given by***

$$D = \{(x, y, z) \mid x^2 + y^2 \le 4,\ 0 \le z \le 1 + x^2 + y^2\}.$$

We compute the geometric volume

$$V = \iiint_D 1\,dV.$$

Step 1: Change to cylindrical coordinates.
Set

$$x = r\cos\theta, \qquad y = r\sin\theta, \qquad z = z,$$

with $r \in [0, 2]$***,*** $\theta \in [0, 2\pi)$***, and*** z ***between the planes. The Jacobian is*** $dV = r\,dr\,d\theta\,dz$***. The upper bound for*** z ***becomes*** $z = 1 + r^2$***.***
Step 2: Set up the triple integral.

$$V = \int_{\theta=0}^{2\pi}\int_{r=0}^{2}\int_{z=0}^{1+r^2} r\,dz\,dr\,d\theta.$$

Step 3: Integrate with respect to z***.***

$$\int_{z=0}^{1+r^2} r\,dz = r\Big(1 + r^2\Big).$$

So

$$V = \int_0^{2\pi}\int_0^2 r\Big(1 + r^2\Big)\,dr\,d\theta = \int_0^{2\pi}\left(\int_0^2 (r + r^3)\,dr\right)d\theta.$$

Step 4: Integrate with respect to r***.***

$$\int_0^2 (r + r^3)\,dr = \left[\frac{r^2}{2} + \frac{r^4}{4}\right]_0^2 = \frac{4}{2} + \frac{16}{4} = 2 + 4 = 6.$$

Step 5: Integrate with respect to θ***.***

$$V = \int_0^{2\pi} 6\,d\theta = 6 \cdot 2\pi = 12\pi.$$

Answer:

$$\boxed{V = 12\pi}$$

The volume of the solid is 12π ***cubic units.***

1.4 Solved Exercises

We recall that, in order to compute a definite integral of a continuous function, one must first compute an antiderivative (primitive) of this function, and then evaluate this antiderivative at the bounds of the integral.

Exercise 1.2 ***(Direct computation)***
Compute the following integrals:

$$\int_0^{\frac{\pi}{2}} \cos(x)\,dx,\quad \int_0^1 \frac{\sqrt[3]{x^2}+2\sqrt{x}+1}{x^4}\,dx,\quad \int_{-1}^{1}\left(3x^2-x+1\right)\,dx.$$

Solution 1.6 1) ***We calculate all the antiderivatives of the function*** $x \longmapsto \cos(x)$***, which are given by the following indefinite integral:***

$$\int \cos(x)\,dx = \sin(x)+k,\quad k\in\mathbb{R},$$

We take these antiderivatives between the bounds of the integral, and we have

$$\int_0^{\frac{\pi}{2}} \cos(x)\,dx = [\sin(x)]_0^{\frac{\pi}{2}} = \sin\left(\frac{\pi}{2}\right) - \sin(0) = 1.$$

2) ***We follow the same previous procedure.***

$$\begin{aligned}
\int \left(3x^2-x+1\right)\,dx &= \int 3x^2 - \int x\,dx + \int\,dx\\
\int_{-1}^{1} \left(3x^2-x+1\right)\,dx &= \int_{-1}^{1} 3x^2 - \int_{-1}^{1} x\,dx + \int_{-1}^{1}\,dx\\
&= 3\left[\frac{x^3}{3}\right]_{-1}^{1} - \left[\frac{x^2}{2}\right]_{-1}^{1} + [x]_{-1}^{1}\\
&= \left[3\frac{x^3}{3} - \frac{x^2}{2} + x\right]_{-1}^{1}\\
&= \left[(1)^3 - \frac{(1)^2}{2} + (1)\right] - \left[(-1)^3 - \frac{(-1)^2}{2} + (-1)\right]\\
&= 4.
\end{aligned}$$

3)
$$\begin{aligned}
\int \frac{\sqrt[3]{x^2}+2\sqrt{x}+1}{x^4}\,dx &= \int \frac{\sqrt[3]{x^2}}{x^4}\,dx + 2\int \frac{\sqrt{x}}{x^4}\,dx + \int \frac{1}{x^4}\,dx\\
&= \int x^{\frac{-10}{3}}\,dx + 2\int x^{\frac{-11}{3}}\,dx + \int x^{-4}\,dx\\
&= \frac{-3}{7}x^{\frac{-7}{3}} - \frac{6}{8}x^{\frac{-8}{3}} - \frac{1}{3}x^{-3} + k, \qquad k\in\mathbb{R}\\
\int_0^1 \frac{\sqrt[3]{x^2}+2\sqrt{x}+1}{x^4}\,dx &= \left[\frac{-3}{7}x^{\frac{-7}{3}} - \frac{6}{8}x^{\frac{-8}{3}} - \frac{1}{3}x^{-3}\right]_0^1\\
&= \frac{-3}{7}(1)^{\frac{-7}{3}} - \frac{6}{8}(1)^{\frac{-8}{3}} - \frac{1}{3}(1)^{-3} - 0\\
&= -\frac{127}{84}.
\end{aligned}$$

Exercise 1.3 ***(Integration by substitution)***
Calculate $\int \frac{x}{\sqrt{1-x^2}}\,dx.$

Solution 1.7 ***We set*** $u = 1 - x^2$, ***then*** $\frac{du}{dx} = -2x$, ***Consequently*** $x\,dx = -\frac{1}{2}du.$

$$\begin{aligned}
\int \frac{x}{\sqrt{1-x^2}}\,dx &= -\frac{1}{2}\int \frac{du}{\sqrt{u}} \\
&= -\frac{1}{2}\int u^{\frac{-1}{2}}\,du \\
&= -\frac{1}{2}\left[\frac{u^{\frac{-1}{2}+1}}{\frac{-1}{2}+1}\right] \\
&= -\sqrt{u} + k, \quad k \in \mathbb{R} \\
&= -\sqrt{1-x^2} + k, \quad k \in \mathbb{R}.
\end{aligned}$$

Exercise 1.4 ***(Integration by substitution)***
Calculate $\int \arccos(x)\,dx.$

Solution 1.8 ***First, we begin by writing the function to be integrated as a product of two functions. Thus, we calculate***

$$\int 1 \times \arccos(x)\,dx,$$

we have

$$\begin{cases} u' = 1 \Longrightarrow u = x \\ v = \arccos(x) \Longrightarrow v' = \dfrac{-1}{\sqrt{1-x^2}}. \end{cases}$$

The principle of integration by parts gives

$$\begin{aligned}
\int 1 \times \arccos(x)\,dx &= [x\arccos(x)] - \int \frac{-x}{\sqrt{1-x^2}}\,dx \\
&= [x\arccos(x)] - \sqrt{1-x^2} + k, \quad k \in \mathbb{R}.
\end{aligned}$$

Exercise 1.5 ***(Use of decomposition)***
Calculate the following integrals

$$I_1 = \int \frac{-x+1}{x^2+2x+5}\,dx, \quad I_2 = \int \frac{\ln(x^2+2x+5)}{(x-1)^2}\,dx.$$

Solution 1.9 ***1) We can easily see that***

$$\int \frac{-x+1}{x^2+2x+5}\,dx = \int \frac{-x+1}{(x+1)^2+4}\,dx,$$

we use the change of variable $u = x + 1$, ***which gives*** $\frac{du}{dx} = 1$ ***and*** $x = u - 1$. ***The integral*** I_1 ***can be written as***

$$I_1 = \int \frac{-u+2}{u^2+4}\,du = \int \frac{2}{u^2+4}\,du - \int \frac{u}{u^2+4}\,dx$$

$$\int \frac{2}{u^2+4}\,du = \int \frac{\frac{1}{2}}{\left(\frac{u}{2}\right)^2+1}\,du = \arctan\left(\frac{u}{2}\right) + k_1,$$

$$\int \frac{u}{u^2+4}\,dx = \frac{1}{2}\int \frac{2u}{u^2+4}\,dx = \ln\left|u^2+4\right| + k_2,$$

Then

$$I_1 = \arctan\left(\frac{u}{2}\right) - \frac{1}{2}\ln\left|u^2+4\right| + k = \arctan\left(\frac{x+1}{2}\right) - \frac{1}{2}\ln\left|(x+1)^2+4\right| + k, \quad k \in \mathbb{R}$$

$$\int \frac{-x+1}{x^2+2x+5}\,dx = \arctan\left(\frac{x+1}{2}\right) - \frac{1}{2}\ln\left|\left(x^2+2x+5\right)\right| + k, \quad k \in \mathbb{R}.$$

2) To calculate I_2, we consider integration by parts.

$$I_2 = \int \frac{\ln\left(x^2+2x+5\right)}{(x-1)^2}\,dx = \int \frac{1}{(x-1)^2}\ln\left(x^2+2x+5\right)\,dx.$$

$$\begin{cases} u' = \dfrac{1}{(x-1)^2} \Longrightarrow u = \displaystyle\int \frac{1}{(x-1)^2}\,dx = \frac{-1}{x-1} \\ v = \ln\left(x^2+2x+5\right) \Longrightarrow v' = \dfrac{2x+2}{x^2+2x+5} \end{cases}$$

The principle of integration by parts gives us

$$\int \frac{\ln\left(x^2+2x+5\right)}{(x-1)^2}\,dx = \left[\frac{-1}{x-1}\ln\left(x^2+2x+5\right)\right] - \int \frac{-1}{x-1}\frac{2x+2}{x^2+2x+5}\,dx$$

$$I_2 = \frac{-\ln\left(x^2+2x+5\right)}{x-1} + \int \frac{2x+2}{(x-1)\left(x^2+2x+5\right)}\,dx.$$

We decompose the fraction $\dfrac{2x+2}{(x-1)\left(x^2+2x+5\right)}$.

$$\begin{aligned} \frac{2x+2}{(x-1)\left(x^2+2x+5\right)} &= \frac{A_1}{x-1} + \frac{A_2x+A_3}{x^2+2x+5}, \\ &= \frac{A_1\left(x^2+2x+5\right) + \left(A_2x+A_3\right)(x-1)}{(x-1)\left(x^2+2x+5\right)}. \end{aligned}$$

After simplification and identification, we find $A_1 = \frac{1}{2},\ A_2 = -\frac{1}{2},\ A_3 = \frac{1}{2}$.

By decomposing the fraction $\dfrac{-x+1}{x^2+2x+5}$, ***we have***

$$\begin{aligned} \int \frac{2x+2}{(x-1)\left(x^2+2x+5\right)}\,dx &= \frac{1}{2}\int \frac{1}{x-1}\,dx + \frac{1}{2}\int \frac{-x+1}{x^2+2x+5}\,dx, \\ &= \frac{1}{2}\ln\left|x-1\right| + \frac{1}{2}\arctan\left(\frac{x+1}{2}\right) - \frac{1}{4}\ln\left|\left(x^2+2x+5\right)\right| + k, \quad k \in \mathbb{R}. \end{aligned}$$

$$I_2 = \frac{-\ln\left(x^2+2x+5\right)}{x-1} + \frac{1}{2}\ln\left|x-1\right| + \frac{1}{2}\arctan\left(\frac{x+1}{2}\right) - \frac{1}{4}\ln\left|\left(x^2+2x+5\right)\right| + k, \quad k \in \mathbb{R}.$$

Exercise 1.6 ***(Bioche's rules)***
Calculate $\int \frac{1}{\sin(2x)}\,dx$, ***and deduce from it*** $\int \frac{1}{\sin(x)}\,dx$.

Solution 1.10 ***We set***

$$f(x) = \frac{1}{\sin(2x)}\,dx,$$

we have

$$f\left(x+\pi\right) = \frac{1}{\sin 2\left(x+\pi\right)}\,d\left(x+\pi\right) = \frac{1}{\sin(2x)}\,dx = f(x).$$

According to Bioche's rules, we use the change of variable $u = \tan(x)$***. Then we have***

$$\frac{du}{dx} = \frac{1}{\cos^2(x)} \qquad \textbf{\textit{and}} \qquad u = \frac{1}{\cos^2(x)}\,dx.$$

On the other hand,

$$\begin{aligned} \frac{1}{\sin(2x)} = \frac{1}{2\sin(x)\cos(x)} &= \frac{1}{2}\frac{\cos(x)}{\sin(x)\cos^2(x)} \\ &= \frac{1}{2}\frac{1}{\tan(x)}\frac{1}{\cos^2(x)}, \end{aligned}$$

hence

$$\begin{aligned} \int \frac{1}{\sin(2x)}\,dx &= \frac{1}{2}\int \frac{1}{\tan(x)}\frac{1}{\cos^2(x)}\,dx \\ &= \frac{1}{2}\int \frac{1}{u}\,du \\ &= \frac{1}{2}\ln|u| + k, \quad k \in \mathbb{R} \\ &= \frac{1}{2}\ln\left|\tan(x)\right| + k \quad k \in \mathbb{R}. \end{aligned}$$

To calculate

$$\int \frac{1}{\sin(x)}\,dx,$$

we use the change of variable $x = 2u$***, which gives*** $u = \frac{x}{2}$ ***and*** $dx = 2\,du$***.***

$$\begin{aligned} \int \frac{1}{\sin(x)}\,dx &= 2\int \frac{1}{\sin(2u)}\,du \\ &= \ln\left|\tan(u)\right| + k, \quad k \in \mathbb{R} \\ &= \ln\left|\tan\left(\frac{x}{2}\right)\right| + k, \quad k \in \mathbb{R}. \end{aligned}$$

a. Double Integrals

Exercise 1.7 ***(Double integral) Compute***

$$I = \iint_{[0,2]\times[0,1]} (3x+2y)\,dx\,dy.$$

Solution 1.11 ***Use iterated integrals:***

$$I = \int_0^1 \int_0^2 (3x + 2y)\, dx\, dy.$$

First integrate with respect to x***:***

$$\int_0^2 (3x + 2y)\, dx = \left[\tfrac{3x^2}{2} + 2yx\right]_0^2 = \tfrac{3\cdot 4}{2} + 2y \cdot 2 = 6 + 4y.$$

Now integrate with respect to y***:***

$$I = \int_0^1 (6 + 4y)\, dy = \left[6y + 2y^2\right]_0^1 = 6 + 2 = 8.$$

Thus, $\boxed{I = 8.}$

Exercise 1.8 ***(Double Integral over a Triangle)***
Calculate the double integral of the function

$$f(x, y) = xy$$

over the triangular domain

$$D = \{(x, y) \mid 0 \leq x \leq 1,\ 0 \leq y \leq 1 - x\}.$$

Solution 1.12 ***The double integral is given by:***

$$\iint_D xy\, dx\, dy = \int_{x=0}^{1} \int_{y=0}^{1-x} xy\, dy\, dx.$$

First, integrate with respect to y***:***

$$\int_{y=0}^{1-x} xy\, dy = x \int_0^{1-x} y\, dy = x \left[\frac{y^2}{2}\right]_0^{1-x} = \frac{x(1-x)^2}{2}.$$

Next, integrate with respect to x***:***

$$\int_0^1 \frac{x(1-x)^2}{2}\, dx = \frac{1}{2} \int_0^1 x(1 - 2x + x^2)\, dx = \frac{1}{2} \int_0^1 (x - 2x^2 + x^3)\, dx.$$

Compute the integral term by term:

$$\frac{1}{2}\left[\frac{x^2}{2} - \frac{2x^3}{3} + \frac{x^4}{4}\right]_0^1 = \frac{1}{2}\left(\frac{1}{2} - \frac{2}{3} + \frac{1}{4}\right) = \frac{1}{2}\left(\frac{6 - 8 + 3}{12}\right) = \frac{1}{2} \cdot \frac{1}{12} = \frac{1}{24}.$$

$$\boxed{\iint_D xy\, dx\, dy = \frac{1}{24}}$$

Exercise 1.9 ***(Polar coordinates) Compute the integral***

$$I = \iint_D (x^2 + y^2)\, dx\, dy,$$

where D ***is the disk*** $x^2 + y^2 \leq 4$ ***(disk radius*** 2***).***

Solution 1.13 ***Switch to polar coordinates:*** $x = \rho\cos\theta,\ y = \rho\sin\theta$***, and*** $x^2+y^2 = \rho^2$***. The Jacobian is*** ρ***. Domain:*** $\rho \in [0, 2],\ \theta \in [0, 2\pi]$***.***

$$I = \int_0^{2\pi}\int_0^2 \rho^2 \cdot \rho \, d\rho \, d\theta = \int_0^{2\pi}\int_0^2 \rho^3 \, d\rho \, d\theta.$$

Integrate in ρ***:***

$$\int_0^2 \rho^3 \, d\rho = \left[\tfrac{\rho^4}{4}\right]_0^2 = \tfrac{16}{4} = 4.$$

Then

$$I = \int_0^{2\pi} 4 \, d\theta = 4 \cdot 2\pi = 8\pi.$$

Thus,

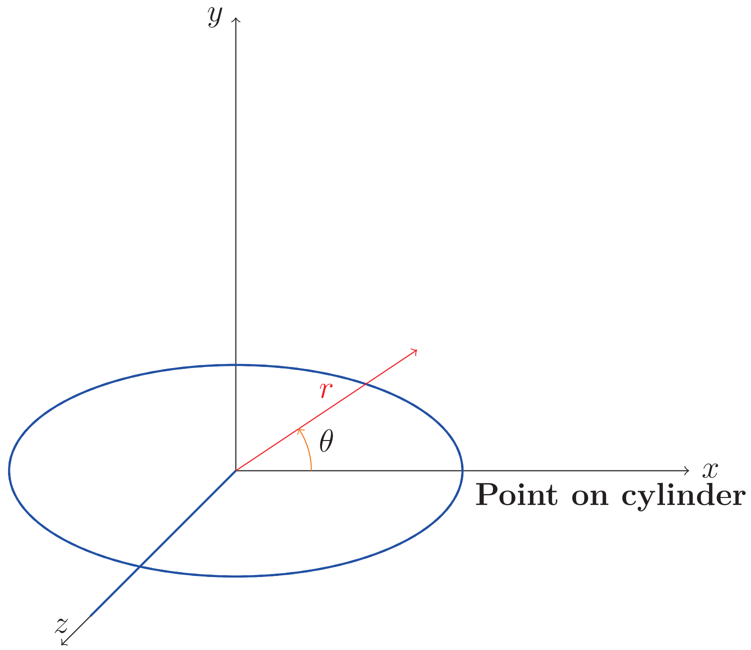
$$\boxed{I = 8\pi.}$$

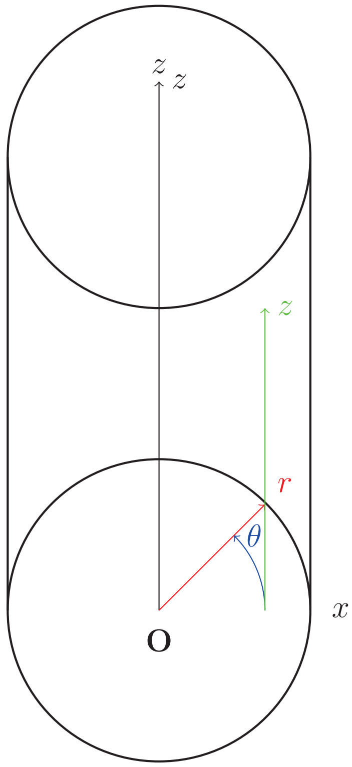
b. Triple integrals

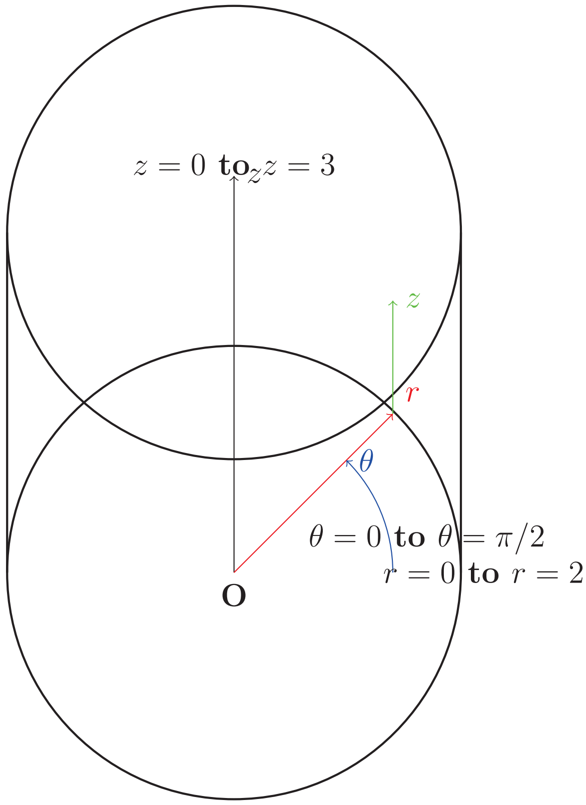
Exercise 1.10 ***(Cylindrical coordinates)***
Compute the triple integral of the function

$$f(x, y, z) = x + y + z$$

over the solid bounded by the cylinder $x^2+y^2 = 4$ ***and the planes*** $z = 0$ ***and*** $z = 3$***.***







Solution 1.14 ***The integral can be simplified using cylindrical coordinates:***

$$x = r\cos\theta, \quad y = r\sin\theta, \quad z = z, \quad \textbf{\textit{with}} \quad dx\,dy\,dz = r\,dr\,d\theta\,dz.$$

The bounds are:

$$0 \le r \le 2, \quad 0 \le \theta \le 2\pi, \quad 0 \le z \le 3.$$

So the integral becomes:

$$\int_0^3 \int_0^{2\pi} \int_0^2 (r\cos\theta + r\sin\theta + z)\, r\, dr\, d\theta\, dz.$$

Then the integral becomes

$$\int_{z=0}^3 \int_{\theta=0}^{2\pi} \int_{r=0}^2 (r\cos\theta + r\sin\theta + z)\, r\, dr\, d\theta\, dz.$$

Step 2: Expand the integrand:

$$(r\cos\theta + r\sin\theta + z)\cdot r = r^2\cos\theta + r^2\sin\theta + rz.$$

Thus the integral is

$$\int_{z=0}^3 \int_{\theta=0}^{2\pi} \int_{r=0}^2 \Big(r^2\cos\theta + r^2\sin\theta + rz\Big) dr\, d\theta\, dz.$$

Step 3: Integrate with respect to r***:***

$$\int_0^2 r^2\, dr = \frac{8}{3}, \qquad \int_0^2 rz\, dr = 2z.$$

Hence, after integrating over r***, we get***

$$\int_{z=0}^3 \int_{\theta=0}^{2\pi} \left(\frac{8}{3}\cos\theta + \frac{8}{3}\sin\theta + 2z\right) d\theta\, dz.$$

Step 4: Integrate with respect to θ***:***

$$\int_0^{2\pi} \cos\theta\, d\theta = 0, \qquad \int_0^{2\pi} \sin\theta\, d\theta = 0,$$

so we are left with

$$\int_{z=0}^3 \int_{\theta=0}^{2\pi} 2z\, d\theta\, dz = \int_{z=0}^3 2z \cdot 2\pi\, dz = 4\pi \int_0^3 z\, dz.$$

Step 5: Integrate with respect to z***:***

$$4\pi \int_0^3 z\, dz = 4\pi \left[\frac{z^2}{2}\right]_0^3 = 4\pi \cdot \frac{9}{2} = 18\pi.$$

Answer: $\boxed{18\pi}$

Exercise 1.11 ***(Cylindrical coordinates) Compute the triple integral of the function***

$$f(x, y, z) = x^2 + y^2 + z$$

over the solid bounded by the cylinder

$$x^2 + y^2 = 4$$

and the planes

$$z = 0 \quad \textbf{\textit{and}} \quad z = 3.$$

Solution 1.15 ***Step 1: Convert to cylindrical coordinates***
We use cylindrical coordinates:

$$x = r\cos\theta, \quad y = r\sin\theta, \quad z = z, \quad r \geq 0, \quad 0 \leq \theta \leq 2\pi$$

with Jacobian $dx\,dy\,dz = r\,dr\,d\theta\,dz$***.***
The function becomes

$$f(r, \theta, z) = r^2 + z$$

since $x^2 + y^2 = r^2$***.***
Step 2: Write the triple integral

$$\int_0^{2\pi} \int_0^2 \int_0^3 (r^2 + z)\, r\, dz\, dr\, d\theta$$

Step 3: Integrate with respect to z

$$\int_0^3 (r^2 + z) dz = \left[r^2 z + \frac{z^2}{2} \right]_0^3 = 3r^2 + \frac{9}{2} = 3r^2 + 4.5$$

Step 4: Integrate with respect to r

$$\int_0^2 (3r^2 + 4.5) \cdot r\, dr = \int_0^2 (3r^3 + 4.5r)\, dr$$

$$\int_0^2 3r^3\, dr = 12, \qquad \int_0^2 4.5r\, dr = 9$$

$$12 + 9 = 21$$

Step 5: Integrate with respect to θ

$$\int_0^{2\pi} 21\, d\theta = 21 \cdot 2\pi = 42\pi$$

Final Answer: $\boxed{42\pi}$

Exercise 1.12 ***(Triple integral in spherical coordinates)***
Compute the volume integral

$$I = \iiint_B z\, dx\, dy\, dz,$$

where B ***is the ball*** $x^2 + y^2 + z^2 \leq R^2$***.***

Solution 1.16 ***Use spherical coordinates:***

$$x = \rho \sin\varphi \cos\theta, \quad y = \rho \sin\varphi \sin\theta, \quad z = \rho \cos\varphi,$$

with $\rho \in [0, R],\ \varphi \in [0, \pi],\ \theta \in [0, 2\pi]$***. The Jacobian is*** $\rho^2 \sin\varphi$***. Thus***

$$I = \int_0^{2\pi} \int_0^{\pi} \int_0^R (\rho\cos\varphi)\, \rho^2 \sin\varphi\, d\rho\, d\varphi\, d\theta = \int_0^{2\pi} \int_0^{\pi} \int_0^R \rho^3 \cos\varphi \sin\varphi\, d\rho\, d\varphi\, d\theta.$$

Integrate in ρ***:***

$$\int_0^R \rho^3\, d\rho = \left[\tfrac{\rho^4}{4}\right]_0^R = \tfrac{R^4}{4}.$$

So

$$I = \tfrac{R^4}{4} \int_0^{2\pi} \int_0^{\pi} \cos\varphi \sin\varphi \, d\varphi \, d\theta.$$

Compute the φ-integral (use substitution $u = \sin\varphi$ or note $\cos\varphi \sin\varphi = \frac{1}{2}\sin(2\varphi)$):

$$\int_0^{\pi} \cos\varphi \sin\varphi \, d\varphi = \left[\tfrac{1}{2}\sin^2\varphi\right]_0^{\pi} = \tfrac{1}{2}(\sin^2\pi - \sin^2 0) = 0.$$

Therefore the whole integral is zero:

$$I = \tfrac{R^4}{4} \cdot 0 \cdot \int_0^{2\pi} d\theta = 0.$$

Interpretation: by symmetry the integral of z over the full ball vanishes (positive and negative parts cancel). Thus $\boxed{I = 0.}$

c. Area and Volume Calculation

1. Area

Exercise 1.13 ***Calculate the area of the domain***

$$D = \{(x, y) \in \mathbb{R}^2 \mid 0 \le x \le 2,\ 0 \le y \le x^2\}.$$

Solution 1.17 ***The area of the domain D can be calculated using a double integral:***

$$\textbf{\textit{Area}}(D) = \int_0^2 \int_0^{x^2} dy \, dx.$$

Step 1: Integrate with respect to y:

$$\int_0^{x^2} dy = y\Big|_0^{x^2} = x^2.$$

Step 2: Integrate with respect to x:

$$\int_0^2 x^2 \, dx = \frac{x^3}{3}\Big|_0^2 = \frac{8}{3}.$$

Answer:

$$\boxed{\textbf{\textit{Area}}(D) = \frac{8}{3}}$$

Exercise 1.14 ***Calculate the area of the domain***

$$D = \{(x, y) \in \mathbb{R}^2 \mid 0 \le x \le 1,\ x \le y \le 2x\}.$$

Solution 1.18 ***The area of the domain D is given by the double integral:***

$$\boldsymbol{Area}(D) = \int_0^1 \int_{y=x}^{y=2x} dy\, dx.$$

Step 1: Integrate with respect to y:

$$\int_{y=x}^{y=2x} dy = y\Big|_{y=x}^{y=2x} = 2x - x = x.$$

Step 2: Integrate with respect to x:

$$\int_0^1 x\, dx = \frac{x^2}{2}\Big|_0^1 = \frac{1}{2}.$$

Answer: $\boxed{\boldsymbol{Area}(D) = \frac{1}{2}}$

2. Volume

Exercise 1.15 ***Compute the volume of the sphere (ball) of radius $R > 0$,***

$$D = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 \leq R^2\},$$

using spherical coordinates.

Solution 1.19 ***We compute the geometric volume***

$$V = \iiint_D 1\, dV.$$

Step 1: Spherical coordinates.
Use the spherical change of variables

$$\begin{aligned} x &= r \sin\phi \cos\theta, \\ y &= r \sin\phi \sin\theta, \qquad r \geq 0,\ \theta \in [0, 2\pi),\ \phi \in [0, \pi]. \\ z &= r \cos\phi, \end{aligned}$$

The volume element (Jacobian) is

$$dV = r^2 \sin\phi\ dr\, d\phi\, d\theta.$$

Step 2: Set up the triple integral.
In these coordinates the ball D corresponds to

$$0 \leq r \leq R, \quad 0 \leq \phi \leq \pi, \quad 0 \leq \theta \leq 2\pi,$$

so

$$V = \int_{\theta=0}^{2\pi} \int_{\phi=0}^{\pi} \int_{r=0}^{R} r^2 \sin\phi\ dr\, d\phi\, d\theta.$$

Step 3: Integrate with respect to r.

$$\int_0^R r^2\,dr = \frac{R^3}{3}.$$

Hence

$$V = \int_0^{2\pi}\int_0^{\pi} \frac{R^3}{3}\sin\phi\,d\phi\,d\theta = \frac{R^3}{3}\left(\int_0^{2\pi} d\theta\right)\left(\int_0^{\pi}\sin\phi\,d\phi\right).$$

Step 4: Compute the angular integrals.

$$\int_0^{2\pi} d\theta = 2\pi, \qquad \int_0^{\pi}\sin\phi\,d\phi = [-\cos\phi]_0^{\pi} = 2.$$

Step 5: Final result.

$$V = \frac{R^3}{3}\cdot 2\pi\cdot 2 = \frac{4}{3}\pi R^3.$$

$$\boxed{V = \frac{4}{3}\pi R^3}$$

Thus the volume of the ball of radius R is $\frac{4}{3}\pi R^3$.

Exercise 1.16 ***Compute the volume of the solid bounded above by the paraboloid***

$$z = 4 - x^2 - y^2$$

and below by the plane $z = 0$.

Solution 1.20 ***We compute the geometric volume***

$$V = \iiint_D 1\,dV,$$

where D is the region in $\mathbb{R}^3$ bounded by $z = 0$ and $z = 4 - x^2 - y^2$.
Step 1: Project onto the xy-plane.
The intersection of the paraboloid with the plane $z = 0$ is given by

$$4 - x^2 - y^2 = 0 \quad\Longrightarrow\quad x^2 + y^2 = 4.$$

Thus the projection on the xy-plane is the disk

$$\{(x, y) \mid x^2 + y^2 \le 4\}.$$

Step 2: Use cylindrical coordinates.
Set

$$x = r\cos\theta, \qquad y = r\sin\theta, \qquad z = z,$$

with $r \in [0, 2]$, $\theta \in [0, 2\pi)$. The Jacobian is $dV = r\,dr\,d\theta\,dz$. In these coordinates the top surface becomes $z = 4 - r^2$ and the bottom is $z = 0$.
Step 3: Set up the triple integral.

$$V = \int_{\theta=0}^{2\pi}\int_{r=0}^{2}\int_{z=0}^{4-r^2} r\,dz\,dr\,d\theta.$$

Step 4: Integrate with respect to z.

$$\int_{z=0}^{4-r^2} r\ dz = r(4-r^2).$$

So

$$V = \int_0^{2\pi}\int_0^2 r(4-r^2)\,dr\,d\theta = \int_0^{2\pi}\left(\int_0^2 (4r-r^3)\,dr\right)d\theta.$$

Step 5: Integrate with respect to r.

$$\int_0^2 (4r-r^3)\,dr = \left[2r^2 - \frac{r^4}{4}\right]_0^2 = \left(2\cdot 4 - \frac{16}{4}\right) - 0 = 8-4 = 4.$$

Step 6: Integrate with respect to θ.

$$V = \int_0^{2\pi} 4\,d\theta = 4\cdot 2\pi = 8\pi.$$

Answer: $\boxed{V = 8\pi}$ ***Thus the volume of the solid is 8π cubic units.***

Chapter 2

Improper Integrals

Introduction

Improper integrals naturally appear in physics, for example in gravitational or electrostatic potentials, quantum mechanics, or Green's functions. An improper integral is an integral where either:

- **one of the bounds is infinite, or**
- **the integrand becomes infinite at some point of the interval.**

2.1 Improper Integrals of Functions Defined on an Unbounded Interval (First Kind)

Definition 2.1 ***Let f be a continuous function. An integral of the form***

$$\int_a^{+\infty} f(x)\,dx, \quad \int_{-\infty}^{b} f(x)\,dx, \quad \textbf{\textit{or}} \quad \int_{-\infty}^{+\infty} f(x)\,dx$$

is called an improper integral of the first kind, because the interval of integration is unbounded. These are defined through limits:

$$\int_a^{+\infty} f(x)\,dx := \lim_{t\to+\infty} \int_a^t f(x)\,dx,$$

$$\int_{-\infty}^{b} f(x)\,dx := \lim_{t\to-\infty} \int_t^b f(x)\,dx,$$

$$\int_{-\infty}^{+\infty} f(x)\,dx := \lim_{A\to-\infty,\,B\to+\infty} \int_A^B f(x)\,dx,$$

provided these limits exist and are finite. If the limit does not exist or is infinite, the integral is said to diverge.

2.1.1 Convergence criteria

1. p-integral test

Theorem 2.1 ***Let p be an arbitrary power, that can be any real number.***

$$\int_1^{+\infty} \frac{1}{x^p}\,dx \quad \textbf{\textit{converges if}}\ p>1, \quad \textbf{\textit{diverges if}}\ p\le 1.$$

Examples 2.1 ***1. Convergent integral:***

$$\int_1^{+\infty} \frac{1}{x^2}\,dx = \lim_{t\to+\infty} \int_1^t \frac{1}{x^2}\,dx = \lim_{t\to+\infty} \left[\frac{1}{x}\right]_1^t = \lim_{t\to+\infty} \left(\frac{1}{t} - 1\right) = 1.$$

We can easily see that this integral converge by p-integral test for $p = 2$.

2. Divergent integral:

$$\int_1^{+\infty} \frac{1}{x}\,dx = \lim_{t\to+\infty} \ln(t) = +\infty.$$

We can easily see that this integral diverge by p-integral test for $p = 1$.

3. Integral over the whole real line:

$$\int_{-\infty}^{+\infty} e^{-x^2}\,dx = \sqrt{\pi}.$$

We want to compute:

$$I = \int_{-\infty}^{+\infty} e^{-x^2}\,dx.$$

Step 1. Square the integral. We consider

$$I^2 = \left(\int_{-\infty}^{+\infty} e^{-x^2}\,dx\right)\left(\int_{-\infty}^{+\infty} e^{-y^2}\,dy\right).$$

This can be written as a double integral over the plane:

$$I^2 = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{-(x^2+y^2)}\,dx\,dy.$$

Step 2. Switch to polar coordinates. Recall that in polar coordinates:

$$x = r\cos\theta, \quad y = r\sin\theta, \quad dx\,dy = r\,dr\,d\theta.$$

Thus,

$$I^2 = \int_0^{2\pi}\int_0^{+\infty} e^{-r^2}\,r\,dr\,d\theta.$$

Step 3. Compute the radial integral. First compute the inner integral:

$$\int_0^{+\infty} e^{-r^2}\,r\,dr.$$

Use the substitution $u = r^2$, so $du = 2r\,dr$ and hence $r\,dr = \frac{1}{2}\,du$. Therefore,

$$\int_0^{+\infty} e^{-r^2}\,r\,dr = \frac{1}{2}\int_0^{+\infty} e^{-u}\,du = \frac{1}{2}.$$

In fact,

$$\int_0^{+\infty} e^{-r^2}\,r\,dr = \frac{1}{2}\int_0^{+\infty} e^{-u}\,du = \frac{1}{2}\lim_{t\to+\infty}\int_0^t e^{-u}\,du = \frac{1}{2}\lim_{t\to+\infty}\left[-e^{-u}\right]_0^t = \frac{1}{2}\lim_{t\to+\infty}\left(1-e^{-t}\right) = \frac{1}{2}\cdot 1$$

Step 4. Compute the angular integral. Now,

$$I^2 = \int_0^{2\pi} \left(\frac{1}{2}\right) d\theta = \pi.$$

Step 5. Conclude. Since $I > 0$, we have

$$I = \sqrt{\pi}.$$

Final Answer

$$\int_{-\infty}^{+\infty} e^{-x^2}\, dx = \sqrt{\pi}.$$

2. Comparison Test (Useful Criterion)

Let f and g be continuous functions on $[a, +\infty)$ such that:
$0 \le f(x) \le g(x)$ for all $x \ge a$.

- **If $\int_a^{+\infty} g(x)\, dx$ converges, then $\int_a^{+\infty} f(x)\, dx$ also converges.**
- **If $\int_a^{+\infty} f(x)\, dx$ diverges, then $\int_a^{+\infty} g(x)\, dx$ also diverges.**

Example 2.1 ***Consider the improper integral***

$$\int_1^{+\infty} \frac{1}{x^2+1}\, dx.$$

For all $x \ge 1$, we have

$$0 \le \frac{1}{x^2+1} \le \frac{1}{x^2}.$$

Now, we know that

$$\int_1^{+\infty} \frac{1}{x^2}\, dx$$

is a convergent improper integral (it is a p-integral with $p = 2 > 1$). Therefore, by the Comparison Test,

$$\int_1^{+\infty} \frac{1}{x^2+1}\, dx$$

also converges.

Example 2.2 ***Consider the improper integral***

$$\int_1^{+\infty} \frac{1}{\sqrt{x}}\, dx.$$

For all $x \ge 1$, we have

$$\frac{1}{\sqrt{x}} \ge \frac{1}{x}.$$

Now, we know that

$$\int_1^{+\infty} \frac{1}{x}\, dx$$

is a divergent improper integral (it is a p-integral with $p = 1$). Therefore, by the Comparison Test,

$$\int_1^{+\infty} \frac{1}{\sqrt{x}}\, dx$$

also diverges.

3. Limit Comparison Test

Let f and g be continuous and positive on $[a, \infty)$. Assume

$$L := \lim_{x \to \infty} \frac{f(x)}{g(x)} \in [0, +\infty].$$

1. **If $0 < L < \infty$, then $\int_a^\infty f(x)\,dx$ converges iff $\int_a^\infty g(x)\,dx$ converges.**

   Remarks 2.1

   $$\int_a^\infty f(x)\,dx \ \textbf{\textit{converges}} \quad \Longleftrightarrow \quad \int_a^\infty g(x)\,dx \ \textbf{\textit{converges}}.$$

   Implications

   - **If $\int_a^\infty g(x)\,dx$ converges, then $\int_a^\infty f(x)\,dx$ converges.**
   - **If $\int_a^\infty f(x)\,dx$ converges, then $\int_a^\infty g(x)\,dx$ converges.**

   Contrapositives (for divergence)

   - **If $\int_a^\infty f(x)\,dx$ diverges, then $\int_a^\infty g(x)\,dx$ diverges.**
   - **If $\int_a^\infty g(x)\,dx$ diverges, then $\int_a^\infty f(x)\,dx$ diverges.**

   (a) **When $0 < L < \infty$, the two integrals always share the same behavior: they both converge or they both diverge.**

   (b) **The explicit writing of implications and their contrapositives avoids ambiguity when dealing with divergence.**

2. **If $L = 0$ and $\int_a^\infty g(x)\,dx$ converges, then $\int_a^\infty f(x)\,dx$ converges.**

3. **If $L = +\infty$ and $\int_a^\infty g(x)\,dx$ diverges, then $\int_a^\infty f(x)\,dx$ diverges.**

Example 2.3 ***Study*** $\displaystyle\int_1^\infty \frac{1}{x^2 + 3x + 1}\,dx.$

Take $f(x) = \dfrac{1}{x^2 + 3x + 1}$ ***and*** $g(x) = \dfrac{1}{x^2}$***. Then***

$$\frac{f(x)}{g(x)} = \frac{x^2}{x^2 + 3x + 1} = \frac{1}{1 + \frac{3}{x} + \frac{1}{x^2}} \xrightarrow[x \to \infty]{} 1.$$

Thus $L = 1 \in (0, \infty)$***. Since*** $\displaystyle\int_1^\infty \frac{1}{x^2}\,dx$ ***converges (p-integral with*** $p = 2 > 1$***), the Limit Comparison Test implies that***

$$\int_1^\infty \frac{1}{x^2 + 3x + 1}\,dx$$

also converges.

Example 2.4 ***Study*** $\int_1^{\infty} \frac{1}{\sqrt{x^2+1}}\,dx.$

Take $f(x) = \frac{1}{\sqrt{x^2+1}}$ ***and*** $g(x) = \frac{1}{x}$***. Then***

$$\frac{f(x)}{g(x)} = \frac{x}{\sqrt{x^2+1}} = \frac{1}{\sqrt{1+\frac{1}{x^2}}} \xrightarrow[x\to\infty]{} 1.$$

Thus $L = 1$***. Since*** $\int_1^{\infty} \frac{1}{x}\,dx$ ***diverges (harmonic integral), the Limit Comparison Test implies that***

$$\int_1^{\infty} \frac{1}{\sqrt{x^2+1}}\,dx$$

also diverges.

4. Absolutely Convergent Integrals

Definition 2.2 ***Let*** $f : [a, +\infty[\to \mathbb{R}$ ***be a measurable function. The improper integral***

$$\int_a^{\infty} f(x)\,dx$$

is said to be absolutely convergent if

$$\int_a^{\infty} |f(x)|\,dx$$

is convergent.

Theorem 2.2 ***If the integral of*** $|f(x)|$ ***is convergent, then the integral of*** $f(x)$ ***is also convergent:***

$$\int_a^{\infty} |f(x)|\,dx \text{ convergent} \quad \Longrightarrow \quad \int_a^{\infty} f(x)\,dx \text{ convergent}.$$

Example 2.5 ***Consider the integral***

$$\int_1^{\infty} \frac{\cos(x)}{x^2}\,dx.$$

We observe that

$$\left|\frac{\cos(x)}{x^2}\right| \le \frac{1}{x^2}.$$

Now, the integral

$$\int_1^{\infty} \frac{1}{x^2}\,dx$$

is convergent.
Therefore,

$$\int_1^{\infty} \left|\frac{\cos(x)}{x^2}\right|\,dx$$

is convergent, and consequently

$$\int_1^{\infty} \frac{\cos(x)}{x^2}\,dx$$

is absolutely convergent.

5. Dirichlet Test for Improper Integrals

Let $f(x)$ and $g(x)$ be functions defined on $[a, \infty)$. Consider the improper integral

$$\int_a^\infty f(x)g(x)\,dx.$$

The integral converges if the following conditions hold:
1. The function $f(x)$ has a bounded primitive:

$$F(x) = \int_a^x f(t)\,dt \quad \textbf{is bounded for all } x \geq a.$$

2. The function $g(x)$ is monotone decreasing and tends to zero:

$$g(x) \textbf{ is monotone decreasing,} \quad \lim_{x\to\infty} g(x) = 0.$$

Remark: This test is particularly useful for integrals of the type

$$\int_1^\infty \frac{\sin(x)}{x^\alpha}\,dx \quad \textbf{or} \quad \int_0^1 \frac{\sin(1/x)}{x^\beta}\,dx.$$

Example 2.6 ***Consider the integral***

$$\int_1^\infty \frac{\sin(x)}{x}\,dx.$$

Step 1: Identify $f(x)$ and $g(x)$:

$$f(x) = \sin(x), \quad g(x) = \frac{1}{x}.$$

Step 2: Check Dirichlet conditions:
1. $f(x) = \sin(x)$ has a bounded primitive:

$$F(X) = \int_1^X \sin(x)dx = [-\cos(x)]_1^X = -\cos(X) + \cos(1),$$

so $|F(X)| \leq 2$.(F(x) is bounded)
2. $g(x) = 1/x$ is monotone decreasing and tends to zero:

$$\lim_{x\to\infty} \frac{1}{x} = 0.$$

Step 3: Conclusion: By Dirichlet's Test, the integral converges:

$$\int_1^\infty \frac{\sin(x)}{x}\,dx \textbf{ converges.}$$

2.2 Integrals of Functions Defined on a Bounded Interval, Infinite at One Endpoint (Second Kind)

Definition 2.3 ***Improper Integrals of the Second Kind (Integration of an Unbounded Integrand)***
Let f be a function defined on an interval $(a, b]$, $[a, b)$, or (a, b), but not necessarily at one or both endpoints because it becomes infinite there. An integral of the form

$$\int_a^b f(x)\,dx$$

is called an improper integral of the second kind if f is unbounded at a, b, or both.
Formally:

- ***If f is unbounded at a:***

$$\int_a^b f(x)\,dx := \lim_{\varepsilon \to 0^+} \int_{a+\varepsilon}^b f(x)\,dx$$

- ***If f is unbounded at b:***

$$\int_a^b f(x)\,dx := \lim_{\varepsilon \to 0^+} \int_a^{b-\varepsilon} f(x)\,dx$$

- ***If f is unbounded at both a and b, split the interval at some $c \in (a, b)$:***

$$\int_a^b f(x)\,dx := \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

  where each integral is defined as above.

Remark 2.1

- ***f is not defined at the singular point(s) because it diverges.***
- ***The convergence depends on the behavior of f near the singularity. For example, if $f(x) \sim (x-a)^{-\alpha}$ near a, then:***
  - ***Convergent if $\alpha < 1$***
  - ***Divergent if $\alpha \geq 1$***
- ***Similarly for singularities near b.***

Method of Calculation

1. **Replace the singular endpoint by a variable $\varepsilon > 0$.**
2. **Compute the integral on the modified interval as a normal (proper) integral.**
3. **Take the limit as $\varepsilon \to 0^+$.**

Example 2.7 ***Convergent integral***

$$\int_0^1 \frac{dx}{\sqrt{x}}$$

Solution:

$$\int_0^1 \frac{dx}{\sqrt{x}} = \lim_{\varepsilon\to 0^+} \int_\varepsilon^1 x^{-1/2}dx = \lim_{\varepsilon\to 0^+} \left[2\sqrt{x}\right]_\varepsilon^1 = \lim_{\varepsilon\to 0^+} (2 - 2\sqrt{\varepsilon}) = 2$$

Example 2.8 ***Divergent integral***

$$\int_0^1 \frac{dx}{x}$$

Solution:

$$\int_0^1 \frac{dx}{x} = \lim_{\varepsilon\to 0^+} \int_\varepsilon^1 \frac{dx}{x} = \lim_{\varepsilon\to 0^+} (-\ln\varepsilon) = +\infty$$

Example 2.9 ***Singular at both endpoints***

$$\int_0^1 \frac{dx}{\sqrt{x(1-x)}}$$

Solution: Split at $c = \frac{1}{2}$:

$$\int_0^{1/2} \frac{dx}{\sqrt{x(1-x)}} + \int_{1/2}^1 \frac{dx}{\sqrt{x(1-x)}}$$

Both integrals converge $\Rightarrow$ total integral convergent.

2.3 Integrals of Functions with Discontinuities on the Interval of Integration (*Third kind of improper integrals*)

We consider integrals of the form

$$I = \int_a^b f(x)\,dx,$$

where the function f is *discontinuous* at one or more points inside the interval $[a, b]$. Such discontinuities prevent us from computing the integral in the usual Riemann sense. Therefore, these are treated as *improper integrals.* This case is often referred to as the *third kind of improper integrals.* If f has a discontinuity at some point $c \in (a, b)$, the integral is defined by splitting at c:

$$\int_a^b f(x)\,dx = \lim_{\substack{\alpha\to c^- \\ \beta\to c^+}} \left(\int_a^\alpha f(x)\,dx + \int_\beta^b f(x)\,dx \right).$$

- **If this limit exists and is finite, the integral is said to converge.**

- **If the limit does not exist or is infinite, the integral is said to diverge.**

Exercise 2.1 ***Study the improper integral***

$$\int_{-1}^{1} \frac{dx}{x}.$$

Solution 2.1 ***The integrand*** $f(x) = 1/x$ ***is not defined at*** $x = 0$***, which lies inside the integration interval*** $[-1, 1]$***. Thus we must treat the integral as an improper integral by splitting at the singular point*** 0***.***

1. Split into one-sided integrals.
By definition, the improper integral (ordinary definition) is

$$\int_{-1}^{1} \frac{dx}{x} = \lim_{\alpha\to 0^-} \int_{-1}^{\alpha} \frac{dx}{x} + \lim_{\beta\to 0^+} \int_{\beta}^{1} \frac{dx}{x},$$

provided both one-sided limits exist and are finite.

2. Compute the one-sided integrals.
For any $a < b$ ***not containing*** 0 ***we have***

$$\int_{a}^{b} \frac{dx}{x} = \ln|x|\Big|_{a}^{b} = \ln|b| - \ln|a|.$$

Apply this to the two pieces:

$$\int_{-1}^{\alpha} \frac{dx}{x} = \ln|\alpha| - \ln 1 = \ln|\alpha|, \qquad (\alpha < 0),$$

and

$$\int_{\beta}^{1} \frac{dx}{x} = \ln 1 - \ln\beta = -\ln\beta, \qquad (\beta > 0).$$

3. Take the limits.
As $\alpha \to 0^-$***,*** $|\alpha| \to 0^+$ ***so*** $\ln|\alpha| \to -\infty$***. As*** $\beta \to 0^+$***,*** $-\ln\beta \to +\infty$***.***
Therefore the left one-sided limit equals $-\infty$ ***and the right one-sided limit equals*** $+\infty$***. Because the two one-sided limits are not finite, the ordinary improper integral does not converge.***

$$\boxed{\textit{The improper integral } \int_{-1}^{1} \frac{dx}{x} \textit{ diverges.}}$$

4. Cauchy principal value (optional but important).
Although the ordinary improper integral diverges, one can consider the **Cauchy principal value** ***(symmetrically approaching the singularity):***

$$\text{p. v.} \int_{-1}^{1} \frac{dx}{x} := \lim_{\varepsilon\to 0^+} \left(\int_{-1}^{-\varepsilon} \frac{dx}{x} + \int_{\varepsilon}^{1} \frac{dx}{x} \right).$$

Compute the combined integral for fixed ε***:***

$$\int_{-1}^{-\varepsilon} \frac{dx}{x} + \int_{\varepsilon}^{1} \frac{dx}{x} = \Big(\ln|-\varepsilon| - \ln 1\Big) + \Big(\ln 1 - \ln\varepsilon\Big) = \ln\varepsilon - \ln\varepsilon = 0.$$

Since the value is 0 ***for every*** $\varepsilon > 0$***, the limit exists and***

$$\text{p. v.} \int_{-1}^{1} \frac{dx}{x} = 0.$$

Remark. ***The principal value is a different notion from the ordinary improper integral:***

- ***The ordinary improper integral requires both one-sided limits to be finite, here they are infinite, so the integral diverges.***
- ***The principal value uses a symmetric limiting process; it can exist even when the ordinary improper integral does not. The principal value is often useful in applications (Fourier transforms, distributions), but it does not restore ordinary convergence.***

5. A symmetry dependence.
If the two limits approaching 0 ***are taken at different rates (non-symmetric), the combined limit*** $\lim_{\alpha\to 0^-,\, \beta\to 0^+} \left(\int_{-1}^{\alpha} 1/x\, dx + \int_{\beta}^{1} 1/x\, dx \right)$ ***need not exist (and typically will diverge). The principal value chooses the symmetric path*** $\alpha = -\varepsilon,\ \beta = \varepsilon$***, which yields the finite value*** 0***.***

Conclusion. The improper integral $\displaystyle\int_{-1}^{1} \frac{dx}{x}$ ***diverges, since the one-sided limits at*** 0 ***are infinite. However, its Cauchy principal value exists and equals*** 0***.***

Example 2.10 ***Decide convergence (improper integrals with an interior singularity) and compute the value when it converges.***

1. $J_1 = \displaystyle\int_{-1}^{1} \frac{dx}{|x|^{1/2}}.$
2. $J_2 = \displaystyle\int_{-1}^{1} \frac{dx}{x^2}.$

Solution 2.2 ***(A)*** $J_1 = \displaystyle\int_{-1}^{1} \frac{dx}{|x|^{1/2}}.$
Step 1 - split at the singularity.

$$J_1 = \int_{-1}^{0} \frac{dx}{|x|^{1/2}} + \int_{0}^{1} \frac{dx}{|x|^{1/2}}.$$

Since $|x|^{-1/2} = (-x)^{-1/2}$ ***on*** $[-1, 0)$ ***and*** $= x^{-1/2}$ ***on*** $(0, 1]$***, both one-sided integrals are identical.***
Step 2 - test near 0. ***Compare with the model*** x^{-p} ***with*** $p = \frac{1}{2}$***. We know*** $\int_0^{\varepsilon} x^{-1/2}\, dx = 2\sqrt{\varepsilon} < \infty$***. Hence each one-sided integral is finite.***
Step 3 - compute. ***Use symmetry:***

$$J_1 = 2 \int_0^1 x^{-1/2}\, dx = 2\Big[2\sqrt{x}\Big]_0^1 = 2 \cdot 2 = 4.$$

$$\boxed{J_1 \text{ \textbf{\textit{converges and}} } J_1 = 4.}$$

(B) $J_2 = \int_{-1}^{1} \frac{dx}{x^2}$.

Step 1 - split at the singularity.

$$J_2 = \lim_{\alpha \to 0^-} \int_{-1}^{\alpha} \frac{dx}{x^2} + \lim_{\beta \to 0^+} \int_{\beta}^{1} \frac{dx}{x^2},$$

provided both one-sided limits are finite.

Step 2 - compute one-sided integrals. ***A primitive of*** x^{-2} ***is*** $-x^{-1}$***. For*** $0 < \beta < 1$***,***

$$\int_{\beta}^{1} \frac{dx}{x^2} = \left[-\frac{1}{x}\right]_{\beta}^{1} = -1 + \frac{1}{\beta} = \frac{1-\beta}{\beta}.$$

As $\beta \to 0^+$***,*** $\frac{1}{\beta} \to +\infty$***, so the right-hand one-sided integral diverges to*** $+\infty$***.***
Similarly the left-hand piece diverges to $+\infty$***.***

Conclusion. ***For the integral***

$$J_2 = \int_{-1}^{1} \frac{dx}{x^2},$$

we split at the singular point $x = 0$***.***

- On the interval $(0, 1]$***:***

$$\int_{\beta}^{1} \frac{dx}{x^2} = \frac{1-\beta}{\beta} \quad \xrightarrow[\beta \to 0^+]{} +\infty.$$

Hence, the right-hand side diverges to $+\infty$***.***

- On the interval $[-1, 0)$***:***

$$\int_{-1}^{\alpha} \frac{dx}{x^2} = \frac{1+\alpha}{|\alpha|} \quad \xrightarrow[\alpha \to 0^-]{} +\infty.$$

Thus, the left-hand side also diverges to $+\infty$***.***
Since both one-sided integrals are infinite, the improper integral cannot converge in the usual sense.
Moreover, the **Cauchy principal value** ***of an improper integral with a singularity inside the interval is defined by taking symmetric limits around the singular point. In our case:***

$$\textbf{\textit{p.v.}} \int_{-1}^{1} \frac{dx}{x^2} = \lim_{\varepsilon \to 0^+} \left(\int_{-1}^{-\varepsilon} \frac{dx}{x^2} + \int_{\varepsilon}^{1} \frac{dx}{x^2} \right).$$

Step 1 - compute the left-hand integral.

$$\int_{-1}^{-\varepsilon} \frac{dx}{x^2} = \left[-\frac{1}{x}\right]_{-1}^{-\varepsilon} = \left(-\frac{1}{-\varepsilon}\right) - \left(-\frac{1}{-1}\right) = \frac{1}{\varepsilon} - 1.$$

As $\varepsilon \to 0^+$***, we have*** $\frac{1}{\varepsilon} \to +\infty$***. Thus, the left-hand part diverges to*** $+\infty$***.***

Step 2 - compute the right-hand integral.

$$\int_{\varepsilon}^{1} \frac{dx}{x^2} = \left[-\frac{1}{x}\right]_{\varepsilon}^{1} = (-1) - \left(-\frac{1}{\varepsilon}\right) = \frac{1}{\varepsilon} - 1.$$

Again, as $\varepsilon \to 0^+$, $\frac{1}{\varepsilon} \to +\infty$. So the right-hand part also diverges to $+\infty$.
Step 3 - combine the two parts.

$$\int_{-1}^{-\varepsilon} \frac{dx}{x^2} + \int_{\varepsilon}^{1} \frac{dx}{x^2} = \left(\frac{1}{\varepsilon} - 1\right) + \left(\frac{1}{\varepsilon} - 1\right) = \frac{2}{\varepsilon} - 2.$$

Taking the limit as $\varepsilon \to 0^+$:

$$\lim_{\varepsilon \to 0^+} \left(\frac{2}{\varepsilon} - 2\right) = +\infty.$$

In this case, both one-sided integrals diverge to $+\infty$, so there is no possibility of cancellation (unlike the case of $\int_{-1}^{1} \frac{dx}{x}$). Therefore, the Cauchy principal value does not exist either.

$$\boxed{J_2 \textit{ diverges and has no finite principal value.}}$$

2.4 Solved Exercises

Improper Integrals

Exercise 2.2 ***Compute***

$$I = \int_0^{\infty} e^{-x}\, dx.$$

Solution 2.3 ***We write the improper integral as a limit:***

$$I = \lim_{R \to \infty} \int_0^R e^{-x}\, dx = \lim_{R \to \infty} \Big[-e^{-x}\Big]_0^R = \lim_{R \to \infty} \left(1 - e^{-R}\right) = 1.$$

Therefore $I = 1$.

Exercise 2.3 ***Compute***

$$I = \int_1^{\infty} \frac{\ln x}{x^2}\, dx.$$

Solution 2.4 ***Use integration by parts on the finite integral and then pass to the limit. For $R > 1$,***

$$\int_1^R \frac{\ln x}{x^2}\, dx$$

let $u = \ln x$ and $dv = x^{-2}dx$. Then $du = \frac{1}{x}dx$ and $v = -\frac{1}{x}$. Thus

$$\int_1^R \frac{\ln x}{x^2}\, dx = \left[-\frac{\ln x}{x}\right]_1^R + \int_1^R \frac{1}{x} \cdot \frac{1}{x}\, dx = -\frac{\ln R}{R} + 0 + \int_1^R \frac{1}{x^2}\, dx.$$

Compute the remaining integral:

$$\int_1^R \frac{1}{x^2}\,dx = \left[-\frac{1}{x}\right]_1^R = 1 - \frac{1}{R}.$$

So

$$\int_1^R \frac{\ln x}{x^2}\,dx = -\frac{\ln R}{R} + 1 - \frac{1}{R}.$$

Let $R \to \infty$***. Since*** $\frac{\ln R}{R} \to 0$ ***and*** $\frac{1}{R} \to 0$***, we get***

$$\int_1^\infty \frac{\ln x}{x^2}\,dx = 1.$$

Exercise 2.4 ***Compute***

$$I = \int_0^1 \frac{1}{\sqrt{x}}\,dx.$$

Solution 2.5 ***This is an improper integral at*** 0***. Write it as a limit:***

$$\int_0^1 \frac{1}{\sqrt{x}}\,dx = \lim_{\varepsilon\to 0^+}\int_\varepsilon^1 x^{-1/2}\,dx = \lim_{\varepsilon\to 0^+}\left[2x^{1/2}\right]_\varepsilon^1 = \lim_{\varepsilon\to 0^+}\left(2 - 2\sqrt{\varepsilon}\right) = 2.$$

Thus the integral converges and equals 2***.***

Exercise 2.5 ***Study the convergence of the improper integral:***

$$J = \int_0^1 \frac{dx}{x^p}, \qquad p \in \mathbb{R}.$$

Solution 2.6 ***The integrand*** $\dfrac{1}{x^p}$ ***is unbounded at*** $x = 0$***. We write***

$$J = \lim_{\varepsilon\to 0^+}\int_\varepsilon^1 \frac{dx}{x^p}.$$

Case 1: $p \neq 1$***. We compute the primitive:***

$$\int \frac{dx}{x^p} = \frac{x^{1-p}}{1-p}, \quad (p \neq 1).$$

Thus,

$$J = \lim_{\varepsilon\to 0^+}\left[\frac{x^{1-p}}{1-p}\right]_\varepsilon^1 = \frac{1}{1-p}\left(1 - \varepsilon^{1-p}\right).$$

- If $p < 1$***, then*** $1 - p > 0$ ***and*** $\varepsilon^{1-p} \to 0$***, so*** $J = \dfrac{1}{1-p}$ ***(finite).***
- If $p > 1$***, then*** $1 - p < 0$ ***and*** $\varepsilon^{1-p} \to +\infty$***, so*** $J = +\infty$ ***(divergent).***
Case 2: $p = 1$***.***

$$\int_\varepsilon^1 \frac{dx}{x} = \ln(1) - \ln(\varepsilon) = -\ln(\varepsilon).$$

As $\varepsilon \to 0^+$***,*** $-\ln(\varepsilon) \to +\infty$***, hence the integral diverges.***

The integral converges if and only if $p < 1$***, with*** $J = \dfrac{1}{1-p}$***.***

Exercise 2.6 ***Determine whether the following integral converges:***

$$K = \int_0^1 \ln(x)\, dx.$$

Solution 2.7 ***The function*** $\ln(x)$ ***is unbounded at*** $x = 0$***. We write***

$$K = \lim_{\varepsilon \to 0^+} \int_\varepsilon^1 \ln(x)\, dx.$$

Integration by parts: set

$$u = \ln(x), \quad dv = dx, \quad du = \frac{dx}{x}, \quad v = x.$$

Then,

$$\int \ln(x)\, dx = x\ln(x) - \int x \cdot \frac{dx}{x} = x\ln(x) - x.$$

Thus,

$$K = \lim_{\varepsilon \to 0^+} \Big[x\ln(x) - x \Big]_\varepsilon^1.$$

At $x = 1$***, we get*** $1 \cdot \ln(1) - 1 = -1$***. At*** $x = \varepsilon$***, we get*** $\varepsilon \ln(\varepsilon) - \varepsilon$***.***
As $\varepsilon \to 0^+$***: -*** $\varepsilon \ln(\varepsilon) \to 0$***, -*** $\varepsilon \to 0$***.***
So the lower limit tends to 0***. Hence,***

$$K = -1 - 0 = -1.$$

The integral converges and its value is -1.

Chapter 3

Differential Equations

3.1 Ordinary Differential Equations

3.1.1 Generalities

Definition 3.1 ***A differential equation is an equation in which the unknown is a function and where some derivatives of the unknown function appear.***

Example 3.1 ***Let*** u ***be a function, the following equations are differential equations.***

1. $u^{'} = 2u$
2. $u^{''} - 3u^{'} + 1 = 0$
3. $u^{(3)} = u$

Definition 3.2 ***Let*** $u = u(x)$ ***be an unknown function of the variable*** x***. An equation of the form***

$$F\Big(x, u, u', u'', \ldots, u^{(n)}\Big) = 0, \tag{3.1}$$

with $n \in \mathbb{N}$, ***is called a differential equation of order*** n. ***Here*** $u', u", \ldots, u^{(n)}$ ***denote the derivatives of*** u ***of orders*** $1, 2, \ldots, n$***, respectively.***

Remarks 3.1

1. ***The equation*** (3.1) ***involves*** $(n+2)$ ***variables.***
2. ***The unknown function may also be denoted by*** $y, t, \ldots$
3. ***In a differential equation, when we write*** $u, u', u'', \ldots, u^{(n)}$***, it is understood that we mean*** $u(x), u'(x), u''(x), \ldots, u^{(n)}(x)$.

Definition 3.3 ***A solution of equation*** (3.1) ***on the interval*** I ***is a function that is*** n ***times differentiable on*** I ***and satisfies*** (3.1)***.***

Example 3.2 ***It can be easily verified that the function*** $u(x) = ce^{4x},\ c \in \mathbb{R}$, ***is a solution of the differential equation*** $u' = 4u$. ***Indeed, it is clear that if*** $u(x) = ce^{4x}$ ***then*** $u'(x) = 4ce^{4x} = 4u(x)$.

Definition 3.4 ***A differential equation of the type $u'f(u) = g(x)$ is called a*** separable variables equation.

Example 3.3 ***The equation $u' = \dfrac{e^{-u}}{x^2}$ can be rewritten as $u'e^u = \dfrac{1}{x^2}$. We can easily find the solutions of this equation. By integrating both sides, we obtain***

$$e^u = -\frac{1}{x} + k, \qquad k \in \mathbb{R}.$$

This yields

$$u(x) = \ln\left| -\frac{1}{x} + k \right|, \qquad k \in \mathbb{R}.$$

Definition 3.5 ***The*** order ***of a differential equation is the order of the highest derivative appearing in the equation.***

Example 3.4

1. $2xu' + u = 0$ ***is a first-order differential equation.***
2. $u'' + u' - 3u = \ln(x)$ ***is a second-order differential equation.***

Definition 3.6

1. ***A differential equation of order n is said to be*** linear ***if it has the form***

$$f_0(x)u + f_1(x)u' + f_2(x)u'' + \cdots + f_n(x)u^{(n)} = f(x), \tag{3.2}$$

where $f_0, f_1, f_2, \ldots, f_n, f$ are real continuous functions on an interval $I \subset \mathbb{R}$.

2. ***If $f(x) = 0$ for all $x \in I$, then equation (3.2) is called*** homogeneous, ***and it has the form***

$$f_0(x)u + f_1(x)u' + f_2(x)u'' + \cdots + f_n(x)u^{(n)} = 0.$$

3. ***Equation (3.2) is said to have*** constant coefficients ***if the functions $f_0, f_1, f_2, \ldots, f_n$ are constants on I. In other words, equation (3.2) can be written as***

$$a_0u + a_1u' + a_2u'' + \cdots + a_nu^{(n)} = f(x),$$

where $a_0, a_1, a_2, \ldots, a_n$ are real constants.

Remark 3.1 ***In a linear differential equation, none of the terms $u, u', u'', \ldots, u^{(n)}$ are raised to a power.***

Example 3.5

1. $e^x u + x^{\frac{1}{2}}u'' = x^2 + 1$ ***is a linear differential equation, and $e^x u + x^{\frac{1}{2}}u'' = 0$ is the associated homogeneous equation.***
2. $2u' - 3u'' + \frac{1}{5}u^{(3)} = x$ ***is a linear differential equation with constant coefficients.***
3. ***The equation $(u')^2 + u'' + 3u = 0$ is*** not ***a linear differential equation.***

Proposition 3.1 ***If*** u_1, u_2 ***are two solutions of the linear homogeneous equation***

$$f_0(x)u + f_1(x)u' + f_2(x)u'' + \cdots + f_n(x)u^{(n)} = 0, \tag{3.3}$$

then $\alpha u_1 + \beta u_2$ ***is also a solution of*** (3.3)***, for any constants*** $\alpha, \beta \in \mathbb{R}$***.***

Consider the linear differential equation (3.2) and its associated homogeneous equation (3.3). The following proposition allows us to find the general solution of equation (3.2).

Proposition 3.2 ***If*** u_0 ***is a particular solution of*** (3.2) ***and*** u_1 ***is a solution of the homogeneous equation*** (3.3)***, then***

$$u = u_1 + u_0$$

is a general solution of (3.2)***.***

Remark 3.2 ***Recall that a particular solution of equation*** (3.2) ***is a function that is*** n ***times differentiable and satisfies*** (3.2)***.***

3.2 First-Order Differential Equation

3.2.1 First-Order Linear Differential Equation Without a Nonhomogeneous Term

Consider the equation

$$a_1(x)u' + a_0(x)u = 0, \quad \text{with} \quad a_1(x) \neq 0, \tag{3.4}$$

which is equivalent to the separable equation

$$u' + a(x)u = 0, \quad \text{where} \quad a(x) = \frac{a_0(x)}{a_1(x)}. \tag{3.5}$$

To solve equation (3.5), we follow the steps:

$$\begin{aligned} u' + a(x)u = 0 &\iff u' = -a(x)u, \\ &\iff \frac{du}{dx} = -a(x)u, \\ &\iff \frac{du}{u} = -a(x)\,dx, \\ &\iff \int \frac{du}{u} = -\int a(x)\,dx, \\ &\iff \ln|u| = -A(x) + k, \\ &\iff |u| = e^{-A(x)+k}, \\ &\iff u = Ke^{-A(x)}, \end{aligned}$$

where $A(x)$ is an antiderivative of $a(x)$ and $K = \pm e^k,\ k \in \mathbb{R}$.

Example 3.6 ***The solution of the equation***

$$u' - \sqrt{x}u = 0, \quad x > 0,$$

is given by

$$u(x) = Ke^{-A(x)},$$

with $K = \pm e^k$ ***and***

$$-A(x) = \int \sqrt{x}\,dx = \frac{2}{3}\sqrt{x^3}.$$

3.2.2 First-Order Linear Differential Equation with a Nonhomogeneous Term

Consider the equation

$$a_1(x)u' + a_0(x)u = f_1(x), \quad \textbf{with} \quad a_1(x) \neq 0, \tag{3.6}$$

which is equivalent to

$$u' + a(x)u = f(x), \quad \textbf{where} \quad a(x) = \frac{a_0(x)}{a_1(x)} \quad \textbf{and} \quad f(x) = \frac{f_1(x)}{a_1(x)}. \tag{3.7}$$

According to Proposition 3.2, the solution of equation (3.7) **is of the form**

$$u(x) = u_0(x) + u_1(x),$$

where $u_1(x) = Ke^{-A(x)}$ **is the solution of the homogeneous equation associated with** (3.7)**, and** $u_0(x)$ **is a particular solution of** (3.7)**.**

Example 3.7 ***Consider the equation***

$$u' - \sqrt{x}u = 1 - x\sqrt{x}, \quad x > 0. \tag{3.8}$$

From the previous example, $u_1(x) = Ke^{\frac{2}{3}\sqrt{x^3}}$ ***is a solution of the homogeneous equation associated with*** (3.8)***. On the other hand, it can be easily verified that*** $u_0(x) = x$ ***is a particular solution of*** (3.8)***.***
Therefore, the general solution of equation (3.8) ***is***

$$u(x) = x + Ke^{\frac{2}{3}\sqrt{x^3}} = x + Ke^{\frac{2}{3}x\sqrt{x}}, \quad K \in \mathbb{R}.$$

The question now is: how can we find a particular solution?

3.2.3 Finding a Particular Solution: The Method of Variation of the Constant

We know that the solution of the homogeneous equation associated with (3.7) **is of the form**

$$u_1(x) = Ke^{-A(x)}, \quad K \in \mathbb{R}.$$

The *method of variation of the constant* consists in looking for a particular solution of (3.7) **of the form**

$$u_0(x) = K(x)e^{-A(x)},$$

where $K(x)$ **is a function of the variable** x **instead of a constant.**
Saying that $u_0(x) = K(x)e^{-A(x)}$ **is a solution of** (3.7) **means that**

$$u_0'(x) + a(x)u_0(x) = f(x), \quad \textbf{with} \quad A'(x) = a(x). \tag{3.9}$$

$$\begin{aligned}
u_0^{'}(x) + a(x)u_0(x) = f(x) &\iff \left(K(x)e^{-A(x)}\right)^{'} + a(x)K(x)e^{-A(x)} = f(x) \\
&\iff K^{'}(x)e^{-A(x)} - K(x)A^{'}(x)e^{-A(x)} + a(x)K(x)e^{-A(x)} = f(x) \\
&\iff K^{'}(x)e^{-A(x)} - K(x)a(x)e^{-A(x)} + a(x)K(x)e^{-A(x)} = f(x) \\
&\iff K^{'}(x)e^{-A(x)} = f(x) \\
&\iff K^{'}(x) = f(x)e^{A(x)} \\
&\iff K(x) = \int f(x)e^{A(x)}\, dx.
\end{aligned}$$

Thus, a particular solution of (3.7) **can be written in the form**

$$u_0(x) = \left(\int f(x)e^{A(x)}\, dx\right) e^{-A(x)}.$$

Exercise 3.1 ***Find the solutions of the equation***

$$u' - 2u = e^{3x+1}. \tag{3.10}$$

Proof. Finding the solution of the homogeneous equation:
The solution of the homogeneous equation

$$u' - 2u = 0$$

associated with equation (3.10) **is given by**

$$u_1(x) = Ke^{2x}, \quad K \in \mathbb{R}.$$

$$\begin{aligned}
u^{'} = 2u &\iff \frac{du}{dx} = 2u \\
&\iff \frac{du}{u} = 2dx \\
&\iff \ln|u| = 2x + k, \quad k \in \mathbb{R} \\
&\iff u = Ke^{2x}, \; K = \pm e^{k}.
\end{aligned}$$

Then

$$u_1(x) = Ke^{2x}$$

Finding the particular solution:

We look for a function $K(x)$ such that a particular solution of equation (3.10) is of the form

$$u_0(x) = K(x)e^{2x}.$$

$u_0(x) = K(x)e^{2x}$.

$$\begin{aligned}
u_0^{'}(x) - 2u_0(x) = e^{3x+1} &\iff \left(K(x)e^{2x}\right)^{'} - 2K(x)e^{2x} = e^{3x+1} \\
&\iff K^{'}(x)e^{2x} + 2K(x)e^{2x} - 2K(x)e^{2x} = e^{3x+1} \\
&\iff K^{'}(x)e^{2x} = e^{3x+1} \\
&\iff K^{'}(x) = e^{3x+1}e^{-2x} \\
&\iff K(x) = \int e^{x+1}\,dx \\
&\implies K(x) = e\int e^{x}\,dx \\
&\implies K(x) = e^{x+1}.
\end{aligned}$$

The solution of equation (3.10) is of the form

$$u(x) = e^{x+1}e^{2x} + Ke^{2x} = \left(e^{x+1} + K\right)e^{2x}, \quad K \in \mathbb{R}.$$

■

3.2.4 First-Order Linear Differential Equation with Constant Coefficients

Consider equations of the form

$$a_1u' + a_0u = f_1(x). \tag{3.11}$$

This is a special case of equations (3.6). Equation (3.11) is solved in the same way as (3.6).

3.2.5 Bernoulli Differential Equation

Definition 3.7 ***Any equation of the form***

$$u' + a(x)u + b(x)u^n = 0 \tag{3.12}$$

is called a Bernoulli equation.

Solving the Bernoulli Equation

1. **If $n = 0$, equation (3.12) becomes of the form (3.7).**
2. **If $n = 1$, equation (3.12) becomes of the form (3.5).**

3. **If $n \neq 0$ and $n \neq 1$, we try to transform equation (3.12) into a first-order linear differential equation. To do this, we follow the following method: we divide by u^n and equation (3.12) becomes**

$$u^{-n}u' + a(x)u^{1-n} + b(x) = 0 \tag{3.13}$$

We set $y = u^{1-n}$, so that

$$\frac{1}{1-n}y' = u^{-n}u'.$$

Therefore, equation (3.13) becomes

$$\frac{1}{1-n}y' + a(x)y + b(x) = 0. \tag{3.14}$$

Equation (3.14) is of the form (3.6).

Example 3.8 ***Solve the following equation:***

$$u' + e^x u + e^x u^3 = 0. \tag{3.15}$$

Proof. Dividing by u^3 we obtain

$$u^{-3}u' + e^x u^{-2} = -e^x. \tag{3.16}$$

By making the change of variable $y = u^{-2}$ and differentiating both sides, we obtain

$$y' = -2u'u^{-3}.$$

In other words,

$$u'u^{-3} = -\frac{1}{2}y.$$

This change of variable allows us to write equation (3.16) in the form

$$-\frac{1}{2}y' + e^x y = -e^x, \tag{3.17}$$

which is a first-order linear equation with a nonhomogeneous term whose solution follows the previous steps. ■

3.2.6 Homogeneous Differential Equation

Let H be a numerical function defined and continuous on a domain $D \subset \mathbb{R}$.

Definition 3.8 ***A differential equation is called*** **homogeneous** ***if it is of the form***

$$F(x, u, u') = 0$$

and remains unchanged when x is replaced by αx and u by αu, while leaving u' unchanged. These equations are of the form

$$u' = H\left(\frac{u}{x}\right). \tag{3.18}$$

The solution of equation (3.18) generally reduces to solving a simple equation using the change of variable $t = \frac{u}{x}$, with $u = tx$ and $u' = t'x + t$. The solutions are in the form (x, u).

Example 3.9 ***Solve the equation***

$$2xuu' = u^2 - x^2.$$

Proof. **When x is replaced by αx and u by αu, leaving u' unchanged, we obtain**

$$2\alpha^2 xuu' = \alpha^2(u^2 - x^2),$$

which is exactly

$$2xuu' = u^2 - x^2.$$

Thus, the equation is homogeneous.
We use the change of variable $t = \frac{u}{x}$, with $u = tx$ and $u' = t'x + t$. The given equation becomes

$$\begin{aligned}
2xuu' = u^2 - x^2 &\iff 2uu' = \frac{u^2}{x} - x \\
&\iff 2tx\left(t'x + t\right) = \left(\frac{u}{x}\right)u - x \\
&\iff 2x^2tt' + 2t^2x = t^2x - x \\
&\iff 2x^2tt' = -\left(t^2 + 1\right) \\
&\iff \frac{2t}{t^2+1}\,dt = -\frac{1}{x}\,dx \\
&\iff \ln\left(t^2 + 1\right) = -\ln|x| + k, \quad k \in \mathbb{R} \\
&\iff \ln\left(t^2 + 1\right)|x| = k \\
&\iff x = \frac{K}{t^2+1},\ u = \frac{Kt}{t^2+1},\ K = \pm e^k.
\end{aligned}$$

■

3.3 Second-Order Differential Equations

We consider equations of the form

$$F(x, u, u', u'') = 0.$$

To solve these equations, we distinguish several cases.

3.3.1 Equations of the form $F(x, u', u'') = 0$

In this type of equation, the function u does not appear in the equation. A technique for solving it consists in using the change of variable $y = u'$, and the equation becomes of the form

$$F(x, y, y') = 0.$$

Example 3.10 ***Find the solutions of the differential equation***

$$xu'' - u' = 1.$$

Proof. Using the change of variable $y = u'$, the previous equation becomes

$$xy' = 1 + y.$$

We then have:

$$\begin{aligned}
xy^{'} = 1 + y &\iff x\frac{dy}{dx} = (1 + y) \\
&\iff \frac{dy}{1 + y} = \frac{1}{x}dx \\
&\iff \ln|y + 1| = \ln|x| + \ln|k| \quad k \in \mathbb{R}^* \\
&\iff 1 + y = kx, \quad k \in \mathbb{R}^* \\
&\iff u^{'} = kx - 1 \\
&\iff u = \frac{k}{2}x^2 - x + c, \quad c \in \mathbb{R}.
\end{aligned}$$

■

3.3.2 Equations of the form $F(x, u'') = 0$

In this equation, u and u' do not both appear in the equation. We have a relation linking x and u''. The technique for solving it is to integrate u'' to find u', then integrate u' to find u.

Example 3.11 ***Solve the equation***

$$(1 + x^2)u'' = 1.$$

Proof.

$$\begin{aligned}
(1 + x^2)u'' = 1 &\iff u'' = \frac{1}{1 + x^2} \\
&\iff u' = \arctan(x) + k, \quad k \in \mathbb{R}. \\
&\iff u = \int \arctan(x)\, dx + kx + c, \quad c \in \mathbb{R}.
\end{aligned}$$

We use integration by parts to calculate $\int \arctan(x)\, dx$. ■

3.3.3 Equations of the form $F(u, u', u'') = 0$

The variable x does not appear in these equations of the form $F(u, u', u'') = 0$. The technique for solving them is to use the change of variable $u' = y$, reducing the problem to a first-order equation.

Proof. Substitution $u' = y$ and Expression for u''

Let us consider a second-order differential equation in which we set

$$u' = y.$$

Step 1: Expressing the second derivative.

By definition, the second derivative is

$$u'' = \frac{d}{dx}(u').$$

Since $u' = y$, we have

$$u'' = \frac{dy}{dx}.$$

Step 2: Using the chain rule when y is a function of u.

Sometimes it is convenient to consider y as a function of u. Then, by the chain rule,

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}.$$

But $\frac{du}{dx} = u' = y$, so that

$$u'' = \frac{dy}{dx} = y\frac{dy}{du}.$$

Step 3: Summary.

- **If $y = u'$ is treated as a function of x, then $u'' = \frac{dy}{dx}$.**
- **If $y = u'$ is treated as a function of u, then**

$$u'' = y\frac{dy}{du}.$$

This substitution is particularly useful in second-order differential equations where x does not appear explicitly, as it reduces the problem to a first-order differential equation in y as a function of u. ■

Example 3.12 ***Solve the equation***

$$2uu^{''} - (u^{'})^2 = 1$$

Proof. Using the change of variable $y = u'$, we have $u'' = y\frac{dy}{du}$. It follows that

$$\begin{aligned} 2uu^{''} - (u^{'})^2 = 1 &\iff 2uy\frac{dy}{du} = 1 + y^2 \\ &\iff \frac{2y}{1+y^2}dy = \frac{1}{u}du \\ &\iff u = k\left(1 + y^2\right), \end{aligned}$$

On the other hand, we have $y = u'$, so that

$$\begin{aligned} y &= \frac{du}{dx} \\ &= \frac{d}{dx}\left(k + ky^2\right) \\ &= 2k\frac{dy}{dx}y, \end{aligned}$$

It follows that

$$\begin{aligned} 2k\frac{dy}{dx} = 1 &\iff 2kdy = dx \\ &\iff y = \frac{x}{2k} + k^{'} \\ &\iff u = k\left(c + \left(\frac{x}{2k} + k^{'}\right)^2\right), \quad k, k^{'}, c \in \mathbb{R}. \end{aligned}$$

■

3.3.4 Second-Order Linear Differential Equations

I/ The case where the coefficients are non-constant
We consider the equation

$$a_2(x)u^{''} + a_1(x)u^{'} + a_0(x)u = f_1(x) \tag{3.19}$$

There are two techniques to solve equation (3.19):

(i) If a particular solution of equation (3.19) is known, then the solution of this equation is of the form $u = u_0 + u_1$, where u_0 is the particular solution and u_1 is the solution of the homogeneous equation associated with (3.19).

(ii) If the particular solution of equation (3.19) is not known, the method of variation of constants is used when we have two linearly independent solutions g_1, g_2 of the homogeneous equation associated with (3.19).

As we have just seen, to find a solution of equation (3.19), we must first solve the homogeneous equation

$$a_2(x)u^{''} + a_1(x)u^{'} + a_0(x)u = 0$$

which can be written in the form

$$u^{''} + b_1(x)u^{'} + b_0(x)u = 0 \tag{3.20}$$

where b_1, b_0, f are continuous functions on a domain $D \subset \mathbb{R}$.

(a) **Finding solutions of the homogeneous equation $u'' + b_1(x)u' + b_0(x)u = 0$**
The general rule is as follows:

Lemma 3.1 ***If g_1 and g_2 are two linearly independent solutions of (3.20), then the general solution of (3.20) is $u_1 = \lambda_1 g_1 + \lambda_2 g_2$, where λ_1, λ_2 are arbitrary real constants.***

Remark 3.3 ***g_1 and g_2 being linearly independent means that they are not proportional. In other words, there is no real λ such that $g_1 = \lambda g_2$.***

We distinguish several possible cases:
First case: We know g_1 and g_2
If g_1 and g_2 are two particular solutions of (3.20), then the general solution of (3.20) is $u_1 = \lambda_1 g_1 + \lambda_2 g_2, \quad \lambda_1, \lambda_2 \in \mathbb{R}$.
Second case: We know only one particular solution of the homogeneous equation
If there exists a non-zero function g on D such that

$$g'' + b_1(x)g' + b_0(x)g = 0, \tag{3.21}$$

then we can reduce the solution of (3.20) to solving a first-order equation by setting $u = gy$. Indeed, when we set $u = gy$ we have $u' = g'y + y'g$ and $u'' = g''y + y'g' + y''g + g'y'$, and thus equation (3.20) becomes

$$gy'' + \left(2g' + b_1(x)g\right)y' + \left(g'' + b_1(x)g' + b_0(x)g\right)y = 0,$$

using (3.21), we find

$$gy'' + \left(2g' + b_1(x)g\right)y' = 0,$$

which is indeed a first-order differential equation that can be easily solved.

$$gy'' + \left(2g' + b_1(x)g\right)y' = 0 \Longleftrightarrow \frac{y''}{y'} = -2\frac{g'}{g} - b_1(x)$$

Another change of variable is necessary: we set $v = y'$, and thus $\frac{dv}{dx} = y''$. It follows that

$$\frac{dv}{v} = -2\frac{g'}{g}\,dx - b_1(x)\,dx.$$

Integrating both sides of the equation gives

$$\ln|v| = -2\ln|g| - B(x) + k.$$

Thus,

$$v = Ke^{-2\ln|g| - B(x)},$$

where $B(x)$ is a primitive of $b_1(x)$. If $G(x)$ is a primitive of $e^{-2\ln|g|-B(x)}$, then

$$v = y^{'} = Ke^{-2\ln|g|-B(x)} \Longrightarrow y = KG(x) + \eta.$$

Now, it is enough to replace y to find

$$u = gy = KgG(x) + \eta g.$$

The general solution of (3.20) is

$$u_1 = \lambda_1 g_1 + \lambda_2 g_2, \quad \lambda_1, \lambda_2 \in \mathbb{R},$$

with $g_1 = g$ and $g_2 = u = gy = KgG(x) + \eta g$.

Third case: We do not know any particular solution of the homogeneous equation

In the case where we do not know any particular solution of (3.20), we look for the general solution u_1 in the form of a product of two unknown functions. We set $u_1 = vg$, and we choose $v(x)$ so that the factor of $g^{'}$ is zero. Starting from these conditions, we have: $u_1 = vg \Longrightarrow u_0^{'} = v^{'}g + g^{'}v$ and thus $u_0^{''} = v^{''}g + g^{''}v + 2v^{'}g^{'}$. Substituting these results into equation (3.20), we find:

$$g^{''}v + \left(b_1(x)v + 2v^{'}\right)g^{'} + \left(v^{''} + b_1(x)v^{'} + b_0(x)v\right)g = 0. \tag{3.22}$$

By choosing

$$\left(b_1(x)v + 2v^{'}\right) = 0, \tag{3.23}$$

we have

$$g^{''}v + \left(v^{''} + b_1(x)v^{'} + b_0(x)v\right) = 0. \tag{3.24}$$

Solving equation (3.23) gives v, and solving (3.24) gives g. Thus the general solution of (3.20) is fully determined.

Last case: We reduce to a homogeneous equation with constant coefficients

The technique is to make a suitable change of variable to transform equation (3.20) into a homogeneous equation with constant coefficients, which is simpler to solve.

Example 3.13 ***Consider the equation***

$$ax^2u^{''} + bxu^{'} + cu = 0 \tag{3.25}$$

where a, b, c are real constants and $a \neq 0$.

We perform the change of variable $x = \alpha e^t$, with $\alpha = 1$ if $x > 0$, and $\alpha = -1$ if $x < 0$. It follows that

$$\frac{dt}{dx} = \frac{1}{\alpha e^t},$$

hence

$$u^{'} = \frac{du}{dx} = \frac{du}{dt}\frac{dt}{dx} = \frac{1}{\alpha e^t}\frac{du}{dt}.$$

$$\begin{aligned}
u^{''} &= \frac{d^2u}{dx^2} = \frac{du^{'}}{dx} = \frac{du^{'}}{dt}\frac{dt}{dx} = \frac{1}{\alpha e^t}\frac{du^{'}}{dt} \\
&= \frac{1}{\alpha e^t}\frac{d}{dt}\left(\frac{1}{\alpha e^t}\frac{du}{dt}\right) \\
&= \frac{1}{\alpha^2 e^t}\left(-\frac{du}{dt}+\frac{d^2u}{dt^2}\right) \\
&= \frac{1}{e^t}\left(-\frac{du}{dt}+\frac{d^2u}{dt^2}\right).
\end{aligned}$$

Substituting these results into (3.25)***, this equation becomes***

$$\frac{d^2u}{dt^2}+\left(\frac{b}{a}-1\right)\frac{du}{dt}+\frac{c}{a}u = 0,$$

which is a linear homogeneous differential equation with constant coefficients.

(b)

Finding a particular solution of the equation $u^{''}+b_1(x)u^{'}+b_0(x)u = f(x)$

If g_1 **and** g_2 **are two non-zero and linearly independent solutions of equation** (3.20)**, the solution of equation** (3.19) **is of the form** $u = g_1u_1+g_2u_2$**, where** u_1, u_2 **are unknown functions that can be determined if certain additional conditions are imposed. We use the method of variation of constants.**
A simple calculation gives

$$u^{'} = g_1^{'}u_1+g_1u_1^{'}+g_2^{'}u_2+g_2u_2^{'},$$

and

$$u^{''} = g_1^{''}u_1+g_1^{'}u_1^{'}+g_1^{'}u_1^{'}+g_1u_1^{''}+g_2^{''}u_2+g_2^{'}u_2^{'}+g_2^{'}u_2^{'}+g_2u_2^{''}.$$

Considering that g_1 **and** g_2 **are solutions of equation** (3.20)**, and that** $u = g_1u_1+g_2u_2$ **is a solution of equation** (3.19)**, we obtain**

$$2\left(g_1^{'}u_1^{'}+g_2^{'}u_2^{'}\right)+b_1(x)\left(g_1u_1^{'}+g_2u_2^{'}\right)+g_1u_1^{''}+g_2u_2^{''} = f(x). \tag{3.26}$$

By imposing the additional condition

$$g_1u_1^{'}+g_2u_2^{'} = 0,$$

and differentiating both sides, we obtain

$$g_1u_1^{''}+g_2u_2^{''} = -\left(g_1^{'}u_1^{'}+g_2^{'}u_2^{'}\right).$$

Substituting this result into equation (3.26)**, we get**

$$g_1^{'}u_1^{'}+g_2^{'}u_2^{'} = f(x).$$

In conclusion, to find $u_1^{'}$ and $u_2^{'}$, we solve the system

$$\begin{cases} g_1 u_1^{'} + g_2 u_2^{'} = 0, \\ g_1^{'} u_1^{'} + g_2^{'} u_2^{'} = f(x). \end{cases}$$

As soon as we have $u_1^{'}$ and $u_2^{'}$, we compute their antiderivatives to obtain u_1 and u_2.

II/ The case where the coefficients are constant
We consider the equation

$$a_2 u^{''} + a_1 u^{'} + a_0 u = f_1(x),$$

which can be written in the form

$$u^{''} + b_1 u^{'} + b_0 u = f(x), \tag{3.27}$$

where a_0, a_1, a_2, b_0, b_1 are real numbers with $a_2 \neq 0$ and f_1, f are continuous functions on a domain $D \subset \mathbb{R}$. Let

$$u^{''} + b_1 u^{'} + b_0 u = 0 \tag{3.28}$$

be the homogeneous equation associated with (3.27). It is clear that the solutions of (3.27) are of the form $u = u_0 + u_1$, where u_0 is the general solution of (3.28) and u_1 is a particular solution of (3.27). Therefore, from now on, we will focus on the determination of u_0 and u_1.

1/ Calculation of the general solution of the homogeneous equation
To determine the solutions of the homogeneous equation (3.28), we define the characteristic equation associated with (3.28), which is given by

$$r^2 + b_1 r + b_0 = 0. \tag{3.29}$$

The solutions of equation (3.29) depend on the sign of the discriminant $\Delta = b_1^2 - 4b_0$. Indeed,

- **If $\Delta > 0$, equation (3.29) has two distinct real solutions:**

$$r_1 = \frac{-b_1 - \sqrt{\Delta}}{2}, \quad r_2 = \frac{-b_1 + \sqrt{\Delta}}{2}.$$

- **If $\Delta < 0$, equation (3.29) has two complex solutions:**

$$r_1 = \frac{-b_1 - i\sqrt{\Delta'}}{2} = \alpha - i\beta, \quad r_2 = \frac{-b_1 + i\sqrt{\Delta'}}{2} = \alpha + i\beta, \quad -\Delta' = \Delta,\ i^2 = -1.$$

- **If $\Delta = 0$, equation (3.29) has a double solution:**

$$r_1 = r_2 = \frac{-b_1}{2}.$$

Let λ_1, λ_2 be two arbitrary real numbers. The following table summarizes the methods for calculating the general solution of the homogeneous equation (3.29)**:**

$\Delta > 0$	$g_1(x) = e^{r_1 x},\ g_2(x) = e^{r_2 x}$	$u_1 = \lambda_1 e^{r_1 x} + \lambda_2 e^{r_2 x}$
$\Delta < 0$	$g_1(x) = e^{\alpha x}\cos(\beta x),\ g_2(x) = e^{\alpha x}\sin(\beta x)$	$u_1 = \lambda_1 e^{\alpha x}\cos(\beta x) + \lambda_2 e^{\alpha x}\sin(\beta x)$
$\Delta = 0$	$g_1(x) = e^{r_1 x},\ g_2(x) = x e^{r_1 x}$	$u_1 = (\lambda_1 + \lambda_2 x) e^{r_1 x}$

2/ Calculation of a particular solution
Now, we aim to determine a particular solution of equation (3.27)**. Generally, the form of f guides us in choosing the particular solution u_0. Several situations are possible:**

- **If $f(x)$ is a polynomial of degree n, we look for u_0 in the form of a polynomial of degree n that satisfies** (3.27)**.**

  Example 3.14 ***Find a particular solution of the equation***

  $$u'' - 3u' - u = 4x^2 + 1 \tag{3.30}$$

  We set $u_0 = ax^2 + bx + c$ with $a \neq 0$. We have $u_0' = 2ax + b$, $u_0'' = 2a$. Substituting these results into* (3.30) *and performing identification, we find the values of a, b, c.

- **If $f(x) = h(x)e^{rx}$, where $h(x)$ is a polynomial of degree n, we look for $u_0 = k(x)e^{rx}$, with $k(x)$ a polynomial of degree m such that**
  - $m = n$ **if** $r^2 + b_1 r + b_0 \neq 0$
  - $m = n + 1$ **if** $r^2 + b_1 r + b_0 = 0$
  - $m = n + 2$ **if r is a double root of** $r^2 + b_1 r + b_0$

  Example 3.15 ***Find a particular solution of***

  $$u'' - 3u' + u = 2e^{3x} \tag{3.31}$$

  Here, $h(x) = 2$, $r = 3$, $n = 0$, $(3)^2 - 3(3) + 1 = 1 \neq 0$, hence $m = 0$, $k(x) = k$, $k \in \mathbb{R}$, and $u_0 = ke^{3x}$. We compute $u_0' = 3ke^{3x}$, $u_0'' = 9ke^{3x}$. Substituting these results into equation* (3.31) *and identifying both sides, we find the value of k.

- **If $f(x) = a\cos(rx) + b\sin(rx)$, we look for u_0 in one of the following forms:**
  - $u_0 = \alpha\cos(rx) + \beta\sin(rx)$,
  - $u_0 = x\left(\alpha\cos(rx) + \beta\sin(rx)\right)$ **if $\cos(rx)$ is a solution of the homogeneous equation.**

  Using the same previous technique, we find α, β.

- **If $f(x) = f_1(x) + f_2(x) + \cdots + f_m(x)$ where $f_1, f_2, \cdots, f_m$ take one of the previous forms, then we look for $u_0 = v_1 + v_2 + \cdots + v_m$ where v_i is a particular solution of**

  $$v_i'' + b_1 v_i' + b_0 v_i = f_i(x), \quad 1 \leq i \leq m. \tag{3.32}$$

- **If $f(x)$ cannot be written in any of the above forms, we use the method of variation of constants. If $u_1 = \lambda_1 g_1 + \lambda_2 g_2,\ (\lambda_1, \lambda_2 \in \mathbb{R})$ is the general solution of the homogeneous equation (3.28), we look for a particular solution of (3.27) in the form $u_0 = \lambda_1(x) g_1 + \lambda_2(x) g_2$, where $\lambda_1(x), \lambda_2(x)$ are unknown continuous functions to be determined. This returns us to the method already seen in the case of a second-order linear differential equation with non-constant coefficients and a non-homogeneous term (Ib).**

3.4 Multivariable Functions

3.4.1 Functions of Several Variables with Real Values

Let n be a non-zero natural number, and E a non-empty subset of $\mathbb{R}^n$.

Definition 3.9 ***A function of n real variables with real values is any function f defined as follows***

$$f : \left| \begin{array}{ccc} E \subset \mathbb{R}^n & \longrightarrow & \mathbb{R} \\ (x_1, x_2, x_3, \cdots, x_n) & \longmapsto & f(x_1, x_2, x_3, \cdots, x_n) = y \end{array} \right.$$

Examples 3.1 ***1. f is a function of two variables x and y***

$$f : \left| \begin{array}{ccc} \mathbb{R}^2 & \longrightarrow & \mathbb{R} \\ (x, y) & \longmapsto & f(x, y) = x^2 + y^2 \end{array} \right.$$

$f(1, 2) = 1^2 + 2^2 = 5.$

2. g is a function of three real variables.

$$f : \left| \begin{array}{ccc} \mathbb{R}^2 \times \mathbb{R}^* & \longrightarrow & \mathbb{R} \\ (x, y, z) & \longmapsto & f(x, y, z) = \dfrac{e^x + 2\sin(y)}{3z^2}. \end{array} \right.$$

1. Domain of Definition of a Function of Two Real Variables

Let f be a function of two real variables x, y defined by

$$f : \left| \begin{array}{ccc} \mathbb{R}^2 & \longrightarrow & \mathbb{R} \\ (x, y) & \longmapsto & f(x, y). \end{array} \right.$$

The domain of definition of f, denoted by D, is the set of pairs $(x, y) \in \mathbb{R}^2$ for which $f(x, y)$ exists.

Example 3.16

$$f : \left| \begin{array}{ccc} \mathbb{R}^2 & \longrightarrow & \mathbb{R} \\ (x, y) & \longmapsto & f(x, y) = \ln(x) + \sin(y). \end{array} \right.$$

For f to be well-defined, both $\ln(x)$ and $\sin(y)$ must be defined simultaneously. Therefore, $x > 0$ and $y \in \mathbb{R}$, so

$$D = \mathbb{R}_+^* \times \mathbb{R}.$$

2. Graphical Representation of a Function of Two Real Variables

Let f be a function defined on a domain D of $\mathbb{R}^2$ such that

$$f : \left| \begin{array}{ccc} D \subset \mathbb{R}^2 & \longrightarrow & \mathbb{R} \\ (x,y) & \longmapsto & f(x,y). \end{array} \right.$$

The graphical representation of f is a surface S_f in $\mathbb{R}^3$ defined by

$$S_f = \left\{ (x,y,z) \in \mathbb{R}^3 \ / \ [z = f(x,y)] \wedge [(x,y) \in D] \right\}.$$

In other words, S_f is the set of points in space with coordinates $M(x,y,f(x,y))$ for $(x,y) \in D$. To each point $(x,y) \in D$ corresponds a point in space lying on the surface S_f.

3.4.2 First-Order Partial Derivatives

In this section, we assume that the notion of the derivative of a function defined from $\mathbb{R}$ to $\mathbb{R}$ is known, and we want to provide its generalization for functions of several variables with values in $\mathbb{R}$. For simplicity, we start with functions of two real variables; the case of functions of three or more real variables follows easily.

Definition 3.10 ***Let $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$. We say that f has a first derivative at $x_0 \in \mathbb{R}^2$ along the vector $u = (u_1, u_2)$ if, and only if, $\psi_u : t \longmapsto f(x_0 + tu)$ is differentiable at 0. In this case, $\psi_u^{'}(0)$ represents the derivative of f at the point x_0 in the direction of u, denoted by $D_u f(x_0)$, and we have***

$$D_u f(x_0) = \lim_{t \to 0} \frac{f(x_0 + tu) - f(x_0)}{t}$$

Example 3.17 $f(x,y) = xy,\ x_0 = (0,0),\ u = (1,1).$ ***Compute $D_u f(x_0)$.***
We have $x_0 + tu = (t,t)$ and $f(t,t) = t^2$. thus

$$\lim_{t \to 0} \frac{f(x_0 + tu) - f(x_0)}{t} = \lim_{t \to 0} \frac{t^2}{t} = \lim_{t \to 0} t = 0.$$

Hence,

$$D_u f(x_0) = 0.$$

Definition 3.11 ***The derivatives of a function $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$, along the vectors $\overrightarrow{i}(1,0)$ and $\overrightarrow{j}(0,1)$, if they exist, correspond respectively to the partial derivatives with respect to x and y, denoted by $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$. We then have***

$$D_{\overrightarrow{i}} f(x,y) = \frac{\partial f}{\partial x}(x,y) \quad \textit{and} \quad D_{\overrightarrow{j}} f(x,y) = \frac{\partial f}{\partial y}(x,y).$$

Now, we generalize the notion of partial derivatives to functions defined on $\mathbb{R}^n$.

Definition 3.12 ***Let*** $f : D \subset \mathbb{R}^n \longrightarrow \mathbb{R}$ ***and*** $a = (a_1, a_2, \cdots, a_n) \in D$. ***For*** $i = 1, 2, \cdots, n$, ***the partial derivative of*** f ***at*** a ***with respect to*** x_i ***is denoted*** $\dfrac{\partial f}{\partial x_i}(a)$ ***and is defined as the derivative of the partial function taken at*** a_i

$$\frac{\partial f}{\partial x_i}(a) = \lim_{x_i \to a_i} \frac{f(a_1, a_2, \cdots, x_i, \cdots, a_n) - f(a_1, a_2, \cdots, a_i, \cdots, a_n)}{x_i - a_i} = f'_{x_i}(a),$$

we can also write

$$\frac{\partial f}{\partial x_i}(a) = \lim_{h_i \to 0} \frac{f(a_1, a_2, \cdots, a_i + h_i, \cdots, a_n) - f(a_1, a_2, \cdots, a_i, \cdots, a_n)}{h_i} = f'_{x_i}(a). \quad (3.33)$$

In particular, if $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$, $a = (a_1, a_2)$,

$$\frac{\partial f}{\partial x}(a_1, a_2) = \lim_{h_1 \to 0} \frac{f(a_1 + h_1, a_2) - f(a_1, a_2)}{h_1} = f'_x(a_1, a_2), \quad (3.34)$$

and

$$\frac{\partial f}{\partial y}(a_1, a_2) = \lim_{h_2 \to 0} \frac{f(a_1, a_2 + h_2) - f(a_1, a_2)}{h_2} = f'_y(a_1, a_2). \quad (3.35)$$

Remark 3.4 ***It should be understood that one can only talk about partial derivatives at*** $a = (a_1, a_2)$ ***if the limits in*** (3.34) ***and*** (3.35) ***exist. When these limits exist, they are denoted by*** $\dfrac{\partial f}{\partial x}(a_1, a_2)$ ***and*** $\dfrac{\partial f}{\partial y}(a_1, a_2)$ ***respectively. This remark remains valid for functions defined on*** $\mathbb{R}^n$, ***where the limit in*** (3.33) ***is required to exist.***

When the partial derivatives exist, how can we compute them?

Example 3.18

$$f(x, y) = 2x^3y^2, \quad a = (-1, 2)$$

1. ***Consider*** y ***as a constant and differentiate with respect to*** x. ***Then,***

$$f'_x(x, y) = \frac{\partial f}{\partial x}(x, y) = 6y^2x^2.$$

2. ***Consider*** x ***as a constant and differentiate with respect to*** y. ***Then,***

$$f'_y(x, y) = \frac{\partial f}{\partial y}(x, y) = 4yx^3.$$

Thus,

$$\frac{\partial f}{\partial x}(-1, 2) = 6(-1)^2(2)^2 = 24,$$
$$\frac{\partial f}{\partial y}(-1, 2) = 4(2)(-1)^3 = -8.$$

Remark 3.5 ***The existence of partial derivatives at the point*** $a = (a_1, a_2)$ ***does not imply that*** f ***is continuous at this point.***

Example 3.19 ***Consider the function*** f ***defined by***

$$f(x,y)=\begin{cases}\dfrac{xy}{x^2+y^2}, & \textbf{\textit{if}}\ (x,y)\neq(0,0),\\ 0, & \textbf{\textit{if}}\ (x,y)=(0,0).\end{cases}$$

Study the existence of partial derivatives at the point $(0,0)$.

$\lim\limits_{h_1\to 0}\dfrac{f(h_1,0)-f(0,0)}{h_1}=\lim\limits_{h_1\to 0}\dfrac{0}{h_1}=0\Longrightarrow\dfrac{\partial f}{\partial x}(0,0)=0,$

$\lim\limits_{h_2\to 0}\dfrac{f(0,h_2)-f(0,0)}{h_2}=\lim\limits_{h_2\to 0}\dfrac{0}{h_2}=0\Longrightarrow\dfrac{\partial f}{\partial y}(0,0)=0.$

The partial derivatives at the point $(0,0)$ ***exist.***

3.4.3 Higher-Order Partial Derivatives

The definition is given for a function f of two variables, and it remains valid for functions of n $(n>2)$ variables.

Definition 3.13 ***If for a function*** $f(x,y)$ ***defined on*** $D\subset\mathbb{R}^2$, ***the partial derivatives*** $\dfrac{\partial f}{\partial x}(x,y)$ ***and*** $\dfrac{\partial f}{\partial y}(x,y)$ ***exist and are themselves functions of*** x ***and*** y, ***then*** $\dfrac{\partial f}{\partial x}$ ***and*** $\dfrac{\partial f}{\partial y}$ ***may have partial derivatives such that:***

$\dfrac{\partial^2 f}{\partial x^2}=\dfrac{\partial}{\partial x}\left(\dfrac{\partial f}{\partial x}\right)=f''_{xx}$ ***we differentiate twice with respect to*** x.

$\dfrac{\partial^2 f}{\partial x\partial y}=\dfrac{\partial}{\partial x}\left(\dfrac{\partial f}{\partial y}\right)=f''_{yx}$ ***we first differentiate with respect to*** y ***and then with respect to*** x.

$\dfrac{\partial^2 f}{\partial y\partial x}=\dfrac{\partial}{\partial y}\left(\dfrac{\partial f}{\partial x}\right)=f''_{xy}$ ***we first differentiate with respect to*** x ***and then with respect to*** y.

$\dfrac{\partial^2 f}{\partial y^2}=\dfrac{\partial}{\partial y}\left(\dfrac{\partial f}{\partial y}\right)=f''_{yy}$ ***we differentiate twice with respect to*** y.

Example 3.20 $f(x,y)=3x^4-2xy^3$

$$\frac{\partial f}{\partial x}=12x^3-2y^3,\quad \frac{\partial f}{\partial y}=-6xy^2.$$

$$\frac{\partial^2 f}{\partial x^2}=36x^2,\quad \frac{\partial^2 f}{\partial y^2}=-12xy.$$

$$\frac{\partial^2 f}{\partial y\partial x}=\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)=\frac{\partial}{\partial y}\left(12x^3-2y^3\right)=-6y^2.$$

$$\frac{\partial^2 f}{\partial x\partial y}=\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)=\frac{\partial}{\partial x}\left(-6xy^2\right)=-6y^2.$$

In Example 3.20, we notice that $\dfrac{\partial^2 f}{\partial x\partial y}=\dfrac{\partial^2 f}{\partial y\partial x}$. Is this a coincidence, or is it always the case? The following theorem answers this question:

Theorem 3.1 ***(Schwarz's Theorem)***
Let $f : D \subset \mathbb{R}^n \longrightarrow \mathbb{R}$ ***be a function continuous on*** D. ***If for all*** $i, j = 1, 2, \cdots, n$, ***the partial derivatives*** $\frac{\partial f}{\partial x_i}, \frac{\partial f}{\partial x_j}$ ***exist and are continuous, and the mixed partial derivatives*** $\frac{\partial}{\partial x_i}\left(\frac{\partial f}{\partial x_j}\right)$ ***exist and are also continuous, then for every*** $a = (a_1, a_2, \cdots, a_n) \in D$, ***and for*** $i, j = 1, 2, \cdots, n$,

$$\frac{\partial^2 f}{\partial x_i \partial x_j}(a) = \frac{\partial^2 f}{\partial x_j \partial x_i}(a).$$

This theorem states that the order of differentiation with respect to the variables is irrelevant in the case where the partial derivatives exist and are continuous in the neighborhood of a point.

3.4.4 Derivatives of a Function Composed of Two Variables

Definition 3.14 ***Let*** $z = f(u, v)$ ***be a function of two variables*** u ***and*** v, ***where*** u ***and*** v ***themselves are functions of*** x ***and*** y, ***i.e.,***

$$u = u(x, y), \quad v = v(x, y).$$

Then z ***can be expressed explicitly as a function of*** x ***and*** y***:***

$$z(x, y) = f(u(x, y), v(x, y)).$$

Theorem 3.2 (Chain Rule for Two Variables) ***If*** f, u, ***and*** v ***are differentiable, the partial derivatives of*** $z(x, y)$ ***are given by:***

$$\frac{\partial z}{\partial x} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial x}, \quad \frac{\partial z}{\partial y} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial y}.$$

Example 3.21 ***Let***

$$z(x, y) = \sin(u(x, y)\, v(x, y)), \quad u(x, y) = x^2 + y, \quad v(x, y) = xy.$$

Compute the partial derivatives $\frac{\partial z}{\partial x}$ ***and*** $\frac{\partial z}{\partial y}$.
Solution: Using the chain rule,

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x} = v(x, y)\cos(uv) \cdot 2x + u(x, y)\cos(uv) \cdot y,$$
$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial y} = v(x, y)\cos(uv) \cdot 1 + u(x, y)\cos(uv) \cdot x.$$

Remark 3.6 ***Recal that for a function*** $u(x, y)$ ***where*** $x = x(t)$ ***and*** $y = y(t)$ ***we have***

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt} = u_x\frac{dx}{dt} + u_y\frac{dy}{dt}.$$

3.4.5 Differential

We know that if a function $f : D \subset \mathbb{R} \longrightarrow \mathbb{R}$ is differentiable on D, then its derivative $f^{'}$ satisfies

$$\forall x \in D: \; f^{'}(x) = \frac{df}{dx} \tag{3.36}$$

and

$$\forall x \in D: \; df = f^{'}(x)\, dx \tag{3.37}$$

df is the differential of f. We generalize this result for functions of several variables.

Definition 3.15 ***Let $f : D \subset \mathbb{R}^n \longrightarrow \mathbb{R}$. We say that f is differentiable at $a \in D$ if there exists a linear map $L : \mathbb{R}^n \longrightarrow \mathbb{R}$, such that in the neighborhood of a we have***

$$f\,(a+h) = f(a) + Lh + o(\|h\|).$$

If such a map exists, it is called the differential of f at the point a and is denoted $df(a)$.

We will admit the following proposition:

Proposition 3.3 ***If a function $f : D \subset \mathbb{R}^n \longrightarrow \mathbb{R}$ is continuous and has partial derivatives that are all continuous, then f is differentiable on D and we have:***

$$df = \frac{\partial f}{\partial x_1} dx_1 + \frac{\partial f}{\partial x_2} dx_2 + \cdots + \frac{\partial f}{\partial x_n} dx_n.$$

1. Total Differentials

Definition 3.16 ***Let $z = f(x, y)$ be a function of two variables. The total differential of f at the point (x, y) is defined as:***

$$dz = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy,$$

where dx and dy are infinitesimal changes in x and y, respectively.

Remark 3.7 ***If $z(x, y) = cst$ then $dz = 0$.***

Example 3.22 ***Let***

$$z(x, y) = x^2 y + \sin(xy).$$

Then the total differential is:

$$\begin{aligned} dz &= \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy \\ &= (2xy + y\cos(xy))\, dx + \left(x^2 + x\cos(xy)\right) dy. \end{aligned}$$

Remark 3.8 ***The total differential*** dz ***gives an approximation of the change in*** z ***for small changes*** dx ***and*** dy***:***

$$\Delta z \approx dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy.$$

Example 3.23

$$f\left(x, y\right) = \sin\left(xy\right)$$

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = y\cos(xy)dx + x\cos(xy)dy.$$

2. Exact Total Differentials

Definition 3.17 ***Let*** $M(x, y)$ ***and*** $N(x, y)$ ***be functions defined on a domain*** $D \subset \mathbb{R}^2$***. The differential expression***

$$\omega = M(x, y)\,dx + N(x, y)\,dy$$

is called a total differential of some function $f(x, y)$ ***if there exists a function*** f ***such that***

$$df(x, y) = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = M(x, y)\,dx + N(x, y)\,dy.$$

In this case, ω ***is said to be exact.***

Theorem 3.3 (Condition for Exactness) ***A differential form***

$$\omega = M(x, y)\,dx + N(x, y)\,dy$$

is exact in a simply connected domain D ***if*** M ***and*** N ***have continuous first partial derivatives and satisfy***

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.$$

Example 3.24 ***Consider the differential form***

$$\omega = (2xy + 3)\,dx + (x^2 + 4y)\,dy.$$

Check if ω ***is exact.***
We have

$$M(x, y) = 2xy + 3; \quad N(x, y) = x^2 + 4y.$$

Solution: Compute the partial derivatives:

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}(2xy + 3) = 2x, \quad \frac{\partial N}{\partial x} = \frac{\partial}{\partial x}(x^2 + 4y) = 2x.$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$***, the differential form is exact.***
Since

$$M(x, y) = \frac{\partial f}{\partial x}, \quad \textbf{and } N(x, y) = \frac{\partial f}{\partial y},$$

to find the potential function $f(x, y)$, we first integrate M with respect to x:

$$f(x, y) = \int M\, dx = \int (2xy + 3)\, dx = x^2 y + 3x + g(y),$$

where $g(y)$ is a function of y.
Differentiate f with respect to y and set it equal to $N(x, y)$:

$$\frac{\partial f}{\partial y} = x^2 + g'(y) = N(x, y) = x^2 + 4y \implies g'(y) = 4y \implies g(y) = 2y^2.$$

Hence, the potential function is

$$f(x, y) = x^2 y + 3x + 2y^2.$$

Particular Case: Finding the Potential Function $f(x, y)$
Consider the differential equation

$$M(x, y)\, dx + N(x, y)\, dy = 0,$$

where $M(x, y)$ and $N(x, y)$ are continuous functions with continuous first partial derivatives.

Definition 3.18 ***If the differential form $M(x, y)\, dx + N(x, y)\, dy$ is exact, there exists a function $f(x, y)$ such that***

$$df(x, y) = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy = M(x, y)\, dx + N(x, y)\, dy.$$

Then the solution of the differential equation can be written as

$$f(x, y) = C,$$

where C is a constant.

Method 1 (Finding $f(x, y)$) ***To find the function $f(x, y)$:***

1. ***Integrate $M(x, y)$ with respect to x:***

$$f(x, y) = \int M(x, y)\, dx + g(y),$$

where $g(y)$ is an arbitrary function of y.

2. ***Differentiate $f(x, y)$ with respect to y:***

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left(\int M(x, y)\, dx\right) + g'(y),$$

and set it equal to $N(x, y)$:

$$\frac{\partial f}{\partial y} = N(x, y) \implies g'(y) = N(x, y) - \frac{\partial}{\partial y}\left(\int M(x, y)\, dx\right).$$

3. ***Integrate*** $g'(y)$ ***to find*** $g(y)$***.***

4. ***Substitute*** $g(y)$ ***back into*** $f(x,y)$ ***to obtain the potential function.***

Example 3.25 *Solve*

$$(2xy + 3)\,dx + (x^2 + 4y)\,dy = 0.$$

Solution:

1. ***Integrate*** $M(x,y) = 2xy + 3$ ***with respect to*** x***:***

$$f(x,y) = \int (2xy + 3)\,dx = x^2y + 3x + g(y).$$

2. ***Differentiate with respect to*** y ***and set equal to*** $N(x,y) = x^2 + 4y$***:***

$$\frac{\partial f}{\partial y} = x^2 + g'(y) = x^2 + 4y \implies g'(y) = 4y \implies g(y) = 2y^2.$$

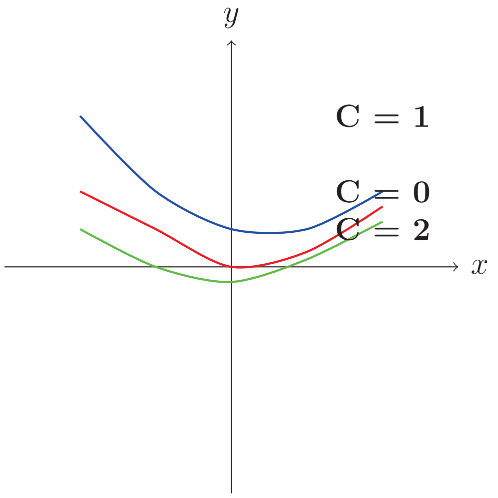
3. ***The potential function is***

$$f(x,y) = x^2y + 3x + 2y^2.$$

4. ***The general solution of the differential equation is***

$$f(x,y) = x^2y + 3x + 2y^2 = C.$$

Level Curves of $f(x,y) = x^2y + 3x + 2y^2$



Clarification

Point 1: What is $f(x,y)$**?**

In our example we have

$$f(x,y) = x^2y + 3x + 2y^2.$$

- **This is a function of two variables,** x **and** y**.**

- **Its value depends on the coordinates (x, y).**
- **For example:**

$$f(1,1) = 1^2 \cdot 1 + 3 \cdot 1 + 2 \cdot 1^2 = 6,$$

  while

$$f(0,1) = 0 + 0 + 2 \cdot 1^2 = 2.$$

- **Therefore, $f(x, y)$ is not a constant; it varies with (x, y).**

Point 2: Where does the constant C come from?
From the exact differential equation

$$(2xy + 3)\, dx + (x^2 + 4y)\, dy = 0,$$

we know that

$$df = 0,$$

where

$$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy.$$

- **The condition $df = 0$ means that the total change of f along a solution is zero.**
- **In other words, along a trajectory that satisfies the differential equation, the value of $f(x, y)$ remains the same.**
- **This is why the solution can be written as**

$$f(x, y) = C,$$

  where C is a constant that depends on the chosen trajectory.

3. Solving a First-Order ODE Using the Exact Differential Method

Consider the first-order ODE:

$$xy' + y = x^2 \tag{3.38}$$

This is of the form

$$a(x)y' + b(x)y = c(x), \quad \textbf{with } a(x) = x,\ b(x) = 1,\ c(x) = x^2.$$

Step 1: Rewrite in differential form
Multiply both sides by dx:

$$x\, dy + y\, dx = x^2\, dx$$

Rewriting:

$$(y - x^2)\,dx + x\,dy = 0$$

where

$$M(x, y) = y - x^2, \quad N(x, y) = x$$

Step 2: Check exactness
The equation is exact if

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Compute:

$$\frac{\partial M}{\partial y} = 1, \quad \frac{\partial N}{\partial x} = 1$$

Thus, the equation is exact.
Step 3: Find $f(x, y)$
We seek $f(x, y)$ such that

$$\frac{\partial f}{\partial x} = M(x, y) = y - x^2, \quad \frac{\partial f}{\partial y} = N(x, y) = x$$

Integrate $M(x, y)$ with respect to x:

$$f(x, y) = \int (y - x^2) dx = xy - \frac{x^3}{3} + g(y)$$

Step 4: Determine $g(y)$
Differentiate $f(x, y)$ with respect to y:

$$\frac{\partial f}{\partial y} = x + g'(y)$$

Compare with $N(x, y) = x$:

$$g'(y) = 0 \implies g(y) = C_0$$

Step 5: Solution
The general solution is:

$$\boxed{f(x, y) = xy - \frac{x^3}{3} = C}$$

where C is an arbitrary constant.

3.5 Elements of Partial Differential Equations (PDEs)

3.5.1 Generalities

1. Introduction
A partial differential equation (PDE) is an equation that relates an unknown function $u(x_1, x_2, \ldots, x_n)$ of several variables to its partial derivatives.
Unlike ordinary differential equations (ODEs), which involve functions of a single variable, PDEs model phenomena depending on multiple variables, often in space and time.
Examples:

- **Heat diffusion: $u(x,t)$ depends on space x and time t.**
- **Wave propagation: $u(x,t)$ represents displacement of a string or membrane.**
- **Electrostatics: potential $V(x,y,z)$ in a spatial domain.**

The general form of a PDE can be written as:

$$F\Big(x_1, \ldots, x_n, u, u_{x_1}, \ldots, u_{x_n}, u_{x_1x_1}, \ldots\Big) = 0$$

2. Order and Degree
Order: the highest order of partial derivative appearing in the equation.
Example:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \Rightarrow \text{second order}$$

Degree: the highest power to which a derivative is raised (if polynomial).
Example:

$$\left(\frac{\partial u}{\partial x}\right)^3 + u = 0 \quad \Rightarrow \text{degree 3}$$

3. Linearity
A PDE is linear if the unknown function u and its derivatives appear linearly: - No products of u and its derivatives - No powers higher than 1
Examples:

- **Linear: $u_t = u_{xx}$ (heat equation)**
- **Non-linear: $u_t = (u_x)^2$**

4. Classification of Second-Order PDEs
For a linear second-order PDE in two variables:

$$Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = G(x,y)$$

Discriminant: $\Delta = B^2 - 4AC$

Type	Condition	Physical Example
Elliptic	$\Delta < 0$	Laplace's equation (potential problems)
Parabolic	$\Delta = 0$	Heat equation
Hyperbolic	$\Delta > 0$	Wave equation

- Elliptic: smooth solutions, equilibrium problems.
- Parabolic: time-evolving solutions, diffusion problems.
- Hyperbolic: wave propagation with finite speed.

5. Initial and Boundary Conditions

To ensure a unique solution:

- Initial conditions: values of u and/or its derivatives at $t = 0$
  Example: $u(x, 0) = f(x)$
- Boundary conditions: values of u on the spatial domain boundary $\partial\Omega$
  - Dirichlet: $u = g(x)$
  - Neumann: $\frac{\partial u}{\partial n} = h(x)$

3.5.2 Methods for Solving PDEs

1. Separation of Variables

Assume the solution can be written as a product of functions, each depending on a single variable:

$$u(x, t) = X(x)\, T(t)$$

Example: Heat Equation

$$u_t = k u_{xx}, \quad 0 < x < L,\ t > 0$$

- Assume $u(x, t) = X(x)T(t)$.
- Substitute into the PDE: $X(x)T'(t) = kX''(x)T(t)$
- Separate variables: $\frac{T'(t)}{kT(t)} = \frac{X''(x)}{X(x)} = -\lambda$
- Solve the resulting ODEs: $\begin{cases} X'' + \lambda X = 0 \\ T' + k\lambda T = 0 \end{cases}$
- Apply boundary conditions to determine possible λ and construct the general solution.

Remark:
The negative sign in $-\lambda$ is a practical convention. It ensures that the spatial function $X(x)$ satisfies the boundary conditions, such as $X(0) = X(L) = 0$.
Using $-\lambda$ gives the harmonic equation:

$$X'' + \lambda X = 0 \quad \Rightarrow \quad X(x) = A\cos(\sqrt{\lambda}x) + B\sin(\sqrt{\lambda}x)$$

which allows for non-trivial solutions satisfying the boundary conditions.
If we had used $+\lambda$, the solution would involve exponentials:

$$X(x) = Ae^{\sqrt{\lambda}x} + Be^{-\sqrt{\lambda}x}$$

which cannot satisfy $X(0) = X(L) = 0$ unless $X \equiv 0$. Therefore, the negative sign simplifies finding physically meaningful solutions.

2. Method of Characteristics

Used for first-order PDEs:

$$a(x, y)u_x + b(x, y)u_y = c(x, y, u)$$

- **Define characteristic curves $(x(s), y(s))$ along which the PDE reduces to an ODE.**
- **Characteristic equations: $\frac{dx}{ds} = a(x, y), \frac{dy}{ds} = b(x, y), \frac{du}{ds} = c(x, y, u)$**
- **Solve these ODEs to find $u(x, y)$.**

Example: $u_x + u_y = 0$
- Characteristics: $y - x =$ constant - Solution: $u(x, y) = f(y - x)$, f determined by initial conditions.

3. Transform Methods (Fourier and Laplace)

- Fourier transform: converts spatial PDE into temporal ODE, useful for infinite or periodic domains. - Laplace transform: converts temporal PDE into algebraic equation, convenient for initial conditions.
Example: Heat equation on $x \in [0, \infty)$ with $u(x, 0) = f(x)$:

- **Apply Laplace transform in t: $U(x, s) = \mathcal{L}\{u(x, t)\}$**
- **PDE becomes ODE in x: $sU(x, s) - f(x) = kU_{xx}(x, s)$**
- **Solve ODE, then apply inverse Laplace transform to obtain $u(x, t)$**

4. Numerical Methods

When analytical solution is not possible:

- **Finite Difference Method (FDM): approximate derivatives using discrete differences.**
- **Finite Element Method (FEM): approximate solution on a mesh and solve linear system.**
- **Finite Volume Method (FVM): conserve mass or energy in fluid mechanics.**

Example: Discretized heat equation on a rod:

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} = k\frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{(\Delta x)^2}$$

Applications
- Heat diffusion in a rod or plate.
- Vibrations of a string or membrane .
- Electrostatic potential and electric fields.
- Fluid flow (simplified Navier-Stokes equations).

3.6 Solved Exercises

Exercise 3.2 ***Solve the following differential equation:***

$$u^2u^{'} + x^2 = 0, \quad u \neq 0, \quad u(0) = 4 \tag{3.39}$$

Solution:
This is a first-order differential equation, homogeneous and separable.

$$\begin{aligned} u^2u^{'} + x^2 = 0 &\iff u^2u^{'} = -x^2 \\ &\iff u^2\frac{du}{dx} = -x^2 \\ &\iff u^2\,du = -x^2\,dx \\ &\implies \int u^2\,du = -\int x^2\,dx \\ &\implies \frac{1}{3}u^3 = -\frac{1}{3}u^3 + k, \quad k \in \mathbb{R}, \\ &\implies u = \sqrt[3]{-x^3 + k}, \quad k \in \mathbb{R}. \end{aligned}$$

$$u(0) = 4 \iff \sqrt[3]{-(0)^3 + k} = 4$$

$$\sqrt[3]{+k} = 4 \implies k = 64.$$

Then $u(x) = \sqrt[3]{-x^3 + 64}$.

Exercise 3.3 ***Solve the following differential equation:***

$$xu^{''} + 2u^{'} = 0, \quad u \neq 0, \quad u(1) = 0,\ u(2) = 1. \tag{3.40}$$

Solution:
This is a second-order homogeneous equation. We reduce it to a first-order equation using the change of variable $y = u'$**, giving**

$$xy' + 2y = 0.$$

$$\begin{aligned}
xy' + 2y = 0 &\iff xy' = -2y \\
&\iff \frac{y'}{y} = -\frac{2}{x} \\
&\iff \frac{1}{y}\frac{dy}{dx} = -\frac{2}{x} \\
&\iff \frac{dy}{y} = -2\frac{dx}{x} \\
&\implies \ln|y| = -2\ln|x| + k, \quad k \in \mathbb{R} \\
&\implies y = \frac{K}{x^2}, \quad K \in \mathbb{R}.
\end{aligned}$$

Then,

$$y = u' \iff u' = \frac{K}{x^2} \implies u = -\frac{K}{x} + \tau, \quad K, \tau \in \mathbb{R}.$$

Using the boundary conditions:

$$\begin{aligned}
u(1) = 0 &\implies -K + \tau = 0 \implies K = \tau, \\
u(2) = 1 &\implies -\frac{K}{2} + \tau = 1 \implies K = 2.
\end{aligned}$$

Thus, the solution is

$$u(x) = 2\left(1 - \frac{1}{x}\right).$$

Exercise 3.4 ***Solve the following differential equation:***

$$u'' + 3u' + 2u = xe^{-x}. \tag{3.41}$$

Solution:

1) Find the general solution of the homogeneous equation

$$u'' + 3u' + 2u = 0 \tag{3.42}$$

The characteristic equation is

$$r^2 + 3r + 2 = 0 \tag{3.43}$$

$$\Delta = 9 - 8 = 1 \implies r_1 = -2, \ r_2 = -1.$$

The general solution of (3.42) **is**

$$u_1 = \lambda_1 e^{-2x} + \lambda_2 e^{-x}.$$

2) Find a particular solution of (3.41)**.**
The nonhomogeneous term is of the form $h(x)e^{-x}$ **with** $h(x)$ **a polynomial of degree** $n = 1$**, and since** $(-1)^2 + 3(-1) + 2 = 0$**, we take**

$$u_0 = k(x)e^{-x},$$

with $k(x)$ **a polynomial of degree** $m = n + 1 = 2$**. Let** $u_0 = (ax^2 + bx + c)e^{-x}$**. Then**

$$u_0' = e^{-x}\left(-ax^2 + (2a - b)x + (b - c)\right), \quad u_0'' = e^{-x}\left(ax^2 + (b - 4a)x + (2a - 2b + c)\right).$$

Substituting into (3.41) **and identifying terms, we find**

$$a = \frac{1}{2}, \quad b = -1, \quad c = 0.$$

Hence,

$$u_0 = \left(\frac{1}{2}x^2 - x\right)e^{-x}.$$

3) The general solution of (3.41) **is**

$$u = \left(\frac{1}{2}x^2 - x\right)e^{-x} + \lambda_1 e^{-2x} + \lambda_2 e^{-x}.$$

Exercise 3.5 ***Let*** $F(f, g) = \sin(fg)$,

1. ***Compute*** dF.
2. ***Suppose that*** f ***and*** g ***are functions such that***

$$f(x, y) = x - 7y; \qquad g(x, y) = x + y.$$

   Compute df ***and*** dg.

3. ***Consider the function*** H ***defined by***

$$H(x, y) = \sin[(x - 7y)(x + y)].$$

   Compute dH, ***and deduce*** $\frac{\partial H}{\partial x}$.

Solution:

⋇ $\frac{\partial F}{\partial f} = g\cos(fg)$, $\frac{\partial F}{\partial g} = f\cos(fg)$, **we obtain:**

$$dF = \frac{\partial F}{\partial f}df + \frac{\partial F}{\partial g}dg = g\cos(fg)\,df + f\cos(fg)\,dg$$

⋇ $\frac{\partial f}{\partial x} = 1$, $\frac{\partial f}{\partial y} = -7$, **and** $\frac{\partial g}{\partial x} = 1$, $\frac{\partial g}{\partial y} = 1$.

It follows that $df = dx - 7dy$, **and** $dg = dx + dy$.

⋇ **We notice that** $H(x, y) = F(f(x, y), g(x, y)) = \sin(f(x, y)\,g(x, y))$.

$$\begin{aligned}
dH &= \frac{\partial F}{\partial f}df + \frac{\partial F}{\partial g}dg, \\
&= g\cos(fg)\,df + f\cos(fg)\,dg, \\
&= (x + y)\cos[(x - 7y)(x + y)](dx - 7dy) + (x - 7y)\cos[(x - 7y)(x + y)](dx + dy) \\
\frac{\partial H}{\partial x}dx + \frac{\partial H}{\partial y}dy &= (2x - 6y)\cos[(x - 7y)(x + y)]\,dx + (-6x - 14y)\cos[(x - 7y)(x + y)]\,dy,
\end{aligned}$$

thus,

$$\frac{\partial H}{\partial x} = (2x - 6y)\cos\left[(x - 7y)(x + y)\right],$$

and

$$\frac{\partial H}{\partial y} = (-6x - 14y)\cos\left[(x - 7y)(x + y)\right].$$

Chapter 4

Series

4.1 Numerical Series

4.1.1 Introduction to Numerical Series

In many problems of physics, we encounter quantities that are obtained by adding infinitely many contributions:

- **the total distance traveled by an oscillating spring with decreasing amplitudes,**
- **the total intensity emitted by a radioactive particle in successive pulses,**
- **the energy stored in an electric circuit with repeated damping,**
- **the Fourier expansion of a periodic signal.**

In such cases, it is essential to understand how to give a precise meaning to an "infinite sum".

4.1.2 Sequences and Numerical Series

Definition 4.1 (Partial sum) ***Let*** $(u_n)_{n\in\mathbb{N}}$ ***be a sequence of real numbers. The partial sum of order*** n ***is defined as***

$$S_n = \sum_{k=0}^{n} u_k = u_0 + u_1 + u_2 + \cdots + u_n.$$

Definition 4.2 (Numerical series) ***The series associated with*** (u_n) ***is the sequence of partial sums*** (S_n)***. We write***

$$\sum_{n=0}^{+\infty} u_n = \lim_{n\to+\infty} S_n,$$

whenever this limit exists.

Definition 4.3 (Convergence and divergence) ***If the sequence of partial sums*** (S_n) ***has a finite limit*** S***, we say that the series*** **converges** ***and we write***

$$\sum_{n=0}^{+\infty} u_n = S.$$

If (S_n) ***does not have a finite limit, the series is said to*** **diverge.**

Example 4.1 (Distance traveled by a moving object) ***A moving object first travels a distance*** d***, then half of the previous distance, then half again, and so on.***
The successive displacements are:

$$d, \quad \frac{d}{2}, \quad \frac{d}{4}, \quad \frac{d}{8}, \quad \dots$$

The n***-th partial sum is:***

$$S_n = d + \frac{d}{2} + \frac{d}{4} + \cdots + \frac{d}{2^n}.$$

This is a geometric sum:

$$S_n = d \cdot \frac{1 - \left(\frac{1}{2}\right)^{n+1}}{1 - \frac{1}{2}} = 2d\left(1 - \frac{1}{2^{n+1}}\right).$$

As $n \to +\infty$***,***

$$S_n \to 2d.$$

Interpretation: ***even though the object makes infinitely many steps, the total distance traveled is finite and equal to*** $2d$***.***

Theorem 4.1 ***If the series*** $\sum_{n=1}^{\infty} u_n$ ***is convergent, then its general term*** u_n ***tends to zero:***

$$\lim_{n\to\infty} u_n = 0.$$

Corollary 4.1 ***If the general term*** u_n ***of the series*** $\sum_{n=1}^{\infty} u_n$ ***does not tend to zero, then the series diverges.***

Example 4.2 ***Consider the series***

$$\sum_{n=1}^{\infty} \frac{n+1}{n}.$$

Here, the general term is

$$u_n = \frac{n+1}{n} = 1 + \frac{1}{n}.$$

Since

$$\lim_{n\to\infty} u_n = 1 \neq 0,$$

the general term does not tend to zero.
By the contrapositive of the Theorem 4.1 this series diverges.

4.1.3 Geometric Series

Definition 4.4 ***A*** geometric series ***is a series of the form***

$$\sum_{n=0}^{\infty} ar^n = a + ar + ar^2 + \cdots,$$

where a ***is the first term and*** r ***is the common ratio.***

Study of the Convergence:

We compute the partial sum of order n**:**

$$S_n = \sum_{k=0}^{n} ar^k = \begin{cases} a\underbrace{(1+1+1+\cdots+1)}_{n+1 \textbf{ times}} = a(n+1), & \textbf{if } r = 1, \\ \dfrac{a(1-r^{n+1})}{1-r}, & \textbf{if } r \neq 1. \end{cases}$$

Now, let us study the convergence of S_n **as** $n \to \infty$**:**

$$\lim_{n\to\infty} S_n = \begin{cases} \lim_{n\to\infty} a(n+1) = +\infty, & \textbf{if } r = 1, \\ \lim_{n\to\infty} \dfrac{a(1-r^{n+1})}{1-r}, & \textbf{if } r \neq 1. \end{cases}$$

For the case $r \neq 1$**, we analyze according to** $|r|$**:**

$$\lim_{n\to\infty} \frac{a(1-r^{n+1})}{1-r} = \begin{cases} \dfrac{a(1-\lim\limits_{n\to\infty} r^{n+1})}{1-r} = \dfrac{a}{1-r}, & & \textbf{if } |r| < 1, \\ \dfrac{a(1-\lim\limits_{n\to\infty} r^{n+1})}{1-r} = \dfrac{a(1-\infty)}{1-r}, & \textbf{diverges}, & \textbf{if } |r| > 1. \end{cases}$$

Example 4.3 ***(successive reflections of light)***
Consider a ray of light (rayon de lumiére) entering a system of two parallel mirrors with a reflection coefficient r***, where*** $0 < r < 1$***. At the first reflection, the intensity of the ray is***

$$I_0 = I.$$

At the second reflection,

$$I_1 = Ir.$$

At the third reflection,

$$I_2 = Ir^2,$$

and so on.
The total intensity after infinitely many reflections is

$$I_{total} = I + Ir + Ir^2 + Ir^3 + \cdots = \sum_{n=0}^{\infty} Ir^n.$$

This is a geometric series with first term $a = I$ ***and common ratio*** r***. Since*** $0 < r < 1$***, the series converges and***

$$I_{total} = \frac{I}{1-r}.$$

4.1.4 Exponential Series

Definition 4.5 (Taylor expansion of a general function) ***Let $f(x)$ be a function infinitely differentiable at $x = 0$. Its*** **Taylor expansion** ***of order N around $x = 0$ is***

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(N)}(0)}{N!}x^N + R_{N+1}(x),$$

where $R_{N+1}(x)$ is the remainder term. As N becomes very large, $R_{N+1}(x) \to 0$.

Example 4.4 (Application to the exponential function) ***Consider $f(x) = e^x$. All derivatives satisfy***

$$f^{(n)}(x) = e^x \quad \Rightarrow \quad f^{(n)}(0) = 1.$$

Thus, the Taylor expansion of order N is

$$e^x = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^N}{N!} + R_{N+1}(x),$$

with $R_{N+1}(x) \to 0$ as $N \to \infty$.

Definition 4.6 ***The exponential series is defined as***

$$\sum_{n=0}^{\infty} \frac{x^n}{n!} := \lim_{N\to\infty} \sum_{n=0}^{N} \frac{x^n}{n!}.$$

Study of the Convergence

We can write the finite sum explicitly up to order N:

$$\sum_{n=0}^{N} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^N}{N!} = \left(e^x - R_{N+1}(x)\right).$$

Taking the limit on both sides, we have

$$\lim_{N\to\infty} \sum_{n=0}^{N} \frac{x^n}{n!} = \lim_{N\to\infty} \left(e^x - R_{N+1}(x)\right),$$

where $R_{N+1}(x)$ is the remainder term from the Taylor expansion of e^x.
For N sufficiently large, the remainder $R_{N+1}(x)$ tends to zero. Therefore,

$$\lim_{N\to\infty} \sum_{n=0}^{N} \frac{x^n}{n!} = e^x.$$

Hence, the exponential series is convergent and converges to e^x.

Example 4.5 ***Consider the exponential series with $x = 1$:***

$$\sum_{n=0}^{\infty} \frac{1}{n!} = \lim_{N\to\infty} \sum_{n=0}^{N} \frac{1}{n!} = e^1 = e \approx 2.71828.$$

Example 4.6 (Small displacement approximation) ***Consider a mass m attached to a spring with spring constant k. For a very small displacement x, the potential energy is***

$$U(x) = \frac{1}{2}kx^2.$$

We want to compute $e^{-kx/m}$ for small x.

Step 1: Use the exponential series

$$e^x = \lim_{N\to\infty} \sum_{n=0}^{N} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

Step 2: Replace x by $-kx/m$

$$e^{-kx/m} = \lim_{N\to\infty} \sum_{n=0}^{N} \frac{(-kx/m)^n}{n!} = 1 - \frac{kx}{m} + \frac{(kx)^2}{2m^2} - \frac{(kx)^3}{6m^3} + \cdots$$

Step 3: Approximation for small x For very small x, higher order terms are negligible:

$$e^{-kx/m} \approx 1 - \frac{kx}{m} + \frac{(kx)^2}{2m^2}.$$

This is widely used in physics for linear approximations.

4.1.5 Series With Non-Negative Terms

Definition 4.7 ***A series $\sum_{n=0}^{\infty} u_n$ is called a*** series with non-negative terms ***if***

$$u_n \geq 0 \quad \textit{for all } n \geq 0.$$

1. Key Property: Monotonicity of the Partial Sums

Proposition 4.1 ***Let $\sum_{n=0}^{\infty} u_n$ be a series with non-negative terms, and let***

$$S_N = \sum_{n=0}^{N} u_n$$

be its sequence of partial sums.
Since each term $u_n \geq 0$, we have

$$S_{N+1} = S_N + u_{N+1} \geq S_N.$$

Hence, the sequence (S_N) is increasing.
Therefore, the convergence of the series reduces to checking whether the sequence (S_N) is bounded above:

$$\textit{The series converges} \iff \textit{the partial sums } S_N \textit{ are bounded.}$$

Example 4.7 ***Consider the series***

$$\sum_{n=0}^{\infty} \frac{1}{2^n}.$$

Here, each term is non-negative:

$$u_n = \frac{1}{2^n} \geq 0 \quad \textbf{\textit{for all}}\ n \geq 0.$$

The partial sums are

$$S_N = \sum_{n=0}^{N} \frac{1}{2^n} = 1 + \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^N}.$$

Since this is a geometric series with first term $a = 1$ ***and ratio*** $r = \frac{1}{2}$***, the sum of the first*** $N+1$ ***terms is***

$$S_N = \frac{1 - r^{N+1}}{1 - r} = \frac{1 - (1/2)^{N+1}}{1 - 1/2} = 2\left(1 - \frac{1}{2^{N+1}}\right).$$

Because $\frac{1}{2^{N+1}} > 0$***, we have***

$$S_N = 2\left(1 - \frac{1}{2^{N+1}}\right) < 2.$$

Thus, the sequence of partial sums (S_N) ***is increasing and bounded above by 2. Therefore, the series converges.***

2. Riemann Series

Definition 4.8 ***A Riemann series is a series of the form***

$$\sum_{n=1}^{\infty} \frac{1}{n^p}, \qquad p > 0.$$

- ***If*** $p > 1$***, the series*** $\sum \frac{1}{n^p}$ ***converges.***
- ***If*** $0 < p \leq 1$***, the series*** $\sum \frac{1}{n^p}$ ***diverges.***

Example 4.8 ***we present two cases:***
a. Convergent series:

$$\sum_{n=1}^{\infty} \frac{1}{n^2} \quad \textbf{\textit{(here}}\ p = 2 > 1,\ \textbf{\textit{so the series converges).}}$$

b. Divergent series:

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} = \sum_{n=1}^{\infty} \frac{1}{n^{1/2}} \quad \textbf{\textit{(here}}\ p = 1/2 \leq 1,\ \textbf{\textit{so the series diverges).}}$$

3. Bertrand Series

Definition 4.9 (Bertrand Series) ***A Bertrand series is a series of positive terms defined by***

$$\sum_{n=2}^{\infty} \frac{1}{n^{\alpha}(\ln n)^{\beta}},$$

where

$$n \geq 2, \quad \alpha > 0, \quad \beta \in \mathbb{R}.$$

Theorem 4.2 (Convergence of Bertrand Series) ***Consider the Bertrand series***

$$\sum_{n=2}^{\infty} \frac{1}{n^{\alpha}(\ln n)^{\beta}}, \quad \alpha > 0,\ \beta \in \mathbb{R}.$$

The convergence depends on α ***and*** β ***as follows:***

1. $\alpha > 1$***: The series converges for all*** $\beta \in \mathbb{R}$***.***
2. $\alpha < 1$***: The series diverges for all*** $\beta \in \mathbb{R}$***.***
3. $\alpha = 1$***: The series reduces to***

$$\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^{\beta}}.$$

In this case:

- ***If*** $\beta > 1$***, the series converges.***
- ***If*** $\beta \leq 1$***, the series diverges.***

Remark 4.1 ***Let us consider the following cases:***

- ***Case*** $\alpha > 1$***: Compare with the convergent Riemann series*** $\sum 1/n^{\alpha}$***. Since***

$$0 < \frac{1}{n^{\alpha}(\ln n)^{\beta}} \leq \frac{1}{n^{\alpha}}, \quad n \geq 2,$$

the series converges by the comparison test for any β***.***

- ***Case*** $\alpha < 1$***: Compare with the divergent Riemann series*** $\sum 1/n^{\alpha}$***. Since***

$$\frac{1}{n^{\alpha}(\ln n)^{\beta}} \geq \frac{1}{n^{\alpha}(\ln 2)^{\beta}} > 0, \quad n \geq 2,$$

the series diverges by the comparison test for any β***.***

- ***Case*** $\alpha = 1$***: Use the integral test with***

$$f(x) = \frac{1}{x(\ln x)^{\beta}}, \quad x \geq 2.$$

Substituting $t = \ln x$ ***(***$dx = x\,dt$***), we get***

$$\int_{2}^{\infty} \frac{dx}{x(\ln x)^{\beta}} = \int_{\ln 2}^{\infty} \frac{dt}{t^{\beta}}.$$

A p-integral $\int_{A}^{\infty} t^{-p} dt$ ***converges if and only if*** $p > 1$***. Hence:***

– $\beta > 1 \implies$ ***series converges.***

– $\beta \leq 1 \implies$ ***series diverges.***

α	β	Convergence
$\alpha > 1$	any β	Convergent
$\alpha < 1$	any β	Divergent
$\alpha = 1$	$\beta > 1$	Convergent
$\alpha = 1$	$\beta \leq 1$	Divergent

Example 4.9 (Simple Bertrand Series) ***Consider the series***

$$\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}.$$

- This is a Bertrand series with $\alpha = 1$ ***and*** $\beta = 2$.
- Since $\alpha = 1$ ***and*** $\beta > 1$***, the series converges by the integral test.***

4. Comparison Convergence Criterion

Theorem 4.3 ***Let*** $\sum u_n$ ***and*** $\sum v_n$ ***be two series with non-negative terms, i.e.,*** $u_n \geq 0$ ***and*** $v_n \geq 0$ ***for all*** $n \geq 1$.

- ***If there exists a constant*** $C > 0$ ***such that*** $u_n \leq C\, v_n$ ***for all sufficiently large*** n ***and if*** $\sum v_n$ ***converges, then*** $\sum u_n$ ***also converges.***

- ***If*** $u_n \geq v_n \geq 0$ ***for all sufficiently large*** n ***and if*** $\sum v_n$ ***diverges, then*** $\sum u_n$ ***also diverges.***

Example 4.10 ***Study the convergence of***

$$\sum_{n=1}^{\infty} \frac{1}{n^2 + 3n + 1}.$$

Solution: For all $n \geq 1$,

$$n^2 + 3n + 1 \geq n^2 \quad \Rightarrow \quad \frac{1}{n^2 + 3n + 1} \leq \frac{1}{n^2}.$$

Since $\sum \frac{1}{n^2}$ ***converges, by the Theorem 4.3, the given series also converges.***

5. Equivalence Convergence Criterion

Definition 4.10 (Equivalent Series) ***Let***

$$\sum_{n=1}^{\infty} u_n \quad \textit{and} \quad \sum_{n=1}^{\infty} v_n$$

be two series with positive terms. We say that the series are equivalent if and only if ***their terms are asymptotically equal, that is,***

$$\lim_{n \to \infty} \frac{u_n}{v_n} = 1,$$

and we write $u_n \sim v_n$ ***as*** $n \to \infty$.

Theorem 4.4 (Equivalence Test) ***If***

$$\sum_{n=1}^{\infty} u_n \quad \textbf{\textit{and}} \quad \sum_{n=1}^{\infty} v_n$$

are equivalent series with positive terms, then

$$\sum_{n=1}^{\infty} u_n \textbf{\textit{ converges}} \iff \sum_{n=1}^{\infty} v_n \textbf{\textit{ converges}}.$$

In other words, equivalent series are simultaneously convergent or divergent.

Example 4.11 ***Study the convergence of the series***

$$\sum_{n=1}^{\infty} u_n, \quad \textbf{\textit{where }} u_n = \frac{n+1}{n^3+n}.$$

Solution: We compare u_n ***with*** $v_n = \dfrac{1}{n^2}$.
Compute the limit:

$$\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{\frac{n+1}{n^3+n}}{\dfrac{1}{n^2}} = \lim_{n\to\infty} \frac{(n+1)n^2}{n^3+n} = \lim_{n\to\infty} \frac{n^3+n^2}{n^3+n} = 1.$$

Since $\sum v_n = \sum \dfrac{1}{n^2}$ ***converges and the limit equals 1, by the equivalence criterion, the series*** $\sum u_n$ ***also converges.***

6. D'Alembert's Convergence Criterion (D'Alembert's Ratio Test)

Theorem 4.5 (D'Alembert's Ratio Test) ***Let*** $\sum u_n$ ***be a series with positive terms. Suppose the limit***

$$L = \lim_{n\to\infty} \frac{u_{n+1}}{u_n}$$

exists. Then:

- ***If*** $L < 1$, ***the series*** $\sum u_n$ ***converges.***
- ***If*** $L > 1$, ***the series*** $\sum u_n$ ***diverges.***
- ***If*** $L = 1$, ***the test is inconclusive.***

Example 4.12 ***Study the convergence of the series***

$$\sum_{n=1}^{\infty} \frac{2^n}{n!}.$$

Solution: Let $u_n = \dfrac{2^n}{n!}$***. Compute the ratio:***

$$\frac{u_{n+1}}{u_n} = \frac{2^{n+1}/(n+1)!}{2^n/n!} = \frac{2^{n+1}\cdot n!}{2^n\cdot(n+1)!} = \frac{2}{n+1}.$$

Take the limit as $n \to \infty$***:***

$$L = \lim_{n\to\infty} \frac{2}{n+1} = 0 < 1.$$

Since $L < 1$***, by the ratio test, the series*** $\sum_{n=1}^{\infty} \frac{2^n}{n!}$ ***converges.***

5. Cauchy's Convergence Criterion

Theorem 4.6 (Cauchy's Root Test for Positive-Term Series) ***Let*** $\sum_{n=1}^{\infty} u_n$ ***be a series with positive terms (***$u_n \geq 0$***). Suppose the limit***

$$L = \lim_{n\to\infty} \sqrt[n]{u_n}\,.$$

- ***If*** $L < 1$***, the series*** $\sum_{n=1}^{\infty} u_n$ ***converges.***
- ***If*** $L > 1$***, the series*** $\sum_{n=1}^{\infty} u_n$ ***diverges.***
- ***If*** $L = 1$***, the test is inconclusive.***

Example 4.13 ***Study the convergence of the series***

$$\sum_{n=1}^{\infty} \frac{3^n}{2^n+1}.$$

Solution: Let

$$u_n = \frac{3^n}{2^n+1}.$$

We apply Cauchy's root test and compute the n-th root of u_n***:***

$$\sqrt[n]{u_n} = \sqrt[n]{\frac{3^n}{2^n+1}} = \frac{\sqrt[n]{3^n}}{\sqrt[n]{2^n+1}}.$$

Step by step:
a. Numerator: $\sqrt[n]{3^n} = 3$.
b. Denominator:

$$\sqrt[n]{2^n+1} = \sqrt[n]{2^n\left(1+\frac{1}{2^n}\right)} = \sqrt[n]{2^n} \cdot \sqrt[n]{1+\frac{1}{2^n}} = 2 \cdot \sqrt[n]{1+\frac{1}{2^n}}.$$

c. Limit of the second factor:

$$\lim_{n\to\infty} \sqrt[n]{1+\frac{1}{2^n}} = 1.$$

d. Hence, the n-th root limit:

$$\lim_{n\to\infty} \sqrt[n]{u_n} = \frac{3}{2} > 1.$$

Since the limit is greater than 1, by Cauchy's root test, the series

$$\sum_{n=1}^{\infty} \frac{3^n}{2^n+1}$$

diverges.

7. Cauchy's Integral Convergence Criterion

Theorem 4.7 (Cauchy's Integral Test) ***Let*** $f : [1, \infty) \to \mathbb{R}$ ***be a continuous, positive, and decreasing function. Consider the series***

$$\sum_{n=1}^{\infty} u_n \quad \textit{with} \quad u_n = f(n).$$

Then the series $\sum_{n=1}^{\infty} u_n$ ***converges if and only if the improper integral***

$$\int_{1}^{\infty} f(x)\,dx$$

converges.

Example 4.14 ***Study the convergence of the series***

$$\sum_{n=1}^{\infty} \frac{1}{n^2}.$$

Solution: Consider the function

$$f(x) = \frac{1}{x^2}, \qquad x \geq 1.$$

- $f(x)$ ***is continuous, positive, and decreasing on*** $[1, \infty)$***.***
- By Cauchy's integral test, we compare the series with the improper integral

$$\int_{1}^{\infty} \frac{dx}{x^2}.$$

Compute the integral:

$$\int_{1}^{\infty} \frac{dx}{x^2} = \lim_{t\to\infty} \int_{1}^{t} x^{-2}dx = \lim_{t\to\infty} \left[-\frac{1}{x}\right]_1^t = \lim_{t\to\infty} \left(-\frac{1}{t} + 1\right) = 1.$$

Since the integral converges, by Cauchy's integral test, the series

$$\sum_{n=1}^{\infty} \frac{1}{n^2}$$

converges.

4.1.6 Alternating Series

Definition 4.11 (Alternating Series) ***A series***

$$\sum_{n=1}^{\infty} u_n$$

is called alternating if the terms alternate in sign, i.e.,

$$u_n \cdot u_{n+1} < 0 \quad \textit{for all } n \geq 1,$$

or equivalently,

$$u_n = (-1)^n a_n \quad \textit{or} \quad u_n = (-1)^{n+1} a_n$$

with $a_n \geq 0$ ***for all*** n***.***

Leibniz Convergence Criterion for Alternating Series

Theorem 4.8 (Leibniz Criterion for Alternating Series) ***Let*** $\sum_{n=1}^{\infty}(-1)^{n+1}a_n$ ***be an alternating series with*** $a_n \geq 0$***. If the sequence*** (a_n) ***satisfies***

$$a_{n+1} \leq a_n \quad \textbf{\textit{(decreasing)}} \quad \textbf{\textit{and}} \quad \lim_{n\to\infty} a_n = 0,$$

then the series $\sum_{n=1}^{\infty}(-1)^{n+1}a_n$ ***converges.***

Example 4.15 ***Study the convergence of the alternating series***

$$\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}.$$

Solution: Let

$$a_n = \frac{1}{n} \geq 0.$$

Check the conditions of Leibniz Criterion:
- Sequence (a_n) ***is decreasing:*** $a_{n+1} = \frac{1}{n+1} \leq \frac{1}{n} = a_n$***.***
- Limit: $\lim_{n\to\infty} a_n = 0$***.***
Since both conditions are satisfied, by Leibniz criterion, the series

$$\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}$$

converges.

4.1.7 Series with Arbitrary Terms

Definition 4.12 ***A series with arbitrary terms is a series of the form***

$$\sum_{n=0}^{\infty} u_n,$$

where the terms u_n ***can be positive, negative, or zero.***

Absolute Convergence Criterion

Proposition 4.2 ***A series***

$$\sum_{n=0}^{\infty} u_n$$

is said to be absolutely convergent if the series of the absolute values

$$\sum_{n=0}^{\infty} |u_n|$$

converges.

Remark 4.2 *If a series $\sum u_n$ is absolutely convergent, then it is also convergent.*
However, the converse is not true: a series may converge conditionally (convergent but not absolutely convergent).

Example 4.16 *Consider the series*

$$\sum_{n=1}^{\infty} \frac{(-1)^n n}{(n^2+1)^2}.$$

Step 1: Consider the absolute value series

$$\sum_{n=1}^{\infty} \left| \frac{(-1)^n n}{(n^2+1)^2} \right| = \sum_{n=1}^{\infty} \frac{n}{(n^2+1)^2}.$$

Step 2: Compare with a simpler series

$$\frac{n}{(n^2+1)^2} < \frac{n}{n^4} = \frac{1}{n^3}, \quad \textit{for all } n \geq 1.$$

Step 3: Use the comparison test
- The series $\sum \frac{1}{n^3}$ is convergent (Riemann series with $p = 3 > 1$). - Therefore, by the comparison test, the series $\sum n/(n^2+1)^2$ converges.
Step 4: Conclude absolute convergence
- Since $\sum |u_n| = \sum n/(n^2+1)^2$ converges, the original series

$$\sum_{n=1}^{\infty} \frac{(-1)^n n}{(n^2+1)^2}$$

is absolutely convergent. Then it converges.

Definition 4.13 *A series*

$$\sum_{n=1}^{\infty} u_n$$

is said to be conditionally convergent if:

- *The series $\sum u_n$ converges, but*
- *The series of absolute values $\sum |u_n|$ diverges.*

Remark 4.3 *Sometimes, a series converges only because of the cancellation of positive and negative terms, not because the absolute values form a convergent series. In this case,*

$$\sum u_n \quad \textit{converges but} \quad \sum |u_n| = +\infty,$$

and we call it conditional convergence.

Example 4.17 ***The* alternating harmonic series**

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$$

is conditionally convergent:

- ***It converges by the alternating series test.***
- ***But the harmonic series***

  $$\sum_{n=1}^{\infty} \left| \frac{(-1)^{n+1}}{n} \right| = \sum_{n=1}^{\infty} \frac{1}{n}$$

  diverges.

Proof of divergence (integral test). Indeed, consider the function $f(x) = \frac{1}{x}$**, which is positive and decreasing for** $x \geq 1$**. We compute**

$$\int_1^{+\infty} \frac{dx}{x} = \lim_{R \to +\infty} \ln(R) = +\infty.$$

By the integral test (or Cauchy's criterion for positive series), if $\int_1^{\infty} f(x)\, dx = +\infty$**, then the series** $\sum_{n=1}^{\infty} f(n)$ **diverges. Hence,**

$$\sum_{n=1}^{\infty} \frac{1}{n} \quad \textbf{diverges}.$$

Therefore, the alternating harmonic series is conditionally convergent.

Remark 4.4 ***When we say that a series converges "by cancellation", we mean that its convergence is due to the compensation between positive and negative terms. Individually, the sums of the positive terms and the negative terms both diverge, but taken together they balance each other and the whole series converges.***

Consider the alternating harmonic series:

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$$

- **The sum of the positive terms**

  $$1 + \frac{1}{3} + \frac{1}{5} + \cdots$$

  diverges.

- **The sum of the negative terms**

  $$-\frac{1}{2} - \frac{1}{4} - \frac{1}{6} - \cdots$$

  also diverges.

- **However, when positive and negative terms are combined, they compensate each other, and the series converges.**

Thus, the convergence of the alternating harmonic series is due to cancellation of terms.

4.1.8 Series with Known Exact Sums

Some series have sums that can be calculated exactly, giving a finite value in closed form. These series are useful examples in analysis.
Geometric Series The geometric series

$$\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}, \quad |r| < 1,$$

has a known exact sum.
Riemann Zeta Series for Even Integers The series

$$\sum_{n=1}^{\infty} \frac{1}{n^2}$$

is convergent and its exact sum is

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

Similarly,

$$\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}.$$

Alternating Harmonic Series The alternating harmonic series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$$

converges and its sum is

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = \ln 2.$$

4.2 Sequences and Series of Functions

4.2.1 Sequences of Functions

- We consider sequences of functions $f_n(x)$ and are interested in their behavior as $n \to \infty$.
- In physics, common examples include:

$$f_n(x) = \left(1 + \frac{x}{n}\right)^n \quad \textbf{or} \quad f_n(x) = x^n.$$

Example:

$$f_n(x) = x^n \quad \textbf{on } [0,1].$$

- For $0 \le x < 1$, $f_n(x) \to 0$. - For $x = 1$, $f_n(1) = 1$.
- Practically, it is sufficient to examine the behavior of powers or exponentials.

4.2.2 Series of Functions

1. Definition and Difference with Numerical Series

- A numerical series is a sum of numbers:

$$\sum_{n=0}^{\infty} a_n, \quad a_n \in \mathbb{R} \text{ or } \mathbb{C}.$$

- A series of functions is a sum of functions depending on a variable x:

$$\sum_{n=0}^{\infty} u_n(x), \quad u_n(x) \text{ is a function of } x.$$

Main difference:
- In numerical series, convergence concerns numbers a_n.
- In series of functions, convergence can depend on x.
- We distinguish pointwise convergence (each x separately) and uniform convergence (all x simultaneously).

2. Examples of Function Series in Physics

1. Exponential series:

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

2. Trigonometric series:

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}, \quad \cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$$

3. Geometric series:

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n, \quad |x| < 1$$

Remark: In all these examples, each term $u_n(x)$ is a function of x, unlike in numerical series where a_n is a fixed number.

4.2.3 Convergence of Function Series

3. Pointwise Convergence

- The series $\sum u_n(x)$ converges pointwise if, for each x, the partial sums

$$S_N(x) = \sum_{n=0}^{N} u_n(x)$$

have a finite limit as $N \to \infty$.
Example:

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \ldots$$

- Converges pointwise for $|x| < 1$.
- Limit: $S(x) = \frac{1}{1-x}$.

4. Uniform Convergence

- The series $\sum u_n(x)$ converges uniformly on an interval I if the partial sums $S_N(x)$ approach the limit uniformly for all $x \in I$.
Practical example:

$$\sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x$$

- Converges uniformly on any bounded segment $[a, b]$.
- Practical consequence: integration or differentiation term by term is allowed.

5. Practical Tips for Convergence

- If terms $u_n(x)$ decrease very fast (factorials, high powers, etc.), the series converges quickly.
- For small x, often only $2 - 3$ terms are needed for a good approximation.
Example: Small x approximations

$$e^x \approx 1 + x + \frac{x^2}{2}, \quad \sin x \approx x - \frac{x^3}{6}, \quad \cos x \approx 1 - \frac{x^2}{2}$$

- These approximations are widely used in physics:
- Small oscillations (harmonic motion)
- Electrodynamics: weak field expansions
- Thermodynamics and statistical mechanics: exponential expansions

4.3 Power Series (*Entire Series*)

Definition 4.14 ***A power series (or*** entire series***) centered at a point $x_0 \in \mathbb{R}$ is a series of the form***

$$\sum_{n=0}^{\infty} a_n (x - x_0)^n,$$

where $a_n \in \mathbb{R}$ or $\mathbb{C}$ are coefficients.
- If $x_0 = 0$, the series is called a Maclaurin series:

$$\sum_{n=0}^{\infty} a_n x^n.$$

- The general term is

$$u_n(x) = a_n (x - x_0)^n.$$

-The set of values of x for which the series converges is called the domain of convergence of the series and is denoted by D

Example 4.18 ***Consider the exponential series***

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

Solution 4.1 ***We study the absolute convergence of the series. The general term is***

$$u_n(x) = \frac{x^n}{n!}.$$

Applying the ratio test to $|u_n(x)|$***:***

$$\lim_{n\to\infty} \frac{|u_{n+1}(x)|}{|u_n(x)|} = \lim_{n\to\infty} \frac{|x|^{n+1}/(n+1)!}{|x|^n/n!} = \lim_{n\to\infty} \frac{|x|}{n+1} = 0.$$

Since this limit is less than 1 for all $x \in \mathbb{R}$***, the series converges absolutely for all*** x***. Then the domain of convergence denoted by*** D ***is*** $\mathbb{R}$ ***and we write*** $D = \mathbb{R}$.

4.3.1 Abel's Lemma for Power Series

Lemma 4.1 (Abel's Lemma for Power Series) ***Let*** $\sum_{n=0}^{\infty} a_n x^n$ ***be a power series. If there exists a real number*** $r > 0$ ***such that the sequence*** $(a_n r^n)_{n\geq 0}$ ***is bounded, then the power series***

$$\sum_{n=0}^{\infty} a_n x^n$$

converges absolutely for all $x \in \mathbb{R}$ ***such that*** $|x| < r$***.***

Radius of Convergence of a Power Series

Definition 4.15 (Radius of Convergence of a Power Series) ***Let*** $\sum_{n=0}^{\infty} a_n x^n$ ***be a power series. The radius of convergence*** R ***is the non-negative real number (or*** $+\infty$***) such that:***

- ***The series converges absolutely for all*** $x \in \mathbb{R}$ ***such that*** $|x| < R$***.***
- ***The series diverges for all*** $x \in \mathbb{R}$ ***such that*** $|x| > R$***.***

The set

$$\{x \in \mathbb{R} \mid |x| < R\}$$

is called the interval of convergence of the series. Convergence at the endpoints $x = \pm R$ ***must be studied separately.***

Determination of the Radius of Convergence

Lemma 4.2 (Hadamard's Lemma) ***For the power series*** $\sum\limits_{n=0}^{\infty} a_n x^n$***, the radius of convergence*** R ***is***

$$R = \frac{1}{\limsup\limits_{n\to\infty} \sqrt[n]{|a_n|}}.$$

Definition 4.16 (Radius of Convergence) ***Let***

$$\sum_{n=0}^{\infty} a_n (x - x_0)^n$$

be a power series. According to Hadamard's Lemma, its radius of convergence R ***is given by***

$$R = \frac{1}{\limsup_{n\to\infty} \sqrt[n]{|a_n|}}.$$

This value R ***can be finite, zero, or infinite.***

Remark 4.5

- ***If*** $0 < R < \infty$***, the series converges absolutely for*** $|x - x_0| < R$ ***and diverges for*** $|x - x_0| > R$***.***
- ***If*** $R = \infty$***, the series converges absolutely for all*** $x \in \mathbb{R}$***.***
- ***If*** $R = 0$***, the series converges only at*** $x = x_0$***.***

Remark 4.6 ***If the limit exists:***
- The radius of convergence can be calculated using the root test***:***

$$R = \frac{1}{\lim_{n\to\infty} \sqrt[n]{|a_n|}}.$$

- Or using the ratio test ***if it applies:***

$$R = \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right|,$$

Example 4.19 ***Determine the radius of convergence of the power series***

$$\sum_{n=1}^{\infty} \frac{n!\, x^n}{n^n}.$$

Hypothesis: We use the Stirling approximation

$$n! \sim \sqrt{2\pi n} \left(\frac{n}{e}\right)^n \quad \textbf{\textit{as}}\ n \to \infty.$$

Solution 4.2 ***Let*** $a_n = \frac{n!}{n^n}$***. By Hadamard's Lemma, the radius of convergence is***

$$R = \frac{1}{\limsup_{n\to\infty} \sqrt[n]{|a_n|}}.$$

Compute $\sqrt[n]{a_n}$ ***using the Stirling approximation:***

$$\sqrt[n]{a_n} = \sqrt[n]{\frac{n!}{n^n}} \sim \frac{\sqrt[n]{\sqrt{2\pi n}} \cdot \frac{n}{e}}{n} = \frac{1}{e} \quad \textbf{\textit{as}}\ n \to \infty.$$

Thus,

$$R = \frac{1}{1/e} = e.$$

The series converges absolutely for $|x| < e$***.***
Convergence at the endpoints $|x| = e$ ***must be checked separately.***
(i) Endpoint $x = e$***:***
The series becomes

$$\sum_{n=1}^{\infty} \frac{n!\, e^n}{n^n} = \sum_{n=1}^{\infty} \frac{n!}{(n/e)^n}.$$

Using Stirling:

$$\frac{n!}{(n/e)^n} \sim \sqrt{2\pi n} \quad \textbf{\textit{as}}\ n \to \infty.$$

Since $\sqrt{2\pi n} \to \infty$***, the general term does not tend to zero.*** $\implies$ ***The series diverges at*** $x = e$***.***
(ii) Endpoint $x = -e$***:***
The series becomes

$$\sum_{n=1}^{\infty} \frac{n!\,(-e)^n}{n^n} = \sum_{n=1}^{\infty} (-1)^n \frac{n!\, e^n}{n^n}.$$

Again,

$$\frac{n!\, e^n}{n^n} \sim \sqrt{2\pi n} \to \infty.$$

The alternating sign does not help because the terms do not tend to zero. Then the series diverges at $x = -e$***.***
Conclusion:
The series converges absolutely for $|x| < e$ ***and diverges for*** $|x| = e$.

4.3.2 Derivatives and Integrals of a Power Series

Definition 4.17 (Term-by-Term Derivative) Let

$$f(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n$$

be a power series with radius of convergence $R > 0$***. Then,*** $f(x)$ ***is differentiable for*** $|x - x_0| < R$***, and its derivative can be computed term by term:***

$$f'(x) = \sum_{n=1}^{\infty} n a_n (x - x_0)^{n-1}.$$

Definition 4.18 (Term-by-Term Integral) Similarly, the indefinite integral of $f(x)$ ***can be computed term by term:***

$$\int f(x)\, dx = \sum_{n=0}^{\infty} \frac{a_n}{n+1} (x - x_0)^{n+1} + C,$$

where C ***is an arbitrary constant. This series has the same radius of convergence*** R ***as the original series.***

Example 4.20 Consider the power series

$$f(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

Solution 4.3 Derivative: Term-by-term differentiation gives

$$f'(x) = \sum_{n=1}^{\infty} \frac{nx^{n-1}}{n!} = \sum_{n=1}^{\infty} \frac{x^{n-1}}{(n-1)!}.$$

Change the index $m = n - 1$***:***

$$f'(x) = \sum_{m=0}^{\infty} \frac{x^m}{m!} = f(x).$$

Radius of convergence:
- Coefficients of the derivative series are $b_m = 1/m!$***.***
- Using Hadamard's formula:

$$R = \frac{1}{\limsup_{m\to\infty} \sqrt[m]{|b_m|}} = \frac{1}{\lim_{m\to\infty} \sqrt[m]{1/m!}}.$$

- Since $\lim_{m\to\infty} \sqrt[m]{1/m!} = 0$***, we get***

$$R = \infty.$$

Integral: Term-by-term integration gives

$$\int f(x)\,dx = C + \sum_{n=0}^{\infty} \frac{x^{n+1}}{(n+1)n!} = C + \sum_{n=0}^{\infty} \frac{x^{n+1}}{(n+1)!}.$$

- Coefficients of the integral series are $u_n = 1/(n+1)!$***.***
- Applying Hadamard's formula:

$$R = \frac{1}{\limsup_{n\to\infty} \sqrt[n]{|u_n|}} = \frac{1}{\lim_{n\to\infty} \sqrt[n]{1/(n+1)!}}.$$

- Using the fact that $\sqrt[n]{(n+1)!} \to \infty$***, we get***

$$\lim_{n\to\infty} \sqrt[n]{|u_n|} = 0 \qquad \Longrightarrow \qquad R = \infty.$$

Conclusion:
- The derivative of $f(x)$ ***is*** $f(x)$ ***itself. - The integral can be expressed as a new power series. - In both cases, the radius of convergence remains*** $R = \infty$***, meaning the series converges absolutely for all*** $x \in \mathbb{R}$***.***

4.3.3 Operations on Power Series

Definition 4.19 (Sum and Difference of Power Series) ***Let***

$$f(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n \quad \textbf{\textit{and}} \quad g(x) = \sum_{n=0}^{\infty} b_n (x - x_0)^n$$

be two power series with radii of convergence R_1 ***and*** R_2***, respectively. Then, for*** $|x - x_0| < \min(R_1, R_2)$***, we can define***

$$f(x) \pm g(x) = \sum_{n=0}^{\infty} (a_n \pm b_n)(x - x_0)^n.$$

Definition 4.20 (Scalar Multiplication) ***For any scalar*** $\lambda \in \mathbb{R}$ ***(or*** $\mathbb{C}$***), we have***

$$\lambda f(x) = \sum_{n=0}^{\infty} (\lambda a_n)(x - x_0)^n \quad \textbf{\textit{for}} \ |x - x_0| < R,$$

where R ***is the radius of convergence of*** $f(x)$***.***

Definition 4.21 (Term-by-Term Multiplication (Cauchy Product)) ***Let***

$$f(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n \quad \textbf{\textit{and}} \quad g(x) = \sum_{n=0}^{\infty} b_n (x - x_0)^n.$$

The Cauchy product of f ***and*** g ***is***

$$(f \cdot g)(x) = \sum_{n=0}^{\infty} c_n (x - x_0)^n, \quad \textbf{\textit{where}} \ c_n = \sum_{k=0}^{n} a_k b_{n-k}.$$

The series converges at least for $|x - x_0| < \min(R_1, R_2)$***.***

Example 4.21 ***Consider***

$$f(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!} \quad \textbf{\textit{and}} \quad g(x) = \sum_{n=0}^{\infty} x^n.$$

Compute $f(x) + g(x)$***,*** $2f(x)$***, and the Cauchy product*** $f(x) \cdot g(x)$ ***as power series.***

Solution 4.4 ***Sum:***

$$f(x) + g(x) = \sum_{n=0}^{\infty} \left(\frac{1}{n!} + 1 \right) x^n.$$

Scalar multiplication:

$$2f(x) = \sum_{n=0}^{\infty} \frac{2x^n}{n!}.$$

Cauchy product:

$$(f \cdot g)(x) = \sum_{n=0}^{\infty} c_n x^n, \quad \textbf{\textit{with}} \ c_n = \sum_{k=0}^{n} \frac{1}{k!}.$$

Remark: The first series has radius $R_1 = \infty$ ***and the second series has radius*** $R_2 = 1$***.***

By the general property of the Cauchy product, the product series converges at least for

$$|x| < \min(R_1, R_2) = \min(\infty, 1) = 1.$$

Thus, we are guaranteed that the Cauchy product converges for $|x| < 1$***.***

4.3.4 Taylor Series

Definition 4.22 (Taylor Series) ***Let $f(x)$ be a function that is infinitely differentiable at a point $x_0 \in \mathbb{R}$. The Taylor series of f centered at x_0 is the power series***

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n,$$

where $f^{(n)}(x_0)$ denotes the n-th derivative of f evaluated at x_0.

Remark 4.7 ***- If $x_0 = 0$, the series is called a Maclaurin series.***
- The Taylor series may converge only for $|x - x_0| < R$, where R is the radius of convergence.
- If the series converges to $f(x)$ for all $|x - x_0| < R$, we say $f(x)$ is analytic at x_0.

Example 4.22 ***Find the Taylor series of $f(x) = e^x$ centered at $x_0 = 0$.***

Solution 4.5 ***- Compute derivatives: $f^{(n)}(x) = e^x \implies f^{(n)}(0) = 1$. - Apply the Taylor formula:***

$$e^x = \sum_{n=0}^{\infty} \frac{1}{n!}x^n = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

- The radius of convergence is $R = \infty$, so the series converges for all $x \in \mathbb{R}$.

Taylor Series of Common Functions

- **Exponential function:** $f(x) = e^x$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots, \quad R = \infty$$

- **Sine function:** $f(x) = \sin x$

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots, \quad R = \infty$$

- **Cosine function:** $f(x) = \cos x$

$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots, \quad R = \infty$$

- **Natural logarithm:** $f(x) = \ln(1 + x)$, $|x| < 1$

$$\ln(1 + x) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots, \quad R = 1$$

- **Geometric series:** $f(x) = \frac{1}{1-x}$, $|x| < 1$

$$\frac{1}{1 - x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \dots, \quad R = 1$$

- **Arctangent:** $f(x) = \arctan x$, $|x| \leq 1$

$$\arctan x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots, \qquad R = 1$$

- **Binomial series:** $(1+x)^\alpha$, $\quad |x| < 1,\ \alpha \in \mathbb{R}$

$$(1+x)^\alpha = \sum_{n=0}^{\infty} \binom{\alpha}{n} x^n = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2!}x^2 + \frac{\alpha(\alpha-1)(\alpha-2)}{3!}x^3 + \dots, \qquad R = 1$$

where $\binom{\alpha}{n} = \frac{\alpha(\alpha-1)\dots(\alpha-n+1)}{n!}$**.**

4.3.5 Differential Equations and Power Series

Definition 4.23 (Power Series Method for Solving Differential Equations) ***Let us consider a differential equation of the form***

$$y'' + P(x)y' + Q(x)y = 0,$$

where $P(x)$ ***and*** $Q(x)$ ***are analytic functions at*** x_0***.***
We look for a solution $y(x)$ ***in the form of a power series:***

$$y(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n.$$

Procedure:

1. ***Substitute the series for*** $y(x)$***,*** $y'(x)$***, and*** $y''(x)$ ***into the differential equation.***
2. ***Align terms with the same powers of*** $(x - x_0)$***.***
3. ***Obtain a recurrence relation for the coefficients*** a_n***.***
4. ***Solve the recurrence to find*** a_n ***and write the solution as a series.***

Example 4.23 ***Solve the differential equation***

$$y' - y = 0$$

using the power series method around $x_0 = 0$***.***

Solution 4.6 ***Step 1: Assume a power series solution:***

$$y(x) = \sum_{n=0}^{\infty} a_n x^n.$$

Step 2: Compute the derivative:

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n.$$

Step 3: Substitute into the differential equation:

$$y' - y = \sum_{n=0}^{\infty} \left[(n+1)a_{n+1} - a_n\right] x^n = 0.$$

Step 4: Recurrence relation:

$$(n+1)a_{n+1} - a_n = 0 \qquad \Longrightarrow \qquad a_{n+1} = \frac{a_n}{n+1}.$$

Step 5: Solve the recurrence:

$$a_1 = \frac{a_0}{1}, \quad a_2 = \frac{a_1}{2} = \frac{a_0}{2!}, \quad a_3 = \frac{a_2}{3} = \frac{a_0}{3!}, \dots$$

Step 6: Series solution:

$$y(x) = \sum_{n=0}^{\infty} \frac{a_0}{n!} x^n = a_0 \sum_{n=0}^{\infty} \frac{x^n}{n!} = a_0 e^x.$$

Radius of convergence: $R = \infty$***, series converges for all*** $x \in \mathbb{R}$***.***

Example 4.24 ***Solve the differential equation***

$$y'' - xy = 0$$

using a power series around $x_0 = 0$***.***

Solution 4.7 ***Step 1: Assume a power series solution:***

$$y(x) = \sum_{n=0}^{\infty} a_n x^n.$$

Step 2: Compute derivatives:

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}, \quad y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}.$$

Step 3: Substitute into the equation:

$$\begin{aligned} y'' - xy &= \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} - x \sum_{n=0}^{\infty} a_n x^n \\ &= \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} - \sum_{n=0}^{\infty} a_n x^{n+1}. \end{aligned}$$

Step 4: Shift indices to align powers:
- For the first sum, let $k = n-2 \implies n = k+2$***:***

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{k=0}^{\infty} (k+2)(k+1) a_{k+2} x^k.$$

- For the second sum, let** $k = n+1 \implies n = k-1$**:

$$\sum_{n=0}^{\infty} a_n x^{n+1} = \sum_{k=1}^{\infty} a_{k-1} x^k.$$

Step 5: Combine sums:

$$\sum_{k=0}^{\infty} \Big[(k+2)(k+1)a_{k+2} - a_{k-1}\Big] x^k = 0,$$

where we set** $a_{-1} = 0$**.
Step 6: Recurrence relation:

$$a_{k+2} = \frac{a_{k-1}}{(k+2)(k+1)}, \qquad k \geq 0.$$

Step 7: Compute first coefficients:
-** a_0, a_1 **are arbitrary constants. -** $a_2 = a_{-1}/(2 \cdot 1) = 0$**,** $a_3 = a_0/(3 \cdot 2) = a_0/6$**,** $a_4 = a_1/(4 \cdot 3) = a_1/12$**, etc.
Step 8: Series solution:

$$y(x) = a_0 \left(1 + \frac{x^3}{6} + \dots\right) + a_1 \left(x + \frac{x^4}{12} + \dots\right).$$

Remark: This gives a non-trivial solution expressed as a power series around** $x = 0$**. The series converges for all** $x \in \mathbb{R}$ **(infinite radius of convergence).

4.4 Fourier Series

Introduction to Fourier Series

A Fourier series is a way to represent a periodic function as an infinite sum of sines and cosines:

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos(nx) + b_n \sin(nx)\right),$$

where a_n **and** b_n **are called the Fourier coefficients.**
Importance in Physics:

- **Fourier series allow us to decompose complex periodic signals into simple harmonic components.**
- **They are widely used in electrodynamics, quantum mechanics, and wave propagation.**
- **They are essential in solving partial differential equations such as the heat equation, wave equation, and Laplace equation.**
- **They help analyze vibrations, sound waves, and signal processing.**

4.4.1 Fourier Series for a Function of Period T

Definitions 4.1 ***Let $f(t)$ be a periodic function with period $T > 0$, and define the fundamental frequency***

$$\omega_0 = \frac{2\pi}{T}.$$

1. Trigonometric Fourier Series (TFS):

$$f(t) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} \Big(a_n \cos(n\omega_0 t) + b_n \sin(n\omega_0 t) \Big),$$

with coefficients

$$a_n = \frac{2}{T} \int_{-T/2}^{T/2} f(t) \cos(n\omega_0 t)\, dt, \quad b_n = \frac{2}{T} \int_{-T/2}^{T/2} f(t) \sin(n\omega_0 t)\, dt, \quad n \geq 1,$$

and

$$a_0 = \frac{2}{T} \int_{-T/2}^{T/2} f(t)\, dt.$$

2. Exponential Fourier Series (EFS):

$$f(t) \sim \sum_{n=-\infty}^{\infty} c_n e^{in\omega_0 t},$$

with coefficients

$$c_n = \frac{1}{T} \int_{-T/2}^{T/2} f(t) e^{-in\omega_0 t}\, dt, \quad n \in \mathbb{Z}.$$

Remark 4.8 ***: Both series are equivalent. The EFS is often more convenient in engineering, while the TFS is more intuitive in physics.***

Example 4.25 ***Find the Fourier series of the periodic function***

$$f(t) = t, \quad -1 < t < 1,$$

with period $T = 2$.

Solution 4.8 ***Step 1: Fundamental frequency***

$$\omega_0 = \frac{2\pi}{T} = \pi.$$

Step 2: Compute a_0:

$$a_0 = \frac{2}{T} \int_{-1}^{1} t\, dt = \frac{2}{2} \int_{-1}^{1} t\, dt = 0.$$

Step 3: Compute a_n:

$$a_n = \frac{2}{2} \int_{-1}^{1} t \cos(n\pi t)\, dt = \int_{-1}^{1} t \cos(n\pi t)\, dt = 0 \quad (\textbf{\textit{odd function}}).$$

Step 4: Compute b_n:

$$b_n = \int_{-1}^{1} t\sin(n\pi t)\,dt = 2\int_{0}^{1} t\sin(n\pi t)\,dt$$

$$b_n = 2\left[-\frac{t\cos(n\pi t)}{n\pi}\Big|_0^1 + \frac{1}{n\pi}\int_0^1 \cos(n\pi t)\,dt\right] = \frac{2(-1)^{n+1}}{n\pi}.$$

Step 5: Fourier series:

$$f(t) \sim \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n\pi}\sin(n\pi t).$$

Step 6: Exponential form (optional):

$$f(t) \sim \sum_{n=-\infty}^{\infty} c_n e^{in\pi t}, \quad c_n = \frac{(-1)^{n+1}}{in\pi},\ n \neq 0, \quad c_0 = 0.$$

4.4.2 Polar (Amplitude-Phase) Form of Fourier Series

Let $f(t)$ have period T and fundamental frequency $\omega_0 = \frac{2\pi}{T}$. If the Fourier series of $f(t)$ is

$$f(t) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n\cos(n\omega_0 t) + b_n\sin(n\omega_0 t)\right),$$

we can rewrite each term as a single cosine with amplitude and phase:

$$a_n\cos(n\omega_0 t) + b_n\sin(n\omega_0 t) = A_n\cos(n\omega_0 t - \varphi_n),$$

where

$$A_n = \sqrt{a_n^2 + b_n^2}, \quad \varphi_n = \arctan\left(\frac{b_n}{a_n}\right).$$

Thus, the Fourier series in polar form becomes:

$$f(t) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} A_n\cos(n\omega_0 t - \varphi_n).$$

Remark 4.9

- ***A_n represents the amplitude of the n-th harmonic.***
- ***φ_n represents the phase shift of the n-th harmonic.***
- ***This form is particularly useful in physics and engineering to analyze signals in terms of **amplitude and phase** rather than separate sine and cosine components.***

Example 4.26 ***Consider the periodic function $f(t) = t$ on $[-1, 1]$ with period $T = 2$. We already know its Fourier series (trigonometric form):***

$$f(t) \sim \sum_{n=1}^{\infty} b_n\sin(n\pi t), \quad b_n = \frac{2(-1)^{n+1}}{n\pi}.$$

Solution 4.9 ***Step 1: Identify a_n and b_n: Since $f(t)$ is an odd function, all $a_n = 0$. Thus,***

$$a_n = 0, \quad b_n = \frac{2(-1)^{n+1}}{n\pi}.$$

Step 2: Compute amplitude A_n and phase φ_n:

$$A_n = \sqrt{a_n^2 + b_n^2} = |b_n| = \frac{2}{n\pi},$$

$$\varphi_n = \arctan\left(\frac{b_n}{a_n}\right) = \arctan(\infty) = \frac{\pi}{2} \quad (\textit{\textbf{since}}\ b_n > 0\ \textit{\textbf{for odd}}\ n).$$

Step 3: Write polar form:

$$f(t) \sim \sum_{n=1}^{\infty} A_n \cos(n\pi t - \varphi_n) = \sum_{n=1}^{\infty} \frac{2}{n\pi} \cos\left(n\pi t - \frac{\pi}{2}\right).$$

4.4.3 Computation of Fourier Coefficients for Even and Odd Functions

Let $f(t)$ be a periodic function with period T and fundamental frequency $\omega_0 = \frac{2\pi}{T}$.

1. Trigonometric Fourier Series (TFS)

$$f(t) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos(n\omega_0 t) + b_n \sin(n\omega_0 t)\right),$$

with

$$a_n = \frac{2}{T} \int_{-T/2}^{T/2} f(t) \cos(n\omega_0 t)\, dt, \quad b_n = \frac{2}{T} \int_{-T/2}^{T/2} f(t) \sin(n\omega_0 t)\, dt.$$

Case 1: $f(t)$ is even ($f(-t) = f(t)$)

$$b_n = 0, \quad a_n = \frac{4}{T} \int_0^{T/2} f(t) \cos(n\omega_0 t)\, dt.$$

Case 2: $f(t)$ is odd ($f(-t) = -f(t)$)

$$a_n = 0, \quad b_n = \frac{4}{T} \int_0^{T/2} f(t) \sin(n\omega_0 t)\, dt.$$

2. Exponential Fourier Series (EFS)

$$f(t) \sim \sum_{n=-\infty}^{\infty} c_n e^{in\omega_0 t}, \quad c_n = \frac{1}{T} \int_{-T/2}^{T/2} f(t) e^{-in\omega_0 t}\, dt.$$

Simplifications:

- **If $f(t)$ is even: $c_{-n} = c_n$ and c_n is real.**
- **If $f(t)$ is odd: $c_{-n} = -c_n$ and c_n is purely imaginary.**

Remark 4.10 ***: Exploiting symmetry reduces the integration interval to*** $[0, T/2]$ ***and simplifies the calculation of Fourier coefficients.***

Example 4.27 ***1. Even function:*** $f(t) = |t|$***, defined on*** $[-1, 1]$ ***with period*** $T = 2$***.***
2. Odd function: $f(t) = t$***, defined on*** $[-1, 1]$ ***with period*** $T = 2$***.***

Solution 4.10 ***Case 1: Even function*** $f(t) = |t|$
Step 1: Compute a_0***:***

$$a_0 = \frac{2}{T} \int_{-1}^{1} |t| \, dt = \int_{-1}^{1} |t| \, dt = 1.$$

Step 2: Compute a_n ***(***$b_n = 0$ ***since function is even):***

$$a_n = 2 \int_{0}^{1} t \cos(n\pi t) \, dt = 2 \int_{0}^{1} t \cos(n\pi t) \, dt$$

$$a_n = 2 \left[\frac{\sin(n\pi t)}{n\pi} + \frac{\cos(n\pi t)}{(n\pi)^2} \Big|_0^1 \right] = \frac{2((-1)^n - 1)}{(n\pi)^2}, \qquad b_n = 0.$$

Step 3: Exponential coefficients c_n***:***

$$c_0 = \frac{a_0}{2} = \frac{1}{2}, \qquad c_n = \frac{a_n}{2}, \; n \neq 0 \quad (\textbf{\textit{real since function is even}}).$$

Case 2: Odd function $f(t) = t$
Step 1: Compute a_0 ***and*** a_n ***(***$a_0 = a_n = 0$***)***
Step 2: Compute b_n***:***

$$b_n = 2 \int_{0}^{1} t \sin(n\pi t) \, dt = 2 \left[-\frac{t \cos(n\pi t)}{n\pi} + \frac{\sin(n\pi t)}{(n\pi)^2} \Big|_0^1 \right] = \frac{2(-1)^{n+1}}{n\pi}.$$

Step 3: Exponential coefficients c_n***:***

$$c_n = \frac{-ib_n}{2} = \frac{(-1)^n i}{n\pi}, \qquad c_{-n} = -c_n, \qquad c_0 = 0.$$

Conclusion:
- Even functions: only cosine terms (a_n***) are nonzero.***
- Odd functions: only sine terms (b_n***) are nonzero.***
- The exponential coefficients c_n ***reflect this symmetry (real for even, imaginary for odd).***

3. Conditions for the Existence of a Fourier Series

Let $f(t)$ be a function of period T. A Fourier series (trigonometric or exponential) exists if $f(t)$ satisfies the following Dirichlet conditions:

1. **$f(t)$ is periodic with period T.**
2. **$f(t)$ is piecewise continuous on one period $[0, T]$ (or $[-T/2, T/2]$), i.e., it has a finite number of finite discontinuities.**

3. $f(t)$ **has a finite number of maxima and minima in one period.**

4. $f(t)$ **is absolutely integrable over one period:**

$$\int_0^T |f(t)|\, dt < \infty.$$

Remark:

- **If these conditions are satisfied, the Fourier series converges to** $f(t)$ **at all points where** f **is continuous.**
- **At points of discontinuity, the series converges to the average of the left- and right-hand limits:**
$$\frac{f(t^+) + f(t^-)}{2}.$$
- **These conditions apply to both trigonometric and exponential Fourier series.**

4.5 Solved Exercises

Exercise 4.1 (Positive Term Series) ***Determine whether the series***

$$\sum_{n=1}^{\infty} \frac{1}{n \ln n}$$

converges or diverges.

Solution 4.11 ***We apply the Cauchy integral test. Consider the function***

$$f(x) = \frac{1}{x \ln x}, \qquad x \geq 2,$$

which is continuous, positive, and decreasing.
Compute the integral:

$$\int_2^{\infty} \frac{dx}{x \ln x}.$$

- Substitute $t = \ln x \Rightarrow dt = \frac{dx}{x}$***. - Then the integral becomes***

$$\int_{\ln 2}^{\infty} \frac{dt}{t} = \lim_{T \to \infty} \int_{\ln 2}^{T} \frac{dt}{t} = \lim_{T \to \infty} (\ln T - \ln(\ln 2)) = \infty.$$

Since the integral diverges, by the integral test, the series

$$\sum_{n=2}^{\infty} \frac{1}{n \ln n}$$

also diverges.
Conclusion: The series diverges.

Exercise 4.2 ***Consider a vibrating string whose deformation in mode n has an amplitude proportional to***

$$a_n = \frac{(-1)^n}{n^{3/2}} \sin\Big(\omega_0 n\Big), \qquad n = 1, 2, 3, \ldots$$

where $\omega_0 > 0$ is a constant. The total energy associated with the amplitudes is formally given by the series

$$E = \sum_{n=1}^{\infty} a_n^2.$$

1) Determine whether the series defining E converges. Justify your answer by applying an appropriate convergence test.
2) Study the convergence of the alternating series

$$S = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^{3/2}} \sin\Big(\omega_0 n\Big).$$

3) For $\omega_0 = \pi$, simplify a_n and conclude about the convergence (absolute or conditional) of the series E and S in this particular case.

Solution 4.12 ***Initial remark. We are given***

$$a_n = \frac{(-1)^n}{n^{3/2}} \sin(\omega_0 n), \qquad n \geq 1,$$

hence

$$a_n^2 = \frac{\sin^2(\omega_0 n)}{n^3}.$$

1) Convergence of $E = \sum_{n=1}^{\infty} a_n^2$.
We study

$$E = \sum_{n=1}^{\infty} \frac{\sin^2(\omega_0 n)}{n^3}.$$

For every integer n we have

$$0 \leq \sin^2(\omega_0 n) \leq 1,$$

so

$$0 \leq \frac{\sin^2(\omega_0 n)}{n^3} \leq \frac{1}{n^3}.$$

The comparison series

$$\sum_{n=1}^{\infty} \frac{1}{n^3}$$

is a p-series with $p = 3 > 1$, therefore it converges. By the comparison test for nonnegative terms, it follows that

$$\sum_{n=1}^{\infty} \frac{\sin^2(\omega_0 n)}{n^3} < \infty.$$

$$\boxed{E \text{ converges (indeed absolutely, since its terms are nonnegative).}}$$

2) Convergence of

$$S = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^{3/2}} \sin(\omega_0 n).$$

We first test for absolute convergence by considering

$$\sum_{n=1}^{\infty} \left| \frac{(-1)^n}{n^{3/2}} \sin(\omega_0 n) \right| = \sum_{n=1}^{\infty} \frac{|\sin(\omega_0 n)|}{n^{3/2}}.$$

Since $0 \le |\sin(\omega_0 n)| \le 1$ ***for every*** n***, we have***

$$0 \le \frac{|\sin(\omega_0 n)|}{n^{3/2}} \le \frac{1}{n^{3/2}}.$$

The series $\sum_{n=1}^{\infty} n^{-3/2}$ ***is a*** p***-series with*** $p = \frac{3}{2} > 1$***, hence it converges. By comparison, the series of absolute values converges:***

$$\sum_{n=1}^{\infty} \frac{|\sin(\omega_0 n)|}{n^{3/2}} < \infty.$$

Therefore the original series S ***converges absolutely, and so it converges.***

$$\boxed{S \text{ converges absolutely.}}$$

3) Special case $\omega_0 = \pi$***.***
For every integer n***,***

$$\sin(\pi n) = 0.$$

Thus

$$a_n = \frac{(-1)^n}{n^{3/2}} \sin(\pi n) = 0 \quad \textit{for all } n.$$

Consequently both series reduce to the zero series:

$$E = \sum_{n=1}^{\infty} 0 = 0, \qquad S = \sum_{n=1}^{\infty} 0 = 0,$$

which trivially converge (absolutely).

$$\boxed{\text{For } \omega_0 = \pi, \ E = 0 \text{ and } S = 0.}$$

Remarks. The decay $n^{-3/2}$ ***of the amplitudes guarantees that the energy series (involving*** a_n^2***) converges because it behaves like a*** p***-series with exponent*** 3***. The absolute convergence of*** S ***follows from comparison with the*** p***-series of exponent*** $3/2$***.***

Exercise 4.3 *study the convergence of the series*

$$\sum_{n=1}^{\infty} u_n(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^2}, \qquad x \in \mathbb{R}.$$

Solution 4.13 ***We study the absolute convergence of the series. The general term is***

$$u_n(x) = \frac{x^n}{n^2}.$$

Applying the ratio test to $|u_n(x)|$***:***

$$\lim_{n\to\infty} \frac{|u_{n+1}(x)|}{|u_n(x)|} = \lim_{n\to\infty} \frac{|x|^{n+1}/(n+1)^2}{|x|^n/n^2} = \lim_{n\to\infty} |x| \cdot \frac{n^2}{(n+1)^2} = |x|.$$

- If $|x| < 1$***, the limit is less than 1, so the series converges absolutely.***
- If $|x| > 1$***, the limit is greater than 1, so the series diverges.***
- If $|x| = 1$***, the series becomes***

$$\sum_{n=1}^{\infty} \frac{1}{n^2},$$

which converges (p-series with $p = 2 > 1$***).***
Domain of Convergence: Denote by D ***the set of values of*** x ***for which the series converges. Then***

$$D = \{x \in \mathbb{R} \mid |x| \leq 1\} = [-1\,;1]$$

Exercise 4.4 ***Solve the differential equation***

$$y'' + x^2 y = 0$$

around $x_0 = 0$ ***using a power series solution.***

Solution 4.14 ***Step 1: Assume a power series solution:***

$$y(x) = \sum_{n=0}^{\infty} a_n x^n.$$

Step 2: Compute derivatives:

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}, \quad y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}.$$

Step 3: Substitute into the differential equation:

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + x^2 \sum_{n=0}^{\infty} a_n x^n = 0 \quad \Longrightarrow \quad \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + \sum_{n=0}^{\infty} a_n x^{n+2} = 0.$$

Step 4: Shift indices to align powers: - For the first sum, let $k = n - 2 \implies n = k + 2$***:***

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{k=0}^{\infty} (k+2)(k+1) a_{k+2} x^k.$$

- For the second sum, let $k = n + 2 \implies n = k - 2$***:***

$$\sum_{n=0}^{\infty} a_n x^{n+2} = \sum_{k=2}^{\infty} a_{k-2} x^k.$$

Step 5: Combine sums:

$$\sum_{k=0}^{\infty} \Big[(k+2)(k+1)a_{k+2} + a_{k-2}\Big] x^k = 0,$$

where we define $a_{-2} = a_{-1} = 0$***.***
Step 6: Recurrence relation:

$$a_{k+2} = -\frac{a_{k-2}}{(k+2)(k+1)}, \qquad k \geq 0.$$

Step 7: Compute first coefficients: - a_0, a_1 ***are arbitrary constants. - Using the recurrence:***

$$a_2 = -\frac{a_{-2}}{2 \cdot 1} = 0, \quad a_3 = -\frac{a_{-1}}{3 \cdot 2} = 0, \quad a_4 = -\frac{a_0}{4 \cdot 3} = -\frac{a_0}{12}, \quad a_5 = -\frac{a_1}{5 \cdot 4} = -\frac{a_1}{20}, \dots$$

Step 8: Series solution:

$$y(x) = a_0 \left(1 - \frac{x^4}{12} + \dots\right) + a_1 \left(x - \frac{x^5}{20} + \dots\right).$$

Remark: This gives a non-trivial solution expressed as a power series around $x = 0$***. The series converges for all*** $x \in \mathbb{R}$ ***(infinite radius of convergence).***

Exercise 4.5 ***Let*** $f(t)$ ***be a periodic function of period*** T ***with trigonometric Fourier series***

$$f(t) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} \Big(a_n \cos(n\omega_0 t) + b_n \sin(n\omega_0 t)\Big), \qquad \omega_0 = \frac{2\pi}{T}.$$

1. ***Using the definition of the exponential Fourier series***

$$f(t) \sim \sum_{n=-\infty}^{\infty} c_n e^{in\omega_0 t}, \qquad c_n = \frac{1}{T}\int_{-T/2}^{T/2} f(t) e^{-in\omega_0 t}\, dt,$$

express c_n ***in terms of*** a_n ***and*** b_n ***for*** $n \geq 1$ ***and*** $n = 0$***. Also express*** c_{-n} ***in terms of*** a_n ***and*** b_n***.***

Solution 4.15 ***Step 1: Use the identity:***

$$e^{-in\omega_0 t} = \cos(n\omega_0 t) - i \sin(n\omega_0 t).$$

Step 2: Substitute into c_n***:***

$$c_n = \frac{1}{T}\int_{-T/2}^{T/2} f(t)(\cos(n\omega_0 t) - i \sin(n\omega_0 t))\, dt = \frac{1}{T}\int_{-T/2}^{T/2} f(t) \cos(n\omega_0 t)\, dt - i\frac{1}{T}\int_{-T/2}^{T/2} f(t) \sin(n\omega_0 t)\, dt.$$

Step 3: Express in terms of a_n ***and*** b_n***:***

$$\frac{1}{T}\int_{-T/2}^{T/2} f(t) \cos(n\omega_0 t)\, dt = \frac{a_n}{2}, \qquad \frac{1}{T}\int_{-T/2}^{T/2} f(t) \sin(n\omega_0 t)\, dt = \frac{b_n}{2}.$$

Step 4: Final relation:

$$c_n = \frac{1}{2}(a_n - ib_n), \quad n \geq 1, \quad c_0 = \frac{a_0}{2}, \quad c_{-n} = \frac{1}{2}(a_n + ib_n), \quad n \geq 1.$$

Chapter 5

Laplace Transform

5.1 Definition and Properties of the Laplace Transform

Definition 5.1 ***The Laplace transform of a function*** $f(t)$***, defined for*** $t \geq 0$***, is the function*** $F(s)$ ***given by***

$$\mathcal{L}\{f(t)\} = F(s) = \int_0^\infty f(t)e^{-st}\,dt,$$

where s ***is a complex number such that the integral converges.***

Remark 5.1 ***The Laplace transform exists for all functions*** $f(t)$ ***of exponential order; that is, there exist constants*** $M, \alpha > 0$ ***such that*** $|f(t)| \leq Me^{\alpha t}$ ***for all*** $t \geq 0$***.***

Example 5.1 ***Find the Laplace transform of*** $f(t) = e^{2t}$***, for*** $t \geq 0$***.***

Solution 5.1 ***By definition of the Laplace transform:***

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{2t}e^{-st}\,dt = \int_0^\infty e^{-(s-2)t}\,dt$$

Step 1: Condition for convergence
Let $s = \sigma + i\omega$***, with*** $\sigma = \Re(s)$ ***and*** $\omega = \Im(s)$***. Then***

$$e^{-(s-2)t} = e^{-(\sigma-2)t}\,e^{-i\omega t}$$

The magnitude of the integrand is:

$$|e^{-(s-2)t}| = |e^{-(\sigma-2)t}e^{-i\omega t}| = e^{-(\sigma-2)t} \quad (\textbf{\textit{since}}\ |e^{-i\omega t}| = 1)$$

For the improper integral to converge as $t \to \infty$***, the exponent must be negative:***

$$-(\sigma - 2) < 0 \quad \Rightarrow \quad \sigma - 2 > 0 \quad \Rightarrow \quad \sigma > 2$$

Step 2: Compute the integral
If $\Re(s) > 2$***:***

$$\int_0^\infty e^{-(s-2)t}\,dt = \left[\frac{e^{-(s-2)t}}{-(s-2)}\right]_0^\infty = \frac{1}{s-2}$$

$$\Rightarrow \mathcal{L}\{e^{2t}\} = \frac{1}{s-2}, \quad \Re(s) > 2$$

5.1.1 Properties of the Laplace Transform

1. Linearity:

$$\mathcal{L}\{\big(af(t)+bg(t)\big)\} = a\mathcal{L}\{\big(f(t)\big)\} + b\mathcal{L}\{\big(g(t)\big)\}$$

Example 5.2 ***Let*** $f(t) = e^t$ ***and*** $g(t) = \sin(t)$***, with*** $a = 2$***,*** $b = 3$***.***
Step 1: Compute $\mathcal{L}\{e^t\}$

$$\mathcal{L}\{e^t\} = \int_0^\infty e^t e^{-st} dt = \int_0^\infty e^{-(s-1)t} dt$$

Converges for $\Re(s) > 1$***:***

$$\int_0^\infty e^{-(s-1)t} dt = \frac{1}{s-1}$$

Step 2: Compute $\mathcal{L}\{\sin(t)\}$

$$\mathcal{L}\{\sin(t)\} = \int_0^\infty \sin(t) e^{-st} dt = \frac{1}{s^2+1}, \quad \Re(s) > 0$$

Step 3: Apply linearity:

$$\mathcal{L}\{2e^t + 3\sin(t)\} = 2 \cdot \frac{1}{s-1} + 3 \cdot \frac{1}{s^2+1} = \frac{2}{s-1} + \frac{3}{s^2+1}$$

2. First Derivative:

$$\mathcal{L}\{f'(t)\} = s\mathcal{L}\{f(t)\} - f(0)$$

Example 5.3 ***Let*** $f(t) = t^2$***. Then*** $f'(t) = 2t$***,*** $f(0) = 0$***.***
Step 1: Compute $\mathcal{L}\{t^2\}$

$$\mathcal{L}\{t^2\} = \int_0^\infty t^2 e^{-st} dt = \frac{2}{s^3}, \quad s > 0$$

Step 2: Use derivative property:

$$\mathcal{L}\{2t\} = \mathcal{L}\{f'(t)\} = s \cdot \frac{2}{s^3} - 0 = \frac{2}{s^2}$$

Step 3: Verify directly:

$$\mathcal{L}\{2t\} = 2\int_0^\infty t e^{-st} dt = 2 \cdot \frac{1}{s^2} = \frac{2}{s^2}$$

3. Second Derivative:

$$\mathcal{L}\{f''(t)\} = s^2\mathcal{L}\{f(t)\} - sf(0) - f'(0)$$

Example 5.4 ***Let*** $f(t) = t^2$***. Then*** $f''(t) = 2$***,*** $f(0) = 0$***,*** $f'(0) = 0$***.***
Step 1: Compute $\mathcal{L}\{t^2\}$

$$\mathcal{L}\{t^2\} = \frac{2}{s^3}, \quad s > 0$$

Step 2: Apply formula:

$$\mathcal{L}\{f''(t)\} = s^2 \cdot \frac{2}{s^3} - 0 - 0 = \frac{2}{s}$$

Step 3: Direct check:

$$\mathcal{L}\{2\} = \int_0^\infty 2e^{-st}dt = \frac{2}{s}, \quad s > 0$$

4. Integration:

$$\mathcal{L}\left\{\int_0^t f(\tau)d\tau\right\} = \frac{\mathcal{L}\{f(t)\}}{s}$$

Example 5.5 ***Let*** $f(t) = e^{2t}$***. Then*** $\int_0^t e^{2\tau}d\tau = \frac{1}{2}(e^{2t} - 1)$***.***
Step 1: Compute $\mathcal{L}\{e^{2t}\} = \frac{1}{s-2}, \quad \Re(s) > 2$
Step 2: Apply integration property:

$$\mathcal{L}\left\{\int_0^t e^{2\tau}d\tau\right\} = \frac{\mathcal{L}\{e^{2t}\}}{s} = \frac{1/(s-2)}{s} = \frac{1}{s(s-2)}, \quad \Re(s) > 2$$

5. Shifting Theorem:

The result of the Shifting Theorem comes directly from the definition of the Laplace transform and a simple change of variable in the integral. Formally:

Theorem 5.1 (Exponential Shift / First Shifting Theorem) ***Let*** $f(t)$ ***be a function whose Laplace transform exists, and let***

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty f(t)e^{-st}\,dt$$

be the Laplace transform of $f(t)$***. Then, for any real constant*** a***:***

$$\mathcal{L}\{f(t)e^{at}\} = \mathcal{L}\{e^{at}f(t)\} = F(s-a)$$

Proof. By definition of the Laplace transform:

$$\mathcal{L}\{e^{at}f(t)\} = \int_0^\infty e^{at}f(t)e^{-st}dt = \int_0^\infty f(t)e^{-(s-a)t}dt = F(s-a)$$

Convergence condition: If $F(s)$ **converges for** $\Re(s) > \alpha$**, then** $F(s-a)$ **converges for** $\Re(s) > \alpha + a$**.** ■

Example 5.6 ***Let*** $f(t) = t$***. Then***

$$F(s) = \mathcal{L}\{t\} = \int_0^\infty te^{-st}dt = \frac{1}{s^2}, \quad s > 0$$

Take $a = 3$***:***

$$\mathcal{L}\{te^{3t}\} = F(s-3) = \frac{1}{(s-3)^2}, \quad \Re(s) > 3$$

Step-by-step explanation:
i. Compute the Laplace transform of $f(t)$ ***without the exponential:*** $F(s) = 1/s^2$***.***
ii. Apply the exponential shift theorem: replace s ***by*** $s - a = s - 3$ ***in*** $F(s)$***.***
iii. Adjust the convergence region: originally $\Re(s) > 0$***, now*** $\Re(s) > 3$***.***

6. Time Shift (Translation in the Time Domain)

Definition 5.2 (Heaviside Step Function) ***The Heaviside step function*** $u(t-a)$ ***is defined as:***

$$u(t-a) = \begin{cases} 0, & t < a \\ 1, & t \geq a \end{cases}$$

It "turns on" a function at $t = a$ ***and ensures causality for the Laplace transform.***

Theorem 5.2 (Time Shift Property) ***Let*** $f(t)$ ***be a function whose Laplace transform exists. For a real constant*** $a > 0$***, the Laplace transform of the shifted function*** $f(t-a)$ ***multiplied by the Heaviside function*** $u(t-a)$ ***is:***

$$\mathcal{L}\{f(t-a)\,u(t-a)\} = e^{-as}\,F(s),$$

where

$$\mathcal{L}\{f(t)\}(s) = F(s) = \int_0^\infty f(t)e^{-st}\,dt, \quad \Re(s) > 0.$$

Remark 5.2 ***The Heaviside function ensures that the shifted function starts at*** $t = a$***:***

$$f(t-a)\,u(t-a) = \begin{cases} 0, & t < a \\ f(t-a), & t \geq a \end{cases}$$

Proof. **By definition,**

$$\mathcal{L}\{f(t-a)u(t-a)\}(s) = \int_0^\infty f(t-a)u(t-a)e^{-st}\,dt.$$

Since $u(t-a) = 0$ **for** $t < a$**, the integral reduces to**

$$= \int_a^\infty f(t-a)e^{-st}\,dt.$$

With the substitution $\tau = t - a$, $t = \tau + a$, $dt = d\tau$**, we obtain**

$$= \int_0^\infty f(\tau)e^{-s(\tau+a)}\,d\tau = e^{-as}\int_0^\infty f(\tau)e^{-s\tau}\,d\tau.$$

Therefore,

$$\mathcal{L}\{f(t-a)u(t-a)\}(s) = e^{-as}F(s).$$

■

Example 5.7 ***Let*** $f(t) = \sin t$***. Its Laplace transform is***

$$F(s) = \mathcal{L}\{\sin t\}(s) = \frac{1}{s^2+1}.$$

Now consider $g(t) = \sin(t-a)u(t-a)$***. By the time shifting property,***

$$\mathcal{L}\{g(t)\}(s) = e^{-as}F(s) = \frac{e^{-as}}{s^2+1}.$$

Particular Case **For** $a = 2$**, we obtain**

$$\mathcal{L}\{\sin(t-2)u(t-2)\}(s) = \frac{e^{-2s}}{s^2+1}.$$

Example 5.8 ***Let*** $f(t) = t$***. Find*** $\mathcal{L}\{f(t-2)u(t-2)\}$***.***

Solution 5.2 We know that $\mathcal{L}\{f(t)\} = \mathcal{L}\{t\} = \frac{1}{s^2}$***.***

$$f(t-2)\,u(t-2) = \begin{cases} 0, & t < 2 \\ f(t-a), & t \geq 2 \end{cases}$$

By the time shift property:

$$\mathcal{L}\{f(t-2)u(t-2)\} = \mathcal{L}\{f(t-2)\} = e^{-2s}\mathcal{L}\{f(t)\} = e^{-2s} \cdot \frac{1}{s^2}, \qquad t \geq 2,$$

where

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{t\} = \frac{1}{s^2}$$

So the final result is:

$$\mathcal{L}\{(t-2)\} = \frac{e^{-2s}}{s^2}, \qquad t \geq 2.$$

Explanation:

- ***For*** $t < 2$***, the function is zero.***
- ***For*** $t \geq 2$***, the function grows linearly starting from zero at*** $t = 2$***.***
- ***The multiplication by*** e^{-2s} ***in the Laplace domain corresponds to the shift by 2 units in the time domain.***

7. Dilation and Concentration Property of the Laplace Transform

Let

$$F(s) = \mathcal{L}\{f(t)\}(s) = \int_0^\infty e^{-st} f(t)\,dt.$$

For $a > 0$,

$$\mathcal{L}\{f(at)\}(s) = \frac{1}{a}F\left(\frac{s}{a}\right).$$

- If $a > 1$: the function $f(at)$ is concentrated in the time domain (compressed toward $t = 0$).
- If $0 < a < 1$: the function $f(at)$ is dilated (stretched in time).
Proof.

$$\mathcal{L}\{f(at)\}(s) = \int_0^\infty e^{-st} f(at)\,dt.$$

$$\text{Let } u = at \quad \Rightarrow \quad t = \tfrac{u}{a},\ dt = \tfrac{du}{a}.$$

$$= \int_0^\infty e^{-s\frac{u}{a}} f(u) \cdot \frac{du}{a}.$$

$$= \frac{1}{a}\int_0^\infty e^{-\frac{s}{a}u} f(u)\,du.$$

$$= \frac{1}{a}F\left(\frac{s}{a}\right).$$

■

Example 5.9 Concentration ($a > 1$)
Take $f(t) = e^{-t}$. Its Laplace transform is:

$$F(s) = \frac{1}{s+1}.$$

Now compute for $f(2t) = e^{-2t}$:

$$\mathcal{L}\{e^{-2t}\}(s) = \frac{1}{2}F\left(\frac{s}{2}\right) = \frac{1}{2}\cdot\frac{1}{\frac{s}{2}+1} = \frac{1}{s+2}.$$

Dilation ($0 < a < 1$)
Take $f(t) = e^{-t}$. Again,

$$F(s) = \frac{1}{s+1}.$$

Now compute for $f\left(\frac{t}{2}\right) = e^{-\frac{t}{2}}$:

$$\mathcal{L}\{e^{-\frac{t}{2}}\}(s) = \frac{1}{\frac{1}{2}}F(2s) = 2\cdot\frac{1}{2s+1} = \frac{2}{2s+1}.$$

Conclusion
- For $a > 1$, the signal is compressed in time (concentration).
- For $0 < a < 1$, the signal is stretched in time (dilation).

8. Derivative of the Laplace Transform

Let $f : [0,\infty) \to \mathbb{C}$ be a function of *exponential order*: there exist constants $M > 0$ and $k \in \mathbb{R}$ such that

$$|f(t)| \leq Me^{kt} \quad \textbf{for all } t \geq 0.$$

Then the Laplace transform

$$F(s) = \mathcal{L}\{f\}(s) = \int_0^\infty f(t)e^{-st}\,dt$$

exists for all s with $\Re(s) > k$. Under these hypotheses (and sufficient regularity conditions on f), we can differentiate $F(s)$ with respect to s by differentiating under the integral sign.

First derivative

Starting from the definition,

$$F(s) = \int_0^\infty f(t)e^{-st}\,dt,$$

we compute the derivative with respect to s:

$$\begin{aligned} F'(s) &= \frac{d}{ds}\int_0^\infty f(t)e^{-st}\,dt = \int_0^\infty f(t)\,\frac{d}{ds}\Big(e^{-st}\Big)\,dt \\ &= \int_0^\infty f(t)\,(-t)e^{-st}\,dt = -\int_0^\infty t\,f(t)\,e^{-st}\,dt. \end{aligned}$$

Thus we obtain the fundamental relation

$$\boxed{F'(s) = -\mathcal{L}\{t\,f(t)\}(s)} \qquad \Big(\Re(s) > k\Big).$$

General formula for the n-th derivative

By iterating this process, for any integer $n \geq 1$ we have

$$\boxed{F^{(n)}(s) = (-1)^n \int_0^\infty t^n f(t)\,e^{-st}\,dt = (-1)^n\,\mathcal{L}\{t^n f(t)\}(s)}$$

or equivalently

$$\boxed{\mathcal{L}\{t^n f(t)\}(s) = (-1)^n F^{(n)}(s)\,.}$$

Example 5.10 ***Compute $\mathcal{L}\{t^2 \sin t\}(s)$ for $\Re(s) > 0$.***

Solution 5.3 ***We use the general formula***

$$\mathcal{L}\{t^n f(t)\}(s) = (-1)^n F^{(n)}(s),$$

with $f(t) = \sin t$ and $F(s) = \mathcal{L}\{\sin t\}(s) = \dfrac{1}{s^2+1}$ (valid for $\Re(s) > 0$).
Compute derivatives of $F(s)$:

$$F(s) = (s^2+1)^{-1}.$$

First derivative:

$$F'(s) = -\frac{2s}{(s^2+1)^2}.$$

Second derivative:

$$F''(s) = -2(s^2+1)^{-2} + 8s^2(s^2+1)^{-3}$$
$$= \frac{-2(s^2+1)+8s^2}{(s^2+1)^3}$$
$$= \frac{-2+6s^2}{(s^2+1)^3}$$
$$= -2\,\frac{1-3s^2}{(s^2+1)^3}.$$

Finally, by the formula with $n=2$***:***

$$\mathcal{L}\{t^2 \sin t\}(s) = (-1)^2 F''(s) = F''(s).$$

Therefore,

$$\boxed{\mathcal{L}\{t^2 \sin t\}(s) = -2\,\frac{1-3s^2}{(s^2+1)^3}, \qquad \Re(s) > 0.}$$

9. Initial and Final Value Theorems

Theorem 5.3 (Initial Value Theorem (IVT)) ***Let*** $f(t)$ ***be a causal function (i.e.,*** $f(t)=0$ ***for*** $t<0$***), piecewise continuous on*** $[0,\infty)$***, and of exponential order (that is, there exist constants*** $M>0$ ***and*** $\alpha \geq 0$ ***such that*** $|f(t)| \leq Me^{\alpha t}$ ***for all*** $t \geq 0$***). If its Laplace transform is***

$$F(s) = \mathcal{L}\{f(t)\}(s) = \int_0^{\infty} f(t)e^{-st}\,dt,$$

then

$$\boxed{\lim_{t\to 0^+} f(t) = \lim_{s\to\infty} sF(s).}$$

Example 5.11 ***Let*** $f(t) = \sin t$***. Then***

$$F(s) = \frac{1}{s^2+1}.$$

Compute:

$$\lim_{s\to\infty} sF(s) = \lim_{s\to\infty} \frac{s}{s^2+1} = 0.$$

On the other hand,

$$\lim_{t\to 0^+} \sin t = 0.$$

Thus, the theorem is verified.

Theorem 5.4 (Final Value Theorem (FVT)) ***Let*** $f(t)$ ***satisfy the same conditions as above (causal, piecewise continuous, and of exponential order, i.e.*** $|f(t)| \leq Me^{\alpha t}$***). If*** $\lim_{t\to\infty} f(t)$ ***exists and all poles of*** $sF(s)$ ***have negative real parts (except possibly at*** $s=0$***), then***

$$\boxed{\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s).}$$

Example 5.12 ***Let*** $f(t) = 1 - e^{-2t}$***. Then***

$$F(s) = \frac{1}{s} - \frac{1}{s+2}.$$

Compute:

$$\lim_{s\to 0} sF(s) = \lim_{s\to 0}\left(1 - \frac{s}{s+2}\right) = 1.$$

On the other hand,

$$\lim_{t\to\infty}\left(1 - e^{-2t}\right) = 1.$$

Thus, the theorem is verified.

10. Laplace Transform of a Convolution

Definition 5.3 (Definition of Convolution) ***Let*** $f(t)$ ***and*** $g(t)$ ***be two causal functions (i.e.,*** $f(t) = g(t) = 0$ ***for*** $t < 0$***). Their convolution is defined by***

$$(f * g)(t) = \int_0^t f(\tau)\, g(t-\tau)\, d\tau, \qquad t \geq 0.$$

Theorem 5.5 (Convolution Theorem) ***If*** $F(s) = \mathcal{L}\{f(t)\}(s)$ ***and*** $G(s) = \mathcal{L}\{g(t)\}(s)$***, then***

$$\boxed{\mathcal{L}\{(f * g)(t)\}(s) = F(s)\, G(s).}$$

Proof. Start from the definition of the Laplace transform:

$$\mathcal{L}\{(f * g)(t)\}(s) = \int_0^\infty \left(\int_0^t f(\tau) g(t-\tau)\, d\tau\right) e^{-st}\, dt.$$

By Fubini's theorem (changing the order of integration), this becomes

$$\int_0^\infty f(\tau)\left(\int_0^\infty g(u) e^{-s(u+\tau)}\, du\right) d\tau = \left(\int_0^\infty f(\tau) e^{-s\tau}\, d\tau\right)\left(\int_0^\infty g(u) e^{-su}\, du\right).$$

Thus,

$$\mathcal{L}\{(f * g)(t)\}(s) = F(s)G(s).$$

■

Example 5.13 ***Let*** $f(t) = 1$ ***and*** $g(t) = t$ ***for*** $t \geq 0$***. Then their convolution is***

$$(f * g)(t) = \int_0^t 1 \cdot (t-\tau)\, d\tau = \int_0^t (t-\tau)\, d\tau = \frac{t^2}{2}.$$

Now apply the Laplace transform:

$$\mathcal{L}\{(f * g)(t)\}(s) = \mathcal{L}\left\{\tfrac{t^2}{2}\right\}(s) = \frac{1}{s^3}.$$

On the other hand, by the theorem:

$$F(s) = \mathcal{L}\{1\}(s) = \frac{1}{s}, \quad G(s) = \mathcal{L}\{t\}(s) = \frac{1}{s^2},$$

so

$$F(s)G(s) = \frac{1}{s} \cdot \frac{1}{s^2} = \frac{1}{s^3}.$$

Both results agree, verifying the theorem.

5.2 Common Laplace Transforms

Time-domain function $f(t)$	**Laplace transform** $F(s) = \mathcal{L}\{f(t)\}(s)$
1 **(constant)**	$\frac{1}{s}, \quad \Re(s) > 0$
t	$\frac{1}{s^2}, \quad \Re(s) > 0$
$t^n,\ n \in \mathbb{N}$	$\frac{n!}{s^{n+1}}, \quad \Re(s) > 0$
e^{at}	$\frac{1}{s-a}, \quad \Re(s) > a$
$\cos(at)$	$\frac{s}{s^2+a^2}, \quad \Re(s) > 0$
$\sin(at)$	$\frac{a}{s^2+a^2}, \quad \Re(s) > 0$
$e^{at}\cos(bt)$	$\frac{s-a}{(s-a)^2+b^2}, \quad \Re(s) > a$
$e^{at}\sin(bt)$	$\frac{b}{(s-a)^2+b^2}, \quad \Re(s) > a$
Unit step $u(t)$	$\frac{1}{s}, \quad \Re(s) > 0$
Shifted step $u(t-a)$	$\frac{e^{-as}}{s}, \quad a > 0,\ \Re(s) > 0$
Dirac impulse $\delta(t)$	1
Shifted impulse $\delta(t-a)$	$e^{-as}, \quad a \geq 0$
$\frac{\sin(at)}{t}$	$\arctan\left(\frac{a}{s}\right), \quad \Re(s) > 0$
$\cosh(at)$	$\frac{s}{s^2-a^2}, \quad \Re(s) > \lvert a\rvert$
$\sinh(at)$	$\frac{a}{s^2-a^2}, \quad \Re(s) > \lvert a\rvert$

5.3 Inverse Laplace Transform

Definition 5.4 ***The inverse Laplace transform is the operation that allows us to recover the original function*** $f(t)$ ***from its Laplace transform*** $F(s)$***. We denote it by***

$$f(t) = \mathcal{L}^{-1}\{F(s)\}.$$

Formally, if $F(s) = \mathcal{L}\{f(t)\}$***, then***

$$\mathcal{L}^{-1}\{F(s)\} = f(t).$$

Remark 5.3 ***The inverse Laplace transform is unique for functions of exponential order, meaning that each Laplace transform corresponds to a single original function*** $f(t)$ ***(up to piecewise continuity).***

Example 5.14 ***Find the inverse Laplace transform of***

$$F(s) = \frac{1}{s-2}, \quad \Re(s) > 2.$$

Solution 5.4 By the definition of the Laplace transform:

$$\mathcal{L}\{e^{2t}\} = \int_0^\infty e^{2t}e^{-st}dt = \frac{1}{s-2}, \quad \Re(s) > 2.$$

Hence, by matching $F(s)$ ***with a known Laplace transform:***

$$\mathcal{L}^{-1}\left\{\frac{1}{s-2}\right\} = e^{2t}.$$

5.3.1 Properties of the Inverse Laplace Transform

Let $(\mathcal{L}f(t) = F(s))$. **Then the inverse Laplace transform is defined by**

$$f(t) = \mathcal{L}^{-1}\left\{F(s)\right\}.$$

1. **Linearity**

$$\mathcal{L}^{-1}\left\{aF(s) + bG(s)\right\} = a\mathcal{L}^{-1}\left\{F(s)\right\} + b\mathcal{L}^{-1}\left\{G(s)\right\}.$$

2. **Inverse of a Shift in the (s)-domain**

$$\mathcal{L}^{-1}\left\{F(s-a)\right\} = e^{at}f(t), \quad t \geq 0.$$

3. **Inverse of Multiplication by (s)**

$$\mathcal{L}^{-1}\left\{sF(s)\right\} = f'(t) + f(0)\delta(t).$$

4. **Inverse of Division by** (s)

$$\mathcal{L}^{-1}\left(\frac{F(s)}{s}\right) = \int_0^t f(\tau), d\tau.$$

5. **Inverse of Multiplication by** (e^{-as})

$$\mathcal{L}^{-1}\left(e^{-as}F(s)\right) = u(t-a).f(t-a).$$

6. **Inverse of Multiplication by** (s^n)

$$\mathcal{L}^{-1}\left(s^nF(s)\right) = f^{(n)}(t) + \textbf{initial conditions (Dirac terms)}.$$

7. **Inverse of Differentiation in (s)**

$$\mathcal{L}^{-1}\left(\frac{dF}{ds}\right) = -tf(t).$$

8. **Inverse of a Product**

$$\mathcal{L}^{-1}\left\{F(s)G(s)\right\} = (f * g)(t) = \int_0^t f(\tau)g(t-\tau), d\tau.$$

Examples 5.1 Example 1: Linearity

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2}+\frac{1}{s^2+1}\right\}=t+\sin t.$$

Example 2: Shift in the s-domain

$$\mathcal{L}^{-1}\left\{\frac{1}{(s-2)^2+1}\right\}=e^{2t}\sin t.$$

Example 3: Time shift

$$\mathcal{L}^{-1}\left\{e^{-3s}\,\frac{1}{s^2}\right\}=u(t-3)(t-3).$$

Example 4: Multiplication by s

$$\mathcal{L}^{-1}\left\{\frac{s}{s^2}\right\}=1.$$

Example 5: Division by s

$$\mathcal{L}^{-1}\left\{\frac{1}{s(s+1)}\right\}=1-e^{-t}.$$

Example 6: Differentiation in s

$$\mathcal{L}^{-1}\left\{-\frac{1}{s^2}\right\}=-t.$$

Example 7: Convolution

$$\mathcal{L}^{-1}\left\{\frac{1}{s(s+1)}\right\}=\int_0^t e^{-(t-\tau)}\,d\tau=1-e^{-t}.$$

5.4 Application of Laplace Transforms to Differential Equations

Laplace transforms are a powerful tool for solving linear differential equations with given initial conditions. The main idea is to transform the differential equation into an algebraic equation in the variable s, solve it, and then apply the inverse Laplace transform to find the solution in the time domain.

General Method

1. **Apply the Laplace transform to both sides of the differential equation.**
2. **Use the properties of the Laplace transform to handle derivatives:**

$$\mathcal{L}\{y'(t)\}=sY(s)-y(0),\quad \mathcal{L}\{y''(t)\}=s^2Y(s)-sy(0)-y'(0).$$

3. **Insert the initial conditions into the equation.**
4. **Solve the resulting algebraic equation for $Y(s)$.**
5. **Apply the inverse Laplace transform to obtain $y(t)$.**

Example 5.15 (Second-order ODE) ***Solve the initial value problem***

$$y''(t) + y(t) = 0, \quad y(0) = 0, \quad y'(0) = 1.$$

Solution 5.5 ***Step 1. Apply the Laplace transform:***

$$\mathcal{L}\{y''(t)\} + \mathcal{L}\{y(t)\} = 0.$$

Step 2. Use the derivative formulas:

$$\Big(s^2 Y(s) - s y(0) - y'(0)\Big) + Y(s) = 0.$$

Step 3. Insert initial conditions $y(0) = 0$***,*** $y'(0) = 1$***:***

$$s^2 Y(s) - 1 + Y(s) = 0.$$

Step 4. Solve for $Y(s)$***:***

$$Y(s)(s^2 + 1) = 1 \quad \Longrightarrow \quad Y(s) = \frac{1}{s^2 + 1}.$$

Step 5. Inverse Laplace transform:

$$y(t) = \mathcal{L}^{-1}\left\{\frac{1}{s^2 + 1}\right\} = \sin t.$$

$$\boxed{y(t) = \sin t}$$

Hence, the solution of the differential equation is $y(t) = \sin t$***.***

5.5 Solved Exercises

Exercise 5.1 ***1. Compute*** $f(t) = \mathcal{L}^{-1}\left\{\frac{1}{s^2 + 1}\right\}$***.***

Solution. We recognize the known transform:

$$\mathcal{L}\{\sin t\}(s) = \frac{1}{s^2 + 1}.$$

Therefore,

$$\boxed{\mathcal{L}^{-1}\left\{\frac{1}{s^2 + 1}\right\} = \sin t.}$$

2. Compute $f(t) = \mathcal{L}^{-1}\left\{\frac{1}{s + a}\right\}$***,*** $a \in \mathbb{R}$***.***

Solution. From the Laplace table we know:

$$\mathcal{L}\{e^{-at}\}(s) = \frac{1}{s + a}, \quad (\Re(s) > -a).$$

Thus,

$$\boxed{\mathcal{L}^{-1}\left\{\frac{1}{s + a}\right\} = e^{-at}.}$$

3. ***Compute*** $f(t) = \mathcal{L}^{-1}\left\{\dfrac{1}{s(s+1)}\right\}$.

Solution. Perform partial fraction decomposition:

$$\frac{1}{s(s+1)} = \frac{A}{s} + \frac{B}{s+1}.$$

Solving: $1 = A(s+1) + Bs$***. Setting*** $s = 0$ ***gives*** $A = 1$***, and*** $s = -1$ ***gives*** $B = -1$***. Hence,***

$$\frac{1}{s(s+1)} = \frac{1}{s} - \frac{1}{s+1}.$$

Inverting term by term:

$$\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} = 1, \qquad \mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} = e^{-t}.$$

So,

$$\boxed{\mathcal{L}^{-1}\left\{\frac{1}{s(s+1)}\right\} = 1 - e^{-t}.}$$

4. ***Compute*** $f(t) = \mathcal{L}^{-1}\left\{e^{-3s}\dfrac{1}{s^2}\right\}$ ***(time-shifting).***

Solution. We know $\mathcal{L}\{t\}(s) = \dfrac{1}{s^2}$***. The inversion rule for the factor*** e^{-as} ***gives:***

$$\mathcal{L}^{-1}\left\{e^{-3s}F(s)\right\} = u(t-3)\, f(t-3),$$

where $f(t) = \mathcal{L}^{-1}\{F(s)\} = t$***. Hence,***

$$\boxed{\mathcal{L}^{-1}\left\{e^{-3s}\frac{1}{s^2}\right\} = u(t-3)\,(t-3).}$$

5. ***Compute*** $f(t) = \mathcal{L}^{-1}\left\{\dfrac{s}{s+1}\right\}$ ***(distributional term).***

Solution. Write the fraction in a useful form:

$$\frac{s}{s+1} = 1 - \frac{1}{s+1}.$$

We use $\mathcal{L}^{-1}\{1\} = \delta(t)$ ***(Dirac impulse) and*** $\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} = e^{-t}$***. Thus,***

$$\boxed{\mathcal{L}^{-1}\left\{\frac{s}{s+1}\right\} = \delta(t) - e^{-t}.}$$

6. ***Compute*** $f(t) = \mathcal{L}^{-1}\left\{\dfrac{1}{s(s+2)}\right\}$ ***(partial fractions / convolution).***

Solution. Decompose:

$$\frac{1}{s(s+2)} = \frac{A}{s} + \frac{B}{s+2}.$$

We get $1 = A(s+2) + Bs$***. For*** $s=0$***:*** $A = \frac{1}{2}$***. For*** $s=-2$***:*** $B = -\frac{1}{2}$***. So,***

$$\frac{1}{s(s+2)} = \frac{1}{2}\left(\frac{1}{s} - \frac{1}{s+2}\right).$$

Inverting term by term:

$$f(t) = \frac{1}{2}\Big(1 - e^{-2t}\Big).$$

Therefore,

$$\boxed{\mathcal{L}^{-1}\left\{\frac{1}{s(s+2)}\right\} = \frac{1}{2}\Big(1 - e^{-2t}\Big).}$$

7. ***Compute*** $f(t) = \mathcal{L}^{-1}\left\{-\frac{1}{s^2}\right\}$ ***(differentiation rule in*** s***).***

   Solution. Recall that if $F(s) = \mathcal{L}\{g(t)\}$ ***then***

$$\frac{dF}{ds} = \mathcal{L}\{-tg(t)\}.$$

   Here $F(s) = \frac{1}{s}$ ***corresponds to*** $g(t) = 1$***. Then,***

$$\frac{d}{ds}\left(\frac{1}{s}\right) = -\frac{1}{s^2} = \mathcal{L}\{-t \cdot 1\}.$$

   Thus,

$$\boxed{\mathcal{L}^{-1}\left\{-\frac{1}{s^2}\right\} = -t.}$$

8. ***Compute*** $f(t) = \mathcal{L}^{-1}\left\{\frac{1}{s(s^2+1)}\right\}$ ***(product*** $F(s)G(s) \to$ ***convolution).***

   Solution. We can write:

$$\frac{1}{s(s^2+1)} = \frac{1}{s} \cdot \frac{1}{s^2+1}.$$

   Let $f_1(t) = 1$ ***(inverse of*** $1/s$***) and*** $f_2(t) = \sin t$ ***(inverse of*** $1/(s^2+1)$***). The inverse of the product is the convolution:***

$$(f_1 * f_2)(t) = \int_0^t f_1(\tau)\, f_2(t-\tau)\, d\tau = \int_0^t \sin(t-\tau)\, d\tau.$$

   Change of variable $u = t - \tau$***, we get:***

$$\int_0^t \sin(t-\tau)\, d\tau = \int_0^t \sin u\, du = 1 - \cos t.$$

   Therefore,

$$\boxed{\mathcal{L}^{-1}\left\{\frac{1}{s(s^2+1)}\right\} = 1 - \cos t.}$$

Exercise 5.2 ***RC Circuit with a Step Input.***
Consider an electrical circuit consisting of a resistor R ***and a capacitor*** C ***in series. The input voltage is a step function*** $E(t) = E_0\, u(t)$***, where*** $E_0 > 0$ ***is a constant and*** $u(t)$ ***is the unit step function.***
The governing equation for the capacitor voltage $v(t)$ ***is***

$$RC\,\frac{dv}{dt} + v(t) = E_0, \qquad v(0) = 0.$$

Tasks:

1. ***Apply the Laplace transform to the differential equation.***
2. ***Solve for*** $V(s)$***, the Laplace transform of*** $v(t)$***.***
3. ***Compute*** $v(t)$ ***by applying the inverse Laplace transform.***

Hint: Recall that

$$\mathcal{L}\left\{\frac{dv}{dt}\right\}(s) = sV(s) - v(0).$$

Solution 5.6 ***Step 1. Apply the Laplace transform.***
The differential equation is

$$RC\,\frac{dv}{dt} + v(t) = E_0\, u(t), \qquad v(0) = 0.$$

Take the Laplace transform of both sides (denote $V(s) = \mathcal{L}\{v(t)\}(s)$***). Using*** $\mathcal{L}\{v'(t)\} = sV(s) - v(0)$ ***and*** $\mathcal{L}\{u(t)\} = \frac{1}{s}$***, we get***

$$RC\Big(sV(s) - v(0)\Big) + V(s) = E_0\,\frac{1}{s}.$$

Step 2. Insert the initial condition and solve for $V(s)$***.***
Given $v(0) = 0$***, the equation becomes***

$$RC\,sV(s) + V(s) = \frac{E_0}{s}.$$

Factor $V(s)$***:***

$$V(s)\Big(RCs + 1\Big) = \frac{E_0}{s}.$$

Thus

$$V(s) = \frac{E_0}{s\Big(RCs + 1\Big)}.$$

It is convenient to rewrite $RCs + 1 = RC\left(s + \frac{1}{RC}\right)$***, so***

$$V(s) = \frac{E_0}{RC} \cdot \frac{1}{s\left(s + \frac{1}{RC}\right)}.$$

Step 3. Partial fraction decomposition.
We decompose

$$\frac{1}{s\left(s+\dfrac{1}{RC}\right)} = \frac{A}{s} + \frac{B}{s+\frac{1}{RC}}.$$

Solve for A, B***:***

$$1 = A\left(s+\frac{1}{RC}\right) + Bs.$$

Setting $s=0$ ***gives*** $1 = A\cdot\frac{1}{RC}$***, hence*** $A = RC$***. Setting*** $s = -\dfrac{1}{RC}$ ***gives*** $1 = B\left(-\dfrac{1}{RC}\right)$***, hence*** $B = -RC$***. Therefore***

$$\frac{1}{s\left(s+\dfrac{1}{RC}\right)} = \frac{RC}{s} - \frac{RC}{s+\frac{1}{RC}}.$$

Substituting back into $V(s)$***:***

$$V(s) = \frac{E_0}{RC}\left(\frac{RC}{s} - \frac{RC}{s+\frac{1}{RC}}\right) = E_0\left(\frac{1}{s} - \frac{1}{s+\frac{1}{RC}}\right).$$

Step 4. Inverse Laplace transform.
Use standard inverses:

$$\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} = 1, \qquad \mathcal{L}^{-1}\left\{\frac{1}{s+\frac{1}{RC}}\right\} = e^{-t/(RC)}.$$

Hence

$$v(t) = E_0\Big(1 - e^{-t/(RC)}\Big), \qquad t \geq 0.$$

$$\boxed{v(t) = E_0\Big(1 - e^{-t/(RC)}\Big)}$$

Remark. This is the well-known charging law of a capacitor in an RC series circuit: the capacitor voltage starts at 0 ***(since*** $v(0)=0$***) and asymptotically approaches*** E_0 ***with time constant*** $\tau = RC$***.***

Chapter 6

Fourier Transform

6.1 Definition and Properties of the Fourier Transform

6.1.1 Definitions

Definition 6.1 ***For a continuous signal $x(t)$ defined for all $t \in \mathbb{R}$ and absolutely integrable, the Fourier transform is defined as:***

$$X(f) = \mathcal{F}\{x(t)\} = \int_{-\infty}^{+\infty} x(t)e^{-j2\pi ft}\,dt, \quad j^2 = -1$$

where:

- $X(f)$ ***is the Fourier transform of*** $x(t)$
- f ***is the frequency in Hz***
- j ***is the imaginary unit (***$j^2 = -1$***)***

The Fourier transform represents a time-domain signal as a superposition of sinusoidal components at different frequencies.

Examples 6.1 Example 1: Causal exponential
Consider the signal $x(t) = e^{-2t}u(t)$, where $u(t)$ is the unit step function defined by:

$$u(t) = \begin{cases} 0, & t < 0 \\ 1, & t \geq 0 \end{cases}$$

and its derivative, is the Dirac delta function:

$$\frac{d}{dt}u(t) = \delta(t)$$

Then the Fourier transform is:

$$X(f) = \int_0^{\infty} e^{-2t}e^{-j2\pi ft}\,dt = \int_0^{\infty} e^{-(2+j2\pi f)t}\,dt = \frac{1}{2 + j2\pi f}.$$

Remark 6.1 ***The unit step function makes the signal*** **causal*, i.e., zero for*** $t < 0$***.***

Example 2: Dirac delta function
Consider $x(t) = \delta(t)$**, the Dirac delta function, defined by:**

$$\delta(t) = \begin{cases} 0, & t \neq 0 \\ \textbf{infinity}, & t = 0 \end{cases}, \quad \textbf{with} \int_{-\infty}^{\infty} \delta(t)dt = 1$$

Theorem 6.1 (Sifting Property of the Dirac Delta) ***Let*** $\delta(t)$ ***be the Dirac delta function, and let*** $\phi(t)$ ***be any continuous function. Then:***

$$\int_{-\infty}^{+\infty} \delta(t)\, \phi(t)\, dt = \phi(0)$$

More generally, for a shifted delta function $\delta(t - t_0)$***:***

$$\int_{-\infty}^{+\infty} \delta(t - t_0)\, \phi(t)\, dt = \phi(t_0)$$

Remark 6.2 This property means that the delta "samples" the value of the function at the point $t = t_0$***.***

The Fourier transform is:

$$\mathcal{F}\{\delta(t)\} = X(f) = \int_{-\infty}^{\infty} \delta(t) e^{-j2\pi f t}\, dt = 1$$

Remark 6.3 ***The Dirac delta contains all frequencies with equal amplitude.***

Example 3: Pure cosine
Consider $x(t) = \cos(3t)$**. Using Euler's formula:**

$$\cos(3t) = \frac{e^{j3t} + e^{-j3t}}{2}$$

Then the Fourier transform is:

$$X(f) = \frac{1}{2} \int_{-\infty}^{\infty} e^{-j(2\pi f - 3)t}\, dt + \frac{1}{2} \int_{-\infty}^{\infty} e^{-j(2\pi f + 3)t}\, dt = \frac{1}{2}\delta(2\pi f - 3) + \frac{1}{2}\delta(2\pi f + 3)$$

Remark 6.4 ***A pure cosine has two frequency components at*** $\pm 3/(2\pi)$ ***Hz.***

6.1.2 Properties of the Fourier Transform

1. Linearity

Theorem 6.2 (Linearity) ***For signals*** $x(t)$ ***and*** $y(t)$***, and constants*** $a, b \in \mathbb{C}$***:***

$$\mathcal{F}\{ax(t) + by(t)\} = aX(f) + bY(f).$$

Where $X(f)$ ***and*** $Y(f)$ ***are respectively the Fourier transform of*** $x(t)$ ***and*** $y(t)$***.***

Example 6.1 ***Compute the Fourier transform of*** $x(t) = 2e^{-t}u(t)$ ***and*** $y(t) = \cos(3t)$ ***combined as*** $x(t) + y(t)$***.***

$$\mathcal{F}\{x(t)\} = \frac{2}{1 + j2\pi f}, \quad \mathcal{F}\{y(t)\} = \frac{1}{2}\delta(2\pi f - 3) + \frac{1}{2}\delta(2\pi f + 3)$$

$$\mathcal{F}\{x(t) + y(t)\} = \frac{2}{1 + j2\pi f} + \frac{1}{2}\delta(2\pi f - 3) + \frac{1}{2}\delta(2\pi f + 3)$$

—

2. Time Shifting

Theorem 6.3 (Time Shifting) ***If*** $x(t)$ ***has Fourier transform*** $X(f)$***, then for any*** $t_0 \in \mathbb{R}$***:***

$$\mathcal{F}\{x(t - t_0)\} = X(f)e^{-j2\pi f t_0}$$

Example 6.2 ***Let*** $x(t) = e^{-t}u(t)$ ***and*** $t_0 = 1$***. Compute*** $\mathcal{F}\{x(t-1)\}$***.***
Solution:
Step 1: Write the shifted signal explicitly:

$$x(t - 1) = e^{-(t-1)}u(t - 1)$$

Here, $u(t - 1)$ ***is the unit step function that ensures the signal is causal, i.e., zero for*** $t < 1$***.***

—

Step 2: Use the definition of the Fourier transform:

$$\mathcal{F}\{x(t - 1)\} = \int_{-\infty}^{+\infty} x(t - 1)e^{-j2\pi f t}\, dt = \int_{-\infty}^{+\infty} e^{-(t-1)}u(t - 1)e^{-j2\pi f t}\, dt$$

—

Step 3: Adjust the limits using $u(t - 1)$***:***

$$u(t - 1) = 0 \textbf{\textit{ for }} t < 1, \quad u(t - 1) = 1 \textbf{\textit{ for }} t \geq 1$$

So the integral becomes:

$$\mathcal{F}\{x(t - 1)\} = \int_{1}^{\infty} e^{-(t-1)}e^{-j2\pi f t}\, dt$$

—

Step 4: Factor terms to simplify the exponent:

$$e^{-(t-1)}e^{-j2\pi f t} = e^{-(t-1)}e^{-j2\pi f(t-1)}e^{-j2\pi f} = e^{-(1+j2\pi f)(t-1)}e^{-j2\pi f}$$

—

Step 5: Change variable $\tau = t - 1 \implies t = \tau + 1, dt = d\tau$***:***

$$\mathcal{F}\{x(t - 1)\} = \int_{0}^{\infty} e^{-(1+j2\pi f)\tau}e^{-j2\pi f}\, d\tau = e^{-j2\pi f}\int_{0}^{\infty} e^{-(1+j2\pi f)\tau}\, d\tau$$

—

Step 6: Evaluate the integral:

$$\int_0^{\infty} e^{-(1+j2\pi f)\tau}\, d\tau = \frac{1}{1+j2\pi f}$$

—

Step 7: Final result:

$$\mathcal{F}\{x(t-1)\} = \frac{1}{1+j2\pi f}\, e^{-j2\pi f}$$

Remark 6.5 ***The unit step*** $u(t-1)$ ***is included in the integral limits, which is why it does not appear explicitly in the final formula, but it is essential to make the signal causal.***

—

3. Frequency Shifting (Modulation)

Theorem 6.4 (Frequency Shifting) ***If*** $x(t)$ ***has Fourier transform*** $X(f)$***, then***

$$\mathcal{F}\{x(t)e^{j2\pi f_0 t}\} = X(f - f_0)$$

Example 6.3 ***Let*** $x(t) = e^{-t}u(t)$ ***and*** $f_0 = 2$***.***
Compute the Fourier transform of $x(t)e^{j2\pi(2)t} = x(t)e^{j4\pi t}$***.***
Solution:
Step 1: Write the signal explicitly with the unit step:

$$x(t)e^{j4\pi t} = e^{-t}u(t)\cdot e^{j4\pi t} = e^{-t}e^{j4\pi t}u(t)$$

Here, $u(t)$ ***ensures the signal is causal (zero for*** $t < 0$***).***
—
Step 2: Use the definition of the Fourier transform:

$$\mathcal{F}\{x(t)e^{j4\pi t}\} = \int_{-\infty}^{+\infty} e^{-t}e^{j4\pi t}u(t)\, e^{-j2\pi f t}\, dt$$

Combine exponentials:

$$e^{j4\pi t}\cdot e^{-j2\pi f t} = e^{-j2\pi f t}\cdot e^{j2\pi(2)t} = e^{-j2\pi(f-2)t}$$

So the integral becomes:

$$\mathcal{F}\{x(t)e^{j4\pi t}\} = \int_0^{\infty} e^{-t}e^{-j2\pi(f-2)t}\, dt$$

- The lower limit 0 ***comes from*** $u(t)$***. - The upper limit is*** $+\infty$ ***because the signal exists for all*** $t \geq 0$***.***
—
Step 3: Combine terms in the exponent:

$$e^{-t}\cdot e^{-j2\pi(f-2)t} = e^{-(1+j2\pi(f-2))t}$$

—

Step 4: Integrate:

$$\int_0^{\infty} e^{-(1+j2\pi(f-2))t}\,dt = \frac{1}{1+j2\pi(f-2)}$$

—

Step 5: Final result:

$$\mathcal{F}\{x(t)e^{j4\pi t}\} = \frac{1}{1+j2\pi(f-2)}$$

Remark: Multiplying by** $e^{j2\pi f_0 t}$ **in time shifts the Fourier transform in frequency by** f_0**, here** $f_0 = 2$**. The unit step** $u(t)$ **ensures the integral starts at** 0**.

—

4. Differentiation in Time

Theorem 6.5 (Time Differentiation) ***If** $x(t)$ **has Fourier transform** $X(f)$**, then***

$$\mathcal{F}\left\{\frac{dx}{dt}\right\} = j2\pi f X(f)$$

More generally,for the** n**-th derivative

$$\mathcal{F}\left\{\frac{d^n x(t)}{dt^n}\right\} = (j2\pi f)^n X(f), \quad n = 1, 2, 3, \ldots$$

Example 6.4 ***Let** $x(t) = e^{-t}u(t)$**.***
1. Compute the Fourier transform of** $\frac{dx}{dt}(t)$ **using the propertie of time differentiation.
2. Compute** $\frac{dx}{dt}(t)$ **and its Fourier transform using an other method.
Solution:
1. Recall the property:

$$\mathcal{F}\left\{\frac{dx}{dt}(t)\right\} = j2\pi f X(f)$$

with** $\mathcal{F}\{e^{-t}u(t)\} = \frac{1}{1+j2\pi f}$**. Then

$$\mathcal{F}\left\{\frac{dx}{dt}(t)\right\} = j2\pi f \frac{1}{1+j2\pi f} = \frac{j2\pi f}{1+j2\pi f}$$

2. Step 1: Recall that** $x(t) = e^{-t}u(t)$**. Using the product rule for derivatives:

$$\frac{d}{dt}[e^{-t}u(t)] = \frac{de^{-t}}{dt}u(t) + e^{-t}\frac{du(t)}{dt}$$

—

Step 2: Compute each term:

1. $\dfrac{de^{-t}}{dt}u(t) = -e^{-t}u(t)$
2. $\dfrac{du(t)}{dt} = \delta(t)$ ***by definition of the Dirac delta***
So the derivative is:

$$\frac{dx}{dt}(t) = -e^{-t}u(t) + e^{-t}\delta(t)$$

But notice that $e^{-t}\delta(t) = \delta(t)$ ***because*** $\delta(t)$ ***"samples" the function at*** $t = 0$***:***

$$\int_{-\infty}^{\infty} e^{-t}\delta(t)\phi(t)dt = \phi(0)e^{0} = \phi(0) = \int_{-\infty}^{\infty} \delta(t)\phi(t)dt$$

That is $e^{-t}\delta(t) = \delta(t)$ ***in distribution sens. Hence, we can write:***

$$\frac{dx}{dt}(t) = \delta(t) - e^{-t}u(t)$$

—

Step 3: Fourier transform of the derivative
Using linearity:

$$\mathcal{F}\left\{\frac{dx}{dt}(t)\right\} = \mathcal{F}\{\delta(t)\} - \mathcal{F}\{e^{-t}u(t)\}$$

—

Step 4: Fourier transforms of each term:
1. $\mathcal{F}\{\delta(t)\} = 1$
2. $\mathcal{F}\{e^{-t}u(t)\} = \dfrac{1}{1 + j2\pi f}$
So we get:

$$\mathcal{F}\left\{\frac{dx}{dt}\right\} = 1 - \frac{1}{1 + j2\pi f}$$

—

Step 5: Simplify the expression:

$$1 - \frac{1}{1 + j2\pi f} = \frac{1 + j2\pi f - 1}{1 + j2\pi f} = \frac{j2\pi f}{1 + j2\pi f}$$

—

Final Result:

$$\frac{dx}{dt} = \delta(t) - e^{-t}u(t), \quad \mathcal{F}\left\{\frac{dx}{dt}\right\} = \frac{j2\pi f}{1 + j2\pi f}$$

Remark 6.6 The delta function appears because of the derivative of the unit step $u(t)$***, and it is crucial to include it for a correct Fourier transform.***

—

5. Convolution in Time

Theorem 6.6 (Convolution) ***If*** $x(t)$ ***and*** $y(t)$ ***have Fourier transforms*** $X(f)$ ***and*** $Y(f)$***, then***

$$\mathcal{F}\{x(t) * y(t)\} = X(f) \cdot Y(f)$$

Example 6.5 ***Let*** $x(t) = e^{-t}u(t)$ ***and*** $y(t) = u(t)$***. Compute the convolution*** $x(t) * y(t)$ ***and its Fourier transform.***
Solution:
Step 1: Recall the definition of convolution:

$$(x * y)(t) = \int_{-\infty}^{+\infty} x(\tau)\, y(t-\tau)\, d\tau$$

Since both $x(t)$ ***and*** $y(t)$ ***are causal (***$u(t)$***), the integral limits reduce to:***

$$(x * y)(t) = \int_{0}^{t} x(\tau)\, y(t-\tau)\, d\tau$$

—

Step 2: Substitute $x(\tau) = e^{-\tau}u(\tau)$ ***and*** $y(t-\tau) = u(t-\tau)$***:***
- $u(\tau) = 1$ ***for*** $\tau \geq 0$
- $u(t-\tau) = 1$ ***for*** $0 \leq \tau \leq t$

$$(x * y)(t) = \int_{0}^{t} e^{-\tau} \cdot 1\, d\tau$$

—

Step 3: Evaluate the integral:

$$\int_{0}^{t} e^{-\tau} d\tau = \Big[-e^{-\tau}\Big]_{0}^{t} = -e^{-t} + e^{0} = 1 - e^{-t}$$

So the convolution result is:

$$x(t) * y(t) = 1 - e^{-t}$$

—

Step 4: Verify with the direct Fourier transform

$$\mathcal{F}\{1 - e^{-t}\} = \mathcal{F}\{1\} - \mathcal{F}\{e^{-t}u(t)\} = \frac{1}{j2\pi f} - \frac{1}{1 + j2\pi f}$$

- This matches $X(f)Y(f)$ ***by the convolution theorem. Indeed, Fourier transforms of*** $x(t)$ ***and*** $y(t)$***:***
1. $x(t) = e^{-t}u(t) \implies X(f) = \frac{1}{1+j2\pi f}$
2. $y(t) = u(t) \implies Y(f) = \frac{1}{j2\pi f} + \pi\delta(f)$
- Often in examples, the $\delta(f)$ ***term is ignored when focusing on*** $f \neq 0$***. Recall the convolution theorem:***

$$\mathcal{F}\{x(t) * y(t)\} = X(f)\, Y(f)$$

we have

$$X(f)Y(f) = \frac{1}{1 + j2\pi f} \cdot \frac{1}{j2\pi f} = \frac{1}{(j2\pi f)(1 + j2\pi f)}$$

Partial fraction decomposition: $\frac{1}{(j2\pi f)(1+j2\pi f)} = \frac{1}{j2\pi f} - \frac{1}{1+j2\pi f}$

$$\implies X(f)Y(f) = \frac{1}{j2\pi f} - \frac{1}{1+j2\pi f} = \mathcal{F}\{1 - e^{-t}\}$$

Remark 6.7 - Convolution in the time domain corresponds to multiplication in the frequency domain.
- The unit step $u(t)$ ***determines the integration limits and ensures causality.***

—

6. Multiplication in Time

Theorem 6.7 (Time Multiplication) ***If*** $x(t)$ ***and*** $y(t)$ ***have Fourier transforms*** $X(f)$ ***and*** $Y(f)$***, then***

$$\mathcal{F}\{x(t)\,y(t)\} = X(f) * Y(f)$$

where $*$ ***denotes convolution in frequency.***

Example 6.6 ***Let*** $x(t) = u(t)$ ***and*** $y(t) = u(t-1)$***. Compute the Fourier transform of the product*** $x(t)y(t)$***.***
Solution:
Step 1: Write the product explicitly using the unit step functions:

$$x(t)y(t) = u(t) \cdot u(t-1)$$

- $u(t) = 0$ ***for*** $t < 0$***,*** 1 ***for*** $t \geq 0$ ***-*** $u(t-1) = 0$ ***for*** $t < 1$***,*** 1 ***for*** $t \geq 1$
So the product is:

$$x(t)y(t) = \begin{cases} 0, & t < 1 \\ 1, & t \geq 1 \end{cases} = u(t-1)$$

—

Step 2: Recall the Fourier transform of a shifted unit step:

$$\mathcal{F}\{u(t-a)\} = \frac{1}{j2\pi f} e^{-j2\pi f a} + \pi\delta(f)$$

Here, $a = 1$***. So:***

$$\mathcal{F}\{u(t-1)\} = \frac{1}{j2\pi f} e^{-j2\pi f} + \pi\delta(f)$$

—

Step 3: Connect to the convolution property
- The Fourier transform of a product of two signals in time is equal to the convolution of their Fourier transforms:

$$\mathcal{F}\{x(t)y(t)\} = X(f) * Y(f)$$

- Here:

$$X(f) = \mathcal{F}\{u(t)\} = \frac{1}{j2\pi f} + \pi\delta(f)$$

$$Y(f) = \mathcal{F}\{u(t-1)\} = \frac{1}{j2\pi f}e^{-j2\pi f} + \pi\delta(f)$$

- The direct computation gave

$$X(f) * Y(f) = \int_{-\infty}^{\infty} X(\lambda)\, Y(f-\lambda)\, d\lambda = \mathcal{F}\{u(t-1)\} = \mathcal{F}\{x(t)y(t)\}$$

Remark 6.8 - This shows that multiplying two signals in time corresponds to convolution in frequency.
- This is useful for modulated signals or windowed signals, where one signal acts as a "window" for the other.

7. Fourier Transform: Frequency Derivative Property

Theorem 6.8 (Frequency Derivative Property) ***Let $x(t)$ be an absolutely integrable signal with Fourier transform***

$$X(f) = \mathcal{F}\{x(t)\}.$$

Then the derivative of $X(f)$ with respect to frequency f is given by:

$$\frac{dX(f)}{df} = \mathcal{F}\{-j2\pi t\, x(t)\}.$$

Proof.

1. **By definition, the Fourier transform of $x(t)$ is**

$$X(f) = \int_{-\infty}^{\infty} x(t)\, e^{-j2\pi f t}\, dt.$$

2. **Differentiating $X(f)$ with respect to f:**

$$\frac{dX(f)}{df} = \int_{-\infty}^{\infty} x(t)\, \frac{d}{df}\left(e^{-j2\pi f t}\right) dt.$$

3. **Since**

$$\frac{d}{df}e^{-j2\pi f t} = -j2\pi t\, e^{-j2\pi f t},$$

we have

$$\frac{dX(f)}{df} = \int_{-\infty}^{\infty} (-j2\pi t\, x(t))\, e^{-j2\pi f t}\, dt.$$

4. **Recognizing this as a Fourier transform:**

$$\frac{dX(f)}{df} = \mathcal{F}\{-j2\pi t\, x(t)\}.$$

■

Example 6.7 ***Let***

$$x(t) = e^{-t}u(t),$$

where $u(t)$ ***is the unit step function. Compute*** $\frac{dX(f)}{df}$***.***
Solution:
Step 1: Fourier transform of $x(t)$

$$X(f) = \mathcal{F}\{x(t)\} = \int_0^\infty e^{-t}e^{-j2\pi ft}dt = \int_0^\infty e^{-(1+j2\pi f)t}dt = \frac{1}{1+j2\pi f}.$$

Step 2: Write the integral

$$\frac{dX(f)}{df} = \int_0^\infty \left(-j2\pi t\, e^{-t}\right) e^{-j2\pi ft}dt = -j2\pi \int_0^\infty t\, e^{-(1+j2\pi f)t}dt$$

Let $\alpha = 1 + j2\pi f$***, then:***

$$\frac{dX(f)}{df} = -j2\pi \int_0^\infty t\, e^{-\alpha t}dt$$

—

Step 3: Use the standard integral

$$\int_0^\infty t\, e^{-\alpha t}dt = \frac{1}{\alpha^2}, \qquad \Re(\alpha) > 0$$

Hence:

$$\frac{dX(f)}{df} = -j2\pi \cdot \frac{1}{(1+j2\pi f)^2} = -\frac{j2\pi}{(1+j2\pi f)^2}$$

Step 4: Verification by direct differentiation

$$\frac{dX(f)}{df} = \frac{d}{df}\left(\frac{1}{1+j2\pi f}\right) = -\frac{j2\pi}{(1+j2\pi f)^2}.$$

Step 5: Conclusion

$$\frac{dX(f)}{df} = \mathcal{F}\{-j2\pi t\, e^{-t}u(t)\} = -\frac{j2\pi}{(1+j2\pi f)^2}.$$

This confirms that multiplication by t ***in the time domain corresponds to differentiation with respect to*** f ***in the frequency domain.***

8. Fourier Transform of the Complex Conjugate

Theorem 6.9 (Fourier Transform of Complex Conjugate) ***Let*** $x(t)$ ***be a signal with Fourier transform***

$$X(f) = \mathcal{F}\{x(t)\}.$$

Then the Fourier transform of the complex conjugate $\overline{x(t)}$ ***is given by:***

$$\mathcal{F}\{\overline{x(t)}\} = \overline{X(-f)}.$$

Proof.

1. **By definition, the Fourier transform of $\overline{x(t)}$ is**

$$\mathcal{F}\{\overline{x(t)}\} = \int_{-\infty}^{\infty} \overline{x(t)}\, e^{-j2\pi ft}\, dt$$

2. **Consider $X(-f)$:**

$$X(-f) = \int_{-\infty}^{\infty} x(t)\, e^{j2\pi ft}\, dt$$

3. **Take the complex conjugate:**

$$\overline{X(-f)} = \int_{-\infty}^{\infty} \overline{x(t)}\, e^{-j2\pi ft}\, dt$$

4. **Hence:**

$$\mathcal{F}\{\overline{x(t)}\} = \overline{X(-f)}$$

■

Example 6.8 ***Let***

$$x(t) = e^{-j2\pi f_0 t}.$$

Step 1: Fourier transform of $x(t)$

$$X(f) = \delta(f - f_0)$$

Step 2: Complex conjugate of $x(t)$

$$\overline{x(t)} = e^{j2\pi f_0 t}$$

Step 3: Fourier transform of $\overline{x(t)}$

$$\mathcal{F}\{\overline{x(t)}\} = \delta(f + f_0) = \overline{X(-f)}$$

This verifies the theorem.

Remark 6.9 ***- For real signals,*** $\overline{x(t)} = x(t)$***.***
- Therefore, $X(-f) = \overline{X(f)}$***, which is the conjugate symmetry property.***

6.2 Inverse Fourier Transform

6.2.1 Definition

If $X(f)$ is the Fourier transform of $x(t)$, then the original signal $x(t)$ can be recovered by the **inverse Fourier transform:**

$$x(t) = \mathcal{F}^{-1}\{X(f)\} = \int_{-\infty}^{\infty} X(f)\, e^{j2\pi ft}\, df$$

- This integral recovers the time-domain signal from its frequency-domain representation.

Example 6.9 ***Given***

$$X(f) = \frac{1}{1 + j2\pi f},$$

find $x(t) = \mathcal{F}^{-1}\{X(f)\}$***.***
Proof. We recognize this as a standard Fourier pair:

$$\mathcal{F}\{e^{-at}u(t)\} = \frac{1}{a + j2\pi f}, \quad a > 0$$

Comparing with $X(f) = \frac{1}{1+j2\pi f}$***, we have*** $a = 1$***.***
Therefore, the inverse Fourier transform is:

$$x(t) = e^{-t}u(t)$$

■

6.2.2 Properties of the Inverse Fourier Transform

Theorem 6.10 (Linearity)

$$\mathcal{F}^{-1}\{aX_1(f) + bX_2(f)\} = ax_1(t) + bx_2(t)$$

Proof. Follows directly from the linearity of the integral:

$$\int (aX_1(f) + bX_2(f))e^{j2\pi ft}df = a\int X_1(f)e^{j2\pi ft}df + b\int X_2(f)e^{j2\pi ft}df$$

■

Example 6.10

$$X(f) = 2X_1(f) - X_2(f) \implies x(t) = 2x_1(t) - x_2(t)$$

Theorem 6.11 (Time Shifting)

$$\mathcal{F}^{-1}\{X(f)e^{-j2\pi ff_0}\} = x(t - f_0)$$

Proof.

$$\mathcal{F}^{-1}\{X(f)e^{-j2\pi ff_0}\} = \int_{-\infty}^{\infty} X(f)e^{-j2\pi ff_0}e^{j2\pi ft}df = \int_{-\infty}^{\infty} X(f)e^{j2\pi f(t-f_0)}df = x(t - f_0)$$

■

Example 6.11 ***If*** $X(f) = \frac{1}{1+j2\pi f}$ ***and*** $f_0 = 1$***, then***

$$x(t) = e^{-(t-1)}u(t - 1)$$

Theorem 6.12 (Frequency Shifting)

$$\mathcal{F}^{-1}\{X(f-f_0)\} = x(t)e^{j2\pi f_0 t}$$

Proof.

$$\mathcal{F}^{-1}\{X(f-f_0)\} = \int X(f-f_0)e^{j2\pi ft}df$$

Change of variable $u = f - f_0$***,*** $du = df$***:***

$$= \int X(u)e^{j2\pi(u+f_0)t}du = e^{j2\pi f_0 t}\int X(u)e^{j2\pi ut}du = x(t)e^{j2\pi f_0 t}$$

■

Example 6.12 ***If*** $X(f) = \frac{1}{1+j2\pi f}$ ***and*** $f_0 = 2$***, then***

$$x(t) = e^{-t}u(t)e^{j4\pi t}$$

Theorem 6.13 (Scaling)

$$\mathcal{F}^{-1}\{X(af)\} = \frac{1}{|a|}x\left(\frac{t}{a}\right), \quad a \neq 0$$

Proof.

$$\mathcal{F}^{-1}\{X(af)\} = \int X(af)e^{j2\pi ft}df$$

Substitute $u = af \implies du = adf$***:***

$$= \int X(u)e^{j2\pi\frac{u}{a}t}\frac{du}{a} = \frac{1}{|a|}\int X(u)e^{j2\pi u\frac{t}{a}}du = \frac{1}{|a|}x\left(\frac{t}{a}\right)$$

■

Example 6.13 ***If*** $X(f) = \frac{1}{1+j2\pi f}$ ***and*** $a = 2$***, then***

$$x(t) = \frac{1}{2}e^{-t/2}u(t/2)$$

—

Theorem 6.14 (Conjugation)

$$\mathcal{F}^{-1}\{\overline{X(f)}\} = \overline{x(-t)}$$

Proof.

$$\mathcal{F}^{-1}\{\overline{X(f)}\} = \int \overline{X(f)}e^{j2\pi ft}df$$

Change variable $u = -f$***,*** $du = -df$***:***

$$= \int \overline{X(-u)}e^{-j2\pi ut}(-du) = \int \overline{X(-u)}e^{-j2\pi ut}du = \overline{x(-t)}$$

■

6.2.3 Parseval's Theorem (Conservation of Energy)

Theorem 6.15 (Parseval's Theorem) ***Let $x(t)$ be a signal with Fourier transform***

$$X(f) = \mathcal{F}\{x(t)\} = \int_{-\infty}^{+\infty} x(t)e^{-j2\pi ft}dt.$$

Then the total energy of the signal is conserved in the frequency domain:

$$\int_{-\infty}^{+\infty} |x(t)|^2 dt = \int_{-\infty}^{+\infty} |X(f)|^2 df.$$

Proof.
Step 1: Energy in time domain
The energy of the signal in the time domain is defined as

$$E = \int_{-\infty}^{+\infty} |x(t)|^2 dt = \int_{-\infty}^{+\infty} x(t)\overline{x(t)}dt.$$

Step 2: Express $x(t)$ in terms of its Fourier transform
To connect the time-domain energy to the frequency domain, we use the inverse Fourier transform:

$$x(t) = \int_{-\infty}^{+\infty} X(f)e^{j2\pi ft}df.$$

- Writing $x(t)$ in terms of $X(f)$ allows us to express the time-domain energy as an integral over the frequency components. - Each frequency component $X(f)$ contributes to $x(t)$, and therefore to the total energy.
Similarly, the complex conjugate is:

$$\overline{x(t)} = \int_{-\infty}^{+\infty} \overline{X(\nu)}e^{-j2\pi\nu t}d\nu.$$

Step 3: Substitute into the energy integral

$$\int_{-\infty}^{+\infty} |x(t)|^2 dt = \int_{-\infty}^{+\infty} \left(\int_{-\infty}^{+\infty} X(f)e^{j2\pi ft}df\right)\left(\int_{-\infty}^{+\infty} \overline{X(\nu)}e^{-j2\pi\nu t}d\nu\right) dt$$

- Now the energy integral is expressed entirely in terms of the frequency-domain representation.
- This is why we first wrote $x(t)$ in terms of $X(f)$.
Step 4: Rearrange as double integral

$$\int_{-\infty}^{+\infty} |x(t)|^2 dt = \int_{-\infty}^{+\infty}\int_{-\infty}^{\infty} X(f)\overline{X(\nu)}\left(\int_{-\infty}^{\infty} e^{j2\pi(f-\nu)t}dt\right) df\, d\nu$$

- The inner integral over t gives the Dirac delta:

$$\int_{-\infty}^{+\infty} e^{j2\pi(f-\nu)t}dt = \delta(f-\nu)$$

Step 5: Apply the sifting property of the Dirac delta

$$\int_{-\infty}^{+\infty} X(f)\overline{X(\nu)}\delta(f-\nu)df = X(\nu)\overline{X(\nu)} = |X(\nu)|^2$$

- The delta function "picks out" the value at $f = \nu$, collapsing the double integral into a single integral.
Step 6: Integrate over ν

$$\int_{-\infty}^{+\infty} |x(t)|^2 dt = \int_{-\infty}^{\infty} |X(\nu)|^2 d\nu = \int_{-\infty}^{+\infty} |X(f)|^2 df$$

- This confirms that the total energy is conserved in the frequency domain.
■

Example 6.14 ***Let***

$$x(t) = e^{-t}u(t),$$

where $u(t)$ is the unit step function. Verify Parseval's theorem.
Step 1: Energy in time domain

$$\int_0^{\infty} |x(t)|^2 dt = \int_0^{\infty} e^{-2t} dt = \frac{1}{2}.$$

Step 2: Fourier transform of $x(t)$

$$X(f) = \int_0^{\infty} e^{-t} e^{-j2\pi f t} dt = \frac{1}{1 + j2\pi f}.$$

Step 3: Energy in frequency domain

$$\int_{-\infty}^{\infty} |X(f)|^2 df = \int_{-\infty}^{\infty} \frac{1}{1 + (2\pi f)^2} df$$

Change of variable $u = 2\pi f$, $df = du/(2\pi)$:

$$\int_{-\infty}^{\infty} |X(f)|^2 df = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{du}{1 + u^2} = \frac{1}{2\pi} \cdot \pi = \frac{1}{2}.$$

Step 4: Verification

$$\int_0^{\infty} |x(t)|^2 dt = \frac{1}{2} = \int_{-\infty}^{\infty} |X(f)|^2 df$$

This confirms Parseval's theorem.

6.2.4 Fourier Series Expansion of a Non-Periodic Function

1. Fourier Series of a Periodic Function

Let $f_T(t)$ be a periodic function with period T. Its Fourier series is:

$$f_T(t) = \sum_{n=-\infty}^{\infty} c_n e^{jn\omega_0 t}, \quad \omega_0 = \frac{2\pi}{T}$$

where the Fourier coefficients are:

$$c_n = \frac{1}{T} \int_{-T/2}^{T/2} f_T(t) e^{-jn\omega_0 t} dt$$

- Each c_n gives the amplitude and phase of the harmonic of frequency $n\omega_0$.

2. Non-Periodic Functions

For a non-periodic function $f(t)$, we consider it as the limit of a periodic function with very large period $T \to \infty$.
- As T increases, $\omega_0 = \frac{2\pi}{T} \to 0$.
- The discrete frequencies $n\omega_0$ become continuous: $\omega = n\omega_0 \in \mathbb{R}$.
- The Fourier series sum transforms into an integral over all frequencies.

3. From Discrete Sum to Integral

Start from the Fourier series:

$$f_T(t) = \sum_{n=-\infty}^{\infty} c_n e^{jn\omega_0 t}, \quad c_n = \frac{1}{T}\int_{-T/2}^{T/2} f_T(\tau)e^{-jn\omega_0\tau}d\tau$$

Define the frequency increment:

$$\Delta\omega = \omega_0 = \frac{2\pi}{T}$$

Then:

$$c_n = \frac{\Delta\omega}{2\pi}\int_{-T/2}^{T/2} f_T(\tau)e^{-jn\Delta\omega\tau}d\tau$$

$$f_T(t) = \sum_{n=-\infty}^{\infty} \frac{\Delta\omega}{2\pi}\int_{-T/2}^{T/2} f_T(\tau)e^{-jn\Delta\omega\tau}d\tau\, e^{jn\Delta\omega t}$$

4. Limit as $T \to \infty$

Taking $T \to \infty$ ($\Delta\omega \to 0$):

$$f(t) = \lim_{T\to\infty}\sum_{n=-\infty}^{\infty} \frac{\Delta\omega}{2\pi}\int_{-T/2}^{T/2} f(\tau)e^{-jn\Delta\omega\tau}d\tau\, e^{jn\Delta\omega t}$$

$$\Rightarrow f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} f(\tau)e^{-j\omega\tau}d\tau\right]e^{j\omega t}d\omega$$

- The term in brackets is the Fourier transform:

$$\boxed{F(\omega) = \int_{-\infty}^{\infty} f(\tau)e^{-j\omega\tau}d\tau}$$

- Therefore, the inverse Fourier transform is:

$$\boxed{f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{j\omega t}d\omega}$$

—

5. **Interpretation**

- **Fourier series decomposes a periodic function into discrete harmonics.**
- **For a non-periodic function, the harmonics become continuous, forming a continuous spectrum.**
- **The Fourier transform is the continuous analogue of the Fourier series.**
- **The coefficients c_n become a continuous function $F(\omega)$ of frequency.**

Example 6.15 ***Let $f(t) = e^{-at}u(t)$, $a > 0$.***

$$\begin{aligned} F(\omega) &= \int_0^{\infty} e^{-at} e^{-j\omega t} dt \\ &= \int_0^{\infty} e^{-(a+j\omega)t} dt \\ &= \left[\frac{-1}{a+j\omega} e^{-(a+j\omega)t} \right]_0^{\infty} \\ &= 0 - \left(\frac{-1}{a+j\omega} \right) = \frac{1}{a+j\omega} \end{aligned}$$

- This result shows that the non-periodic function $e^{-at}u(t)$ has a **continuous spectrum**, which is the limit of the Fourier series coefficients as $T \to \infty$.

6.3 Application of the Fourier Transform to Solving Differential Equations

The Fourier transform is a powerful tool to solve linear differential equations. It converts derivatives in the time domain into algebraic multiplication in the frequency domain, simplifying the solution process.

6.3.1 General Principle

Consider a linear differential equation of order n with constant coefficients:

$$y^{(n)}(t) + a_{n-1} y^{(n-1)}(t) + \cdots + a_1 y'(t) + a_0 y(t) = f(t),$$

where $f(t)$ is a known input signal and $y(t)$ is the unknown output.
Steps to solve using Fourier transform:

1. **Apply the Fourier transform to each term. For derivatives:**

$$\mathcal{F}\{y'(t)\} = j2\pi f\, Y(f), \quad \mathcal{F}\{y''(t)\} = (j2\pi f)^2\, Y(f), \quad \ldots$$

2. **The differential equation becomes an algebraic equation in the frequency domain:**

$$\Big((j2\pi f)^n + a_{n-1}(j2\pi f)^{n-1} + \cdots + a_1(j2\pi f) + a_0 \Big) Y(f) = F(f),$$

   where $F(f) = \mathcal{F}\{f(t)\}$.

3. Solve for $Y(f)$:

$$Y(f) = \frac{F(f)}{(j2\pi f)^n + a_{n-1}(j2\pi f)^{n-1} + \cdots + a_1(j2\pi f) + a_0}.$$

4. Apply the inverse Fourier transform to recover $y(t)$:

$$y(t) = \mathcal{F}^{-1}\{Y(f)\}.$$

Remark 6.10 ***Differentiation in time corresponds to multiplication by $(j2\pi f)^n$ in frequency, simplifying the solution of linear differential equations.***

Example 6.16 ***1. Example: First-Order Differential Equation***
Solve:

$$y'(t) + y(t) = e^{-t}u(t),$$

where $u(t)$ is the unit step function.
Proof. Step 1: Fourier Transform

$$\mathcal{F}\{y'(t)\} + \mathcal{F}\{y(t)\} = \mathcal{F}\{e^{-t}u(t)\}$$

$$(j2\pi f)Y(f) + Y(f) = \frac{1}{1 + j2\pi f} \quad \Rightarrow \quad Y(f)(1 + j2\pi f) = \frac{1}{1 + j2\pi f}$$

$$\Rightarrow Y(f) = \frac{1}{(1 + j2\pi f)^2}.$$

Step 2: Inverse Fourier Transform
Using the known Fourier pair:

$$\mathcal{F}\{te^{-t}u(t)\} = \frac{1}{(1 + j2\pi f)^2} \quad \Rightarrow \quad y(t) = te^{-t}u(t).$$

Solution:

$$y(t) = te^{-t}u(t).$$

■

3. Advantages

- **Converts differential equations into algebraic equations in frequency.**
- **Handles signals defined for all $t \in (-\infty, \infty)$.**
- **Useful for analyzing linear time-invariant (LTI) systems and frequency response.**
- **Simplifies convolutions, since convolution in time corresponds to multiplication in frequency.**

6.4 Solved Exercises

Exercise 6.1 ***Calculate the Fourier transform of the following common signals:***

1. ***Unit step function:*** $u(t)$
2. ***Exponential decay:*** $e^{-at}u(t)$, $a > 0$
3. ***Sine function:*** $\sin(2\pi f_0 t)$
4. ***Cosine function:*** $\cos(2\pi f_0 t)$

Solution 6.1 ***The Fourier transforms of the above signals are summarized in the following table:***

Signal $x(t)$	***Fourier Transform*** $X(f)$
$u(t)$	$\frac{1}{j2\pi f} + \pi\delta(f)$
$e^{-at}u(t)$	$\frac{1}{a + j2\pi f}$
$\sin(2\pi f_0 t)$	$\frac{1}{2j}[\delta(f - f_0) - \delta(f + f_0)]$
$\cos(2\pi f_0 t)$	$\frac{1}{2}[\delta(f - f_0) + \delta(f + f_0)]$

We compute the Fourier transform of each signal .
1. Unit step: $x(t) = u(t)$

$$X(f) = \int_{-\infty}^{\infty} u(t)e^{-j2\pi ft}dt = \int_{0}^{\infty} e^{-j2\pi ft}dt$$

$$X(f) = \lim_{A\to\infty} \int_{0}^{A} e^{-j2\pi ft}dt = \lim_{A\to\infty} \frac{1 - e^{-j2\pi fA}}{j2\pi f}$$

Using the distribution property:

$$X(f) = \frac{1}{j2\pi f} + \pi\delta(f)$$

2. Exponential decay: $x(t) = e^{-at}u(t)$, $a > 0$

$$X(f) = \int_{0}^{\infty} e^{-at}e^{-j2\pi ft}dt = \int_{0}^{\infty} e^{-(a+j2\pi f)t}dt$$

$$X(f) = \left[\frac{e^{-(a+j2\pi f)t}}{-(a + j2\pi f)}\right]_0^{\infty} = \frac{1}{a + j2\pi f}$$

3. Sine function: $x(t) = \sin(2\pi f_0 t)$

$$\sin(2\pi f_0 t) = \frac{e^{j2\pi f_0 t} - e^{-j2\pi f_0 t}}{2j}$$

$$X(f) = \frac{1}{2j}\int_{-\infty}^{\infty}\Big(e^{j2\pi f_0 t} - e^{-j2\pi f_0 t}\Big)e^{-j2\pi f t}dt$$

$$X(f) = \frac{1}{2j}\left[\int_{-\infty}^{\infty} e^{-j2\pi(f-f_0)t}dt - \int_{-\infty}^{\infty} e^{-j2\pi(f+f_0)t}dt\right] = \frac{1}{2j}\Big[\delta(f-f_0) - \delta(f+f_0)\Big]$$

4. Cosine function: $x(t) = \cos(2\pi f_0 t)$

$$\cos(2\pi f_0 t) = \frac{e^{j2\pi f_0 t} + e^{-j2\pi f_0 t}}{2}$$

$$X(f) = \frac{1}{2}\int_{-\infty}^{\infty}\Big(e^{j2\pi f_0 t} + e^{-j2\pi f_0 t}\Big)e^{-j2\pi f t}dt$$

$$X(f) = \frac{1}{2}\left[\int_{-\infty}^{\infty} e^{-j2\pi(f-f_0)t}dt + \int_{-\infty}^{\infty} e^{-j2\pi(f+f_0)t}dt\right] = \frac{1}{2}\Big[\delta(f-f_0) + \delta(f+f_0)\Big]$$

Exercise 6.2 ***Calculate the inverse Fourier transform of the following signals:***

1. $X(f) = 1$

2. $X(f) = \delta(f - f_0)$

3. $X(f) = \frac{1}{a+j2\pi f}$, $a > 0$

4. $X(f) = \frac{1}{2}[\delta(f - f_0) + \delta(f + f_0)]$

5. $X(f) = \frac{1}{2j}[\delta(f - f_0) - \delta(f + f_0)]$

Solution 6.2 ***Inverse Fourier transforms table:***

Frequency Signal $X(f)$	***Inverse Fourier Transform*** $x(t)$
1	$\delta(t)$
$\delta(f - f_0)$	$e^{j2\pi f_0 t}$
$\frac{1}{a+j2\pi f}$	$e^{-at}u(t)$
$\frac{1}{2}[\delta(f - f_0) + \delta(f + f_0)]$	$\cos(2\pi f_0 t)$
$\frac{1}{2j}[\delta(f - f_0) - \delta(f + f_0)]$	$\sin(2\pi f_0 t)$

We use the inverse Fourier transform definition:

$$x(t) = \int_{-\infty}^{\infty} X(f)e^{j2\pi f t}df$$

1. $X(f) = 1$

$$x(t) = \int_{-\infty}^{\infty} e^{j2\pi f t}df = \delta(t)$$

2. $X(f) = \delta(f - f_0)$

$$x(t) = \int_{-\infty}^{\infty} \delta(f - f_0) e^{j2\pi ft} df = e^{j2\pi f_0 t}$$

3. $X(f) = \frac{1}{a+j2\pi f}$

$$x(t) = \int_{-\infty}^{\infty} \frac{e^{j2\pi ft}}{a + j2\pi f} df = e^{-at} u(t)$$

4. $X(f) = \frac{1}{2}[\delta(f - f_0) + \delta(f + f_0)]$

$$x(t) = \frac{1}{2} \int \delta(f - f_0) e^{j2\pi ft} df + \frac{1}{2} \int \delta(f + f_0) e^{j2\pi ft} df = \cos(2\pi f_0 t)$$

5. $X(f) = \frac{1}{2j}[\delta(f - f_0) - \delta(f + f_0)]$

$$x(t) = \frac{1}{2j} \int \delta(f - f_0) e^{j2\pi ft} df - \frac{1}{2j} \int \delta(f + f_0) e^{j2\pi ft} df = \sin(2\pi f_0 t)$$

Exercise 6.3 ***Solve the following differential equation using the Fourier transform method.***

$$\frac{dy}{dt} + ay(t) = u(t), \quad a > 0, \quad y(0) = 0$$

where $u(t)$ ***is the unit step function (Heaviside function) defined by:***

$$u(t) = \begin{cases} 0, & t < 0 \\ 1, & t \geq 0 \end{cases}$$

Remark 6.11 - $u(t)$ ***represents a signal that "switches on" at*** $t = 0$***.***
- Its Fourier transform is:

$$\mathcal{F}\{u(t)\} = \frac{1}{j2\pi f} + \pi\delta(f)$$

- Its derivative is the Dirac delta function:

$$\frac{du(t)}{dt} = \delta(t)$$

Solution 6.3 ***Step 0: Recall the Fourier transform properties***
- The Fourier transform of a derivative:

$$\mathcal{F}\left\{\frac{dy}{dt}\right\} = j2\pi f Y(f)$$

- The Fourier transform of the unit step function $u(t)$***:***

$$\mathcal{F}\{u(t)\} = \frac{1}{j2\pi f} + \pi\delta(f)$$

Step 1: Take the Fourier transform of both sides
The differential equation is:

$$\frac{dy}{dt} + ay(t) = u(t)$$

Apply Fourier transform term by term:

$$j2\pi f Y(f) + aY(f) = \frac{1}{j2\pi f} + \pi\delta(f)$$

—

Step 2: Solve for $Y(f)$

$$Y(f)(a + j2\pi f) = \frac{1}{j2\pi f} + \pi\delta(f)$$

$$\boxed{Y(f) = \frac{\frac{1}{j2\pi f} + \pi\delta(f)}{a + j2\pi f}}$$

—

Step 3: Split the fraction for easier inversion

$$Y(f) = \frac{1}{j2\pi f(a + j2\pi f)} + \frac{\pi\delta(f)}{a + j2\pi f}$$

Step 4: Inverse Fourier transform

1. The first term corresponds to the causal response:

$$\mathcal{F}^{-1}\left\{\frac{1}{j2\pi f(a + j2\pi f)}\right\} = \frac{1}{a}\left(1 - e^{-at}\right)u(t)$$

2. The second term involves the delta function:

$$\mathcal{F}^{-1}\left\{\frac{\pi\delta(f)}{a + j2\pi f}\right\} = \int_{-\infty}^{\infty} \frac{\pi\delta(f)}{a + j2\pi f} e^{j2\pi f t} df = \frac{\pi}{a}$$

In fact:

$$\begin{aligned}
\mathcal{F}^{-1}\left\{\frac{\pi\delta(f)}{a + j2\pi f}\right\} &= \int_{-\infty}^{\infty} \frac{\pi\delta(f)}{a + j2\pi f}\, e^{j2\pi f t}\, df \quad \textbf{\textit{(definition of inverse Fourier transform)}} \\
&= \pi \int_{-\infty}^{\infty} \delta(f) \cdot \frac{e^{j2\pi f t}}{a + j2\pi f}\, df \quad \textbf{\textit{(factor out the constant }} \pi) \\
&= \pi \cdot \frac{e^{j2\pi(0)t}}{a + j2\pi(0)} \quad \textbf{\textit{(sifting property of }} \delta(f) : \int \delta(f) g(f) df = g(0)) \\
&= \pi \cdot \frac{1}{a} \cdot 1 \quad \textbf{\textit{(since }} e^0 = 1 \textbf{\textit{ and }} j2\pi \cdot 0 = 0) \\
&= \frac{\pi}{a}
\end{aligned}$$

- In a causal system with $y(0) = 0$***, this constant is already accounted for by the*** $1 - e^{-at}$ ***term.***

- Therefore, for $t > 0$***, the*** $\delta(f)$ ***term does not contribute beyond the first term, ensuring the solution satisfies the initial condition.***

Step 5: Final solution in time domain

$$\boxed{y(t) = \frac{1}{a}\left(1 - e^{-at}\right)u(t)}$$

- ***At*** $t = 0$***,*** $y(0) = 0$***.***
- ***As*** $t \to \infty$***,*** $y(t) \to 1/a$***, the steady-state value.***
- ***The system response is the classic first-order exponential rise due to a step input.***

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