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Université Djilali Bounaama, Khemis Miliana
Faculté des Sciences de la Matière et d'informatique
Conseil Scientifique de la Faculté

جامعة جيلالي بونعامة خميس مليانة
كلية علوم المادة والإعلام الآلي
المجلس العلمي للكلية

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EXTRAIT DU PV
DE LA REUNION ORDINAIRE DU CONSEIL SCIENTIFIQUE
Du 02/02/2026

Objet : : Expertise de polycopié pédagogique

En l'an deux mille vingt-six (2026), le lundi 02 février à 13 h 30, une réunion ordinaire du Conseil Scientifique de la Faculté des Sciences de la Matière et de l'Informatique s'est tenue dans la salle de réunion de la faculté (Bloc B).

Suite aux rapports favorables reçus de la part des experts cités ci-après concernant l'expertise du polycopié pédagogique, le CSF a prononcé favorablement pour la conformité du polycopié pédagogique en vue de préparer son professorat.

- **Auteur du polycopié** : Dr. HOUASNI Mohamed (MCA)
- **Intitulé du polycopié** : Course and Exercises In Algebra 2.
- **Destiné aux étudiants de** : 1ére année Licence Mathématiques.
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Ministry of Higher Education and Scientific Research

Djilali Bounaama University Khemis Miliana
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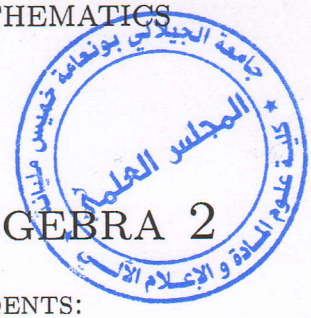

Handout on Algebra 2

Intended for students:
Level: L1 Mathematics and Computer Science
Specialization: Mathematics

Prepared by:

HOUASNI Mohamed

Course and Exercises in Algebra 2

ACADEMIC YEAR: 2025–2026

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Author

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PREFACE

Students generally seek simplified references to understand the mathematical concepts they have learned in a way that suits their scientific abilities. Therefore, we provided them with this handout, which contains lessons and corrected exercises compatible with the curriculum of the Ministry of Higher Education and Scientific Research.

This handout is composed of four chapters:

1. Vector spaces.
2. Linear applications.
3. Matrices.
4. Systems of equations.

To fully understand this content, the reader should be familiar with concepts related to algebraic structures (groups, rings, and fields).

At the end of each chapter, the reader will find a series of exercises with detailed answers.

This course was presented to students at Djilali Bounâama University in Khemis Miliana, Ain Defla, Algeria.

We also invite our colleagues and students who would like to send me their comments, suggestions, and criticisms. Please do not hesitate to send them to us so that we can improve this document.

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CHAPTER 1

VECTOR SPACES

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Geometry predominated in Greek mathematics and it was not until Descartes in the 17th century that the link was made, thanks to the notion of reference, between geometric notions: of points on the plane or in space, of curves and algebraic ones, of pairs or triplets of real numbers and equations. This approach proved fruitful for both geometers and analysts. It offered the former all the power of analysis to deal with geometric problems and, to the latter, the representations of geometry to visualize and state the phenomena of analysis. The generalization of the geometric notions of the plane $\mathbb{R}^2$ and of space $\mathbb{R}^3$ Higher-dimensional spaces were not immediately apparent. The formalism to address this problem was lacking. It was the self-taught German mathematician Hermann Grassmann who, in the 19th century, outlined the notions of vector space and dimension. His work was difficult to understand, and it was thanks to the Italian mathematician Giuseppe Peano that these concepts were refined and took their definitive form. Linear algebra has become the framework for the study of many theories, particularly in analysis. Today, with the development of computing tools, linear algebra is more easily implemented using matrix calculus.

This chapter covers some fundamental concepts required for this document. Mastery of the material in this chapter is essential. The concepts covered are:

—Vector spaces.

—Vector subspaces.

—Basis and dimension of a vector space.

—Sum of vector subspaces.

1.1 Vector space

Throughout this chapter $\mathbb{k}$ designates the field $\mathbb{R}$ or $\mathbb{C}$. We will give the axioms that define a vector space.

Definition 1.1 ($\mathbb{k}$-espace vectoriel) *Let $(\mathbb{k}, +, \times)$ be a field. We call a vector space over the field $\mathbb{k}$ any set E equipped with an internal composition law $+$ (addition)*

$$\circledast : \begin{cases} E \times E & \to & E \\ (x, y) & \mapsto & x \circledast y \end{cases}$$

and an external composition law $\cdot$ (multiplication by a scalar)

$$. : \begin{cases} \mathbb{k} \times E & \to & E \\ (\lambda, y) & \mapsto & \lambda.x \end{cases}$$

such that

1. *$(E, \circledast)$ is a commutative group. We denote 0_E its neutral element.*
2. *For all $(\alpha, \beta) \in \mathbb{k}^2$, and for all $(x, y) \in E^2$, we have*

- $(\alpha + \beta).x = \alpha.x \circledast \beta.x$ *Axiom 1*
- $(\alpha \times \beta).x = \alpha.(\beta.x)$ *Axiom 2*
- $\alpha.(x \circledast y) = \alpha.x \circledast \alpha.y$ *Axiom 3*
- $1_{\mathbb{k}}.x = x$ *Axiom 4*

We then say that $(E, \circledast, \cdot)$ is a K-vector space. The elements of $\mathbb{k}$ are called scalars, those of E, vectors. The neutral element of $(E, \circledast)$, 0_E is called the null vector.

Example 1.1 *The commutative field $\mathbb{k}$, is a vector space on itself, for addition and the product existing on $\mathbb{k}$.*

Example 1.2 *Let $n \in \mathbb{N}^*$ be a non-zero integer, we define the Cartesian product $\mathbb{k}^n$ as follows*
If $n = 1$, $\mathbb{k}^1 = \mathbb{k}$.
If $n = 2$, $\mathbb{k}^2 = \mathbb{k} \times \mathbb{k}$ is the set of pairs formed from elements of E.
If $n = 3$, $\mathbb{k}^3 = \mathbb{k}^2 \times \mathbb{k}$ by definition, it is therefore the set of couples of the type (a, z) with a in $\mathbb{k}^2$ so of the type $a = (x, y)$ with x and y in $\mathbb{k}$. Instead of noting $((x, y), z)$ the generic element of $\mathbb{k}^3$, we denote it (x, y, z) and we talk about the triplet, (x, y, z) of $\mathbb{k}^3$.

More generally, $\mathbb{k}^n$ will be, by rare definition, the Cartesian product of $\mathbb{k}^{n-1}$ by $\mathbb{k}$, let $\mathbb{k}^n = \mathbb{k}^{n-1} \times \mathbb{k}$, and for convenience, we will denote $(x_1, x_2, ..., x_n)$ the generic element of $\mathbb{k}^n$, (with each $x_i \in \mathbb{k}$) called n-tuple.

Formally, we therefore define $\mathbb{k}^n$ from the couples. On $\mathbb{k}^n$, we define a vector space structure by posing:

for $X = (x_1, x_2, ..., x_n)$ and $Y = (y_1, y_2, ..., y_n)$ in $\mathbb{k}^n$, and λ in $\mathbb{k}$,

$$\begin{aligned} X + Y &= (x_1 + y_1, ..., x_n + y_n), \\ \lambda.X &= (\lambda x_1, \lambda x_2, ..., \lambda x_n), \end{aligned}$$

and it is easy to verify that we have a vector space structure.

Example 1.3 *The set $\mathbb{R}_n[x]$ polynomial functions with coefficients in $\mathbb{R}$ of degree $\leq n$, that's to say :*

$$\mathbb{R}_n[x] = \{P : \mathbb{R} \to \mathbb{R} |\ P(x) = a_0 + a_1 x + ... + a_n x^n,\ a_i \in \mathbb{R}\}$$

is a vector space over $\mathbb{R}$ for the laws:

$$\begin{aligned} (a_0 + a_1 x + ... + a_n x^n) + (b_0 + b_1 x + ... + b_n x^n) &:= (a_0 + b_0) + ... + (a_n + b_n) x^n \\ \lambda.(a_0 + a_1 x + ... + a_n x^n) &:= \lambda a_0 + \lambda a_1 x + ... + \lambda a_n x^n \end{aligned}$$

More generally, the set $\mathbb{R}[x]$ polynomial functions of all possible degrees with coefficients in $\mathbb{R}$ is a vector space over $\mathbb{R}$ with the laws:

$$\begin{aligned} \sum_k a_k x^k + \sum_k b_k x^k &:= \sum_k (a_k + b_k) x^k \\ \lambda . \sum_k a_k x^k &:= \sum_k (\lambda a_k) x^k \end{aligned} \tag{1.1}$$

(the sums only contain a finite number of non-zero terms).

Example 1.4 *Let A be a non-empty set and E a vector space over the field $\mathbb{k}$. The set E^A applications of A in E is equipped with a vector structure on $\mathbb{k}$ in the following way.*

If f and g are two functions from A to E, and λ a scalar in $\mathbb{k}$ we define the function $f + g$ by

$$\forall x \in A,\ (f + g)(x) = f(x) + g(x),$$

and the function $\lambda.f$ by

$$\forall x \in A, (\lambda.f)(x) = \lambda.f(x),$$

and here again it is a exercise to verify that we have a vector structure for the space of applications of A in the vector space E. Its zero vector is the identically zero function on A with values in E,

$$\begin{aligned} 0_{E^A} :\ A &\to E \\ x &\mapsto 0_E \end{aligned}$$

Example 1.5 *Let $E_1, ..., E_n$. n vector spaces over the commutative field $\mathbb{k}$, by proceeding by recurrence as for $\mathbb{k}^n$, we define the Cartesian product $E = E_1 \times E_2 \times ... \times E_n$ still noted $E = \prod_{i=1}^{n} E_i$, whose elements are the n-tuples $X = (X_1, ..., X_n)$, with $\forall i = 1...n$, $X_i \in E_i$.*

It is easy to verify that for the addition $+$ defined by

$$(X_1, ..., X_n) + (Y_1, ..., Y_n) = (X_1 + Y_1, ..., X_i + Y_i, ..., X_n + Y_n)$$

E is an additive group, and that with the product

$$\lambda.X = (\lambda X_1, \lambda X_2, ..., \lambda X_n),$$

we provide E with a vector structure.

1.2 Calculation rules

Proposition 1.1 *Let $(E, +, \cdot)$ be a $\mathbb{k}$-vector space. For all scalars $\alpha, \beta, \lambda \in \mathbb{k}$ and for all vectors $x, y \in E$, we have*

1. $0_{\mathbb{k}} \cdot x = 0_E$
2. $(-1) \cdot x = -x$
3. $(-\lambda) \cdot x = -(\lambda \cdot x) = \lambda \cdot (-x)$
4. $(\alpha - \beta).x = \alpha.x - \beta.x$
5. $\lambda.(x - y) = \lambda.x - \lambda.y$
6. $\lambda \cdot 0_E = 0_E$
7. $\lambda \cdot x = 0_E \Longleftrightarrow (\lambda = 0_{\mathbb{k}} \quad or \quad x = 0_E)$.

Proof. 1. We have

$$\begin{aligned} 0_{\mathbb{k}} \cdot x + 0_E &= 0_{\mathbb{k}} \cdot x \text{ because } (E, +) \text{ is a group} \\ &= (0_{\mathbb{k}} + 0_{\mathbb{k}}) \cdot x \text{ because } \mathbb{k} \text{ is a field} \\ &= 0_{\mathbb{k}} \cdot x + 0_{\mathbb{k}} \cdot x. \end{aligned}$$

From where $0_{\mathbb{k}} \cdot x = 0_E$ by subtracting $0_{\mathbb{k}} \cdot x$ to the right of the two members of this equality.
2. We have :

$$\begin{aligned} x + (-1) \cdot x &= 1.x + (-1) \cdot x \text{ thanks to the axiom 4} \\ &= (1 + (-1)) .x \text{ thanks to the axiom 1} \\ &= 0_{\mathbb{k}} \cdot x \text{ because } \mathbb{k} \text{ is a field} \\ &= 0_E \text{ according to 1 previous ones.} \end{aligned}$$

then $(-1).x$ is the opposite of x. We can then write : $-x = (-1).x$.

3. We have:

$$\begin{aligned}(-\lambda)x &= (-1.\lambda)x \text{ because } \mathbb{k} \text{ is a field}\\ &= (-1).(\lambda.x) \text{ thanks to the axiom 2}\\ &= -\lambda.x \text{ according to 2 previous ones.}\end{aligned}$$

4. We have:

$$\begin{aligned}(\alpha-\beta).x &= (\alpha+(-\beta)).x \text{ because } \mathbb{k} \text{ is a field}\\ &= \alpha.x+(-\beta).x \text{ thanks to the axiom 1}\\ &= \alpha.x-\beta.x \text{ according to 3 previous ones}\end{aligned}$$

5. We have:

$$\begin{aligned}\lambda.(x-y) &= \lambda.(x+(-y)) \text{ because } (E,+) \text{ is a group}\\ &= \lambda.x+\lambda.(-y) \text{ thanks to the axiom 3}\\ &= \lambda.x+\lambda.(-1y) \text{ according to 2 previous ones}\\ &= \lambda.x+(\lambda.(-1)).y \text{ thanks to the axiom 2}\\ &= \lambda.x+(-\lambda).y \text{ because } \mathbb{k} \text{ is a field}\\ &= \lambda.x-\lambda.y \text{ according to 3 previous ones}\end{aligned}$$

6. On a:

$$\begin{aligned}\lambda\cdot 0_E &= \lambda\cdot(x-x) \text{ car } (E,+) \text{ is a group}\\ &= \lambda\cdot x+\lambda\cdot(-x) \text{ hanks to the axiom 1}\\ &= \lambda\cdot x-\lambda\cdot x \text{ according to 3 previous ones}\\ &= 0_E \text{ car } (E,+) \text{ is a group.}\end{aligned}$$

7. Let's suppose that $\lambda\cdot x = 0_E$, if $\lambda = 0_{\mathbb{k}}$ so according to 3, λ. Otherwise, if $\lambda \neq 0_{\mathbb{k}}$ the as $\mathbb{k}$ is a field λ^{-1} exists and

$$x = 1.x = \left(\lambda.\lambda^{-1}\right).x = \lambda^{-1}.(\lambda.x) = \lambda^{-1}.0_E = 0_E,$$

and so $x = 0_E$. The converse is obvious. □

1.3 Vector subspaces

Definition 1.2 (Vector subspace) *We call a vector subspace of a vector space E over the field $\mathbb{k}$, any subset F of E which is an additive subgroup of E is such that $\forall\lambda \in \mathbb{k}$, $\forall x \in F$, $\lambda.x \in F$.*

Remark 1.1 *It is then obvious that F is a vector space on $\mathbb{k}$, the conditions of the definition 1.1 being verified.*

Example 1.6 *$\{0_E\}$ and E are vector subspaces of E.*

Definition 1.3 (Linear combination) ∘ *Let $x_1, x_2, ..., x_n$ n vectors of a $\mathbb{k}$-vector space E. We call any vector a linear combination of these n vectors $x \in E$ of the form*

$$x = \lambda_1 x_1 + ... + \lambda_p x_n = \sum_{k=1}^{n} \lambda_k x_k$$

where $(\lambda_1, ..., \lambda_n) \in \mathbb{k}^n$.

∘ If A is a part of E, we call a linear combination of elements of A any linear combination of a finite number of elements of A.

In principle, to show that F is a vector subspace, one would have to verify the eight axioms of the Definition 1.1 In fact, it is sufficient to check the "stability" of the composition laws as stated in the following proposition:

Proposition 1.2 *A part F of a vector space E on $\mathbb{k}$ is a vector subspace of E if and only if*

1. $F \neq \varnothing$
2. $\forall (\alpha, \beta) \in \mathbb{k}^2$, $\forall (x, y) \in F^2$, $\alpha.x + \beta.y \in F$.

Proof. $\Longrightarrow$) First if F is a subspace of E, as an additive subgroup F is non-empty because it contains 0, the null element of E, then if x et y are in F and α and β in the field $\mathbb{k}$, $\alpha.x$ and $\beta.y \in F$ (see definition 1.2) which is stable for addition, therefore $\alpha.x + \beta.y \in F$.

$\Longleftarrow$) If 1. and 2. are verified, with $\alpha = 1$ and $\beta = -1$ and x and y in F we have:

$$(F \neq \varnothing) \text{ et } \left(\forall (x, y) \in F^2, x - y \in F\right),$$

which already justifies that F is an additive subgroup of E, then 2. with $\beta = 0$ gives

$$\forall \lambda \in \mathbb{k}, \forall x \in F, \lambda.x \in F.$$

We have F vector subspace of E. □

Remark 1.2 *As we saw during the proof, if F is a vector subspace, then F necessarily contains the null vector.*

1.3.1 Fundamental examples of vector subspaces

1. Vector line:

Let $v \in E$, $v \neq 0$, then :

$$F = \{y \in E | \ \exists \lambda \in \Bbbk : y = \lambda v\}$$

is a vector subspace of E called a vector line generated by v.

Indeed $F \neq \emptyset$, because $v \in F$. What's more, F is stable for the laws of E, because if $x, y \in F$ (that's to say: $x = \lambda v$, $y = \mu v$), we have:

$$x + y = \lambda v + \mu v = (\lambda + \mu)v \in F$$

Likewise, if $x \in F$ (that's to say $x = \lambda v$), we have: $\mu x = \mu(\lambda v) = (\mu\lambda)v \in F$.

2. Vector plane:

Let $x_1, x_2 \in E$ then :

$$F = \{y \in E | \ \exists \lambda_1, \lambda_2 \in \Bbbk : y = \lambda_1 x_1 + \lambda_2 x_2\}$$

F is a vector subspace of E, called the subspace generated by x_1, x_2. If x_1 and x_2 are not null and x_2 does not belong to the vector line generated by x_1, F is said to be a vector plane generated by x_1 and x_2.

3. Generated subspace:

More generally, if $x_1, x_2, ..., x_p \in E$ then :

$$F = \{y \in E | \ \exists \lambda_1, ..., \lambda_p \in \Bbbk : y = \lambda_1 x_1 + ... + \lambda_p x_p\}$$

is a vector subspace of E denoted $Span\{x_1, x_2, ..., x_p\}$, said sub-space occupied by $x_1, ..., x_p$, or also space of linear combinations of $x_1, x_2, ..., x_p$. We will see later that, basically, all vector subspaces are of this type, that is to say obtained by "linear combinations" of a family of elements of E.

Remark 1.3 *Let E be the vector space of vectors with origin O. A vector line is a line passing through O. Similarly, a vector plane is a plane passing through O. More generally, a vector subspace of $\mathbb{R}^n$ can be visualized as a "p-dimensional plane" passing through O. One could give a precise meaning to the notion of "p-dimensional plane", but this is not necessary. Let us retain for the moment the fact that it must pass through O, because every vector subspace must contain the null vector. Thus, for example, a line not passing through O is not a vector subspace: the points of the line are the endpoints of the vectors from O and the null vector is not among them.*

Example 1.7 *Let*

$$F = \{(x, y, z) \in \mathbb{R}^3 |\ 3x + y + 2z = 0\}.$$

F is a vector subspace of $\mathbb{R}^3$. Indeed, let us $v_1 = (x_1, y_1, z_1)$ and $v_2 = (x_2, y_2, z_2) \in F$; we have:

$$3x_1 + y_1 + 2z_1 = 0 \text{ and } 3x_2 + y_2 + 2z_2 = 0$$

from which, by adding: $3(x_1 + x_2) + (y_1 + y_2) + 2(z_1 + z_2) = 0$, that's to say

$$v_1 + v_2 = (x_1 + x_2, y_1 + y_2, z_1 + z_2) \in F.$$

Similarly, we see that if $\lambda \in \mathbb{k}$ and $v \in F$ we have: $\lambda v \in F$.

Example 1.8 *We have:*

$$G = \{(x, y, z) \in \mathbb{R}^3 |\ x + 4y + z = 1\},$$

is not a vector subspace of $\mathbb{R}^3$ because $0_{\mathbb{R}^3} = (0, 0, 0) \notin G$ $(0 + 4.0 + 0 \neq 1)$.

Proposition 1.3 *Let F and G be two vector subspaces of E.*

1. *$F \cap G$ is a vector subspace of E.*
2. *$F \cup G$ is not in general a vector subspace of E.*
3. *The complementary $E \backslash F$ of a vector subspace F is not a vector subspace of E.*

Proof. **1.** We have first $F \cap G \neq \varnothing$, because $0_E \in F \cap G$.

Let $x, y \in F \cap G$, we have: $x, y \in F$ then $x + y \in F$. Likewise, if $x, y \in G$, $x + y \in G$ and consequently $x + y \in F \cap G$.

If $\lambda \in \mathbb{k}$ and $x \in F \cap G$, we have: $x \in F$, then $\lambda x \in F$, and $x \in G$, then $\lambda x \in G$, from where: $\lambda x \in F \cap G$.

2. This is because in general $F \cup G$ is not stable by the sum. For example, let $E = \mathbb{R}^2$, F the vector line generated by $(1, 0)$ and G the vector line generated by $(0, 1)$. We have : $(1, 0) \in F$ then $(1, 0) \in F \cup G$. $(0, 1) \in G$ then $(0, 1) \in F \cup G$ but: $w = (1, 0) + (0, 1) = (1, 1) \notin F \cup G$.

3. $E \backslash F$ does not contain 0_E, so it is not a vector subspace (Remark 1.2). □

1.4 Basis (in finite dimension)

Definition 1.4 (Generating family) *A family of vectors $\{v_1, ..., v_p\}$ of a vector space E is said to be generating, if $E = Span\{v_1, ..., v_p\}$, which means that everything $\forall x \in E$, decomposes on vectors v_i, or even that everything $x \in E$ is a linear combination of vectors v_i,*

$$x = \lambda_1 v_1 + ... + \lambda_p v_p = \sum_{k=1}^{p} \lambda_k v_k.$$

Remark 1.4 *Such a (finite) family does not always exist. Consider for example* $\mathbb{R}[x]$ *awith the vector space structure defined by the laws (1.1) and let* $\{P_1, ..., P_n\}$ *a finite family of polynomials. It cannot be generating because, by performing linear combinations, we will only obtain polynomials of* $\deg < Sup\{deg\ of\ (P_i)\}$.

Example 1.9 *In* $\mathbb{R}^2$, *let* $v_1 = (1, 1)$ *and* $v_2 = (1, -1)$. *Let us show that* $\{v_1, v_2\}$ *is generating. Let* $x = (a, b) \in \mathbb{R}^2$ *with* a, b *arbitrary: it is a question of showing that it exists* $x_1, x_2 \in \mathbb{R}$ *such that* $x = x_1v_1 + x_2v_2$, *that's to say:*

$$x = (a, b) = (x_1, x_1) + (x_2, -x_2) = (x_1 + x_2, x_1 - x_2)$$

This means that $\forall (a, b) \in \mathbb{R}^2$, $\exists\ x_1, x_2 \in \mathbb{R}$ *checking the system:*

$$\begin{cases} x_1 + x_2 = & a \\ x_1 - x_2 = & b, \end{cases}$$

By solving, we find in fact:

$$x_1 = \frac{a+b}{2}\ et\ x_2 = \frac{a-b}{2},$$

solution defined for arbitrary a, b. *Therefore* $\{v_1, v_2\}$ *is generative.*

Definition 1.5 *A vector space is said to be finite-dimensional if there exists a finite generating family; otherwise, it is said to be infinite-dimensional.*

Definition 1.6 (Linearly independent family) *Let* $\{v_1, ..., v_p\}$, *a finite family of elements of* E. *We say that it is free, if:*

$$\lambda_1v_1 + ... + \lambda_pv_p = 0 \Longrightarrow \lambda_1 = \lambda_2 = ... = \lambda_p = 0.$$

We also say that vectors $v_1, ..., v_p$ *are linearly independent. A family that is not free is said to be linked (its vectors are also said to be linked or linearly dependent).*

Example 1.10 *In* $\mathbb{R}^3$, *vectors* $v_1 = (1, 1, -1)$, $v_2 = (0, 2, 1)$ *and* $v_3 = (0, 0, 5)$ *are linearly independent. Indeed, suppose that there exist real numbers* λ_1, λ_2, λ_3 *so that* $\lambda_1v_1+\lambda_2v_2+\lambda_3v_3 = 0_{\mathbb{R}^3}$, *that's to say:*

$$\lambda_1(1, 1, -1) + \lambda_2(0, 2, 1) + \lambda_3(0, 0, 5) = 0_{\mathbb{R}^3}$$

We obtain

$$(\lambda_1, \lambda_2, -\lambda_1 + \lambda_2 + 5\lambda_3) = 0_{\mathbb{R}^3},$$

then

$$\begin{cases} \lambda_1 = & 0 \\ \lambda_2 = & 0 \\ -\lambda_1 + \lambda_2 + 5\lambda_3 = & 0 \end{cases}$$

which immediately gives $\lambda_1 = \lambda_2 = \lambda_3 = 0$.

Example 1.11 *In $\mathcal{F}(\mathbb{R},\mathbb{R})$ the family $\{\sin,\cos,\exp\}$ is linearly independent. Let $\alpha,\beta,\gamma\in\mathbb{R}$ such that $\alpha\sin+\beta\cos+\gamma\exp=0$. Then for all $x\in\mathbb{R}$, $\alpha\sin(x)+\beta\cos(x)+\gamma\exp(x)=0$, which is also written*

$$\alpha\frac{\sin(x)}{\exp(x)}+\beta\frac{\cos(x)}{\exp(x)}+\gamma=0.$$

It is easily shown using the gendarmes' theorem that

$$lim_{+\infty}\frac{\sin(x)}{\exp(x)}=lim_{+\infty}\frac{\cos(x)}{\exp(x)}=0.$$

We deduce that $\gamma=0$. We then have $\forall x\in\mathbb{R}$, $\alpha\sin(x)+\beta\cos(x)+\gamma\exp(x)=0$. If we do $x=0$ we obtaint $\beta=0$ and if we do $x=\pi/2$,it comes that $\alpha=0$. It was then clearly shown that $\alpha=\beta=\gamma=0$.

Proposition 1.4 *A family $\{v_1,...,v_p\}$ is linked if and only if at least one of the vectors v_i is written as a linear combination of the other vectors in the family.*

Proof. $\Longrightarrow$) : If $\{v_1,...,v_p\}$ is linearly dependent family, there exists λ_1,..., λ_p not all null such that $\lambda_1v_1+...+\lambda_pv_p=0$. If, for example $\lambda_1\neq0$, we can write:

$$v_1=-\frac{\lambda_2}{\lambda_1}v_2+...+\frac{-\lambda_p}{\lambda_1}v_p$$

$\Longleftarrow$) : suppose for example, that v_1 is a linear combination of vectors $\lambda_2v_2,...,\lambda_pv_p$, so there exists $\mu_2,...,\mu_p\in\Bbbk$ such thqt $v_1=\lambda_2v_2+...+\lambda_pv_p$, that's to say:

$$v_1-\lambda_2v_2-...-\lambda_pv_p=0.$$

So there exists a linear combination of vectors $\{v_1,...,v_p\}$ which is zero, without the coefficients all being zero. So the family is linked (linearly dependent). □

Proposition 1.5 *Let $\{v_1,...,v_p\}$ a linearly independent family and x any vector in the space generated by the vectors v_i (that is, x is a linear combination of the v_i). So the decomposition of x on the v_i is unique.*

Proof. Let

$$\begin{aligned}x&=\lambda_1v_1+...+\lambda_pv_p,\\x&=\beta_1v_1+...+\beta_pv_p,\end{aligned}$$

two decompositions of x. Taking the difference we find:

$$(\lambda_1-\beta_1)v_1+...+(\lambda_p-\beta_p)v_p=0,$$

Since the family is free, we have $(\lambda_1-\beta_1)=...=(\lambda_p-\beta_p)=0$, that's to say: $\lambda_1=\beta_1,...,\lambda_p=\beta_p$. □

Definition 1.7 (Basis) *A family that is both generating and linearly independent is called a basis.*

Proposition 1.6 *A familY $\{v_1, ..., v_p\}$ is a basis of E if and only if every $x \in E$ decomposes in a unique way on v_i. That's to say :*

$\forall x \in E$ there is a unique one n-scalars $(\lambda_1, ..., \lambda_n) \in \mathbb{k}^n$ so that:

$$x = \lambda_1 v_1 + ... + \lambda_n v_n.$$

Proof. The existence of decomposition for all $x \in E$ is equivalent to the fact that the family is generative; uniqueness to the fact that the family is linearly independent. □

Example 1.12 (Canonical basis of $\mathbb{R}^n$) *Let the vectors be:*

$$e_1 = (1, 0, ..., 0), ..., e_i = (0, ..., \underbrace{1}_{i^e rang}, ..., 0), ..., e_n = (0, 0, ..., 1).$$

We already know that they form a generating family. Let us show that it is free. We have:

$$\lambda_1 e_1 + ... + \lambda_i e_i + ... + e_n v_n = 0_{\mathbb{R}^n},$$

that's to say:

$$\lambda_1(1, 0, ..., 0) + ... + \lambda_i(0, ..., \underbrace{1}_{i^e rang}, ..., 0) + ... + \lambda_n(0, 0, ..., 1) = 0_{\mathbb{R}^n},$$

then

$$(\lambda_1, \lambda_2, ..., \lambda_n) = 0_{\mathbb{R}^n}.$$

Therefore $\{e_1, ..., e_n\}$ is a basis of $\mathbb{R}^n$, called canonical basis.

Example 1.13 *(Canonical basis of $\mathbb{R}_n[x]$) The family $\mathcal{B} = \{1, x, ..., x^n\}$ is a basis of $\mathbb{R}_n[x]$, indeed, any $P(x) = a_0 + a_1 x + ... + a_n x^n$, $a_i \in \mathbb{R}$; B is therefore generative. Moreover :*

$$\lambda_0 1 + \lambda_1 x + ... + \lambda_n x^n = 0 \Longrightarrow \lambda_0 = \lambda_1 = ... = \lambda_n = 0.$$

Example 1.14 *Let $F = \{(x, y, z) \in \mathbb{R}^3 |\ 2x + y + 2z = 0\}$. Find a basis of F. We saw that F is a vector subspace of $\mathbb{R}^3$, we have: $(x, y, z) \in F \Leftrightarrow y = -2x - 2z$ then:*

$$u \in F \Leftrightarrow u = (x, -2x - 2z, z) \Leftrightarrow u = x(1, -2, 0) + z(0, -2, 1).$$

Therefore the vectors $v_1 = (1, 2, 0), v_2 = (0, -3, 1)$ form a generating family of F. On the other hand:

$$\lambda_1 v_1 + \lambda_2 v_2 = 0 \Leftrightarrow \lambda_1(1, -2, 0) + \lambda_2(0, -2, 1) = (0, 0, 0),$$

which is equivalent to $\lambda_1 = \lambda_2 = 0$. Then $\{v_1, v_2\}$ is linearly independent and therefore it is a basis of F.

Proposition 1.7 *We have :*

1. $\{x\}$ *is a linearly independent family* $\iff x \neq 0_E$.
2. *Any family containing a generating family is generating.*
3. *Any subfamily of a linearly independent family is linearly independent.*
4. *Any family containing a linearly dependent family is linearly dependent.*
5. *Any family* $\{v_1, ..., v_p\}$ *one of the vectors of which* v_i *is null, is linearly dependent.*

Proof.

1. $\Longleftarrow$) According to Proposition 1.1 (7), $\lambda \cdot x = 0_E \iff (\lambda = 0_{\mathbb{k}}$ or $x = 0_E)$. Then, if $x \neq 0$, $\lambda x = 0$ implies that $\lambda = 0$, which means that $\{x\}$ is a linearly independent family.

   $\Longrightarrow$) Suppose that $\{x\}$ is a linearly independent family. So, according to the definition of linearly independent family, if $\lambda x = 0$ we necessarily have $\lambda = 0$, which means, always according to the proposition 1.1 (7), that $x \neq 0$.

2. Let $\{v_1, ..., v_p\}$ a generative family and $x = \lambda_1 v_1 + ... + \lambda_p v_p$ an arbitrary element of E. We can also write:

$$x = \lambda_1 v_1 + ... + \lambda_p v_p + 0w_1 + ... + 0w_q, \quad w_1, ..., w_q \in E.$$

   Then any $x \in E$ is a linear combination of $v_1, ..., v_p, w_1, ..., w_q$.

3. Let $\mathcal{T} = \{v_1, ..., v_p\}$ is a linearly independent family and $\mathcal{T}'$ a subfamily of $\mathcal{T}$. If we are going to change the numbering, we can assume that $\mathcal{T}\prime = \{v_1, ..., v_k\}$ (with $k < p$). If $\mathcal{T}\prime$ was linearly dependent, one of the vectors $v_1, ..., v_k$ would be a linear combination of the others. There would therefore exist an element of $\mathcal{T}$ which would be written as a linear combination of certain elements of $\mathcal{T}$. But this is impossible because $\mathcal{T}$ is linearly independent family (see Proposition 1.4).

4. Let $\mathcal{F} = \{v_1, ..., v_p\}$ a linearly dependent family and $\mathcal{G} = \{v_1, ..., v_p, w_1, ..., w_q\}$. According to proposition 1.4, one of v_i is a linear combination of the others. Now, vectors v_i belong to $\mathcal{G}$; so one of the elements of $\mathcal{G}$ is a linear combination of the others, and therefore $\mathcal{G}$ is linearly dependent.

5. Obvious from 4., because it is a family containing $\{0\}$, and $\{0\}$ is a linearly dependent, from 1.

□

1.5 Dimension of a vector space

Definition 1.8 (Dimension of a vector space) *If $E = \{0\}$, we say that E is of dimension 0 and we denote $\dim E = 0$. Otherwise, if E is a vector space on $\mathbb{k}$ of finite dimension not reduced to $\{0\}$,we call dimension of E the cardinality of a basis of E and we denote it $\dim_{\mathbb{k}} E$.*

Example 1.15 $\dim \mathbb{R}^n = n$, $\dim \mathbb{R}_n[x] = n + 1$.

Remark 1.5 *A family f of at least $n+1$ vectors in a space E of dimension n is always linearly dependent. Indeed, if it were linearly independent then we would have a linearly independent family f of cardinality greater than that of any basis $\mathcal{B}$ of E. So, $\mathcal{B}$ is a generating family of E and the cardinality of a linearly independent family is always smaller than that of a generating family.*

1.6 Dimension of a vector subspace

Proposition 1.8 *Let E a finite-dimensional vector space n and F a vector subspace of E. We have:*

1. *F is of finite dimension and $\dim F \leq \dim E$.*
2. *$(\dim F = \dim E) \Leftrightarrow F = E$*

Remark 1.6 *To verify that two vector subspaces F and G are equal*
- *We show that $F \subset G$.*
- *We show that $\dim F = \dim G$.*

Example 1.16 *Let $E = \mathbb{R}^4$ and*

$$\begin{aligned} F &= Vect((1,1,\alpha,3),(0,1,1,2)), \\ G &= \{(x,y,z,t) \in \mathbb{R}^4 |\ x-y+z=0, x+2y-t=0\}. \end{aligned}$$

Let us look at what condition on $\alpha \in \mathbb{R}$ do we have $F = G$?

Let us first assume that $F = G$. Then $(1,1,\alpha,3) \in G$ and must satisfy in particular the equation $x - y + z = 0$. We then find that $\alpha = 0$.

Let us show that if $\alpha = 0$ then $F = G$. We know that $F = Vect((1,1,0,3),(0,1,1,2))$. The vectors $(1,1,0,3),(0,1,1,2)$ generate F and they are non-collinear so they form a linearly independent family. Therefore, it is a basis of F and $\dim F = 2$. Moreover

$$\begin{aligned} G &= \{(x,y,z,t) \in E |\ x-y+z=0, x+2y-t=0\} \\ &= \{(x,y,y-x,x+2y) |\ x,y \in \mathbb{R}\} \\ &= Vect((1,0,-1,1),(0,1,1,2)), \end{aligned}$$

we then show in the same way as before that $\dim G = 2$. *What's more* $(1,1,0,3)$ *and* $(0,1,1,2)$ *satisfies the system*

$$\begin{cases} x - y + z = 0 \\ x + 2y - t = 0 \end{cases}$$

then $(1,1,0,3),(0,1,1,2) \in G$ *and as* G *is a subspace,*

$$F = Vect((1,1,\alpha,3),(0,1,1,2) \subset G.$$

Finally, as $\dim F = \dim G$, *we obtain* $F = G$.

1.7 Sum of vector subspaces

Definition 1.9 (Sum of two vector subspaces) *Let* F *and* G *two vector subspaces of a* $\mathbb{k}$*-vector space* E. *We call the sum of* F *and* G *and we denote by* $F+G$ *the vector subspace of* E *given by*

$$F + G = \{x + y \mid (x,y) \in F \times G\}.$$

Remark 1.7 *The subset* $F+G$ *is indeed a vector subspace of* E. *Indeed,* $F+G \subset E$ *because* E *is stable for addition. Moreover,* $F+G$ *is non-empty because* F *and* G *are. Finally, if* $u = x+y \in F+G$ *and* $u' = x'+y' \in F+G$ *with* $x, x' \in F$ *and* $y, y' \in G$ *then,for* $\alpha, \beta \in \mathbb{k}$.

$$\alpha u + \beta u' = \underbrace{\alpha x + \beta x'}_{\in F} + \underbrace{\alpha y + \beta y'}_{\in G} \in F + G$$

because F *and* G *are vector subspaces.*

Proposition 1.9 *Let* F *and* G *two vector subspaces of a* $\mathbb{k}$*-vector space* E. *Then* $F+G$ *is the smallest vector subspace of* E *container* $F \cup G$.

Proof. We proved in the previous remark that $F+G$ is a vector subspace of E. It contains F and G because 0_E is an element of F and G and therefore $F = F + 0_E \subset F+G$ and $G = 0_E + G \subset F+G$. Moreover, if we consider a subspace H of E which contains $F \cup G$ so let's show that $F+G \subset H$. Let $x+y \in F+G$ with $x \in F$ and $y \in G$. As $F \cup G \subset H$, we also have $x, y \in H$ and since H is a vector subspace, it follows that $x+y \in H$. Then $F+G \subset H$ and $F+G$ is the smallest vector subspace of E containing F and G. □

Example 1.17 *In* $\mathcal{F}(\mathbb{R},\mathbb{R})$, *Let* $F = Vect(\sin)$ *and* $G = Vect(\exp)$ *then:*

$$F + G = Vect\,(\sin, \exp) = \{x \mapsto \alpha \sin(x) + \beta \exp(x) \mid \alpha, \beta \in \mathbb{R}\}$$

Proposition 1.10 *Let A and B be two parts of a $\mathbb{k}$-vector space E, then*

$$Vect(A) + Vect(B) = Vect(A \cup B).$$

Example 1.18 *In space $\mathbb{R}^3$, we consider the parties $F = \{(x, 0, 0) \mid x \in \mathbb{R}\}$ and $G = \{(x, x, 0) \mid x \in \mathbb{R}\}$. Let us show that these are vector subspaces of $\mathbb{R}^3$ and determine the subspace $F + G$. We have $F = Vect(1, 0, 0)$ and $G = Vect(1, 1, 0)$ then F and G are vector subspaces of $\mathbb{R}^3$. Moreover $F + G = Vect((1, 0, 0), (1, 1, 0))$ and we recognize that $F + G$ is the vector plane of $\mathbb{R}^3$ generated by $(1, 0, 0)$ and $(1, 1, 0)$.*

1.7.1 Direct sum, supplementary subspaces

Definition 1.10 (Direct sum) *We say that two vector subspaces F and G of E are in direct sum if $F \cap G = \{0_E\}$. We then note $F \oplus G$ their sum.*

In other words:

$$\zeta = F \oplus G \Leftrightarrow \begin{cases} \zeta = F + G \\ et \\ F \cap G = \{0_E\} \end{cases}$$

Example 1.19 *In $\mathbb{C}$, the vector subspaces $F = \mathbb{R}$ and $G = i\mathbb{R}$ are in direct sum:*

Let $x \in F \cap G$, then $x \in F$ thus x is real, and $x \in G$ then x is pure imaginary. x is a complex that is both real and pure imaginary, so $x = 0_E$.

Example 1.20 *In $E = \mathcal{F}(\mathbb{R}, \mathbb{R})$, we consider*

$$F = \{f \in E \mid f(0) = 0\}\} \text{ et } G = Vect(x \mapsto 1).$$

It is clear that F and G are vector subspaces of E. G is the set of constant applications of $\mathbb{R}$ in $\mathbb{R}$.

Let $f \in F \cap G$. $f \in F$ then $f(0) = 0$. Likewise $f \in G$ so there exists $a \in \mathbb{R}$ such that $f(x) = a$. But $a = f(0) = 0$ then $f = 0_E$. Thus F and G are in direct sum.

Proposition 1.11 *Let F and G be two vector subspaces of the $\mathbb{k}$-espace vectoriel E. F and G are in direct sum if and only if $\forall x \in F + G, \exists!(x_1, x_2) \in F \times G : x = x_1 + x_2$ (that is, the decomposition of x is unique).*

Proof. $\Longrightarrow$) Suppose that F and G are in direct sum and that $x \in F + G$. By definition, there exists $x_1 \in F$ and $x_2 \in G$ such that $x = x_1 + x_2$. Let's assume that there exists $x_1' \in F$ and $x_2' \in G$ such that we still have $x = x_1' + x_2'$. As $x = x_1 + x_2 = x_1' + x_2'$, we have the equality : $x_1 - x_1' = x_2 - x_2'$. Let us denote this vector by y. As F and G are vector subspaces of E,

$y = x_1 - x'_1 \in F$ and $y = x_2 - x'_2 \in G$. Therefore $y \in F \cap G$. But F and G being in direct sum, we have $F \cap G = \{0_E\}$ then $y = 0$. Therefore, $x_1 = x'_1$ and $x_2 = x'_2$ and then uniqueness.

$\Longleftarrow$) Let $x \in F \cap G$. There are then two pairs of $F \times G$ allowing to decompose x into a vector of F and a vector of G:$(x, 0)$ et $(0, x)$. By hypothesis, they are equal: $(x, 0) = (0, x)$. Therefore $x = 0$ and the subspaces F and G are in direct sum. □

Definition 1.11 (Supplementary subspaces) *Let E be a vector space and F, G two vector subspaces of E. We say that F and G are supplementary (or that G is a supplementary of F), if $E = F \oplus G$.*

Proposition 1.12 *Let E be a vector space and F, G two vector subspaces of E. Then $E = F \oplus G$ (F et G are supplementary) if and only if for any basis $\mathcal{B}_1$ of F and for any basis $\mathcal{B}_2$ of G, $\{\mathcal{B}_1, \mathcal{B}_2\}$ is a basis of E.*

Proof. $\Longleftarrow$) Let $\mathcal{B}_1 = \{v_\alpha\}_{\alpha \in A}$ and $\mathcal{B}_2 = \{w_\beta\}_{\beta \in B}$ basis of F and G respectively and suppose that $\{v_\alpha, w_\beta\}_{(\alpha,\beta) \in A \times B}$ is a basis of E. Then all $x \in E$ is written in a unique way:

$$x = \lambda_1 v_{\alpha_1} + ... + \lambda_p v_{\alpha_p} + \mu_1 w_{\beta_1} + ... + \mu_q w_{\beta_q},$$

that is to say everything $x \in E$ is written in a unique way $x = x_1 + x_2$ with $x_1 \in F$ and $x_2 \in G$, then $E = F \oplus G$.

$\Longrightarrow$) If $E = F \oplus G$, any $x \in E$ decomposes in a unique way on F and G and, therefore, on the family $\mathcal{B} = \{\mathcal{B}_1, \mathcal{B}_2\}$. We deduce that $\mathcal{B}$ is a basis of E. □

Corollary 1.1 *Let E be a vector space. For any vector subspace F, there always exists a supplementary. The supplementary of F is not unique, but if E is finite-dimensional, all supplementary spaces of F have the same dimension.*

Theorem 1.1 *Let E be a finite-dimensional vector space and F, G two vector subspaces of E. Then*

$$\dim(F + G) \Leftrightarrow \dim F + \dim G - \dim F \cap G.$$

Proof. Let's suppose that $\dim F = p$, $\dim G = q$ and $\dim F \cap G = r$. Note that, since $F \cap G$ is a vector subspace of F and G, we have $r < p$ and $r < q$. Consider a basis $\{a_1, ..., a_r\}$ of $F \cap G$. Since the family $\{a_1, ..., a_r\}$ is linearly independent, we can complete it in a base of F and also in a base of G. We can therefore construct: a basis of F of the type $\{a_1, ..., a_r, e_{r+1}, ..., e_p\}$ and a basis of G of the type $\{a_1, ..., a_r, f_{r+1}, ..., f_q\}$.

We know that any vector of $F+G$ is written as the sum of a vector of F, and a vector of G and therefore it is of the form:

$$\begin{aligned} x &= \lambda_1 a_1 + ... + \lambda_r a_r + \lambda_{r+1} e_{r+1} + ... + \lambda_p e_p + \\ &\quad \mu_1 a_1, ... + \mu_r a_r + \mu_{r+1} f_{r+1}, ... + \mu_q f_q, \end{aligned}$$

that is, by posing $\tau_i = \lambda_i + \mu_i$, for $i = 1, ..., r$:

$$x = \tau_1 a_1 + ... + \tau_r a_r + \lambda_{r+1} e_{r+1} + ... + \lambda_p e_p + \mu_{r+1} f_{r+1}, ... + \mu_q f_q. \tag{1.2}$$

Therefore, the family $\{a_1, ..., a_r, e_{r+1}, ..., e_p, f_{r+1}, ..., f_q\}$ egene-in-gender $F+G$. Let us show that it is free. Let us consider a null linear combination:

$$\underbrace{\tau_1 a_1 + ... + \tau_r a_r}_{\alpha \in F \cap G} + \underbrace{\lambda_{r+1} e_{r+1} + ... + \lambda_p e_p}_{\beta \in F} + \underbrace{\mu_{r+1} f_{r+1}, ... + \mu_q f_q}_{\gamma \in G} = 0.$$

We have $\alpha + \beta + \gamma = 0$, then $\gamma = -(\alpha + \beta)$. Thus $\gamma \in g$ and $\alpha + \beta \in F$, then $\gamma \in F \cap G$. Therefore, γ can be written as a linear combination of the a_i:

$$\mu_{r+1} f_{r+1}, ... + \mu_q f_q = \delta_1 a_1 + ... + \delta_r a_r.$$

But $\{a_1, ..., a_r, \mu_{r+1} f_{r+1}, ... + \mu_q f_q\}$ is a basis of G so all coefficients of this linear combination must be null. In particular, $\mu_{r+1} = 0, ..., \mu_q = 0$. Likewise $\lambda_{r+1} = 0, ..., \lambda_q = 0$. From (1.2) we then deduce that:

$$\tau_1 a_1 + ... + \tau_r a_r = 0$$

Now the family $\{a_1, ..., a_r\}$ is linearly independent, so $\tau_1 = 0, ..., \tau_r = 0$. So the family

$$\{a_1, ..., a_r, e_{r+1}, ..., e_p, f_{r+1}, ..., f_q\}$$

is linearly independent and therefore it is a basis of $F+G$. We deduce that:

$$\begin{aligned} \dim(F+G) &= r + (p-r) + (q-r) = p + q - r \\ &= \dim E + \dim G + \dim F \cap G. \end{aligned}$$

□

Corollary 1.2 *Let E be a finite-dimensional vector space and F, G two vector subspaces of E. Then*

$$E = F \oplus G \Leftrightarrow \begin{cases} F \cap G = \{0_E\} \\ \dim E = \dim F + \dim G \end{cases}$$

Example 1.21 *In $E = \mathbb{C}$, $F = Vect(\{1\})$ et $G = Vect(\{i\})$. We know that F and G are vector subspaces of E. Si $x \in F \cap G$ then x is both real and pure imaginary therefore $x = 0$ and $F \cap G = \{0_E\}$. Moreover*

$$F + G = Vect(1) + Vect(i) = Vect(1, i) = \mathbb{C}$$

then $E = F + G$. We then showed that F and G are supplementary.

Example 1.22 *In $\mathbb{R}^3$, Let π a vector plane and v a vector not contained in this plane. We have:*

$$\mathbb{R}^3 = \pi \oplus Vect(v)$$

because if $\{e_1, e_2\}$ is a basis of π, then $\{e_1, e_2, v\}$ is a basis of $\mathbb{R}^3$.

1.8 Exercises

Exercise 1.

1. Is the subset $E = \{(x, y) \in \mathbb{R}^2, y = 2x\}$ of $\mathbb{R}^2$, equipped with the usual laws of the vector space $\mathbb{R}^2$, a vector space on $\mathbb{R}$?

2. Is the subset $F = \{(x, y, z) \in \mathbb{R}^3, y^2 = 2x, z = 0\}$ of $\mathbb{R}^3$, equipped with the usual laws of the vector space $\mathbb{R}^3$, a subspace of $\mathbb{R}^3$?

Exercise 2.

Let $u_1 = (1, 2, 3, 4)$ and $u_2 = (1, -2, 3, -4)$ be two vectors of $\mathbb{R}^4$.

Can one determine x and y so that $(x, 1, y, 1) \in Vect\,(u_1, u_2)$? And so that $(x, 1, 1, y) \in$ Vect (u_1, u_2)?

Exercise 3.

In $\mathbb{R}^4$, let's consider the sub-set E of vectors $(x_1, x_2, x_3, x_4) \in \mathbb{R}^4$ satisfying $x_1+x_2+x_3+x_4 = 0$.

Is the subset E a subspace of $\mathbb{R}^4$? If it is, give a basis.

Exercise 4.

Let E be a vector space on $\mathbb{R}$ and v_1, v_2, v_3 and v_4 a linearly independent family of elements of E, are the following families linearly independent?

1. $(v_1, 2v_2, v_3)$
2. (v_1, v_3)
3. $(v_1, v_1 + 2, v_4)$
4. $(3v_1 + v_3, v_3, v_2 + v_3)$.
5. $(2v_1 + v_2, v_1 - 3v_2, v_4, v_2 - v_1)$

Exercise 5.

In $\mathbb{R}^4$, compare the following subspaces F and G:

$$F = Vect((1, 0, 1, 1), (-1, -2, 3, -1), (-5, -3, 1, 5))$$

$$G = \text{Vect}((-1, -1, 1, -1), (4, 1, 2, 4))$$

Exercise 6.

In $\mathcal{F}(\mathbb{R}, \mathbb{R})$, are the following three functions $x \mapsto \sin(x), x \mapsto \sin(2x)$ and $x \mapsto \sin(3x)$ linearly independent?

Exercise 7.

Let $f(x) = \cos(x), g(x) = \cos(x)\cos(2x)$ and $h(x) = \sin(x)\sin(2x)$. Determine Vect(f, g, h).

Exercise 8.

Let $a = (2, 3, -1), b = (1, -1, -2), c = (3, 7, 0)$ and $d = (5, 0, -7)$ be four vectors of $\mathbb{R}^3$.

Let $E = \text{Vect}(a, b)$ and $F = \text{Vect}(c, d)$ be subspaces of $\mathbb{R}^3$. Prove that $E = F$.

Exercise 9.

Let $E = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \mid x_1 - x_2 = 0 \text{ and } x_3 - x_4 = 0\}$

We admit that F is a subspace of $\mathbb{R}^4$.

1. Determine a basis of E.
2. Complete this basis of E to a basis of $\mathbb{R}^4$.

Exercise 10.

Let $E = \{P \in \mathbb{R}_2[X], P(1) = 0\}$.

1. Prove that E is a subspace of $\mathbb{R}_2[X]$.
2. Give a basis of E and deduce its dimension.

Exercise 11.

Let $E = \{P \in \mathbb{R}_3[X], P(-1) = 0 \text{ and } P(1) = 0\}$

1. Prove that E is a subspace of $\mathbb{R}_3[X]$.
2. Determine a basis and the dimension of E.

Exercise 12.

Let's consider the vectors $v_1 = (1, 0, 0, 1), v_2 = (0, 0, 1, 0), v_3 = (0, 1, 0, 0), v_4 = (0, 0, 0, 1)$ and $v_5 = (0, 1, 0, 1)$ of $\mathbb{R}^4$.

1. Is Vect (v_1, v_2) supplementary to Vect (v_3) in $\mathbb{R}^4$?
2. Same question for Vect (v_1, v_3, v_4) and Vect (v_2, v_5).
3. Same question for Vect (v_1, v_2) and Vect (v_3, v_4, v_5).

1.9 Solutions

Exercise 1.

1. $0 = 2 \times 0$ so $0_{\mathbb{R}^2} \in E$.
   Let $u = (x, y) \in E, y = 2x$ and $u' = (x', y') \in E, y' = 2x'$
   For all λ and λ' two real numbers

$$\lambda u + \lambda' u' = (\lambda x + \lambda' x', \lambda y + \lambda' y') = (X, Y)Y = \lambda y + \lambda' y' = \lambda 2x + \lambda' 2x' = 2(\lambda x + \lambda' x') = 2X$$

   So $\lambda u + \lambda' u' \in E$. This shows that E is a subspace of $\mathbb{R}^2$.
2. Let $u = (2, 2, 0) \in F$ because $y^2 = 2^2 = 2 \times 2 = 2x$ and $z = 0$
   $2u = (4, 4, 0) y^2 = 4^2 = 16 \neq 2 \times 4 = 8$ so $2u \notin F$ therefore F is not a vector subspace.

Exercise 2.

The problem is to determine x and y such that there exist α and β satisfying $(x, 1, y, 1) = \alpha u_1 + \beta u_2$

$$\begin{cases} x = \alpha + \beta \\ 1 = 2\alpha - 2\beta \\ y = 3\alpha + 3\beta \\ 1 = 4\alpha - 4\beta \end{cases} \iff \begin{cases} x = \alpha + \beta \\ -4\beta = 1 - 2x \\ y = y - 3x \\ -8\beta = 1 - 4x \end{cases}$$

$$\Longleftrightarrow \begin{cases} x = \alpha + \beta \\ -4\beta = 1 - 2x \\ y = y - 3x \\ 0 = -1 \end{cases}$$

The last line implies that there is no solution.
The problem is to determine x and y such that there exist α and β satisfying $(x, 1, 1, y) = \alpha u_1 + \beta u_2$

$$\begin{cases} x = \alpha + \beta \\ 1 = 2\alpha - 2\beta \\ 1 = 3\alpha + 3\beta \\ y = 4\alpha - 4\beta \end{cases} \Longleftrightarrow \begin{cases} x = \alpha + \beta \\ -4\beta = 1 - 2x \\ 0 = 1 - 3x \\ -8\beta = y - 4x \end{cases} \Longleftrightarrow \begin{cases} x = \alpha + \beta \\ -4\beta = 1 - 2x \\ x = \frac{1}{3} \\ 0 = y - 4x - 2(1 - 2x) \end{cases}$$

$$\Leftrightarrow \begin{cases} x = \alpha + \beta \\ -4\beta = 1 - 2x \\ x = \frac{1}{3} \\ y = 2 \end{cases} \Longleftrightarrow \begin{cases} \frac{1}{3} = \alpha + \beta \\ \beta = \frac{-1}{12} \\ x = \frac{1}{3} \\ y = 2 \end{cases}$$

$$\Longleftrightarrow \begin{cases} \alpha = \frac{5}{12} \\ \beta = \frac{-1}{12} \\ x = \frac{1}{3} \\ y = 2 \end{cases}$$

$$\left(\frac{1}{3}, 1, 1, 2\right) = \frac{5}{12}u_1 - \frac{1}{12}u_2$$

Exercise 3.
A vector of E is written

$$\begin{aligned} x &= (-x_2 - x_3 - x_4, x_2, x_3, x_4) \\ &= x_2(-1, 1, 0, 0) + x_3(-1, 0, 1, 0) + x_4(-1, 0, 0, 1) \end{aligned}$$

So $E = Vect((-1, 1, 0, 0), (-1, 0, 1, 0), (-1, 0, 0, 1))$, E is a vector subspace of $\mathbb{R}^4$.
To find a basis, it remains to show that $((-1, 1, 0, 0), (-1, 0, 1, 0), (-1, 0, 0, 1))$ is free (Since this family is already generating).

$$\begin{aligned} \alpha(-1, 1, 0, 0) + \beta(-1, 0, 1, 0) + \gamma(-1, 0, 0, 1) &= (0, 0, 0, 0) \\ &\Rightarrow \begin{cases} -\alpha - \beta - \gamma = 0 \\ \alpha = 0 \\ \beta = 0 \\ \gamma = 0 \end{cases} \end{aligned}$$

This family is indeed free, it's a basis of E.

Exercise 4.

1. Yes obviously, otherwise

$$\begin{aligned} \alpha v_1 + 2\beta v_2 + \gamma v_3 &= 0_E \Rightarrow \begin{cases} \alpha = 0 \\ 2\beta = 0 \\ \gamma = 0 \end{cases} \\ &\Rightarrow \begin{cases} \alpha = 0 \\ \beta = 0 \\ \gamma = 0 \end{cases} \end{aligned}$$

2. A subfamily of a free family is free.

3.

$$2v_1 - (2v_1 + v_4) + v_4 = 0_{\mathbb{R}^n}$$

There exists a non-trivially zero linear combination of these three vectors, the family is dependent. 4.

$$\begin{aligned} \alpha(3v_1 + v_3) + \beta v_3 + \gamma(v_2 + v_3) &= 0_{\mathbb{R}^n} \\ &\Rightarrow 3\alpha v_1 + \gamma v_2 + (\alpha + \beta + \gamma)v_3 = 0_E \\ &\Rightarrow \begin{cases} 3\alpha = 0 \\ \gamma = 0 \\ \alpha + \beta + \gamma = 0 \end{cases} \\ &\Rightarrow \begin{cases} \alpha = 0 \\ \beta = 0 \\ \gamma = 0 \end{cases} \end{aligned}$$

The family is free.

5. There are three vectors $2v_1 + v_2, v_1 - 3v_2, v_2 - v_1$ in the plane $Vect(v_1, v_2)$ so these three vectors form a dependent family, adding v_4 doesn't change anything, the family is dependent.

Exercise 5.

Comparing two sets means we must find if one is included in the other (or vice versa) or if the sets are equal.

We will first characterize F using one (or several) Cartesian equation(s), then it will be simple to know if the vectors that generate G are in F.

$u = (x, y, z, t) \in F \Leftrightarrow$ there exist α, β, γ real numbers such that

$$u = \alpha(1, 0, 1, 1) + \beta(-1, -2, 3, -1) + \gamma(-5, -3, 1, 5)$$

$$\begin{cases} \alpha - \beta - 5\gamma = x \\ -2\beta - 3\gamma = y \\ \alpha + 3\beta + \gamma = z \\ \alpha - \beta + 5\gamma = t \end{cases} \Leftrightarrow \begin{cases} \alpha - \beta - 5\gamma = x \\ -2\beta - 3\gamma = y \\ 3\beta + 6\gamma = -x + z \\ 10\gamma = -x + t \end{cases}$$

$$\Leftrightarrow \begin{cases} \alpha - \beta - 5\gamma = x \quad ... \quad \text{L1} \\ -2\beta - 3\gamma = y \quad ... \text{ L2} \\ -x + z + 2y = 0 \quad ... \quad \text{L3} \\ 10\gamma = -x + t \quad ... \quad \text{L4} \end{cases}$$

α, β, γ are given by equations L_1, L_2 and L_4 so

$$F = \{(x, y, z, t) \in \mathbb{R}^4, -x + 2y + z = 0\}$$
$$-(-1) + 2(-1) + 1 = 0 \Rightarrow (-1, -1, 1, -1) \in F$$
$$-4 + 2 \times 1 + 2 = 0 \Rightarrow (4, 1, 2, 4) \in F$$

This shows that $G \subset F$.
Manifestly $\dim(G) = 2$ because the two vectors that generate G are not collinear (so they form a basis of G).

If we knew more we would know that $\dim(F) = 3$, but we are not supposed to know that. We must show that the three vectors that generate F are free, they will form a basis and the dimension of F will be 3.

Let's recalculate $u = \alpha(1, 0, 1, 1) + \beta(-1, -2, 3, -1) + \gamma(-5, -3, 1, 5)$ with $u = (0, 0, 0, 0)$ We find

$$\begin{cases} \alpha - \beta - 5\gamma = x \\ -2\beta - 3\gamma = y \\ 0 = -x + z + 2y \\ 10\gamma = -x + t \end{cases} \Leftrightarrow \begin{cases} \alpha - \beta - 5\gamma = 0 \\ -2\beta - 3\gamma = 0 \\ 0 = -0 + 0 + 2 \times 0 \\ 10\gamma = -0 + 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} \alpha = 0 \\ \beta = 0 \\ \gamma = 0 \end{cases}$$

It's good, $\dim(F) = 3$.

$$F \subset G \dim(G) < \dim(F) \Rightarrow G \subsetneq F$$

In other words G is included in F but G is not equal to F.

Exercise 6.

$$\forall x \in \mathbb{R}, \alpha \sin(x) + \beta \sin(2x) + \gamma \sin(3x) = 0.$$

For $x = \frac{\pi}{3}$,

$$\alpha \sin\left(\tfrac{\pi}{3}\right) + \beta \sin\left(\tfrac{2\pi}{3}\right) + \gamma \sin(\pi) = 0$$
$$\Leftrightarrow \tfrac{\alpha}{2} + \tfrac{\beta}{2} = 0 \Leftrightarrow \beta = -\alpha$$
$$\forall x \in \mathbb{R}, \alpha \sin(x) - \alpha \sin(2x) + \gamma \sin(3x) = 0.$$

For $x = \frac{\pi}{2}$,

$$\alpha \sin\left(\tfrac{\pi}{2}\right) - \alpha \sin(\pi) + \gamma \sin\left(\tfrac{3\pi}{2}\right) = 0 \Leftrightarrow \gamma = \alpha$$
$$\forall x \in \mathbb{R}, \alpha \sin(x) - \alpha \sin(2x) + \alpha \sin(3x) = 0.$$

For $x = \frac{2\pi}{3}$

$$\alpha \sin\left(\tfrac{2\pi}{3}\right) - \alpha \sin\left(\tfrac{4\pi}{3}\right) + \alpha \sin(2\pi) = 0$$
$$\Leftrightarrow \alpha\left(\tfrac{1}{2} - \left(-\tfrac{1}{2}\right)\right) = 0 \Leftrightarrow \alpha = 0.$$

So $\alpha = \beta = \gamma = 0$

This family is free.

Exercise 7.

First method

$F \in \text{Vect}(f, g, h) \Leftrightarrow$ there exist α, β and γ such that $\forall x \in \mathbb{R}$,

$$\begin{aligned} F(x) &= \alpha\cos(x) + \beta\cos(x)\cos(2x) + \gamma\sin(x)\sin(2x) \\ &= \alpha\cos(x) + \beta\cos(x)\left(1 - 2\cos^2(x)\right) + 2\gamma\sin^2(x)\cos(x) \\ &= \alpha\cos(x) + \beta\cos(x) - 2\beta\cos^3(x) + 2\gamma\left(1 - \cos^2(x)\right)\cos(x) \\ &= (\alpha + \beta + 2\gamma)\cos(x) + (-2\beta - 2\gamma)\cos^3(x). \end{aligned}$$

So $F \in \text{Vect}\left(\cos, \cos^3\right)$

This means that $\text{Vect}(f, g, h) \subset \text{Vect}\left(\cos, \cos^3\right)$, the inclusion in the other direction is obvious so

$$\text{Vect}(f, g, h) = \text{Vect}\left(\cos, \cos^3\right).$$

Which is obviously a vector space of dimension 2.

Second method

We want to know if the family (f, g, h) is free, if it is, there is not much to say about $\text{Vect}(f, g, h)$ except that it is a space of dimension 3.

$$\forall x \in \mathbb{R}, \quad \alpha\cos(x) + \beta\cos(x)\cos(2x) + \gamma\sin(x)\sin(2x) = 0.$$

For $x = \frac{\pi}{4}$

$$\alpha\cos\left(\frac{\pi}{4}\right) + \beta\cos\left(\frac{\pi}{4}\right)\cos\left(\frac{\pi}{2}\right) + \gamma\sin\left(\frac{\pi}{4}\right)\sin\left(\frac{\pi}{2}\right) = 0 \Leftrightarrow \alpha + \gamma = 0.$$

For $x = 0$

$$\alpha\cos(0) + \beta\cos(0)\cos(0) + \gamma\sin(0)\sin(0) = 0 \Leftrightarrow \alpha + \beta = 0.$$

So $\gamma = -\alpha$ and $\beta = -\alpha$

$$\forall x \in \mathbb{R}, \quad \alpha\cos(x) - \alpha\cos(x)\cos(2x) + \alpha\sin(x)\sin(2x) = 0.$$

Then, no matter what particular values of x we try, we find $0 = 0$.

$$\begin{aligned} &\forall x \in \mathbb{R}, \alpha\cos(x) - \alpha\cos(x)\cos(2x) + \alpha\sin(x)\sin(2x) = 0 \\ &\Leftrightarrow \forall x \in \mathbb{R}, \alpha(\cos(x) - \cos(x)\cos(2x) + \sin(x)\sin(2x)) = 0 \\ &\Leftrightarrow \forall x \in \mathbb{R}, \alpha(\cos(x)(1 - \cos(2x)) + \sin(x)2\cos(x)\sin(x)) = 0 \\ &\Leftrightarrow \forall x \in \mathbb{R}, \quad \alpha\cos(x)\left(1 - \cos(2x) + 2\sin^2(x)\right) = 0. \end{aligned}$$

Because $\cos(2x) = 1 - 2\sin^2(x)$.

The family is therefore dependent, f and g are not proportional so the family is free and

$$\text{Vect}(f, g, h) = \text{Vect}(f, g).$$

And, $\dim(\text{Vect}(f, g, h)) = 2$.

Exercise 8.

$$c = 2a - b \in \text{Vect}\,(a, b) = E$$
$$d = a + 3b \in \text{Vect}\,(a, b) = E.$$

So $F \subset E$, now a and b are not proportional so (a, b) is a basis of E and $\dim(E) = 2$, similarly c and d are not proportional so (c, d) is a basis of F and $\dim(F) = 2$.

I glossed over the fact that (a, b) is a generating family of E and that (c, d) is a generating family of F.

$$\begin{cases} E \subset F \\ \dim(E) = \dim(F) \end{cases} \Rightarrow E = F$$

There are other ways to do it, for example by finding for E and F a Cartesian equation characterizing these spaces.

Exercise 9.

1. We have:

$$\begin{aligned} x &= (x_1, x_2, x_3, x_4) \in E \Leftrightarrow \begin{cases} x = (x_1, x_2, x_3, x_4) \\ x_1 - x_2 = 0 \\ x_3 - x_4 = 0 \end{cases} \\ &\Leftrightarrow \begin{cases} x = (x_2, x_2, x_4, x_4) \\ x_1 = x_2 \\ x_3 = x_4 \end{cases} \\ x &= (x_2, x_2, x_4, x_4) = x_2(1, 1, 0, 0) + x_4(0, 0, 1, 1) \end{aligned}$$

Let $a = (1, 1, 0, 0)$ and $b = (0, 0, 1, 1)$

$E = \text{Vect}(a, b)$ which implies that $\{a, b\}$ is a generating family of E, and on the other hand $\{a, b\}$ is a free family because these vectors are not proportional, so $\{a, b\}$ is a basis of E.

2. Let $c = (1, 0, 0, 0) \notin E$ because the components of c do not satisfy the equations characterizing E. $\{a, b\}$ is free in E and $c \notin E$ so $\{a, b, c\}$ is free.

$$\begin{aligned} x &= (x_1, x_2, x_3, x_4) \in \text{Vect}(a, b, c) \\ &\Leftrightarrow \exists(\alpha, \beta, \gamma) \in \mathbb{R}^3, x = \alpha a + \beta b + \gamma c, \\ x &= \alpha a + \beta b + \gamma c \\ &\Leftrightarrow (x_1, x_2, x_3, x_4) = \alpha(1, 1, 0, 0) + \beta(0, 0, 1, 1) + \gamma(1, 0, 0, 0) \end{aligned}$$

$$\Leftrightarrow \begin{cases} x_1 = \alpha + \gamma \\ x_2 = \alpha \\ x_3 = \beta \\ x_4 = \beta \end{cases}$$

$$\Leftrightarrow \begin{cases} x_1 = \alpha + \gamma \\ x_2 = \alpha \\ x_3 = \beta \\ x_4 - x_3 = 0 \end{cases}$$

So $\text{Vect}(a,b,c) = \{x = (x_1, x_2, x_3, x_4) \in \mathbb{R}^4, x_4 - x_3 = 0\}$

Let $d = (0,0,0,1), d \notin \text{Vect}(a,b,c)$ because the components of d do not satisfy $x_3 - x_4 = 0$. $\{a,b,c\}$ is free and $d \notin \text{Vect}(a,b,c)$ so $\{a,b,c,d\}$ is a free family, it has 4 elements, it's a basis of $\mathbb{R}^4$.

Exercise 10.

1. The zero vector of $\mathbb{R}_2[X]$ is the zero polynomial, at 1 this polynomial equals 0 , the zero vector of $\mathbb{R}_2[X]$ is in E.

Let $P_1 \in E$ and $P_2 \in E$, so $P_1(1) = 0$ and $P_2(1) = 0$.

For all λ_1 and λ_2 two real numbers,

$$(\lambda_1 P_1 + \lambda_2 P_2)(1) = \lambda_1 P_1(1) + \lambda_2 P_2(1) = \lambda_1 \times 0 + \lambda_2 \times 0 = 0$$

So $\lambda_1 P_1 + \lambda_2 P_2 \in E$

E is a vector subspace of E.

2. Let $P = aX^2 + bX + c \in E$,

$$P(1) = 0 \Leftrightarrow a \times 1^2 + b \times 1 + c = 0 \Leftrightarrow c = -a - b$$

So

$$P = aX^2 + bX - a - b = a\left(X^2 - 1\right) + b(X - 1)$$

$X^2 - 1$ and $X - 1$ are two non-proportional polynomials, they form a free family that generates E, it's a basis of $E \cdot \dim(E) = 2$.

Exercise 11.

1. The zero polynomial Θ satisfies $\Theta(-1) = \Theta(1) = 0$, so $\Theta \in E$.

Let P_1 and P_2 be two polynomials of E and let λ_1 and λ_2 be two real numbers

$$(\lambda_1 P_1 + \lambda_2 P_2)(-1) = \lambda_1 P_1(-1) + \lambda_2 P_2(-1) = 0$$

Because $P_1(-1) = 0$ and $P_2(-1) = 0$,

$$(\lambda_1 P_1 + \lambda_2 P_2)(1) = \lambda_1 P_1(1) + \lambda_2 P_2(1) = 0$$

Because $P_1(1) = 0$ and $P_2(1) = 0$,
so $\lambda_1 P_1 + \lambda_2 P_2 \in E$, which shows that E is a vector subspace of $\mathbb{R}_3[X]$.
2. -1 and 1 are roots of P so there exists Q such that $P = (X-1)(X+1)Q = (X^2-1)\,Q$.
The degree of Q is 1, so there exist two real numbers a and b such that

$$P = \left(X^2-1\right)(aX+b) = aX\left(X^2-1\right) + b\left(X^2-1\right)$$

$(X\,(X^2-1), X^2-1)$ is a free family (the degrees are different) and it generates E, it's a basis of E and $\dim(E) = 2$.

Exercise 12.

1. $\text{Vect}\,(v_1, v_2) + \text{Vect}\,(v_3)$ is a direct sum if and only if $\text{Vect}\,(v_1, v_2) \cap \text{Vect}\,(v_3) = \{0\}$.
Let $u \in \text{Vect}\,(v_1, v_2) \cap \text{Vect}\,(v_3)$, so there exist α, β, γ such that

$$u = \alpha v_1 + \beta v_2 = \gamma v_3$$

$$\Rightarrow \alpha(1,0,0,1) + \beta(0,0,1,0) = \gamma(0,1,0,0)$$

$$\Rightarrow (\alpha, 0, \beta, \alpha) = (0, \gamma, 0, 0)$$

$$\Rightarrow \begin{cases} \alpha = 0 \\ \gamma = 0 \\ \beta = 0 \\ \alpha = 0 \end{cases}$$

So $u = 0$, the sum is direct. But is it a supplementary sum?

$$\dim\left(\text{Vect}\,(v_1, v_2)\right) = 2, \dim\left(\text{Vect}\,(v_3)\right) = 1$$

$$\dim\left(\text{Vect}\,(v_1, v_2) \oplus \text{Vect}\,(v_3)\right) = 3 \neq 4$$

So no, they are not supplementary in $\mathbb{R}^4$.

2. Let's show that the sum is direct. Let $u \in \text{Vect}\,(v_1, v_3, v_4) \cap \text{Vect}\,(v_2, v_5)$, so there exist $\alpha, \beta, \gamma, \delta, \eta$ such that

$$u = \alpha v_1 + \beta v_3 + \gamma v_4 = \delta v_2 + \eta v_5$$

$$\Rightarrow \alpha(1,0,0,1) + \beta(0,1,0,0) + \gamma(0,0,0,1) = \delta(0,0,1,0) + \eta(0,1,0,1)$$

$$\Rightarrow (\alpha, \beta, 0, \alpha+\gamma) = (0, \eta, \delta, \eta)$$

$$\Rightarrow \begin{cases} \alpha = 0 \\ \beta = \eta \\ 0 = \delta \\ \alpha + \gamma = \eta \end{cases} \Rightarrow \begin{cases} \alpha = 0 \\ \delta = 0 \\ \gamma = \eta - \alpha = \eta \\ \beta = \eta \end{cases}$$

So $u = \eta v_3 + \eta v_4 = \eta(v_3 + v_4)$
$v_3 + v_4 = (0,1,0,0) + (0,0,0,1) = (0,1,0,1) = v_5$

So $u = \eta v_5$, and $u \in \text{Vect}(v_2, v_5)$, so u is a vector of $\text{Vect}(v_2, v_5)$, but we already knew that, this doesn't give a contradiction.

So the sum is not direct, in fact $v_5 = v_3 + v_4 \in \text{Vect}(v_1, v_3, v_4) \cap \text{Vect}(v_2, v_5)$, and $v_5 \neq 0$.

So the sum is not direct, and a fortiori not supplementary.

3. Let's show that the sum is direct. Let $u \in \text{Vect}(v_1, v_2) \cap \text{Vect}(v_3, v_4, v_5)$, so there exist $\alpha, \beta, \gamma, \delta, \eta$ such that

$$u = \alpha v_1 + \beta v_2 = \gamma v_3 + \delta v_4 + \eta v_5$$

$$\Rightarrow \alpha(1,0,0,1) + \beta(0,0,1,0) = \gamma(0,1,0,0) + \delta(0,0,0,1) + \eta(0,1,0,1)$$

$$\Rightarrow (\alpha, 0, \beta, \alpha) = (0, \gamma + \eta, 0, \delta + \eta)$$

$$\Rightarrow \begin{cases} \alpha = 0 \\ 0 = \gamma + \eta \\ \beta = 0 \\ \alpha = \delta + \eta \end{cases} \Rightarrow \begin{cases} \alpha = 0 \\ \beta = 0 \\ \gamma = -\eta \\ 0 = \delta + \eta \end{cases}$$

So $u = 0$, the sum is direct. Now, is it supplementary?

$$\dim(\text{Vect}(v_1, v_2)) = 2, \dim(\text{Vect}(v_3, v_4, v_5)) = ?$$

Let's determine the dimension of $\text{Vect}(v_3, v_4, v_5)$

$$\gamma v_3 + \delta v_4 + \eta v_5 = 0 \Rightarrow \gamma(0,1,0,0) + \delta(0,0,0,1) + \eta(0,1,0,1) = (0, \gamma + \eta, 0, \delta + \eta) = (0,0,0,0)$$

$$\Rightarrow \begin{cases} \gamma + \eta = 0 \\ \delta + \eta = 0 \end{cases}$$

So $\gamma = -\eta$ and $\delta = -\eta$, so the family (v_3, v_4, v_5) is not free, for example $v_3 + v_4 - v_5 = 0$, so $\dim(\text{Vect}(v_3, v_4, v_5)) \leq 2$.

But v_3 and v_4 are not proportional, so $\dim(\text{Vect}(v_3, v_4, v_5)) = 2$.

So the sum of the dimensions is $2 + 2 = 4$, and the sum is direct, so they are supplementary.

CHAPTER 2

LINEAR APPLICATIONS

In linear algebra, we are interested in applications that preserve the structure of vector spaces, that is, applications from one vector space to another that preserve linear combinations. In this chapter, which is somewhat the focus of the rest of the document, we will essentially provide basic definitions and elementary results.

The concepts covered in this chapter are:

— Definitions: Linear application, Kernel, image, and rank of a linear application.

2.1 Definitions

Definition 2.1 *Let E and E' be two vector spaces over the same field $\mathbb{k}$ and f an application of E in E'. We say that f is linear, if:*

1. $f(v+w) = f(v) + f(w)$, $\forall v, w \in E$,
2. $f(\lambda v) = \lambda f(v), \forall v \in E, \forall \lambda \in \mathbb{k}$.

The set of linear applications from E to E' is denoted $\mathcal{L}_{\mathbb{k}}(E, E')$ or, more simply, $\mathcal{L}(E, E')$.
If a linear application f from E to E (same starting and ending space) we say that f is an endomorphism of E. The set of endomorphisms of E is denoted $End_{\mathbb{k}}(E)$ or, more simply, $End(E)$.
If a linear application f is bijective, we say that f is an isomorphism of vector spaces.

Remark 2.1

1. *If f is linear, we have: $f(0) = 0$. Just take $\lambda = 0$ in $f(\lambda x) = \lambda f(x)$.*
2. *According to 1) and 2), an application $f : E \to E'$ is linear, if and only if, for all $\lambda \in \mathbb{k}$ and for all $x, y \in E$, we have*
$$f(\lambda x + y) = \lambda f(x) + f(y).$$
3. *In some references an application is also called "map".*

Example 2.1 *Let E be a $\mathbb{k}$-vector space and F a vector subspace of E. We call canonical injection of F into E, the application $i : F \to E$ defined by,*
$$\forall x \in F,\ i(x) = x,$$
So i is a linear application.

Example 2.2
$$\begin{array}{rccc} f: & \mathbb{R}^3 & \to & \mathbb{R}^2 \\ & (x,y,z) & \mapsto & (2x+y, y-z), \end{array}$$
is a linear application.

If $v = (x, y, z)$ and $w = (x', y', z')$, we have:
$$\begin{array}{rcl} f(v+w) & = & f(x+x', y+y', z+z') \\ & = & (2(x+x') + (y+y'), y+y'-z-z') \\ & = & (2x+y, y-z) + (2x'+y', y'-z') \\ & = & f(v) + f(w), \\ f(\lambda v) & = & f(\lambda x, \lambda y, \lambda z) \\ & = & (2\lambda x + \lambda y, \lambda y - \lambda z) \\ & = & \lambda(2x+y, y-z) \\ & = & \lambda f(v). \end{array}$$

As this example illustrates, the linearity of f stems from the fact that the components x, y, z in the codomain (here $\mathbb{R}^3$) all appear to the power of 1. More precisely, each component in the codomain is a homogeneous polynomial of degree 1 in x, y, z. We will examine this in more detail later.

Thus, for example, the application

$$\begin{array}{rccc} f: & \mathbb{R}^3 & \to & \mathbb{R}^2 \\ & (x,y,z) & \mapsto & (x^2-y, y+z) \end{array}$$

is not linear neither 1), nor 2) of the definition 2.1 are not satisfied because of the squared term.

Example 2.3 *Let $\mathcal{C}([0,1],\mathbb{R})$ and $\mathcal{C}^1([0,1],\mathbb{R})$ vector spaces of applications $f : [0,1] \to \mathbb{R}$ respectively continuous and continuous with continuous derivative. The application:*

$$\begin{array}{rccc} D: & \mathcal{C}^1([0,1],\mathbb{R}) & \to & \mathcal{C}([0,1],\mathbb{R}) \\ & f & \mapsto & f' \end{array}$$

is a linear application, since:

$$\begin{array}{lcll} D(f+g) & = & (f+g)' = f'+g' & = Df + Dg \\ D(\lambda f) & = & (\lambda f)' & = \lambda f' \end{array}$$

if $\lambda \in \mathbb{R}$, and f and $g \in \mathcal{C}^1([0,1],\mathbb{R})$.

Example 2.4 *Let $v_0 \neq 0_E$ a vector of E, the translation application defined by*

$$\begin{array}{rccc} t: & E & \to & E \\ & v & \mapsto & v + v_0 \end{array}$$

is not linear (note, for example, that: $t(0) = v_0 \neq 0_E$).

Example 2.5 *Let E be a $\Bbbk$-vector space of finite-dimension n and $(e_1, e_2, ..., e_n)$ a basis of E. Then the application f defined by,*

$$\begin{array}{rccc} t: & \Bbbk^n & \to & E \\ & (\alpha_1, ..., \alpha_n) & \mapsto & f((\alpha_1, ..., \alpha_n)) = \sum_{i=1}^n \alpha_i e_i. \end{array}$$

is an isomorphism of vector spaces. We therefore deduce that any $\Bbbk$-vector space of finite dimension n is isomorphic to $\Bbbk^n$.

Proposition 2.1 *i) Let E, F and G be three vector spaces on the same field $\Bbbk$. $f : E \to F$ and $g : F \to G$ two linear applications. Then $g \circ f$ is a linear application.*

ii) Let $f : E \to F$ be a an isomorphism of vector spaces, then $f^{-1} : F \to E$ is also an isomorphism of vector spaces.

iii) Two finite-dimensional vector spaces of the same dimension are isomorphic.

Proof. *i*) Let $x \in E, y \in E$ and $\alpha \in \Bbbk$, then we have

$$\begin{aligned}(g \circ f)(x+y) &= g(f(x+y)) \\ &= g(f(x)+f(y)) \quad (\text{ car } f \text{ est linéaire }) \\ &= g(f(x))+g(f(y)) \quad (\text{ car } g \text{ est linéaire }) \\ &= (g \circ f)(x)+(g \circ f)(y).\end{aligned}$$

We also have

$$\begin{aligned}(g \circ f)(\alpha x) &= g(f(\alpha x)) \\ &= g(\alpha f(x)) \quad (\text{ car } f \text{ est linéaire }) \\ &= \alpha g(f(x)) \quad (\text{ car } g \text{ est lincaire }) \\ &= \alpha(g \circ f)(x).\end{aligned}$$

Then $g \circ f$ is linear.

ii) Let's suppose that $f : E \longrightarrow F$ is an isomorphism of vector spaces. Let $x \in F, y \in F$ and $\alpha \in \Bbbk$. Let $a \in E$ and $b \in E$, such that $f(a) = x$ and $f(b) = y$. As f is linear, then we have $f(a+b) = x+y$, so we will have

$$f^{-1}(x+y) = f^{-1}(f(a+b)) = a+b = f^{-1}(x)+f^{-1}(y).$$

We also have

$$f^{-1}(\alpha x) = f^{-1}(\alpha f(a)) = f^{-1}(f(\alpha a)) = \alpha a = \alpha f^{-1}(x).$$

Then f^{-1} is linear

iii) Let E and F be two vector spaces of the same dimension n, then according to the previous example, E and F are isomorphic to $\Bbbk^n$. Then if $\varphi : E \longrightarrow \Bbbk^n$ and $\psi : F \longrightarrow \Bbbk^n$ are two isomorphisms of vector spaces, then $\psi^{-1} \circ \varphi : E \longrightarrow F$ is an isomorphism of vector spaces. □

Proposition 2.2 *Let E and E' be two $\Bbbk$-vector spaces. For f and g two elements of $L_{\Bbbk}(E, E')$ and for α element of $\Bbbk$, we define $f+g$ and $\alpha.f$, by*

$$\forall x \in E, \quad (f+g)(x) = f(x)+g(x) \text{ and } (\alpha \cdot f)(x) = \alpha \cdot f(x).$$

Then $(L_{\Bbbk}(E, E'), +, \cdot)$ is a $\Bbbk$-vector space.

Proof. Just check that $L_{\Bbbk}(E, E')$ is a vector subspace of E'^E the $\Bbbk$-vector space of all applications of E to E'. □

Theorem 2.1 *Let E and E' be two $\mathbb{k}$-vector spaces of finite dimension. Then $L_{\mathbb{k}}(E, E')$ is of finite dimension and we have*

$$\dim\left(L_K(E, E')\right) = \dim(E) \times \dim(E').$$

Proof. Let $m = \dim(E), n = \dim(E'), (e_1, e_2, \ldots, e_m)$ be a basis of E and $(e'_1, e'_2, \ldots, e'_n)$ a basis of E'. For $(i, j) \in \mathbb{N}_m \times \mathbb{N}_n$, where for everything $p \in \mathbb{N}^*$, we pose $\mathbb{N}_p = \{1, 2, \ldots, p\}$, we define the application $f_{ij} : E \longrightarrow E'$ by,

$$\forall x \in E, \quad f_{ij}(x) = x_j e'_i \quad \text{where } x = \sum_{j=1}^{m} x_j e_j.$$

Then, $B = \{f_{ij} : (i, j) \in \mathbb{N}_m \times \mathbb{N}_n\}$ forms a basis of $L_K(E, E')$. Indeed,
Let $f \in L_{\mathbb{k}}(E, E')$, so for each $j \in \mathbb{N}_m$,, we have $f(e_j) = \sum_{i=1}^{n} \alpha_{ij} e'_i$.
So for any $x \in E$ with $x = \sum_{j=1}^{m} x_j e_j$, we have

$$\begin{aligned} f(x) &= \sum_{j=1}^{m} x_j f(e_j) = \sum_{j=1}^{m} x_j \left(\sum_{i=1}^{n} \alpha_{ij} e'_i \right) \\ &= \sum_{i=1}^{n} \sum_{j=1}^{m} \alpha_{ij} x_j e'_i = \sum_{i=1}^{n} \sum_{j=1}^{m} \alpha_{ij} f_{ij}(x). \end{aligned}$$

Then B is a finite generating part of $L_K(E, E')$.
It is easy to verify that B is a linearly independent part, noting that

$$\forall k \in \mathbb{N}_m, \quad f_{ij}(e_k) = \begin{cases} e'_i & \text{if } k = j \\ 0 & \text{if } k \neq j. \end{cases}$$

$\text{Card}(B) = \text{Card}(\mathbb{N}_m \times \mathbb{N}_n) = m \times n$, then, $\dim\left(L_K(E, E')\right) = m \times n$. □

2.2 Kernel, Image and Rank

Proposition 2.3 *Let E and F be two $\mathbb{k}$-vector spaces and $f : E \longrightarrow F$ a linear application. Then,*

i) The image by f of a vector subspace of E is a vector subspace of F. In particular, $f(E)$ is a vector subspace of F, called the image of f and denoted $\text{Im}\, f$. *Sa dimension est appelée rang de f et est notée*

$$rg\ f = \dim\left(\text{Im}\, f\right).$$

ii)The inverse image by f of a vector subspace of F is a vector subspace of E. In particular, $f^{-1}(\{0_F\})$ is a vector subspace of E, called the kernel of f and denoted $\ker(f)$.

Proof. i) Let A be a vector subspace of E. Then, $f(A) \neq \emptyset$, since $0_F = f(0_E)$, then $0_F \in f(A)$.

If $x \in A$, $y \in A$ and $\alpha \in K$, we have

$$f(x) + f(y) = f(x+y) \text{ and } \alpha \cdot f(x) = f(\alpha \cdot x).$$

as $x + y \in A$ and $\alpha \cdot x \in A$, therefore $f(x) + f(y) \in f(A)$ and $\alpha \cdot f(x) \in f(A)$.

ii) Now let B be a vector subspace of F. Then $f^{-1}(B) \neq \emptyset$, since $f(0_E) = 0_F$ and $0_F \in B$, then $0_E \in f^{-1}(B)$.

If $x \in f^{-1}(B), y \in f^{-1}(B)$ and $\alpha \in K$, therefore we have $f(x) \in B$ and $f(y) \in B$ and as B is a vector subspace of F and f linear, then $f(x+y) \in f(B)$ and $f(\alpha \cdot x) \in f(B)$, then $x + y \in f^{-1}(B)$ and $\alpha \cdot x \in f^{-1}(B)$. Let us remember that

$$z \in f^{-1}(B) \Longleftrightarrow f(z) \in B.$$

□

Remark 2.2 *Let E and F be two $\mathbb{k}$-vector spaces and $f : E \longrightarrow F$ a linear application. So*

1.
$$\ker(f) = \{x \in E : f(x) = 0_F\}$$
$$x \in \ker(f) \Longleftrightarrow f(x) = 0_F.$$

2.
$$\text{Imf} = \{f(x) : x \in E\}$$
$$y \in \text{Imf} \Longleftrightarrow \exists x \in E : y = f(x).$$

Proposition 2.4 *SLet E and F be two $\mathbb{k}$-vector spaces and $f : E \longrightarrow F$ a linear application. So*

i) f is injetive $\Longleftrightarrow \ker(f) = \{0_E\}$.

ii) f is surjetive $\Longleftrightarrow \text{Im}\, f = F$.

Proof. i) $\Longrightarrow$) Suppose that f is injective and that $x \in \ker(f)$. We have $f(x) = 0_F$ and as f is linear, then $f(0_E) = 0_F$, then $f(x) = f(0_E)$ and since f is injective, then $x = 0_E$. So, $\ker(f) = \{0_E\}$.

$\Longleftarrow$) Suppose that $\ker(f) = \{0_E\}$.

Let $x \in E$ and $y \in E$, such that $f(x) = f(y)$, do we have $x = y$?

As f is linear and $f(x) = f(y)$, so we have $f(x-y) = 0_F$, then $x - y \in \ker(f)$, then as $\ker(f) = \{0_F\}$, then we have $x = y$ and consequently, f is injective.

ii) Trivial, because an application $f : E \longrightarrow F$ is surjective, if and only if, $f(E) = F$. □

Example 2.6 *Let:*

$$D: \begin{array}{ccc} \mathbb{R}[x] & \longrightarrow & \mathbb{R}[x] \\ P & \mapsto & P'. \end{array}$$

The kernel of D is formed by the constant polynomials. On the other hand, $\operatorname{Im} D = \mathbb{R}[x]$, *because if* $P \in \mathbb{R}[x], Q(x) := \int_0^x P(t)dt$ *is a polynomial and we have* $Q' = P$ *that's to say* $DQ = P$.

Example 2.7 *Let:*

$$f : \mathbb{R}^3 \longrightarrow \mathbb{R}^3 (x, y, z) \longmapsto (x', y', z') \qquad \textit{where :} \qquad \begin{cases} x' = x + y - z \\ y' = 2x + y - 3z \\ z' = 3x + 2y - 4z. \end{cases}$$

$ker(f)$ is the set of triplets $(x, y, z) \in \mathbb{R}^3$ which satisfies the system:

$$\begin{cases} x + y - z = 0 \\ 2x + y - 3z = 0 \\ 3x + 2y - 4z = 0. \end{cases}$$

It is easy to find $x = 2\lambda, y = -\lambda, z = \lambda$; that's to say Ker f is the vector line generated by the vector $(2, -1, 1)$.

Regarding Im f, we have :

$(x', y', z') \in \operatorname{Im} f$, if and only if, it exists $(x, y, z) \in \mathbb{R}^3$ satisfies the system:

$$2x + y - 3z = y' 3x + 2y - 4z = z'$$

It is therefore a question of knowing for what values of x', y', z'. This system is compatible. By scaling, we find :

$$\begin{cases} x + y - z = x' \\ -y - z = y' - 2x' \\ -y - z = z' - 3x' \end{cases} \Longrightarrow \begin{cases} x - y - z = x' \\ y + z = 2x' - y' \\ 2x' - y' + z' - 3x' = 0, \end{cases}$$

the compatibility condition is $2x' - y' + z' - 3x' = 0$ so, $x' + y' - z' = 0$. The image of f is therefore the plan of $\mathbb{R}^3$ of the équation $x' + y' - z' = 0$.

Proposition 2.5 *Let* $f \in \mathcal{L}(E, E')$ *and* $\{v_i\}_{i \in I}$ *a family of vectors of* E.

1. *If f is injective and the family of E $\{v_i\}_{i \in I}$ is linearly independent, then the family* $\{f(v_i)\}_{i \in I}$ *of E' is linearly independent.*

2. *If f is surjective and the family* $\{v_i\}_{i \in I}$ *is generator of E then the family* $\{f(v_i)\}_{i \in I}$ *is generator of E'.*

In particular if f is bijective the image of a basis of E is a basis of E'.

Proof. 1. Let's assume the family $\{v_i\}_{i\in I}$ is linearly independent and let f is injective. For any extracted family $\{v_{\alpha_1},\ldots,v_{\alpha_q}\}$, the relation

$$\lambda_1 f(v_{\alpha_1})+\cdots+\lambda_q f\left(v_{\alpha_q}\right)=0$$

implies that $f\left(\lambda_1 v_{\alpha_1}+\cdots+\lambda_q v_{\alpha_q}\right)=0$, so, $\lambda_1 v_{\alpha_1}+\cdots+\lambda_q v_{\alpha_q}\in \operatorname{Ker} f$. But, $\operatorname{Ker} f=\{0\}$, then

$$\lambda_1 v_{\alpha_1}+\cdots+\lambda_q v_{\alpha_q}=0$$

and since the family $\{v_i\}_{i\in I}$ is linearly independent, we have $\lambda_1=0,\ldots,\lambda_q=0$. Then the family $\{f(v_i)\}_{i\in I}$ is linearly independent.
2. Let $y\in E'$ any; since f is surjective, there exists $x\in E$ such that $y=f(x)$. On the other hand, the family $\{v_i\}_{i\in I}$ is a generator, so x is of the form

$$x=\lambda_1 v_{\alpha_1}+\cdots+\lambda_p v_{\alpha_p},$$

thus : $f(x)=\lambda_1 f(v_{\alpha_1})+\cdots+\lambda_p f\left(v_{\alpha_p}\right).y$ is therefore a linear combination of elements of the family $\{f(v_i)\}_{i\in I}$ and, since it is chosen arbitrarily in E', the family $\{f(v_i)\}_{i\in I}$ is generator. □

Theorem 2.2 *Two finite-dimensional vector spaces are isomorphic if and only if they have the same dimension.*

Proof. Indeed, if there exists an isomorphism $f:E\longrightarrow E'$, the image by f of a basis of E is a basis of E', then E and E' have the same dimension. Conversely, suppose that $\dim E=\dim E'$ and let $\{e_1,\ldots,e_n\},\{e'_1,\ldots,e'_n\}$ two bases respectively of E and E'. Consider the application $f:E\longrightarrow E'$ constructed as follows:
- for $k=1,\ldots,n$ we pose: $f(e_k)=e'_k$;
- if $x=\sum_{k=1}^n x_k e_k$ we pose : $f(x)=\sum_{k=1}^n x_k f(e_k)=\sum_{k=1}^n x_k e'_k$,

(in other words, we define f on the basis of E and extend it by linearity over the whole of E). We easily verify that f is linear and bijective (the verification is left as an exercise). □

Remark 2.3 *As can be seen from the proof, the isomorphism of E onto E' depends on the choice of bases in E and in E and in general there is no canonical isomorphism.*

Proposition 2.6 *Let E and F be two K-vector spaces and $f:E\longrightarrow F$ a linear application. So*

a) For any vector subspace G of E, we have

$$f^{-1}(f(G)) = G + \ker(f).$$

b) f eis injective, if and only if, for any vector subspace G of E, we have

$$f^{-1}(f(G)) = G.$$

c) For any vector subspace H of F, we have

$$f\left(f^{-1}(H)\right) = H \cap \operatorname{Im} f.$$

d) f is surjective, if and only if, for any vector subspace H of F, we have

$$f\left(f^{-1}(H)\right) = H.$$

Proof. *a*)Let $x \in E$, then we have

$$\begin{aligned}
&x \in f^{-1}(f(G)) \Longleftrightarrow f(x) \in f(G)\\
&\Longleftrightarrow \exists a \in G : f(x) = f(a)\\
&\Longleftrightarrow \exists a \in G : f(x-a) = 0\\
&\Longleftrightarrow \exists a \in G : x - a \in \ker(f)\\
&\Longleftrightarrow \exists a \in G, \exists b \in \ker(f) : x = a + b\\
&\Longleftrightarrow x \in G + \ker(f).
\end{aligned}$$

b) $\Longrightarrow$) If we assume that f is injective, then $\ker(f) = \{0\}$, so, according to a), for any vector subspace G of E,

$$f^{-1}(f(G)) = G.$$

$\Longleftarrow$) If we assume that for any vector subspace G of E, we have $f^{-1}(f(G)) = G$, so in particular, we have

$$f^{-1}\left(f\left(\{0_E\}\right)\right) = \{0_{E^3}\}.$$

But, $f\left(\{0_E\}\right) = \{f\left(0_E\right)\} = \{0_F\}$, then

$$\ker(f) = f^{-1}\left(\{0_F\}\right) = \{0_E\}.$$

c) $\subset$) Let $y \in f\left(f^{-1}(H)\right)$. so there exists $x \in f^{-1}(H)$, such that $y = f(x)$. As $x \in f^{-1}(H)$, then $f(x) \in H$, thus $y \in H \cap \operatorname{Im} f$.

$\supset$) Let $y \in H \cap \operatorname{Im} f$, then we have

$$\begin{aligned}
&y \in H \cap \operatorname{Im} f \Longrightarrow y \in H \text{ and } y \in \operatorname{Im} f\\
&\Longrightarrow y \in H \text{ and } \exists x \in E, : y = f(x)\\
&\Longrightarrow x \in f^{-1}(H) \text{ and } f(x) \in H\\
&\Longrightarrow f(x) \in f\left(f^{-1}(H)\right)\\
&\Longrightarrow y \in f\left(f^{-1}(H)\right).
\end{aligned}$$

$d) \Longrightarrow)$ Suppose f is surjective, then $\operatorname{Im} f = F$, so for any subspace H of E, we have

$$f\left(f^{-1}(H)\right) = H \cap F = H.$$

$\Longleftarrow)$ Suppose that for any vector subspace H of F we have

$$f\left(f^{-1}(H)\right) = H,$$

so in particular, we will have $f\left(f^{-1}(F)\right) = F$. But, $f^{-1}(F) = E$, then $f(E) = F$, consequently, f is surjective. □

In the case where the spaces E and E' are finite-dimensional, the dimensions of the kernel and the image of f are related by the relation given in the following theorem, one of the most important in Linear Algebra:

Theorem 2.3 (Rank theorem) *Let E and E' be two finite-dimensional vector spaces and $f : E \longrightarrow E'$ a linear application. We then have:*

$$\dim E = \operatorname{rg} f + \dim(\operatorname{Ker} f).$$

Proof. Let's suppose $\dim E = n$, $\dim\operatorname{Ker} f = r$ and show that $\dim(\operatorname{Im} f) = n - r$. Let $\{w_1, \ldots, w_r\}$ be a basis of $\operatorname{Ker} f$, and $\{v_1, \ldots, v_{n-r}\}$ a family of vectors such that $\{w_1, \ldots, w_r, v_1, \ldots, v_{n-r}\}$ be a basis of E. Let $\mathcal{B} = \{f(v_1), \ldots, f(v_{n-r})\}$. We show that $\mathcal{B}$ is a basis of $\operatorname{Im} f$.

- $\mathcal{B}$ generates $\operatorname{Im} f$. Let $y = f(x) \in \operatorname{Im} f$. As $x \in E$, x is of the form $x = a_1 w_1 + \cdots + a_r w_r + b_1 v_1 + \cdots + b_{n-r} v_{n-r}$. Then we have :

$$y = a_1 f(w_1) + \cdots + a_r f(w_r) + b_1 f(v_1) + \cdots + b_{n-r} f(v_{n-r}),$$

$$= b_1 f(v_1) + \cdots + b_{n-r} f(v_{n-r})$$

which shows that $\mathcal{B}$ generates $\operatorname{Im} f$.

- $\mathcal{B}$ is linearly independent. Suppose that $\lambda_1 f(v_1) + \cdots + \lambda_{n-r} f(v_{n-r}) = 0$; we will obtain

$$f(\lambda_1 v_1 + \cdots + \lambda_{n-r} v_{n-r}) = 0,$$

then:

$$\lambda_1 v_1 + \cdots + \lambda_{n-r} v_{n-r} \in \operatorname{Ker} f.$$

Therefore, there is $a_1, \ldots, a_r \in \mathbb{k}$ such that:

$$\lambda_1 v_1 + \cdots + \lambda_{n-r} v_{n-r} = a_1 w_1 + \cdots + a_r w_r,$$

so,

$$\lambda_1 v_1 + \cdots + \lambda_{n-r} v_{n-r} - a_1 w_1 - \cdots - a_r w_r = 0.$$

Since the family $\{v_1, \ldots, v_{n-r}, w_1, \ldots, w_r\}$ is linearly independent, the coefficients of this linear combination are all zero; in particular : $\lambda_1 = 0, \ldots, \lambda_{n-r} = 0$, so, $\mathcal{B}$ is linearly independent. □

To show that a linear map is bijective, it is necessary to show that it is injective and surjective; however, in the finite-dimensional case, if the dimension of the starting space and that of the arrival space are the same, it is sufficient to demonstrate one of the two properties - either injectivity or surjectivity, hence this important corollary.

Corollary 2.1 *Let $f \in \mathcal{L}(E, E')$, E, E' being two vector spaces of the same finite dimension (in particular, for example, if $f \in End\ E$, with E of finite dimension). Then the following properties are equivalent:*

1. *f is injective.*
2. *f is surjective.*
3. *f is bijective.*

Proof. It is sufficient, of course, to show that 1 is equivalent to 2. As we have seen, f is injective if and only if Ker $f = \{0\}$. Since $\dim E = \operatorname{rg} f + \dim(\ker f)$, f is injective if and only if $\dim E = \operatorname{rg} f$, so, $\dim E = \dim(\operatorname{Im} f)$. Now, by hypothesis, $\dim E = \dim E'$, then f is injective if and only if $\dim(\operatorname{Im} f) = \dim E'$. Since $\operatorname{Im} f \subset E'$ this is equivalent to $\operatorname{Im} f = E'$, then f is surjective. □

Remark 2.4 *This result is false in infinite dimension, a counterexample: the application:*

$$D : \begin{array}{ccc} \mathbb{R}[x] & \longrightarrow & \mathbb{R}[x] \\ P & \mapsto & P' \end{array}$$

is surjective and not injective.

Theorem 2.4 *Let E be a K-finite-dimensional vector space and $u : E \longrightarrow E$ an endomorphism of E. Then the following propositions are equivalent:*

i) $E = \ker u \oplus \operatorname{Im} u$.
ii) $\operatorname{Im} u = \operatorname{Im} u^2$.
iii) $\ker u = \ker u^2$.
iv) $\ker u \cap \operatorname{Im} u = \{0\}$.

Proof. *i)* $\Longrightarrow$ *ii)* Let's suppose that $E = \ker u \oplus \operatorname{Im} u$ and show that $\operatorname{Im} u = \operatorname{Im} u^2$. To do this, let us first note that everything $u \in L_K(E)$, we have $\operatorname{Im} u^2 \subseteq \operatorname{Im} u$. So, it is enough to show that $\operatorname{Im} u \subseteq \operatorname{Im} u^2$. For this, let $y \in \operatorname{Im} u$, so there exists $x \in E$, such that $y = u(x)$. Since $E = \ker u \oplus \operatorname{Im} u$, then $x = x_1 + u(x_2)$, with $x_1 \in \ker u$, then $y = u^2(x)$, consequently, $y \in \operatorname{Im} u^2$.

ii) $\Rightarrow$ *iii)* Let's suppose that $\mathrm{Im}(u) = \mathrm{Im}\, u^2$ and show that $\ker u = \ker u^2$. For this, let us also note that for any $u \in L_K(E)$, we have $\ker u \subseteq \ker u^2$. So, it is enough to show that $\ker u^2 \subseteq \ker u$. According to the rank theorem, we have

$$\dim(E) = \dim(\ker u) + \dim(\mathrm{Im}\, u) = \dim\left(\ker u^2\right) + \dim\left(\mathrm{Im}\, u^2\right).$$

Since $\mathrm{Im}\, u = \mathrm{Im}\, u^2$ then $\dim(\ker u) = \dim\left(\ker u^2\right)$, consequently, we will have $\ker u = \ker u^2$.

iii) $\Longrightarrow$ *iv)* Let's suppose that $\ker u = \ker u^2$ and show that $\ker u \cap \mathrm{Im}\, u = \{0\}$. Let $y \in E$, so we have

$$\begin{array}{lll} y \in \ker u \cap \mathrm{Im}\, u \Leftrightarrow u(y) = 0 \text{ and } \exists x \in E : y = u(x) & & \\ \Rightarrow u^2(x) = u(y) = 0 & \Rightarrow x \in \ker u^2 & \\ \Rightarrow x \in \ker u & \Rightarrow u(x) = 0 & \Rightarrow y = 0 \end{array}$$

iv $\Rightarrow$ *i)* Trivial, because we know that

$$E = \ker u \oplus \mathrm{Im}\, u \Leftrightarrow \begin{cases} \dim(\ker u) + \dim(\mathrm{Im}\, u) = \dim(E) \\ \text{ et} \\ \ker u \cap \mathrm{Im}\, u = \{0\}. \end{cases}$$

□

2.3 Exercises

Exercise 1. Let $f : \mathbb{R}^4 \to \mathbb{R}^3$ be the map defined for any $u = (x, y, z, t) \in \mathbb{R}^4$ by :

$$f((x, y, z, t)) = (x + y, z + t, x + y + z + t)$$

1. Prove that f is a linear map.
2. Determine a basis of $\ker(f)$.
3. Determine a basis of $\mathrm{Im}(f)$.

Exercise 2. Let $u : \mathbb{R}^3 \to \mathbb{R}^3$ the map defined by:

$$u((x_1, x_2, x_3)) = (-2x_1 + 4x_2 + 4x_3, -x_1 + x_3, -2x_1 + 4x_2 + 4x_3)$$

1. Prove that u is linear.
2. Determine a basis of $\ker(u)$ and a basis of $\mathrm{Im}(u)$.
3. Do we have $\ker(u) \oplus \mathrm{Im}(u) = \mathbb{R}^3$?

Exercise 3. Let $\beta = \{e_1, e_2, e_3\}$ be the canonical basis of $\mathbb{R}^3$. Let u be an endomorphism of $\mathbb{R}^3$ defined by :

$$u(e_1) = 2e_1 + e_2 + 3e_3; \quad u(e_2) = e_2 - 3e_3; \quad u(e_3) = -2e_2 + 2e_3.$$

1. Let $x = (x_1, x_2, x_3) \in \mathbb{R}^3$ a vector. Determine the image by u of the vector x. (Calculate $u(x)$).

2. Let $E = \{x \in \mathbb{R}^3, u(x) = 2x\}$ and $F = \{x \in \mathbb{R}^3, u(x) = -x\}$. Prove that E and F are subspaces of $\mathbb{R}^3$.

3. Determine a basis of E and a basis of F.
4. Do we have $E \oplus F = \mathbb{R}^3$?

Exercise 4. Let $\beta = \{e_1, e_2, e_3\}$ be the canonical basis of $\mathbb{R}^3$. Let $f : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear map such that :

$$\begin{aligned} f(e_1) &= -\frac{1}{3}e_1 + \frac{2}{3}e_2 + \frac{2}{3}e_3 = \frac{1}{3}(-e_1 + 2e_2 + 2e_3), \\ f(e_2) &= \frac{2}{3}e_1 - \frac{1}{3}e_2 + \frac{2}{3}e_3 = \frac{1}{3}(2e_1 - e_2 + 2e_3) \text{ and} \\ f(e_3) &= \frac{2}{3}e_1 + \frac{2}{3}e_2 - \frac{1}{3}e_3 = \frac{1}{3}(2e_1 + 2e_2 - e_3). \end{aligned}$$

Let $E_{-1} = \{u \in \mathbb{R}^3 \mid f(u) = -u\}$ and $E_1 = \{u \in \mathbb{R}^3 \mid f(u) = u\}$.

1. Prove that E_{-1} and E_1 are subspaces of $\mathbb{R}^3$.
2. Demonstrate that $e_1 - e_2$ and $e_1 - e_3$ are belong to E_{-1} and that $e_1 + e_2 + e_3$ is belong to E_1.
3. What can we deduce about the dimensions of E_{-1} and E_1 ?
4. Determine $E_{-1} \cap E_1$.

5. Do we have $E_{-1} \oplus E_1 = \mathbb{R}^3$?
6. Calculate $f^2 = f \circ f$ and deduce that f is bijective and determine f^{-1}.

Exercise 5. Let $\beta = \{e_1, e_2\}$ be the canonical basis of $\mathbb{R}^2$. Let u an endomorphism of $\mathbb{R}^2$ so that $u(e_1) = e_1 + e_2$ with $\dim(\ker(u)) = 1$

1. Determine $u(e_2)$ depending on a parameter $a \in \mathbb{R}$.
2. Determine the image of a vector $x = (x_1, x_2) \in \mathbb{R}$ depending on a.
3. Determine a basis of the kernel $\ker(u)$.

Exercise 6 **.**

Let $f : \mathbb{R}^4 \to \mathbb{R}$ be the map defined for any $x = (x_1, x_2, x_3, x_4) \in \mathbb{R}^4$ by:

$$f(x) = x_1 + x_2 + x_3 + x_4$$

Let $\beta = \{e_1, e_2, e_3, e_4\}$ be the canonical basis of $\mathbb{R}^4$.

1. Calculate the images of vectors of the canonical basis by f. Deduce the dimension of $\mathrm{Im}(f)$.
2. Determine the dimension of $\ker(f)$ and give a basis for it.

Exercise 7.

Let u be a linear map from E to E, E being a vector space of dimension n with n even. Show that the following two assertions are equivalent:

(a) $u^2 = O_E$ (où O_E is the null linear application) and $n = 2\dim(\mathrm{Im}(u))$

(b) $\mathrm{Im}(u) = \ker(u)$

Exercise 8.

Course Question

Let u be a linear map from E to E. Show that: u is injective if and only if $\ker(u) = \{0_E\}$.

Exercise 9.

Let $u : E \to E$ be a linear application and λ a real.

1. Let $E_\lambda = \ker(u - \lambda id_E)$.

- Calculate $u(x)$ for $x \in E_\lambda$
- Show that is a vector subspace of E.

2. Let $F \subset E$ be a vector subspace of E, show that $u(F)$ is a vector subspace of E.
3. If $\lambda \neq 0$, prove that $u(E_\lambda) = E_\lambda$

Exercise 10.

Let E and F be two vector spaces of respective dimensions n and p. Let $u : E \to F$ a linear application

1. Show that if $n < p$ then u is not surjective.
2. Show that if $n > p$ then u is not injective.

Exercise 11.

Let $f : E \to F$ be a linear map. Prove that:

$$\ker(f) \cap \mathrm{im}(f) = f\left(\ker\left(f^2\right)\right).$$

Exercise 12.

Let f and g be two endomorphisms of $\mathbb{R}^n$. Prove that

$$f(\ker(g \circ f)) = \ker(g) \cap \mathrm{Im}(f).$$

Exercise 13.

Let u be an endomorphism of a vector space E on a field $\mathbb{k}$.

1. Prove that $\ker(u) \subset \ker(u^2)$.
2. Prove that $\mathrm{Im}(u^2) \subset \mathrm{Im}(u)$.

Exercise 14.

Let u be an endomorphism of a vector space E on a field $\mathbb{k}$.

Show that the following assertions are equivalent:

(i) $\ker(u) \cap \mathrm{im}(u) = \{0_E\}$.

(ii) $\ker(u) = \ker(u \circ u)$.

2.4 Solutions

Exercise 1.

1. Let $u = (x, y, z, t)$ and $u' = (x', y', z', t')$ be two vectors of $\mathbb{R}^4$. Let λ and λ' be two real numbers.

$$\lambda u + \lambda' u' = \lambda(x, y, z, t) + \lambda'(x', y', z', t') = (\lambda x + \lambda' x', \lambda y + \lambda' y', \lambda z + \lambda' z', \lambda t + \lambda' t')$$

so

$$\begin{aligned}
&f(\lambda u + \lambda' u') = f(\lambda x + \lambda' x', \lambda y + \lambda' y', \lambda z + \lambda' z', \lambda t + \lambda' t') \\
&= (\lambda x + \lambda' x') + (\lambda y + \lambda' y'), (\lambda z + \lambda' z') + (\lambda t + \lambda' t'), (\lambda x + \lambda' x') + (\lambda y + \lambda' y') \\
&+ (\lambda z + \lambda' z') + (\lambda t + \lambda' t') \\
&= (\lambda(x + y) + \lambda'(x' + y'), \lambda(z + t) + \lambda'(z' + t'), \lambda(x + y + z + t) \\
&+ \lambda'(x' + y' + z' + t') \\
&= \lambda(x + y, z + t, x + y + z + t) + \lambda'(x' + y', z' + t', x' + y' + z' + t') \\
&= \lambda f(u) + \lambda f(u')
\end{aligned}$$

f is linear.

We have:

$$\begin{aligned}
u \in \ker(f) &\Leftrightarrow f(u) = 0_{\mathbb{R}^3} \\
&\Leftrightarrow (x + y, z + t, x + y + z + t) = (0, 0, 0)
\end{aligned}$$

$$\Leftrightarrow \begin{cases} x+y=0 \\ z+t=0 \\ x+y+z+t=0 \end{cases}$$

$$\Leftrightarrow \begin{cases} y=-x \\ t=-z \end{cases} \Leftrightarrow u=(x,-x,z,-z)=x(1,-1,0,0)+z(0,0,1,-1)\ .$$

Let $a=(1,-1,0,0)$ and $b=(0,0,1,-1)$, a and b span $\ker(f)$, moreover these vectors are not proportional, so they form a linearly independent family, finally (a,b) is a basis of ker (f).

3. First method

$$\begin{aligned} &\mathrm{Im}(f)=\mathrm{Vect}\left(f\left(e_1\right),f\left(e_2\right),f\left(e_3\right),f\left(e_4\right)\right) \\ &f\left(e_1\right)=(1,0,1);f\left(e_2\right)=(1,0,1);f\left(e_3\right)=(0,1,1);f\left(e_4\right)=(0,1,1). \end{aligned}$$

Since $f\left(e_1\right)=f\left(e_2\right)$ and $f\left(e_3\right)=f\left(e_4\right)$

$$\mathrm{Im}(f)=\mathrm{Vect}\left(f\left(e_1\right),f\left(e_3\right)\right)$$

$f\left(e_1\right)$ and $f\left(e_3\right)$ are not proportional, they form a linearly independent family, and since this family spans $\mathrm{Im}(f)$, it is a basis of $\mathrm{Im}(f)$.

Second method

By the rank-nullity theorem

$$\begin{aligned} \dim(\ker(f))+\dim(\mathrm{Im}(f)) &= \dim\left(\mathbb{R}^4\right) \\ &\Leftrightarrow 2+\dim(\mathrm{Im}(f))=4\Leftrightarrow\dim(\mathrm{Im}(f))=2. \end{aligned}$$

Then we look for two non-proportional vectors of $\mathrm{Im}(f)$, for example $f\left(e_1\right)$ and $f\left(e_3\right)$, they form a linearly independent family in a space of dimension 2, so it is a basis.

Exercise 2.

1. Let $x=(x_1,x_2,x_3)$ and $y=(y_1,y_2,y_3)$ be two vectors of $\mathbb{R}^3$. Let λ and μ be two real numbers.

$$\begin{aligned} &\lambda x+\mu y=(\lambda x_1+\mu y_1,\lambda x_2+\mu y_2,\lambda x_3+\mu y_3) \\ &u(\lambda x+\mu y)=(-2\left(\lambda x_1+\mu y_1\right)+4\left(\lambda x_2+\mu y_2\right)+4\left(\lambda x_3+\mu y_3\right),-\left(\lambda x_1+\mu y_1\right) \\ &+\lambda x_3+\mu y_3,-2\left(\lambda x_1+\mu y_1\right)+4\left(\lambda x_2+\mu y_2\right)+4\left(\lambda x_3+\mu y_3\right)) \\ &=(\lambda\left[-2x_1+4x_2+4x_3\right]+\mu\left[-2y_1+4y_2+4y_3\right],\lambda\left[-x_1+x_3\right] \\ &+\mu\left[-y_1+y_3\right],\lambda\left[-2x_1+4x_2+4x_3\right]+\mu\left[-2y_1+4y_2+4y_3\right]) \\ &=\lambda\left(-2x_1+4x_2+4x_3,-x_1+x_3,-2x_1+4x_2+4x_3\right) \\ &+\mu\left(-2y_1+4y_2+4y_3,-y_1+y_3,-2y_1+4y_2+4y_3\right)=\lambda u(x)+\mu u(y). \end{aligned}$$

So u is linear.

2. Let $x=(x_1,x_2,x_3)\in\ker(u)$ and $X=\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}$ its coordinates in the canonical basis.

$$\begin{aligned}x \in \ker(u) &\Leftrightarrow \begin{cases}-2x_1+4x_2+4x_3=0\\-x_1+x_3=0\\-2x_1+4x_2+4x_3=0\end{cases} \Leftrightarrow \begin{cases}-x_1+2x_2+2x_3=0\\x_1=x_3\end{cases}\\ &\Leftrightarrow \begin{cases}2x_2+x_3=0\\x_1=x_3\end{cases} \Leftrightarrow \begin{cases}x_2=-\frac{1}{2}x_3\\x_1=x_3\end{cases}\\ x &= \left(x_3,-\frac{1}{2}x_3,x_3\right)=\frac{x_3}{2}(2,-1,2).\end{aligned}$$

$a=(2,-1,2)=2e_1-e_2+2e_3$ is a non-zero vector that spans ker (u), it is a basis of ker (u).

$$\mathrm{Im}(u)=\mathrm{Vect}\left(u\left(e_1\right),u\left(e_2\right),u\left(e_3\right)\right)$$

By the rank-nullity theorem,

$$\begin{aligned}\dim(\ker(u))+\dim(\mathrm{Im}(u)) &= \dim\left(\mathbb{R}^3\right)\\ &\Leftrightarrow 1+\dim(\mathrm{Im}(u))=3\Leftrightarrow\dim(\mathrm{Im}(u))=2\end{aligned}$$

$u\left(e_1\right)=-2e_1-e_2-2e_3=(-2,-1,-2)$ and $u\left(e_2\right)=4e_1+4e_3=(4,0,4)$, these two vectors are not proportional, they form a linearly independent family of $\mathrm{Im}(u)$ which is of dimension $2,\left(u\left(e_1\right),u\left(e_2\right)\right)$ is a basis of $\mathrm{Im}(u)$.

3. $\ker(u)\oplus\mathrm{Im}(u)=\mathbb{R}^3\Leftrightarrow\left(a,u\left(e_1\right),u\left(e_2\right)\right)$ is a basis of $\mathbb{R}^3$.

It is almost obvious that

$$u\left(e_1\right)+u\left(e_3\right)=a.$$

Otherwise, we calculate $\alpha a+\beta u\left(e_1\right)+\gamma u\left(e_3\right)=0_{\mathbb{R}^3}$ and we see that $\alpha=1,\beta=-1$ and $\gamma=-1$ is a non-zero solution.

$\left(a,u\left(e_1\right),u\left(e_2\right)\right)$ is not a basis, so we do not have $\ker(u)\oplus\mathrm{Im}(u)=\mathbb{R}^3$

Exercise 3.

1.

$$\begin{aligned}&u(x)=u(x_1e_1+x_2e_2+x_3e_3)=x_1u\left(e_1\right)+x_2u\left(e_2\right)+x_3u\left(e_3\right)\\&=x_1\left(2e_1+e_2+3e_3\right)+x_2\left(e_2-3e_3\right)+x_3\left(-2e_2+2e_3\right)\\&=2x_1e_1+\left(x_1+x_2-2x_3\right)e_2+\left(3x_1-3x_2+2x_3\right)e_3\\&=\left(2x_1,x_1+x_2-2x_3,3x_1-3x_2+2x_3\right).\end{aligned}$$

2. $f\left(0_{\mathbb{R}^3}\right)=0_{\mathbb{R}^3}=2\times0_{\mathbb{R}^3}\Rightarrow0_{\mathbb{R}^3}\in E$

Let x and y be two vectors of E, then $u(x)=2x$ and $u(y)=2y$

Let λ and μ be two real numbers

$$u(\lambda x+\mu y)=\lambda u(x)+\mu u(y)=\lambda(2x)+\mu(2y)=2(\lambda x+\mu y).$$

So $\lambda x + \mu y \in E$ and E is a vector subspace of $\mathbb{R}^3$
$f(0_{\mathbb{R}^3}) = 0_{\mathbb{R}^3} = -0_{\mathbb{R}^3} \Rightarrow 0_{\mathbb{R}^3} \in F$
Let x and y be two vectors of F, then $u(x) = -x$ and $u(y) = -y$
Let λ and μ be two real numbers

$$u(\lambda x + \mu y) = \lambda u(x) + \mu u(y) = \lambda(-x) + \mu(-y) = -(\lambda x + \mu y).$$

So $\lambda x + \mu y \in F$ and F is a vector subspace of $\mathbb{R}^3$.
3.

$$x \in E \Leftrightarrow u(x) = 2x \Leftrightarrow (2x_1, x_1 + x_2 - 2x_3, 3x_1 - 3x_2 + 2x_3) = 2(x_1, x_2, x_3)$$
$$\Leftrightarrow \begin{cases} 2x_1 = 2x_1 \\ x_1 + x_2 - 2x_3 = 2x_2 \\ 3x_1 - 3x_2 + 2x_3 = 2x_3 \end{cases} \Leftrightarrow \begin{cases} x_1 + x_2 - 4x_3 = 0 \\ 3x_1 - 3x_2 = 0 \end{cases} \Leftrightarrow \begin{cases} x_1 = x_2 \\ x_3 = 0. \end{cases}$$

So $x = (x_1, x_1, x_3) = x_1(1, 1, 0) = x_1(e_1 + e_2)$.
$e_1 + e_2 \neq 0_{\mathbb{R}^3}$, it is a basis of E.

$$x \in F \Leftrightarrow u(x) = -x \Leftrightarrow (2x_1, x_1 + x_2 - 2x_3, 3x_1 - 3x_2 + 2x_3) = -(x_1, x_2, x_3)$$
$$\Leftrightarrow \begin{cases} 2x_1 = -x_1 \\ x_1 + x_2 - 2x_3 = -x_2 \\ 3x_1 - 3x_2 + 2x_3 = -x_3 \end{cases} \Leftrightarrow \begin{cases} x_1 + 2x_2 - 2x_3 = 0 \\ 3x_1 - 3x_2 + 3x_3 = 0 \end{cases} \Leftrightarrow \begin{cases} x_1 = 0 \\ x_2 = x_3. \end{cases}$$

So $x = (0, x_3, x_3) = x_3(0, 1, 1) = x_3(e_2 + e_3)$.
$e_2 + e_3 \neq 0_{\mathbb{R}^3}$, it is a basis of F.
4.

$$\dim(E) + \dim(F) = 1 + 1 = 2.$$

So there is no direct sum.

Exercise 4.

1. Let u, u' be two vectors of E_{-1}, then $f(u) = -u$ and $f(u') = -u'$. Let λ, λ' be two real numbers.

$$f(\lambda u + \lambda' u') = \lambda f(u) + \lambda f(u') = \lambda(-u) + \lambda(-u') = -(\lambda u + \lambda' u').$$

The first equality because f is linear, the second because u and u' are in E_{-1},
The third shows that $\lambda u + \lambda' u' \in E_{-1}$

$$f(0_{\mathbb{R}^3}) = 0_{\mathbb{R}^3} = -0_{\mathbb{R}^3}.$$

The first equality because the image of the zero vector by a linear map is always the zero vector, the second equality shows that $0_{\mathbb{R}^3} \in E_{-1}$.

E_{-1} is a vector subspace of $\mathbb{R}^3$.
Let u, u' be two vectors of E_1, then $f(u) = u$ and $f(u') = u'$. Let λ, λ' be two real numbers.

$$f(\lambda u + \lambda' u') = \lambda f(u) + \lambda f(u') = \lambda u + \lambda u' = \lambda u + \lambda' u'.$$

The first equality because f is linear, the second because u and u' are in E_1,
The third shows that $\lambda u + \lambda' u' \in E_1$

$$f\left(0_{\mathbb{R}^3}\right) = 0_{\mathbb{R}^3}.$$

So $0_{\mathbb{R}^3} \in E_1$.
E_1 is a vector subspace of $\mathbb{R}^3$.
2.

$$f(e_1 - e_2) = f(e_1) - f(e_2) = \tfrac{1}{3}(-e_1 + 2e_2 + 2e_3) - \tfrac{1}{3}(2e_1 - e_2 + 2e_3)$$
$$= \tfrac{1}{3}(-3e_1 + 3e_2) = -(e_1 - e_2).$$

So $e_1 - e_2 \in E_{-1}$.
Similarly

$$f(e_1 - e_3) = f(e_1) - f(e_3) = \tfrac{1}{3}(-e_1 + 2e_2 + 2e_3) - \tfrac{1}{3}(2e_1 + 2e_2 - e_3)$$
$$= \tfrac{1}{3}(-3e_1 + 3e_3) = -(e_1 - e_3).$$

So $e_1 - e_3 \in E_{-1}$.
And

$$f(e_1 + e_2 + e_3) = f(e_1) + f(e_2) + f(e_3)$$
$$= \tfrac{1}{3}(-e_1 + 2e_2 + 2e_3) + \tfrac{1}{3}(2e_1 - e_2 + 2e_3) + \tfrac{1}{3}(2e_1 + 2e_2 - e_3)$$
$$= \tfrac{1}{3}(3e_1 + 3e_2 + 3e_3) = e_1 + e_2 + e_3.$$

So $e_1 + e_2 + e_3 \in E_1$.
3. We have two non-proportional vectors in E_{-1}: $e_1 - e_2$ and $e_1 - e_3$ (they are not proportional because one is not a scalar multiple of the other). So $\dim(E_{-1}) \geq 2$.
We have a non-zero vector in E_1: $e_1 + e_2 + e_3$, so $\dim(E_1) \geq 1$.
Moreover, E_{-1} and E_1 are in $\mathbb{R}^3$ which is of dimension 3, so $\dim(E_{-1}) + \dim(E_1) \leq 3$.
We deduce that $\dim(E_{-1}) = 2$ and $\dim(E_1) = 1$.
4. Let $x \in E_{-1} \cap E_1$, then $f(x) = -x$ and $f(x) = x$, so $-x = x \Rightarrow 2x = 0_{\mathbb{R}^3} \Rightarrow x = 0_{\mathbb{R}^3}$.
So $E_{-1} \cap E_1 = \{0_{\mathbb{R}^3}\}$.
5. We have $\dim(E_{-1}) + \dim(E_1) = 2 + 1 = 3 = \dim(\mathbb{R}^3)$ and $E_{-1} \cap E_1 = \{0_{\mathbb{R}^3}\}$, so $E_{-1} \oplus E_1 = \mathbb{R}^3$.
6.

$$f^2(x) = f(f(x))$$
$$f^2(e_1) = f(f(e_1)) = f\left(\tfrac{1}{3}(-e_1 + 2e_2 + 2e_3)\right) = \tfrac{1}{3}(-f(e_1) + 2f(e_2) + 2f(e_3))$$
$$= \tfrac{1}{3}\left(-\tfrac{1}{3}(-e_1 + 2e_2 + 2e_3) + \tfrac{2}{3}(2e_1 - e_2 + 2e_3) + \tfrac{2}{3}(2e_1 + 2e_2 - e_3)\right)$$
$$= \tfrac{1}{9}(e_1 - 2e_2 - 2e_3 + 4e_1 - 2e_2 + 4e_3 + 4e_1 + 4e_2 - 2e_3)$$
$$= \tfrac{1}{9}(9e_1 + 0e_2 + 0e_3) = e_1.$$

Similarly, we show that $f^2(e_2) = e_2$ and $f^2(e_3) = e_3$.
So $f^2 = id_{\mathbb{R}^3}$, so $f \circ f = id_{\mathbb{R}^3}$, which means that f is bijective and $f^{-1} = f$.

Exercise 5.

1. Let $u(e_2) = ae_1 + be_2$.

We know that $\dim(\ker(u)) = 1$.

We have

$$u(e_1) = e_1 + e_2$$

The matrix of u in the basis β is

$$A = \begin{pmatrix} 1 & a \\ 1 & b \end{pmatrix}$$

We have $\dim(\ker(u)) = 1 \Leftrightarrow \text{rank}(u) = 2 - 1 = 1 \Leftrightarrow \text{rank}(A) = 1$.

So the two columns of A are proportional, so there exists $\lambda \in \mathbb{R}$ such that

$$\begin{pmatrix} a \\ b \end{pmatrix} = \lambda \begin{pmatrix} 1 \\ 1 \end{pmatrix} \Rightarrow a = \lambda, b = \lambda.$$

So $u(e_2) = \lambda e_1 + \lambda e_2$.

2. Let $x = (x_1, x_2) \in \mathbb{R}^2$

$$\begin{aligned} u(x) &= x_1 u(e_1) + x_2 u(e_2) = x_1(e_1 + e_2) + x_2(\lambda e_1 + \lambda e_2) \\ &= (x_1 + \lambda x_2) e_1 + (x_1 + \lambda x_2) e_2 = (x_1 + \lambda x_2, x_1 + \lambda x_2). \end{aligned}$$

3.

$$\begin{aligned} &x \in \ker(u) \Leftrightarrow u(x) = 0_{\mathbb{R}^2} \Leftrightarrow (x_1 + \lambda x_2, x_1 + \lambda x_2) = (0,0) \\ &\Leftrightarrow x_1 + \lambda x_2 = 0 \Leftrightarrow x_1 = -\lambda x_2 \Leftrightarrow x = (-\lambda x_2, x_2) = x_2(-\lambda, 1). \end{aligned}$$

So a basis of $\ker(u)$ is $((-\lambda, 1))$ if $\lambda \neq 0$ and if $\lambda = 0$ then $x_1 = 0$ and $x = (0, x_2) = x_2(0,1)$, so a basis is $((0,1))$.

Exercise 6.

1.

$$\begin{aligned} &f(e_1) = 1; f(e_2) = 1; f(e_3) = 1; f(e_4) = 1. \\ &\text{Im}(f) = \text{Vect}(f(e_1), f(e_2), f(e_3), f(e_4)) = \text{Vect}(1) = \mathbb{R}. \end{aligned}$$

So $\dim(\text{Im}(f)) = 1$.

2. By the rank-nullity theorem:

$$\dim(\ker(f)) + \dim(\text{Im}(f)) = \dim\left(\mathbb{R}^4\right) = 4 \Rightarrow \dim(\ker(f)) = 3.$$

Let $x = (x_1, x_2, x_3, x_4) \in \ker(f) \Leftrightarrow x_1 + x_2 + x_3 + x_4 = 0 \Leftrightarrow x_1 = -x_2 - x_3 - x_4$.

So $x = (-x_2 - x_3 - x_4, x_2, x_3, x_4) = x_2(-1,1,0,0) + x_3(-1,0,1,0) + x_4(-1,0,0,1)$.

Let $a = (-1,1,0,0), b = (-1,0,1,0), c = (-1,0,0,1)$.

These three vectors are linearly independent, and since $\dim(\ker(f)) = 3$, they form a basis of $\ker(f)$.

Exercise 7.

$(b) \Rightarrow (a)$

Assume $\text{Im}(u) = \ker(u)$.

Let $x \in E$, then $u(x) \in \text{Im}(u) = \ker(u)$, so $u(u(x)) = 0_E$, so $u^2(x) = 0_E$ for all $x \in E$, so $u^2 = O_E$.

Moreover, by the rank-nullity theorem:

$$\dim(\ker(u)) + \dim(\mathrm{Im}(u)) = n.$$

But $\mathrm{Im}(u) = \ker(u)$, so $\dim(\ker(u)) = \dim(\mathrm{Im}(u))$, so $2\dim(\mathrm{Im}(u)) = n$.

$(a) \Rightarrow (b)$

Assume $u^2 = O_E$ and $n = 2\dim(\mathrm{Im}(u))$.

Let $y \in \mathrm{Im}(u)$, then there exists $x \in E$ such that $y = u(x)$, then $u(y) = u(u(x)) = u^2(x) = 0_E$, so $y \in \ker(u)$, so $\mathrm{Im}(u) \subset \ker(u)$.

By the rank-nullity theorem:

$$\dim(\ker(u)) + \dim(\mathrm{Im}(u)) = n = 2\dim(\mathrm{Im}(u)).$$

So $\dim(\ker(u)) = \dim(\mathrm{Im}(u))$.

We have $\mathrm{Im}(u) \subset \ker(u)$ and they have the same dimension, so $\mathrm{Im}(u) = \ker(u)$.

Exercise 8.

Course.

Exercise 9.

1. Let $x \in E_\lambda = \ker(u - \lambda id_E)$, then $(u - \lambda id_E)(x) = 0_E \Rightarrow u(x) - \lambda x = 0_E \Rightarrow u(x) = \lambda x$.

So $u(x) = \lambda x$.

Let $x, y \in E_\lambda$ and $\alpha, \beta \in \mathbb{R}$, then $u(x) = \lambda x$ and $u(y) = \lambda y$.

Then $u(\alpha x + \beta y) = \alpha u(x) + \beta u(y) = \alpha(\lambda x) + \beta(\lambda y) = \lambda(\alpha x + \beta y)$.

So $\alpha x + \beta y \in E_\lambda$.

E_λ is a vector subspace of E.

2. Let F be a vector subspace of E.

Let $y_1, y_2 \in u(F)$, then there exist $x_1, x_2 \in F$ such that $y_1 = u(x_1)$ and $y_2 = u(x_2)$.

Let $\alpha, \beta \in \mathbb{R}$, then $\alpha y_1 + \beta y_2 = \alpha u(x_1) + \beta u(x_2) = u(\alpha x_1 + \beta x_2)$.

Since F is a vector subspace, $\alpha x_1 + \beta x_2 \in F$, so $\alpha y_1 + \beta y_2 \in u(F)$.

$u(F)$ is a vector subspace of E.

3. Let $\lambda \neq 0$.

Let $y \in u(E_\lambda)$, then there exists $x \in E_\lambda$ such that $y = u(x) = \lambda x$.

Then $x = \frac{1}{\lambda}y$, and since $x \in E_\lambda$, we have $u(x) = \lambda x$, but $u(x) = y$, so $y = \lambda\left(\frac{1}{\lambda}y\right)$, which is true.

But we want to show that $y \in E_\lambda$, i.e., $u(y) = \lambda y$.

We have $u(y) = u(u(x)) = u(\lambda x) = \lambda u(x) = \lambda y$.

So $y \in E_\lambda$.

We have shown that $u(E_\lambda) \subset E_\lambda$.

Conversely, let $y \in E_\lambda$, then $u(y) = \lambda y$.

Since $\lambda \neq 0$, we have $y = u\left(\frac{1}{\lambda}y\right)$.

Let $x = \frac{1}{\lambda}y$, then $u(x) = u\left(\frac{1}{\lambda}y\right) = \frac{1}{\lambda}u(y) = \frac{1}{\lambda}(\lambda y) = y$.

And $x \in E_\lambda$ because $u(x) = y = \lambda x$? We need to check that $x \in E_\lambda$.

We have $u(x) = y$ and $y = \lambda x$? Not necessarily.

We have $y \in E_\lambda$ means $u(y) = \lambda y$.
But $u(x) = y$, so $u(x) = y$.
We want to show that $x \in E_\lambda$, i.e., $u(x) = \lambda x$.
But $u(x) = y$ and we want $\lambda x = y$, which is true because $x = \frac{1}{\lambda} y$.
So $u(x) = y = \lambda x$, so $x \in E_\lambda$.
So $y = u(x)$ with $x \in E_\lambda$, so $y \in u(E_\lambda)$.
So $E_\lambda \subset u(E_\lambda)$.
Therefore, $u(E_\lambda) = E_\lambda$.

Exercise 10.

1. Assume $n < p$.
By the rank-nullity theorem: $\dim(\ker(u)) + \dim(\mathrm{Im}(u)) = n$.
So $\dim(\mathrm{Im}(u)) \leq n < p$.
But $\mathrm{Im}(u)$ is a subspace of F which is of dimension p, so $\mathrm{Im}(u) \neq F$, so u is not surjective.
2. Assume $n > p$.
By the rank-nullity theorem: $\dim(\ker(u)) + \dim(\mathrm{Im}(u)) = n$.
But $\dim(\mathrm{Im}(u)) \leq p$, so $\dim(\ker(u)) \geq n - p > 0$, so $\ker(u) \neq \{0_E\}$, so u is not injective.

Exercise 11.

Let $y \in \ker(f) \cap \mathrm{im}(f)$, then $f(y) = 0_F$ and there exists $x \in E$ such that $y = f(x)$.
Then $f(y) = f(f(x)) = f^2(x) = 0_F$, so $x \in \ker(f^2)$.
And $y = f(x)$, so $y \in f(\ker(f^2))$.
Conversely, let $y \in f(\ker(f^2))$, then there exists $x \in \ker(f^2)$ such that $y = f(x)$.
Since $x \in \ker(f^2)$, we have $f^2(x) = f(f(x)) = f(y) = 0_F$, so $y \in \ker(f)$.
And $y = f(x) \in \mathrm{im}(f)$.
So $y \in \ker(f) \cap \mathrm{im}(f)$.
Therefore, $\ker(f) \cap \mathrm{im}(f) = f(\ker(f^2))$.

Exercise 12.

Let $y \in f(\ker(g \circ f))$, then there exists $x \in \ker(g \circ f)$ such that $y = f(x)$.
Since $x \in \ker(g \circ f)$, we have $g(f(x)) = g(y) = 0_F$, so $y \in \ker(g)$.
And $y = f(x) \in \mathrm{Im}(f)$.
So $y \in \ker(g) \cap \mathrm{Im}(f)$.
Conversely, let $y \in \ker(g) \cap \mathrm{Im}(f)$, then $g(y) = 0_F$ and there exists $x \in E$ such that $y = f(x)$.
Then $g(f(x)) = g(y) = 0_F$, so $x \in \ker(g \circ f)$.
And $y = f(x)$, so $y \in f(\ker(g \circ f))$.
Therefore, $f(\ker(g \circ f)) = \ker(g) \cap \mathrm{Im}(f)$.

Exercise 13.

1. Let $x \in \ker(u)$, then $u(x) = 0_E$, so $u^2(x) = u(u(x)) = u(0_E) = 0_E$, so $x \in \ker(u^2)$.
So $\ker(u) \subset \ker(u^2)$.
2. Let $y \in \mathrm{Im}(u^2)$, then there exists $x \in E$ such that $y = u^2(x) = u(u(x))$.
Let $z = u(x)$, then $y = u(z)$, so $y \in \mathrm{Im}(u)$.
So $\mathrm{Im}(u^2) \subset \mathrm{Im}(u)$.

Exercise 14.

$(i) \Rightarrow (ii)$

Assume $\ker(u) \cap \text{im}(u) = \{0_E\}$.

We already know that $\ker(u) \subset \ker(u^2)$.

Let $x \in \ker(u^2)$, then $u^2(x) = u(u(x)) = 0_E$, so $u(x) \in \ker(u)$.

But $u(x) \in \text{im}(u)$, so $u(x) \in \ker(u) \cap \text{im}(u) = \{0_E\}$, so $u(x) = 0_E$, so $x \in \ker(u)$.

So $\ker(u^2) \subset \ker(u)$.

Therefore, $\ker(u) = \ker(u^2)$.

$(ii) \Rightarrow (i)$

Assume $\ker(u) = \ker(u^2)$.

Let $y \in \ker(u) \cap \text{im}(u)$, then $u(y) = 0_E$ and there exists $x \in E$ such that $y = u(x)$.

Then $u^2(x) = u(u(x)) = u(y) = 0_E$, so $x \in \ker(u^2) = \ker(u)$, so $u(x) = 0_E$, but $u(x) = y$, so $y = 0_E$.

Therefore, $\ker(u) \cap \text{im}(u) = \{0_E\}$.

CHAPTER 3

MATRICES

In the case where $\mathcal{L}_{\mathbb{k}}(E, E')$ is finite-dimensional, say of dimension r, by choosing a basis we can associate to every linear map f an r-tuple of elements of $\mathbb{k}$, the components of f. For reasons we will see later, these components are arranged not on a line but in a table with a certain number of rows and columns, which is called the matrix associated with the linear map f.

The notions covered in this chapter are:

— Matrices associated with linear maps.
— Product of two matrices.
— Matrix of the inverse of a map.
— Calculation of the inverse of a matrix.
— Change of basis.
— Rank of a matrix.

Definition 3.1 *A matrix of type (p,n) with coefficients in $\mathbb{k}$ is a table A of pn elements of $\mathbb{k}$ arranged in p rows and n columns:*

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{p1} & a_{p2} & \dots & a_{pn} \end{pmatrix} \quad \textit{or, abbreviated}: A = (a_{ik}), \textit{ or also}: A = \|a_{ik}\|.$$

The set of matrices with p rows and n columns is denoted $\mathcal{M}_{p,n}(\mathbb{k})$. If $n = p$, $\mathcal{M}_{n,n}(\mathbb{k})$ is denoted: $\mathcal{M}_n(\mathbb{k})$.

Example 3.1

$$\begin{pmatrix} 1 & 3 & -1 \\ 0 & 1 & 2 \end{pmatrix} \in \mathcal{M}_{2,3}(\mathbb{R}) \qquad \begin{pmatrix} 1 & 2-i & 3+i \\ 0 & 1+i & i \\ -i & 2 & 1 \end{pmatrix} \in \mathcal{M}_3(\mathbb{C})$$

Note that in the notation we have just adopted, a_{ik} denotes the element in the i-th row and the k-th column.

Another notation that we will also use later is the notation by columns:

$$A = \|c_1, \dots, c_n\|, \quad \text{where} \quad c_k = \begin{pmatrix} a_{1k} \\ a_{2k} \\ \vdots \\ a_{pk} \end{pmatrix} \quad \text{is the } k^{\text{th}} \text{ column}$$

On the set $\mathcal{M}_{p,n}(\mathbb{k})$ we define the operations:

- Addition: if $A = (a_{ik}), B = (b_{ik})$, we denote $C = A + B$ the matrix (c_{ik}) such that:

$$c_{ik} = a_{ik} + b_{ik}, \quad \forall i, k$$

- Multiplication by a scalar: if $A = (a_{ik})$ and $\lambda \in \mathbb{k}$ we denote λA the matrix (λa_{ik}) that is-to say the matrix obtained by multiplying all elements by λ.

Example 3.2

$$\begin{pmatrix} 2 & -1 & 0 & 3 \\ 1 & 2 & 1 & -1 \end{pmatrix} + \begin{pmatrix} 1 & 2 & 0 & 4 \\ 3 & 1 & -1 & 2 \end{pmatrix} = \begin{pmatrix} 3 & 1 & 0 & 7 \\ 4 & 3 & 0 & 1 \end{pmatrix}$$

$$5\begin{pmatrix} 2 & -1 & 0 & 3 \\ 1 & 2 & 1 & -1 \end{pmatrix} = \begin{pmatrix} 10 & -5 & 0 & 15 \\ 5 & 10 & 5 & -5 \end{pmatrix}.$$

It is easy to see that, endowed with these operations, $\mathcal{M}_{p,n}(\mathbb{k})$ is a vector space over $\mathbb{k}$. The neutral element is the matrix whose all elements are zero, called the zero matrix, denoted 0 . The opposite of the matrix (a_{ik}) is the matrix $(-a_{ik})$.

Proposition 3.1 *We have:*

$$\dim_{\mathbb{k}} \mathcal{M}_{p,n}(\mathbb{k}) = pn$$

Proof. One easily verifies that the pn matrices, called *elementary matrices*

$$E_{11} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & \cdots & \cdots & 0 \\ \vdots & & & \vdots \\ 0 & \cdots & \cdots & 0 \end{pmatrix}, \ldots, E_{ik} = \begin{pmatrix} 0 & \cdots & 0 & \cdots & 0 \\ \vdots & & \vdots & & \vdots \\ 0 & \cdots & 1 & \cdots & 0 \\ \vdots & & \vdots & & \vdots \\ 0 & \cdots & 0 & \cdots & 0 \end{pmatrix} \leftarrow \text{i}^{\text{th}} \text{ row}, \ldots$$

$$E_{pn} = \begin{pmatrix} 0 & \cdots & 0 & \\ 0 & \cdots & \cdots & 0 \\ \vdots & & \vdots & \\ 0 & \cdots & \cdots & 1 \end{pmatrix},$$

form a basis of $\mathcal{M}_{p,n}(\mathbb{k})$ called the canonical basis. □

3.1 Matrix of a linear map

Let E and E' be two vector spaces over $\mathbb{k}$, of dimension n and p respectively, and $f : E \longrightarrow E'$ a linear map. Let us choose a basis $\{e_1, \ldots, e_n\}$ of E and a basis $\{\varepsilon_1, \ldots, \varepsilon_p\}$ of E'. The images by f of the vectors $e_1, \ldots, e_n$ are decomposed on the basis $\{\varepsilon_1, \ldots, \varepsilon_p\}$:

$$\begin{aligned} f(e_k) &= a_{11}\varepsilon_1 + a_{21}\varepsilon_2 + \cdots + a_{p1}\varepsilon_p \\ f(e_k) &= a_{12}\varepsilon_1 + a_{22}\varepsilon_2 + \cdots + a_{p2}\varepsilon_p \\ &\cdots\cdots\cdots\cdots \\ f(e_n) &= a_{1n}\varepsilon_1 + a_{2n}\varepsilon_2 + \cdots + a_{pn}\varepsilon_p. \end{aligned}$$

Definition 3.2 *We call the matrix of f in the bases $\{e_1, \ldots, e_n\}, \{\varepsilon_1, \ldots, \varepsilon_p\}$ the matrix denoted $M(f)_{e_i,\varepsilon_j}$ belonging to $\mathcal{M}_{p,n}(\mathbb{k})$ whose columns are the components of the vectors $f(e_1), \ldots, f(e_n)$ in the basis $\{\varepsilon_1, \ldots, \varepsilon_p\}$:*

$$M(f)_{e_i,\varepsilon_j} = \begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & & & \vdots \\ a_{p1} & a_{p2} & \ldots & a_{pn} \end{pmatrix}.$$

We will also use the notation: $\|f(e_1), \ldots, f(e_n)\|_{\varepsilon_j}$.

If there is no possible ambiguity, we will also write $M(f)$ instead of $M(f)_{e_i,\varepsilon_j}$, but it is clear that the matrix associated with f depends on the choice of bases of E and E'.

In the case where f is an endomorphism, we can choose the same basis in E considered as the starting and arrival space. In this case, we will denote $M(f)_{e_i}$ instead of $M(f)_{e_i,e_j}$.

Proposition 3.2 *Let E and E' be two vector spaces over $\mathbb{k}$ of dimension n and p respectively, $\{e_i\}$ and $\{\varepsilon_j\}$ bases of E and E'. Then the map:*

$$\begin{array}{rccc} M: & \mathcal{L}_K(E,E') & \longrightarrow & \mathcal{M}_{p,n}(K) \\ & f & \longmapsto & M(f)_{e_i,\varepsilon_j} \end{array}$$

is a vector space isomorphism, that is:

$$\begin{cases} M(f+g) = M(f) + M(g) \\ M(\lambda f) = \lambda M(f), \end{cases}$$

and M is bijective.

In particular: $\dim_{\mathbb{k}} \mathcal{L}(E,E') = np$.

Proof. We indeed have:

$$\begin{aligned} M(f+g)_{e_i,\varepsilon_j} &= \|(f+g)(e_1), \ldots, (f+g)(e_n)\|_{\varepsilon_j} \\ &= \|f(e_1) + g(e_1), \ldots, f(e_n) + g(e_n)\|_{\varepsilon_j} \\ &= \|f(e_1), \ldots, f(e_n)\|_{\varepsilon_j} + \|g(e_1), \ldots, g(e_n)\|_{\varepsilon_j} \end{aligned}$$

by the definition of matrix addition, that is:

$$M(f+g)_{e_i,\varepsilon_j} = M(f)_{e_i,\varepsilon_j} + M(g)_{e_i,\varepsilon_j}.$$

Similarly, if $\lambda \in \mathbb{k}$:

$$\begin{aligned} M(\lambda f)_{e_i,\varepsilon_j} &= \|(\lambda f)(e_1), \ldots, (\lambda f)e_n\|_{\varepsilon_j} \\ &= \|\lambda f(e_1), \ldots, \lambda f(e_n)\|_{e_j} = \lambda \|f(e_1), \ldots, f(e_n)\|_{\varepsilon_j} \end{aligned}$$

So M is linear.

On the other hand M is surjective. Indeed, let:

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ a_{21} & \cdots & a_{2n} \\ \vdots & & \vdots \\ a_{p1} & \cdots & a_{pn} \end{pmatrix} \in \mathcal{M}_{pn}(K)$$

and $f \in \mathcal{L}(E,F)$ defined as follows. First set:

$$\begin{aligned} f(e_1) &= a_{11}\varepsilon_1 + a_{21}\varepsilon_2 + \cdots + a_{p1}\varepsilon_p \\ &\cdots\cdots\cdots\cdots \\ f(e_n) &= a_{1n}\varepsilon_1 + a_{2n}\varepsilon_2 + \cdots + a_{pn}\varepsilon_p \end{aligned}$$

Then, extend f by linearity to E, that is, if:

$$x = \lambda_1 e_1 + \cdots + \lambda_n e_n \in E,$$

set:

$$f(x) = \lambda_1 f(e_1) + \cdots + \lambda_n f(e_n).$$

It is easy to verify that f is linear and that $A = M(f)_{e_i,\varepsilon_j}$. Finally M is injective. Let indeed $f \in \operatorname{Ker} M$:

$$M(f) = \begin{pmatrix} 0 & \cdots & 0 \\ 0 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & 0 \end{pmatrix}$$

This means that $f(e_1) = 0, \ldots, f(e_n) = 0$. So, if $x = \lambda_1 e_1 + \cdots + \lambda_n e_n \in E$, we will have $f(x) = \lambda_1 f(e_1) + \cdots + \lambda_n f(e_n) = 0$, that is $f = 0$. By proposition 2.4, f is injective. □

Example 3.3 *Let E be of dimension n and:*

$$\begin{array}{rccc} \mathrm{id}_E : & E & \longrightarrow & E \\ & x & \mapsto & x \end{array}$$

Consider a basis $\{e_i\}$. We have: $id_E(e_i) = e_i$. So:

$$M\left(\underset{E}{\mathrm{id}}\right)_{e_i} = \begin{pmatrix} 1 & 0 & \ldots & 0 \\ 0 & 1 & 0 & \vdots \\ \vdots & 0 & \ddots & 0 \\ 0 & \ldots & 0 & 1 \end{pmatrix} \quad (1 \textit{ is the unit element of } \mathbb{k})$$

This matrix is denoted I_n or simply I and is called the unit matrix of $\mathcal{M}_n(\mathbb{k})$.

$f(e_1) = a_{11}\varepsilon_1 + a_{21}\varepsilon_2 + \cdots + a_{p1}\varepsilon_p \quad f(e_n) = a_{1n}\varepsilon_1 + a_{2n}\varepsilon_2 + \cdots + a_{pn}\varepsilon_p f(e_n) = a_{1n}\varepsilon_1 + a_{2n}\varepsilon_2 + \cdots +$

Example 3.4 *Let $\{\epsilon_1, \epsilon_2\}$ be the canonical basis of $\mathbb{R}^2$ and $\{e_1, e_2, e_3\}$ the canonical basis of $\mathbb{R}^3$. Consider the linear map:*

$$\begin{array}{rccc} f : & \mathbb{R}^3 & \to & \mathbb{R}^2 \\ & (x, y, z) & \mapsto & (x - y, z - y) \end{array}$$

We have

$$\begin{aligned} f(e_1) &= f(1,0,0) = (1,0) = \epsilon_1 \\ f(e_2) &= f(0,1,0) = (-1,-1) = -\epsilon_1 - \epsilon_2 \\ f(e_3) &= f(0,0,1) = (0,1) = \epsilon_2 \end{aligned}$$

so

$$M(f)_{e_i,\varepsilon_j} = \begin{pmatrix} 1 & -1 & 0 \\ 0 & -1 & 1 \end{pmatrix}.$$

Example 3.5 *Consider the map*

$$\begin{array}{rccc} w: & \mathbb{R}^n & \to & \mathbb{R} \\ & (x_1,...,x_n) & \mapsto & a_1x_1+...+a_nx_n \end{array}$$

By endowing $\mathbb{R}^n$ with the canonical basis $\{e_1,...,e_n\}$ and $\mathbb{R}$ with the canonical basis $\{1\}$, we have:

$$\begin{aligned} f(e_1) &= f(1,...,0) = a_1 = a_1\epsilon_1 \\ f(e_2) &= f(0,1,...,0) = a_2 = a_2\epsilon_2 \\ &..... \\ f(e_n) &= f(0,0,...,1) = a_n = a_n\epsilon_n \end{aligned}$$

Then

$$M(w)_{e_i,\varepsilon_j} = (a_1,...,a_n)\,.$$

3.2 Product of two matrices

In the previous paragraph, we defined the operations of addition and multiplication by a scalar on matrices. By virtue of proposition 3.2, these operations correspond to the analogous operations on linear maps, that is we have:

$$\begin{aligned} M(f+g) &= M(f)+M(g) \\ M(\lambda f) &= \lambda M(f). \end{aligned}$$

We will now define a new operation, the product of matrices. As we will see (cf. proposition 3.5), it corresponds to the composition of maps, in the sense that:

$$M(f\circ g) = M(f)\cdot M(g).$$

First, note that the composition of maps cannot be done for any pair of maps, but only if the arrival space of g is included in the starting space of f. This situation will be found for matrices: the product can only be performed between matrices of a certain type.

Definition 3.3 *We call the product of matrices the map:*

$$\begin{array}{ccc} \mathcal{M}_{p,n}(K)\times\mathcal{M}_{n,q}(K) & \longrightarrow & \mathcal{M}_{p,q}(K) \\ (a_{ji})\,,(b_{mk}) & \longmapsto & (c_{jk}) \end{array}$$

where:

$$c_{jk} = a_{j1}b_{1k}+a_{j2}b_{2k}+...+a_{jn}b_{nk}.$$

In other words, the element c_{jk} of the j$^{\text{th}}$ row and k$^{\text{th}}$ column of the product $C=AB$ is the sum of the products of the elements of the j-th row of A by the elements of the same rank of the k$^{\text{th}}$ column of B. Briefly, we say that the product of two matrices is performed rows by columns. Here is the diagram of this definition.

Remark 3.1 *The product AB can only be performed if the number of columns of A is equal to the number of rows of B.*

Example 3.6 *Let*

$$A = \begin{pmatrix} 2 & -1 & 1 \\ 1 & 0 & 2 \end{pmatrix}, B = \begin{pmatrix} 2 & 0 & 1 & 0 \\ 1 & 1 & 2 & 2 \\ 1 & 2 & 1 & 0 \end{pmatrix}$$

We have:

$$\underbrace{\begin{pmatrix} 2 & -1 & 1 \\ 1 & 0 & 2 \end{pmatrix}}_{A} \times \underbrace{\begin{pmatrix} 2 & 0 & 1 & 0 \\ 1 & 1 & 2 & 2 \\ 1 & 2 & 1 & 0 \end{pmatrix}}_{B} = \underbrace{\begin{pmatrix} 4 & 1 & 1 & -2 \\ 4 & 4 & 3 & 0 \end{pmatrix}}_{C=AB}$$

Example 3.7 *Let*

$$A = \begin{pmatrix} 2 & 3 \\ 4 & 1 \\ 0 & 2 \end{pmatrix} \text{ and } B = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

we have:

$$\underbrace{\begin{pmatrix} 2 & 3 \\ 4 & 1 \\ 0 & 2 \end{pmatrix}}_{A} \times \underbrace{\begin{pmatrix} 1 \\ 3 \end{pmatrix}}_{B} = \underbrace{\begin{pmatrix} 11 \\ 7 \\ 6 \end{pmatrix}}_{AB}$$

The following remarks are important:

Remark 3.2 *1. We can have $AB = 0$ without A or B being zero.*

2. $AB = AC$ with $A \neq 0$ does not necessarily imply $B = C$ (that is, in general we cannot "simplify" by A, even if $A \neq 0$).

3. In general we have $AB \neq BA$ (that is: multiplication between matrices is not commutative).

Example 3.8 *Let*

$$A = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \ B = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \text{ and } C = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}.$$

By performing the products, we find:

$$AB = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{(which shows 1.)}$$

$$BA = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad \text{so } BA \neq AB\text{(which shows 3.)}$$

and $AB = AC$ (which shows 2. since we have $B \neq C$).

Proposition 3.3 *Multiplication is associative, that is:*

$$A(BC) = (AB)C, \quad (\forall A \in \mathcal{M}_{p,n}, \forall B \in \mathcal{M}_{n,q}, \forall C \in \mathcal{M}_{q,m})\,.$$

Multiplication is distributive on the left and right with respect to addition, that is:

$$A(B+C) = AB + AC, \quad (A+D)B = AB + DB, \quad (\forall A, D \in \mathcal{M}_{p,n}, \forall B, C \in \mathcal{M}_{n,q})$$

Proof. Left as an exercise. □

Note finally that multiplication is an internal law on the set $\mathcal{M}_n(K)$ of square matrices of order n, that is it is a map:

$$\mathcal{M}_n(K) \times \mathcal{M}_n(K) \longrightarrow \mathcal{M}_n(K).$$

One immediately verifies that the matrix I_n is the neutral element of multiplication, that is: $\forall A \in \mathcal{M}_n(K) : \quad I_n A = A I_n = A.$

3.3 Matrix of a vector

Definition 3.4 *Let E be a vector space of dimension $n, \{e_1, \ldots, e_n\}$ a basis of E and $x = x_1 e_1 + \cdots + x_n e_n$ a vector of E. We call the matrix of x in the basis $\{e_i\}$ the column matrix of the components of x in the basis $\{e_i\}$:*

$$M(x)_{e_i} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}.$$

(also denoted $M(x)$).

Remark 3.3 *This definition is consistent with the definition of the matrix associated with a linear map. Indeed, we can identify any vector of E with a linear map from K into E: to every x of E we associate the linear map*

If we write the matrix of f in the basis $\varepsilon = 1$ of K and $\{e_i\}$ of E, we have:

$$f(\varepsilon) = x = x_1 e_1 + \cdots + x_n e_n,$$

so:

$$M(f)_{\varepsilon,e_i} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}.$$

$$\begin{array}{rccc} f: & \mathbb{k} & \longrightarrow & E \\ & \lambda & \longmapsto & \lambda x. \end{array}$$

Proposition 3.4 *Let E, F be two vector spaces over $K, \{e_1, \ldots, e_n\}$ and $\{\varepsilon_1, \ldots, \varepsilon_p\}$ two bases of E and F respectively. For any map $f \in \mathcal{L}_K(E,F)$ and for any $x \in E$, we have:*

$$M(f(x))_{\varepsilon_j} = M(f)_{e_i,\varepsilon_j} M(x)_{e_i},$$

or more briefly:

$$M(f(x)) = M(f)M(x).$$

Proof. Let

$$M(f)_{e_i,\varepsilon_j} = \begin{pmatrix} a_{11} & \ldots & a_{1n} \\ \vdots & & \vdots \\ a_{p1} & \ldots & a_{pn} \end{pmatrix},$$

which means that: $f(e_j) = a_{1j}\varepsilon_1 + \cdots + a_{pj}\varepsilon_p = \sum_{k=1}^{p} a_{kj}\varepsilon_k$. We have:

$$\begin{aligned} f(x) &= f(x_1 e_1 + \cdots + x_n e_n) = \sum_{j=1}^{n} x_j f(e_j) \\ &= \sum_{j=1}^{n} x_j \sum_{k=1}^{p} a_{kj}\varepsilon_k = \sum_{k=1}^{p} \underbrace{\left(\sum_{j=1}^{n} a_{kj} x_j \right)}_{y_k} \varepsilon_k \\ &= \sum_{k=1}^{p} y_k \varepsilon_k. \end{aligned}$$

So:

$$M(f(x))_{\varepsilon_j} = \begin{pmatrix} y_1 \\ \vdots \\ y_p \end{pmatrix},$$

with

$$y_k = \sum_{j=1}^{n} a_{kj} x_j.$$

On the other hand:

$$M(f)_{e_i,\varepsilon_j} \cdot M(x)_{e_i} = \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & & \vdots \\ a_{p1} & \dots & a_{pn} \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$
$$= \begin{pmatrix} \sum_{j=1}^{n} a_{1j}x_j \\ \vdots \\ \sum_{j=1}^{n} a_{pj}x_j \end{pmatrix} = \begin{pmatrix} y_1 \\ \vdots \\ y_p \end{pmatrix}$$

so:

$$M(f)_{e_i,\varepsilon_j} \cdot M(x)_{e_i} = M(f(x))_{\varepsilon_j}.$$

□

Example 3.9 *Let the plane $\mathbb{R}^2$ referred to its canonical basis. Determine the image of the vector $x = (3, 2)$ by the rotation with center O and angle $\pi/6$.*

We have:

$$M(f(x)) = M(f) \cdot M(x) = \begin{pmatrix} \cos \pi/6 & -\sin \pi/6 \\ \sin \pi/6 & \cos \pi/6 \end{pmatrix} \begin{pmatrix} 3 \\ 2 \end{pmatrix}$$
$$= \begin{pmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2}\frac{\sqrt{3}}{2} & \end{pmatrix} \begin{pmatrix} 3 \\ 2 \end{pmatrix} = \begin{pmatrix} \frac{3\sqrt{3}-2}{2} \\ \frac{3+2\sqrt{3}}{2} \end{pmatrix}.$$

3.4 Product of matrices

Proposition 3.5 *Let E, F, G be three vector spaces over K of dimensions n, p, q respectively. Let $\{e_i\}, \{\varepsilon_j\}, \{\eta_k\}$ be bases of E, F, G respectively. Let $f \in \mathcal{L}_K(E, F)$ and $g \in \mathcal{L}_K(F, G)$. Then:*

$$M(g \circ f)_{e_i,\eta_k} = M(g)_{\varepsilon_j,\eta_k} M(f)_{e_i,\varepsilon_j}.$$

Proof. Let

$$M(f)_{e_i,\varepsilon_j} = (a_{ji}), \quad M(g)_{\varepsilon_j,\eta_k} = (b_{kj}).$$

This means that:

$$f(e_i) = \sum_{j=1}^{p} a_{ji}\varepsilon_j, \quad g(\varepsilon_j) = \sum_{k=1}^{q} b_{kj}\eta_k.$$

Then:

$$\begin{aligned}(g\circ f)(e_i) &= g(f(e_i)) = g\left(\sum_{j=1}^{p} a_{ji}\varepsilon_j\right)\\ &= \sum_{j=1}^{p} a_{ji} g(\varepsilon_j) = \sum_{j=1}^{p} a_{ji}\sum_{k=1}^{q} b_{kj}\eta_k\\ &= \sum_{k=1}^{q}\left(\sum_{j=1}^{p} b_{kj}a_{ji}\right)\eta_k.\end{aligned}$$

So, the element of the k^{th} row and i^{th} column of $M(g\circ f)_{e_i,\eta_k}$ is:

$$\sum_{j=1}^{p} b_{kj}a_{ji},$$

which is the element of the k^{th} row and i^{th} column of the product $M(g)_{\varepsilon_j,\eta_k}\cdot M(f)_{e_i,\varepsilon_j}$. □

Example 3.10 *Let $f:\mathbb{R}^2\to\mathbb{R}^3$ and $g:\mathbb{R}^3\to\mathbb{R}^2$ defined by:*

$$\begin{aligned}f(x,y) &= (x-y, x+y, 2x)\\ g(x,y,z) &= (x+2y-z, 2x-y+z)\end{aligned}$$

Let us determine the matrix of $g\circ f$ in the canonical bases.

We have:

$$M(f)_{can} = \begin{pmatrix}1 & -1\\ 1 & 1\\ 2 & 0\end{pmatrix},\quad M(g)_{can} = \begin{pmatrix}1 & 2 & -1\\ 2 & -1 & 1\end{pmatrix}$$

$$M(g\circ f)_{can} = M(g)_{can}\cdot M(f)_{can}$$

$$= \begin{pmatrix}1 & 2 & -1\\ 2 & -1 & 1\end{pmatrix}\begin{pmatrix}1 & -1\\ 1 & 1\\ 2 & 0\end{pmatrix} = \begin{pmatrix}1+2-2 & -1+2-0\\ 2-1+2 & -2-1+0\end{pmatrix} = \begin{pmatrix}1 & 1\\ 3 & -3\end{pmatrix}.$$

3.5 Matrix of the inverse of a map

Proposition 3.6 *Let E and F be two vector spaces over K of the same dimension n. Let $f\in\mathcal{L}_K(E,F)$ be an isomorphism. Then the matrix $M(f)_{e_i,\varepsilon_j}$ is invertible, and we have:*

$$\left(M(f)_{e_i,\varepsilon_j}\right)^{-1} = M(f^{-1})_{\varepsilon_j,e_i}.$$

Proof. We have: $f\circ f^{-1} = \mathrm{id}_F$ and $f^{-1}\circ f = \mathrm{id}_E$. By proposition 3.5, we have:

$$\begin{aligned}M(f)_{e_i,\varepsilon_j}M(f^{-1})_{\varepsilon_j,e_i} &= M(f\circ f^{-1})_{\varepsilon_j,\varepsilon_j} = M(\mathrm{id}_F)_{\varepsilon_j} = I_n\\ M(f^{-1})_{\varepsilon_j,e_i}M(f)_{e_i,\varepsilon_j} &= M(f^{-1}\circ f)_{e_i,e_i} = M(\mathrm{id}_E)_{e_i} = I_n.\end{aligned}$$

So $M(f)_{e_i,\varepsilon_j}$ is invertible and its inverse is $M(f^{-1})_{\varepsilon_j,e_i}$. □

3.6 Inverse of a matrix

Definition 3.5 *A matrix $A \in \mathcal{M}_n(K)$ is said to be invertible if there exists a matrix $B \in \mathcal{M}_n(K)$ such that:*

$$AB = BA = I_n.$$

This matrix B is then unique, it is called the inverse of A and is denoted A^{-1}.

Example 3.11 *The matrix* $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ *is invertible and its inverse is* $A^{-1} = \begin{pmatrix} -2 & 1 \\ 3/2 & -1/2 \end{pmatrix}$.

Proposition 3.7 *Let E be a vector space of dimension n, $\{e_i\}$ a basis of E and $f \in \mathcal{L}_K(E,E)$ an endomorphism. Then f is an isomorphism if and only if the matrix $M(f)_{e_i}$ is invertible.*

Proof. If f is an isomorphism, then by proposition 3.6, $M(f)_{e_i}$ is invertible and $M(f)_{e_i}^{-1} = M(f^{-1})_{e_i}$.

Conversely, if $M(f)_{e_i}$ is invertible, let $g \in \mathcal{L}_K(E,E)$ such that $M(g)_{e_i} = M(f)_{e_i}^{-1}$. Then:

$$\begin{aligned} M(f \circ g)_{e_i} &= M(f)_{e_i} M(g)_{e_i} = M(f)_{e_i} M(f)_{e_i}^{-1} = I_n \\ M(g \circ f)_{e_i} &= M(g)_{e_i} M(f)_{e_i} = M(f)_{e_i}^{-1} M(f)_{e_i} = I_n \end{aligned}$$

so $f \circ g = g \circ f = \mathrm{id}_E$, that is f is an isomorphism and $f^{-1} = g$. □

3.7 Change of basis

Let E be a vector space of dimension n and $\{e_1, \ldots, e_n\}$ and $\{e'_1, \ldots, e'_n\}$ two bases of E. The change of basis is the passage from the first basis to the second. It is determined by the matrix of the vectors of the new basis $\{e'_i\}$ in the old basis $\{e_i\}$, that is:

$$M(\underset{E}{\mathrm{id}})_{e'_i, e_i} = \|e'_1, \ldots, e'_n\|_{e_i}.$$

This matrix is called the change of basis matrix from $\{e_i\}$ to $\{e'_i\}$.

If we denote $P = (p_{ij})$ this matrix, we have:

$$e'_j = p_{1j}e_1 + p_{2j}e_2 + \cdots + p_{nj}e_n = \sum_{i=1}^{n} p_{ij}e_i.$$

Proposition 3.8 *The change of basis matrix is invertible. Its inverse is the matrix of the identity map from the basis $\{e'_i\}$ to the basis $\{e_i\}$, that is:*

$$P^{-1} = M(\underset{E}{\mathrm{id}})_{e_i, e'_i}.$$

Proof. By proposition 3.6, we have:

$$M(\underset{E}{\mathrm{id}})^{-1}_{e'_i,e_i} = M(\overset{-1}{\underset{E}{\mathrm{id}}})_{e_i,e'_i} = M(\underset{E}{\mathrm{id}})_{e_i,e'_i}.$$

□

Let us now see how the components of a vector and the matrix of an endomorphism are transformed by a change of basis.

Proposition 3.9 *Let $x \in E$ and $X = M(x)_{e_i}, X' = M(x)_{e'_i}$. Then:*

$$X = PX', \quad \textit{that is: } X' = P^{-1}X.$$

Proof. We have:

$$M(x)_{e_i} = M(\underset{E}{\mathrm{id}}(x))_{e_i} = M(\underset{E}{\mathrm{id}})_{e'_i,e_i} M(x)_{e'_i} = PX'.$$

□

Proposition 3.10 *Let $f \in \mathcal{L}_K(E,E)$ and $A = M(f)_{e_i}, A' = M(f)_{e'_i}$. Then:*

$$A' = P^{-1}AP.$$

Proof. We have:

$$\begin{aligned} A' = M(f)_{e'_i} &= M(\mathrm{id}_E \circ f \circ \mathrm{id}_E)_{e'_i,e'_i} \\ &= M(\mathrm{id}_E)_{e_i,e'_i} M(f)_{e_i} M(\mathrm{id}_E)_{e'_i,e_i} = P^{-1}AP. \end{aligned}$$

□

3.8 Rank of a matrix

Definition 3.6 *Let $A \in \mathcal{M}_{p,n}(K)$. We call the rank of A the dimension of the subspace of K^p generated by the columns of A. We denote it* $\mathrm{rg}(A)$.

Proposition 3.11 *Let E and F be two vector spaces over K of dimensions n and p respectively, $\{e_i\}$ and $\{\varepsilon_j\}$ bases of E and F respectively. Let $f \in \mathcal{L}_K(E,F)$. Then:*

$$\mathrm{rg}(f) = \mathrm{rg}\left(M(f)_{e_i,\varepsilon_j}\right).$$

Proof. We have:

$$\mathrm{rg}(f) = \dim \mathrm{Im}(f) = \dim\left[\mathrm{vect}\left(f(e_1), \ldots, f(e_n)\right)\right].$$

But the components of the vectors $f(e_1), \ldots, f(e_n)$ in the basis $\{\varepsilon_j\}$ are the columns of $M(f)_{e_i,\varepsilon_j}$. So the subspace generated by the columns of $M(f)_{e_i,\varepsilon_j}$ is isomorphic to $\mathrm{Im}(f)$, hence the result. □

Proposition 3.12 *Let $A \in \mathcal{M}_{p,n}(K)$ and $B \in \mathcal{M}_{n,q}(K)$. Then:*

$$\operatorname{rg}(AB) \leq \min\left(\operatorname{rg}(A), \operatorname{rg}(B)\right).$$

Proof. Let $f : K^n \to K^p$ and $g : K^q \to K^n$ be the linear maps whose matrices in the canonical bases are A and B respectively. Then $AB = M(f \circ g)$. But:

$$\operatorname{rg}(f \circ g) \leq \operatorname{rg}(f) \quad \text{and} \quad \operatorname{rg}(f \circ g) \leq \operatorname{rg}(g),$$

so:

$$\operatorname{rg}(AB) \leq \min\left(\operatorname{rg}(A), \operatorname{rg}(B)\right).$$

□

3.9 Exercises

Exercise 1.

Let

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$$

1. Compute A^2 and A^3. Compute $A^3 - A^2 + A - I$.
2. Express A^{-1} in terms of A^2, A and I.
3. Express A^4 in terms of A^2, A and I.

Exercise 2.

Let A be the matrix

$$A = \begin{pmatrix} 3 & 0 & 1 \\ -1 & 3 & -2 \\ -1 & 1 & 0 \end{pmatrix}$$

Compute $(A-2I)^3$, then deduce that A is invertible and determine A^{-1} in terms of I, A and A^2.

Exercise 3.

Let $\beta = (e_1, e_2, e_3)$ be the canonical basis of $\mathbb{R}^3$.

Let u be the linear map that associates to a vector $x = (x_1, x_2, x_3) \in \mathbb{R}^3$ the vector

$$u(x) = (x_2 - 2x_3, 2x_1 - x_2 + 4x_3, x_1 - x_2 + 3x_3)$$

1. Determine the matrix A of u in the canonical basis.
2. Determine a basis (a, b) of $\ker(u - Id)$.
3. Give a vector c such that $\ker(u) = \text{vect}(c)$.
4. Show that $\beta' = (a, b, c)$ is a basis of $\mathbb{R}^3$.
5. Determine the matrix D of u in the basis β'.
6. Show that $\text{Im}(u) = \ker(u - Id)$
7. Show that $\ker(u) \oplus \text{Im}(u) = \mathbb{R}^3$.

Exercise 4.

Let $u : \mathbb{R}_2[X] \to \mathbb{R}[X]$ be defined by $u(P) = P + (1 - X)P'$

Let $\beta = (1, X, X^2)$ be the canonical basis of $\mathbb{R}_2[X]$

1. Show that u is an endomorphism of $\mathbb{R}_2[X]$.
2. Determine the matrix of u in β.
3. Determine the kernel and the image of u.

3.10 Solutions

Exercise 1.

1. We have

$$A^2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

$$A^3 = A^2A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$A^3 - A^2 + A - I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} - \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

$$+ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} - \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = O$$

2. $A^3 - A^2 + A - I = O \Leftrightarrow A\left(A^2 - A + I\right) = I$ so $A^{-1} = A^2 - A + I$
3. $A^3 = A^2 - A + I$ so

$$A^4 = A\left(A^2 - A + I\right) = A^3 - A^2 + A = \left(A^2 - A + I\right) - A^2 + A = I$$

Exercise 2. We have:

$$\begin{aligned} A - 2I &= \begin{pmatrix} 1 & 0 & 1 \\ -1 & 1 & -2 \\ -1 & 1 & -2 \end{pmatrix} \\ (A-2I)^2 &= \begin{pmatrix} 1 & 0 & 1 \\ -1 & 1 & -2 \\ -1 & 1 & -2 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ -1 & 1 & -2 \\ -1 & 1 & -2 \end{pmatrix} = \begin{pmatrix} 0 & 1 & -1 \\ 0 & -1 & 1 \\ 0 & -1 & 1 \end{pmatrix} \\ (A-2I)^3 &= (A-2I)(A-2I)^2 = \begin{pmatrix} 1 & 0 & 1 \\ -1 & 1 & -2 \\ -1 & 1 & -2 \end{pmatrix} \begin{pmatrix} 0 & 1 & -1 \\ 0 & -1 & 1 \\ 0 & -1 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \end{aligned}$$

Which implies that

$$A^3 - 3 \times 2A^2 + 3 \times 2^2A - 2^3I = 0$$

Because A and I commute.
Which is equivalent to

$$A^3 - 6A^2 + 12A - 8I = 0$$

Or again

$$A^3 - 6A^2 + 12A = 8I$$

Then by dividing by 8 and factoring A

$$A\left(\frac{1}{8}A^2 - \frac{3}{4}A + \frac{3}{2}I\right) = I$$

Which shows that A is invertible and that

$$A^{-1} = \frac{1}{8}A^2 - \frac{3}{4}A + \frac{3}{2}I$$

Exercise 3.

1. The coordinates of $u(x)$ in the canonical basis are

$$\begin{pmatrix} x_2 - 2x_3 \\ 2x_1 - x_2 + 4x_3 \\ x_1 - x_2 + 3x_3 \end{pmatrix} = \begin{pmatrix} 0 & 1 & -2 \\ 2 & -1 & 4 \\ 1 & -1 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

So the matrix of u in the canonical basis is

$$A = \begin{pmatrix} 0 & 1 & -2 \\ 2 & -1 & 4 \\ 1 & -1 & 3 \end{pmatrix}$$

2. Let $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ be the coordinates of a vector $x = (x_1, x_2, x_3)$ in the canonical basis

$$\begin{aligned} x &= (x_1, x_2, x_3) \in \ker(u - Id) \Leftrightarrow (A - I)X = 0 \\ &\Leftrightarrow \begin{pmatrix} -1 & 1 & -2 \\ 2 & -2 & 4 \\ 1 & -1 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \\ &\Leftrightarrow \begin{cases} -x_1 + x_2 - 2x_3 = 0 \\ 2x_1 - 2x_2 + 4x_3 = 0 \Leftrightarrow x_1 - x_2 + 2x_3 = 0 \Leftrightarrow x_1 = x_2 - 2x_3 \\ x_1 - x_2 + 2x_3 = 0 \end{cases} \end{aligned}$$

So $x = (x_2 - 2x_3, x_2, x_3) = x_2(1, 1, 0) + x_3(-2, 0, 1)$
Let $a = (1, 1, 0)$ and $b = (-2, 0, 1)$, (a, b) is a family of two non-proportional vectors, so linearly independent, which span $\ker(u - Id)$, it is a basis of $\ker(u - Id)$.

3. Let $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ be the coordinates of a vector $x = (x_1, x_2, x_3)$ in the canonical basis

$$\begin{aligned} x = (x_1, x_2, x_3) \quad &\in \quad \ker(u) \Leftrightarrow AX = 0 \\ &\Leftrightarrow \quad \begin{pmatrix} 0 & 1 & -2 \\ 2 & -1 & 4 \\ 1 & -1 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \\ &\Leftrightarrow \quad \begin{cases} x_2 - 2x_3 = 0 \\ 2x_1 - x_2 + 4x_3 = 0 \\ x_1 - x_2 + 3x_3 = 0 \end{cases} \Leftrightarrow \begin{cases} x_2 = 2x_3 \\ 2x_1 - 2x_3 + 4x_3 = 0 \\ x_1 - 2x_3 + 3x_3 = 0 \end{cases} \\ &\Leftrightarrow \quad \begin{cases} x_2 = 2x_3 \\ 2x_1 + 2x_3 = 0 \\ x_1 + x_3 = 0 \end{cases} \Leftrightarrow \begin{cases} x_1 = -x_3 \\ x_2 = 2x_3 \end{cases} \end{aligned}$$

So $x = (-x_3, 2x_3, x_3)$, if we let $c = (-1, 2, 1)$ then $\ker(u) = \text{Vect}(c)$

4. We have:

$$\begin{aligned} \det(a, b, c) &= \begin{vmatrix} 1 & -2 & -1 \\ 1 & 0 & 2 \\ 0 & 1 & 1 \end{vmatrix} = \begin{vmatrix} 0 & 2 \\ 1 & 1 \end{vmatrix} - \begin{vmatrix} -2 & -1 \\ 1 & 1 \end{vmatrix} \\ &= -2 - (-2 + 1) = -1 \neq 0 \end{aligned}$$

By expanding along the first column, so (a, b, c) is a basis of $\mathbb{R}^3$.

5. $u(a) - a = 0_{\mathbb{R}^3} \Rightarrow u(a) = a$, similarly $u(b) = b$ and $u(c) = 0_{\mathbb{R}^3}$ so

$$D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

6. According to the matrix of u in the basis $\beta', \text{Im}(u) = \text{Vect}(a, b) = \ker(u - Id)$

7. By the rank-nullity theorem

$$\dim(\ker u) + \dim(\text{Im}\, u) = \dim\left(\mathbb{R}^3\right)$$

It remains to show that the intersection of $\ker(u)$ and $\text{Im}(u)$ is the zero vector.

$$\begin{aligned} x \quad &\in \quad \ker(u) \cap \text{Im}(u) \Leftrightarrow \begin{cases} x \in \text{Im}(u) \\ x \in \ker(u) \end{cases} \Leftrightarrow \begin{cases} x \in \ker(u) \\ x \in \ker(u - Id) \end{cases} \\ &\Leftrightarrow \quad \begin{cases} u(x) = 0_{\mathbb{R}^3} \\ u(x) - x = 0_{\mathbb{R}^3} \end{cases} \Leftrightarrow \begin{cases} u(x) = 0_{\mathbb{R}^3} \\ u(x) = x \end{cases} \Leftrightarrow x = 0_{\mathbb{R}^3} \end{aligned}$$

We thus have $\ker(u) \oplus \mathrm{Im}(u) = \mathbb{R}^3$

Exercise 4.

1. We have:

$$\begin{aligned} &u(\alpha P + \beta Q) = \alpha P + \beta Q + (1-X)(\alpha P + \beta Q)' \\ &= \alpha P + \beta Q + (1-X)\left(\alpha P' + \beta Q'\right) \\ &= \alpha\left(P + (1-X)P'\right) + \beta\left(Q + (1-X)Q'\right) = \alpha u(P) + \beta u(Q) \end{aligned}$$

So u is a linear map

$$d^\circ P \leq 2 \Rightarrow d^\circ u(P) \leq 2$$

It goes from $\mathbb{R}_2[X]$ to $\mathbb{R}_2[X]$ so it is an endomorphism of $\mathbb{R}_2[X]$.

2. We have:

$$\begin{aligned} &u(1) = 1 + (1-X) \times 0 = 1 \\ &u(X) = X + (1-X) \times 1 = 1 \\ &u\left(X^2\right) = X^2 + (1-X) \times 2X = 2X - X^2 \\ &A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & -1 \end{pmatrix} \end{aligned}$$

3. We have: $P \in \ker(u)$

$$\begin{aligned} u(P) &= 0 \Leftrightarrow u\left(aX^2 + bX + c\right) = au\left(X^2\right) + bu(X) + cu(1) \\ &= a\left(2X - X^2\right) + b + c = 0 \\ &\Leftrightarrow -aX^2 + 2aX + b + c = 0 \Leftrightarrow \begin{cases} a = 0 \\ c = -b \end{cases} \\ P &= bX - b = b(X-1) \end{aligned}$$

So $\ker(u)$ is the vector line spanned by the polynomial $X - 1$.

$$\begin{aligned} \mathrm{Im}(u) &= \mathrm{Vect}\left(u(1), u(X), u\left(X^2\right)\right) = \mathrm{Vect}\left(1, 1, 2X - X^2\right) \\ &= \mathrm{Vect}\left(1, 2X - X^2\right) \end{aligned}$$

These two polynomials are not proportional so they form a linearly independent family (and generating) of $\mathrm{Im}(u)$ so a basis of $\mathrm{Im}(u)$.

CHAPTER 4

SOLVING SYSTEMS OF EQUATIONS

Determinants provide an efficient and indispensable tool for discussing linear systems: they allow us to have compatibility conditions in the form of relations linking the coefficients and also provide formulas that explicitly give the solution (**Cramer's** formulas).

The essential notions covered in this chapter are:

— Determinant.
— Permutations.
— Notation convention.
— Systems of linear equations
— Definitions and interpretations.
— Matrix expression and rank of a system.
— Vector expression.
— Interpretation in terms of linear maps.
— Cramer's systems.

4.1 Determinant

4.1.1 Permutations

To define the determinant of a square matrix, we must start by introducing permutations.

A permutation is a bijection $\sigma : S \to S$, where S is a set. We usually denote a permutation in the form of a two-row array such that each element in the bottom row represents the image of the element immediately above it. Using this notation, $\sigma = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$ is a permutation such that $\sigma(1) = 3, \sigma(2) = 1$, and $\sigma(3) = 2$. (The notation for permutations should not be confused with that of matrices.) The set of all permutations of the set $\{1, 2, \ldots, n\}$ is denoted by S_n. Note that $|S_n| = n!$. The elements of S_2 are

$$\begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix} \text{ and } \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix},$$

and those of S_3 are

$$\begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix},$$
$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}, \text{ and } \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}.$$

The permutation $\sigma \in S_n$ such that $\sigma(i) = i$ for $i = 1, \ldots, n$ is the identity permutation.

Let $\sigma, \gamma \in S_n$. It is not difficult to see that $\gamma \circ \sigma \in S_n$.

Example 4.1 *Let*

$$\sigma = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} \text{ and } \gamma = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}.$$

What is $\gamma \circ \sigma$?

Note that

$$(\gamma \circ \sigma)(1) = \gamma(\sigma(1)) = \gamma(3) = 1,$$
$$(\gamma \circ \sigma)(2) = \gamma(\sigma(2)) = \gamma(1) = 3,$$
$$(\gamma \circ \sigma)(3) = \gamma(\sigma(3)) = \gamma(2) = 2.$$

Thus, $\gamma \circ \sigma$ is the permutation $\begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}$.

If $\gamma = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$, then $\gamma \circ \sigma$ is the identity permutation. We call γ the inverse permutation (or simply the inverse) of σ. Every permutation has an inverse.

We will see later that the definition of the determinant of an $n \times n$ matrix A is a sum of terms, each containing a product of the form $a_{1,\sigma(1)}a_{2,\sigma(2)} \cdots a_{n,\sigma(n)}$, for a permutation $\sigma \in S_n$, where $a_{i,j}$ denotes the (i, j) element of $\mathbf{A}$.

Recall that we use the first index for rows and the second for columns. Thus, the product $a_{1,\sigma(1)}a_{2,\sigma(2)}\cdots a_{n,\sigma(n)}$ contains exactly one element from each row of A. Since $\sigma(1),\ldots,\sigma(n)$ is a permutation of the numbers $1,\ldots,n$, the product

$$a_{1,\sigma(1)}a_{2,\sigma(2)}\cdots a_{n,\sigma(n)}$$

also contains exactly one element from each column of A.

Example 4.2 *Let*

$$\mathbf{A}=\begin{bmatrix} a & b & c\\ d & e & f\\ u & v & w\end{bmatrix} \text{ and } \sigma=\begin{pmatrix} 1 & 2 & 3\\ 3 & 1 & 2\end{pmatrix}.$$

Then $a_{1,\sigma(1)}a_{2,\sigma(2)}a_{3,\sigma(3)}=a_{1,3}a_{2,1}a_{3,2}=cdv$

Exercise 4.1 *1. For each of the following permutations, give the inverse permutation.*

$$a.\begin{pmatrix} 1 & 2 & 3 & 4\\ 1 & 3 & 2 & 4\end{pmatrix},\ b.\begin{pmatrix} 1 & 2 & 3 & 4\\ 4 & 3 & 2 & 1\end{pmatrix},\ c.\begin{pmatrix} 1 & 2 & 3 & 4 & 5\\ 5 & 1 & 4 & 2 & 3\end{pmatrix}$$

2. Let

$$\mathbf{A}=\begin{bmatrix} 6 & 2 & 3\\ 0 & 1 & -1\\ -5 & 7 & 4\end{bmatrix}.$$

Let $\sigma=\begin{pmatrix} 1 & 2 & 3\\ 3 & 2 & 1\end{pmatrix}$. *What value does* $a_{1,\sigma(1)}a_{2,\sigma(2)}a_{3,\sigma(3)}$ *take?*

Solution. 1. a. The inverse permutation is

$$\begin{pmatrix} 1 & 2 & 3 & 4\\ 1 & 3 & 2 & 4\end{pmatrix}.$$

b. The inverse permutation is

$$\begin{pmatrix} 1 & 2 & 3 & 4\\ 4 & 3 & 2 & 1\end{pmatrix}.$$

c. The inverse permutation is

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5\\ 2 & 4 & 5 & 3 & 1\end{pmatrix}.$$

2. $a_{1,\sigma(1)}a_{2,\sigma(2)}a_{3,\sigma(3)}=a_{1,3}a_{2,2}a_{3,1}=3\cdot 1\cdot(-5)=-15.$ □

Before addressing the determinant, we must also introduce the notion of an inversion of a permutation.

Definition 4.1 *Let $\sigma \in S_n$. The pair (i, j) is an inversion of σ if $i < j$ and $\sigma(i) > \sigma(j)$. (Not to be confused with the inverse permutation discussed earlier.) The total number of inversions of σ is denoted by* $\text{inv}(\sigma)$.

Example 4.3 *Consider*

$$\sigma = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}.$$

The pair $(1, 2)$ is an inversion of σ because we have $1 < 2$ and $\sigma(1) = 3 > 1 = \sigma(2)$. The other inversion is given by the pair $(1, 3)$. So $\text{inv}(\sigma) = 2$.

Example 4.4 *Let $\sigma \in S_n$ for which there exist $i, j \in \{1, \ldots, n\}, i < j$ such that $\sigma(i) = j$, $\sigma(j) = i$, and $\sigma(k) = k$ for all other indices. We want to determine* $\text{inv}(\sigma)$.

Note that for each $k = i + 1, \ldots, j - 1, (i, k)$ is an inversion because $i < k$ and $\sigma(i) = j > k = \sigma(k)$.

Moreover, for each $k = i + 1, \ldots, j - 1, (k, j)$ is an inversion because $k < j$ and $\sigma(k) = k > i = \sigma(j)$.

Finally, the pair (i, j) is also an inversion. Thus,

$$\text{inv}(\sigma) = 2(j - 1 - (i + 1) + 1) + 1 = 2(j - i) - 1,$$

which is always odd.

For example, the inversions of

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 4 & 3 & 2 & 5 \end{pmatrix}$$

are $(2, 3), (3, 4)$ and $(2, 4)$.

Definition 4.2 *If $A = (a_{ij})$, the determinant of* $\mathbf{A}$*, denoted by* $\det(\mathbf{A})$*, is defined by*

$$\sum_{\sigma \in S_n} (-1)^{\text{inv}(\sigma)} a_{1,\sigma(1)} \cdots a_{n,\sigma(n)},$$

where $a_{i,j}$ denotes the element in the i-th row and j-th column of $\mathbf{A}$.

Calculating the determinant as defined requires the sum of $n!$ terms. Each of these terms depends on a permutation of S_n and the product of n elements of $\mathbf{A}$, and the sign depends on the parity of the number of inversions of the permutation. (The parity of an integer indicates whether the number is even or odd.)

Note that even when the size of the matrix is small, the number of terms to compute quickly becomes too high. For example, if $n = 5$, there are already $5 \times 4 \times 3 \times 2 \times 1 = 120$ terms. Fortunately, there are efficient methods for computing the determinant that do not require so many terms.

4.1.2 Determinant of small matrices

In the case of 2×2 or 3×3 matrices, there are formulas to simplify the calculations.

Let

$$\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

The two permutations of S_2 are

$$\sigma_1 = \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix} \text{ and } \sigma_2 = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}.$$

Since $\text{inv}(\sigma_1) = 0$ and $\text{inv}(\sigma_2) = 1$, we have

$$\det(\mathbf{A}) = (-1)^{\text{inv}(\sigma_1)} a_{1,\sigma_1(1)} a_{2,\sigma_1(2)} + (-1)^{\text{inv}(\sigma_2)} a_{1,\sigma_2(1)} a_{2,\sigma_2(2)} = ad - bc.$$

Let

$$\mathbf{A} = \begin{bmatrix} p_1 & p_2 & p_3 \\ q_1 & q_2 & q_3 \\ r_1 & r_2 & r_3 \end{bmatrix}.$$

The six permutations of S_3 are listed in the following table.

i	1	2	3
σ_i	$\begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}$	$\begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}$	$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$
$\text{inv}(\sigma_i)$	0	1	1
$(-1)^{\text{inv}(\sigma_i)}$	1	-1	-1
i	4	5	6
σ_i	$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$	$\begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$
$\text{inv}(\sigma_i)$	2	2	3
$(-1)^{\text{inv}(\sigma_i)}$	1	1	-1

Then

$$\begin{aligned} &\det(A) = \textstyle\sum_{i=1}^{6} (-1)^{\text{inv}(\sigma_i)} a_{1,\sigma_i(1)} a_{2,\sigma_i(2)} a_{3,\sigma_i(3)} \\ &= p_1 q_2 r_3 - p_1 q_3 r_2 - p_2 q_1 r_3 + p_2 q_3 r_1 + p_3 q_1 r_2 - p_3 q_2 r_1 \end{aligned}$$

4.1.3 Notation Convention

Instead of writing

$$\det\left(\begin{bmatrix} a & b & c \\ d & e & f \\ p & q & r \end{bmatrix}\right),$$

we simply write

$$\begin{vmatrix} a & b & c \\ d & e & f \\ p & q & r \end{vmatrix}.$$

Exercise 4.2 *1. For each of the following permutations, give the number of inversions.*

$$a.\begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 2 & 4 \end{pmatrix},\ b.\begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix},\ c.\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 1 & 4 & 2 & 3 \end{pmatrix}$$

2. Compute the determinant of each of the following matrices:

$$a.\begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix},\ b.\begin{bmatrix} 1 & 0 & 0 \\ 0 & a & b \\ 0 & c & d \end{bmatrix},\ c.\begin{bmatrix} 0 & 0 & p \\ q & 0 & 0 \\ 0 & r & 0 \end{bmatrix}$$

3. Let $\mathbf{A}$ *and* $\mathbf{B}$ *be* $n \times n$ *matrices such that* $\mathbf{B}$ *is obtained from* $\mathbf{A}$ *by multiplying a row of* A *by the scalar* α*. Show that* $\det(\mathbf{B}) = \alpha \det(\mathbf{A})$.

Solution. 1. a. The number of inversions is 1, b. The number of inversions is 6, c. The number of inversions is 6.

2. a. 1, b. $ad - bc$, c. pqr.

3. Suppose $\mathbf{B}$ is obtained from $\mathbf{A}$ by multiplying the i-th row by α. Then

$$\begin{aligned} \det(\mathbf{B}) &= \textstyle\sum_{\sigma \in S_n} (-1)^{\mathrm{inv}(\sigma)} \prod_{p=1}^{n} b_{p,\sigma(p)} \\ &= \textstyle\sum_{\sigma \in S_n} (-1)^{\mathrm{inv}(\sigma)} \left(\prod_{p \neq i} a_{p,\sigma(p)} \right) \left(\alpha a_{i,\sigma(i)} \right) \\ &= \alpha \left(\textstyle\sum_{\sigma \in S_n} (-1)^{\mathrm{inv}(\sigma)} \prod_{p=1}^{n} a_{p,\sigma(p)} \right) \\ &= \alpha \det(\mathbf{A}). \end{aligned}$$

□

4.1.4 Special Matrices

Matrices with a zero row or column

Let A be a square matrix having a zero row or column. Then $\det(\mathbf{A}) = 0$. Indeed, note that each term in the definition $\det(\mathbf{A})$ is a product of the form $a_{1,\sigma(1)}a_{2,\sigma(2)} \cdots a_{n,\sigma(n)}$, for a permutation σ. Thus, each term contains exactly one element from each row and each column of $\mathbf{A}$, which implies that each term is zero, which implies that $\det(\mathbf{A}) = 0$.

Permutation matrices

A permutation matrix of size $n \times n$ is a matrix obtained from the identity matrix $\mathbf{I}_n$ by permuting its rows. For example,

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

is a permutation matrix.

As its name suggests, such a matrix directly encodes a permutation of the set $\{1, \ldots, n\}$. The interpretation is as follows: if σ is encoded by the permutation matrix, then $\sigma(i)$ is the column index of the element containing 1 in the i-th row. In the example above, the corresponding permutation is

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 1 & 3 \end{pmatrix}$$

because in row 1, column 2 contains 1; in row 2, column 4 contains 1; in row 3, column 1 contains 1; in row 4, column 3 contains 1.

For a given permutation matrix $\mathbf{P}$ that encodes σ, the determinant of $\mathbf{P}$ is simply $(-1)^{\text{inv}(\sigma)}$. In the example above, there are three inversions. Therefore, the determinant is $(-1)^3 = -1$.

This result can be seen directly from the definition of the determinant: each term of the sum contains a factor $(-1)^{\text{inv}(\sigma')}$which multiplies the product of $n$elements, exactly one from each row and one from each column. The only way to obtain a non-zero term is to use a permutation that extracts the non-zero element from each row. There is only one permutation for which this is the case, the one encoded by σ.

Triangular matrices

Let $\mathbf{A}$be a square upper triangular matrix (i.e., $a_{i,j} = 0$for all $i > j$). For example, the matrix

$$\begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 2 & 5 & 6 \\ 0 & 0 & 3 & 7 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

is upper triangular.

In this case, $\det(\mathbf{A})$ is the product of the elements on the diagonal. Here, the determinant is $1 \cdot 2 \cdot 3 \cdot 1 = 6$.

Let's see why this is the case. Let $\sigma \in S_n$. We start by showing that if σ is not the identity permutation, then $\prod_{i=1}^{n} a_{i,\sigma(i)} = 0$.

Suppose that $\sigma(1) \neq 1$. Then there must be an $i \geq 2$ such that $\sigma(i) = 1$. This gives $a_{i,\sigma(i)} = 0$ since $\mathbf{A}$ is upper triangular and $i > \sigma(i)$. So, $\prod_{i=1}^{n} a_{i,\sigma(i)} = 0$ if $\sigma(1) \neq 1$.

Suppose that $\sigma(1) = 1$ but $\sigma(2) \neq 2$. Then there must be an $i \neq 2$ such that $\sigma(i) = 2$. But $i \neq 1$ since we already have $\sigma(1) = 1$. So, $i \geq 3$. This again gives $a_{i,\sigma(i)} = 0$, since $i > \sigma(i)$. So, $\prod_{i=1}^{n} a_{i,\sigma(i)} = 0$ if $\sigma(1) = 1$ and $\sigma(2) \neq 2$.

We can continue in this manner to show that if $\sigma(i) = i$ and $\sigma(i+1) \neq i+1$, then $\prod_{i=1}^{n} a_{i,\sigma(i)} = 0$. Thus, the only term in $\det(\mathbf{A})$ that can be non-zero is the one for which $\sigma(i) = i$ for all $i = 1, \ldots, n$, which implies that $\det(\mathbf{A}) = a_{1,1}a_{2,2}\cdots a_{n,n}$.

Using a similar argument, we can conclude that the determinant of a lower triangular matrix (a matrix where all elements below the diagonal are zero) is also given by the product of the elements on the diagonal.

Exercise 4.3 *1. Compute the determinant of each of the following matrices:*

$$a.\begin{bmatrix} 2 & 3 \\ 0 & 2 \end{bmatrix}, \; b.\begin{bmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{bmatrix} \text{ and } c.\begin{bmatrix} 2-i & 0 \\ 3 & 1+i \end{bmatrix}.$$

Solution. a. 4, b. adf, c. $(2-i)(1+i) = 3+i$. □

4.1.5 Fundamental Properties

Proposition 4.1 *Let A, $B \in S^{n\times n}, S$ a ring. Then*

$$\det(\mathbf{AB}) = \det(\mathbf{A})\det(\mathbf{B})$$

An immediate and useful consequence of the previous proposition is that

$$\det\left(\mathbf{A}^k\right) = \det(\mathbf{A})^k$$

for every natural number k.

Example 4.5 *Let $\mathbf{A}, \mathbf{B} \in \mathbb{R}^{3\times 3}$ such that $\det(\mathbf{A}) = 3$ and $\det(\mathbf{B}) = -2$. Then $\det(\mathbf{AB}) = \det(\mathbf{A})\det(\mathbf{B}) = -6$.*

Example 4.6 *Let*

$$\mathbf{A} = \begin{bmatrix} -2 & 1 \\ -1 & 1 \end{bmatrix}.$$

Then $\det\left(\mathbf{A}^{100}\right) = \det(\mathbf{A})^{100} = (-1)^{100} = 1$.

The determinant also satisfies the following properties:

- If $\mathbf{B}$ is obtained from $\mathbf{A}$ by adding a multiple of one row of $\mathbf{A}$ to another row of $\mathbf{A}$, then $\det(\mathbf{B}) = \det(\mathbf{A})$.

- If $\mathbf{B}$ is obtained from $\mathbf{A}$ by exchanging two rows of $\mathbf{A}$, then $\det(\mathbf{B}) = -\det(\mathbf{A})$.
- $\det\left(\mathbf{A}^\top\right) = \det(\mathbf{A})$.
- If $\mathbf{A}$ is invertible, then $\det\left(\mathbf{A}^{-1}\right) = \det(\mathbf{A})^{-1}$.
- If α is a scalar and $\mathbf{A}$ is of size $n \times n$, $\det(\alpha\mathbf{A}) = \alpha^n \det(\mathbf{A})$.

Example 4.7 *Let* $\mathbf{A} = \begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix}$. *Note that* $\det(\mathbf{A}) = 3$.

- $\det\left(\mathbf{A}^\top\right) = \det(\mathbf{A}) = 3$.
- *Note that* $\mathbf{A}$ *is invertible. So,* $\det\left(\mathbf{A}^{-1}\right) = \frac{1}{\det(\mathbf{A})} = \frac{1}{3}$.
- $\det(2\mathbf{A}) = 2^2 \det(\mathbf{A}) = 4 \cdot 3 = 12$ *since* $\mathbf{A}$ *is* 2×2.
- $\det(-\mathbf{A}) = \det((-1)\mathbf{A}) = (-1)^2 \det(\mathbf{A}) = \det(\mathbf{A}) = 3$. *Note that in this case,* $\det(-\mathbf{A}) = \det(\mathbf{A})$.

Exercise 4.4 *1. Prove that if* $\mathbf{A}$ *is invertible, then* $\det\left(\mathbf{A}^{-1}\right) = \frac{1}{\det(\mathbf{A})}$.

2. Prove that if $\mathbf{A} \in \mathbb{R}^{4\times 4}$, then $\det(-\mathbf{A}) = \det(\mathbf{A})$.

Solution. 1. Note that, in this case,

$$\det\left(\mathbf{A}^{-1}\right)\det(\mathbf{A}) = \det\left(\mathbf{A}^{-1}\mathbf{A}\right) = \det(\mathbf{I}) = 1$$

hence $\det\left(\mathbf{A}^{-1}\right) = \frac{1}{\det(\mathbf{A})}$.

2. $\det(-\mathbf{A}) = \det((-1)\mathbf{A}) = (-1)^4 \det(\mathbf{A}) = \det(\mathbf{A})$. □

4.1.6 Determinant of a block matrix

Let $\mathbf{A}$ be a square matrix that can be written in block form as

$$\mathbf{A} = \begin{bmatrix} \mathbf{B} & \mathbf{C} \\ \mathbf{0} & \mathbf{D} \end{bmatrix},$$

where $\mathbf{B}$ and $\mathbf{D}$ are square matrices. Then $\det(\mathbf{A}) = \det(\mathbf{B})\det(\mathbf{D})$.

For example, the determinant of

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 5 & 6 \\ 0 & 0 & 2 & 7 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

is $1 \cdot 1 \cdot 2 \cdot 3 = 6$.

4.1.7 Computing the determinant by cofactor expansion

Let $\mathbf{A}$ be an $n \times n$ matrix. Let $\mathbf{A}_{i,j}$ be the $(n-1) \times (n-1)$ matrix obtained by removing the i-th row and the j-th column from $\mathbf{A}$. The (i,j)-cofactor of $\mathbf{A}$ is defined by

$$C_{i,j} = (-1)^{i+j} \det\left(\mathbf{A}_{i,j}\right).$$

The determinant of $\mathbf{A}$ can be computed by expanding along any row or any column. For example, expanding along the i-th row gives

$$\det(\mathbf{A}) = \sum_{j=1}^{n} a_{i,j}C_{i,j}.$$

Similarly, expanding along the j-th column gives

$$\det(\mathbf{A}) = \sum_{i=1}^{n} a_{i,j}C_{i,j}.$$

Example 4.8 *Let*

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 1 & 0 & 6 \end{bmatrix}.$$

We can compute $\det(\mathbf{A})$ *by expanding along the first row. Then*

$$\begin{aligned}
&\det(\mathbf{A}) = a_{1,1}C_{1,1} + a_{1,2}C_{1,2} + a_{1,3}C_{1,3} \\
&= 1 \cdot (-1)^{1+1} \det\left(\begin{bmatrix} 4 & 5 \\ 0 & 6 \end{bmatrix}\right) + 2 \cdot (-1)^{1+2} \det\left(\begin{bmatrix} 0 & 5 \\ 1 & 6 \end{bmatrix}\right) + 3 \cdot (-1)^{1+3} \det\left(\begin{bmatrix} 0 & 4 \\ 1 & 0 \end{bmatrix}\right) \\
&= 1 \cdot (24 - 0) + 2 \cdot (-1)(0 - 5) + 3 \cdot (0 - 4) \\
&= 24 + 10 - 12 = 22.
\end{aligned}$$

We can also expand along the second column. Then

$$\begin{aligned}
&\det(\mathbf{A}) = a_{1,2}C_{1,2} + a_{2,2}C_{2,2} + a_{3,2}C_{3,2} \\
&= 2 \cdot (-1)^{1+2} \det\left(\begin{bmatrix} 0 & 5 \\ 1 & 6 \end{bmatrix}\right) + 4 \cdot (-1)^{2+2} \det\left(\begin{bmatrix} 1 & 3 \\ 1 & 6 \end{bmatrix}\right) + 0 \cdot C_{3,2} \\
&= 2 \cdot (-1)(0 - 5) + 4 \cdot (6 - 3) + 0 \\
&= 10 + 12 = 22.
\end{aligned}$$

Exercise 4.5 *1. Compute the determinant of*

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

by expanding along the first row.

2. Compute the determinant of

$$\begin{bmatrix} 1 & 2 & 0 & 0 \\ 3 & 4 & 0 & 0 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \end{bmatrix}$$

by first using the property of block matrices.

Solution. 1. We have:

$$\begin{aligned}\det(\mathbf{A}) &= 1\cdot\det\left(\begin{bmatrix}5&6\\8&9\end{bmatrix}\right)-2\cdot\det\left(\begin{bmatrix}4&6\\7&9\end{bmatrix}\right)+3\cdot\det\left(\begin{bmatrix}4&5\\7&8\end{bmatrix}\right)\\ &= 1\cdot(45-48)-2\cdot(36-42)+3\cdot(32-35)=-3+12-9=0.\end{aligned}$$

2. The matrix is of the form $\begin{bmatrix}\mathbf{B}&\mathbf{0}\\\mathbf{C}&\mathbf{D}\end{bmatrix}$, where $\mathbf{B}=\begin{bmatrix}1&2\\3&4\end{bmatrix}$, $\mathbf{C}=\begin{bmatrix}5&6\\9&10\end{bmatrix}$, and $\mathbf{D}=\begin{bmatrix}7&8\\11&12\end{bmatrix}$. Then $\det(\mathbf{A})=\det(\mathbf{B})\det(\mathbf{D})=(4-6)\cdot(84-88)=(-2)\cdot(-4)=8$. □

4.1.8 The adjugate matrix and Cramer's rule

Let $\mathbf{A}$ be an $n\times n$ matrix. The adjugate matrix of $\mathbf{A}$ is the $n\times n$ matrix whose (i,j)-th element is $C_{j,i}$, where $C_{j,i}$ is the (j,i)-cofactor of $\mathbf{A}$.

Note the reversal of indices: the element in the (i,j)-th position of the adjugate matrix is the (j,i)-cofactor of $\mathbf{A}$.

The adjugate matrix of $\mathbf{A}$ is denoted by $\operatorname{adj}(\mathbf{A})$.

It can be shown that

$$\mathbf{A}\cdot\operatorname{adj}(\mathbf{A})=\operatorname{adj}(\mathbf{A})\cdot\mathbf{A}=\det(\mathbf{A})\mathbf{I}.$$

If $\det(\mathbf{A})\neq 0$, then $\mathbf{A}$ is invertible and

$$\mathbf{A}^{-1}=\frac{1}{\det(\mathbf{A})}\operatorname{adj}(\mathbf{A}).$$

This gives a formula for the inverse of $\mathbf{A}$.

Example 4.9 *Let*

$$\mathbf{A}=\begin{bmatrix}a&b\\c&d\end{bmatrix}.$$

Then $\operatorname{adj}(\mathbf{A})=\begin{bmatrix}d&-b\\-c&a\end{bmatrix}$, *and* $\mathbf{A}^{-1}=\frac{1}{ad-bc}\begin{bmatrix}d&-b\\-c&a\end{bmatrix}$.

Example 4.10 *Let*

$$\mathbf{A}=\begin{bmatrix}1&2&3\\0&4&5\\1&0&6\end{bmatrix}.$$

We have already computed $\det(\mathbf{A}) = 22$. *The cofactors are:*

$$\begin{aligned}
C_{1,1} &= (-1)^{1+1}\det\left(\begin{bmatrix} 4 & 5 \\ 0 & 6 \end{bmatrix}\right) = 24,\\
C_{1,2} &= (-1)^{1+2}\det\left(\begin{bmatrix} 0 & 5 \\ 1 & 6 \end{bmatrix}\right) = 5,\\
C_{1,3} &= (-1)^{1+3}\det\left(\begin{bmatrix} 0 & 4 \\ 1 & 0 \end{bmatrix}\right) = -4,\\
C_{2,1} &= (-1)^{2+1}\det\left(\begin{bmatrix} 2 & 3 \\ 0 & 6 \end{bmatrix}\right) = -12,\\
C_{2,2} &= (-1)^{2+2}\det\left(\begin{bmatrix} 1 & 3 \\ 1 & 6 \end{bmatrix}\right) = 3,\\
C_{2,3} &= (-1)^{2+3}\det\left(\begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}\right) = 2,\\
C_{3,1} &= (-1)^{3+1}\det\left(\begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}\right) = -2,\\
C_{3,2} &= (-1)^{3+2}\det\left(\begin{bmatrix} 1 & 3 \\ 0 & 5 \end{bmatrix}\right) = -5,\\
C_{3,3} &= (-1)^{3+3}\det\left(\begin{bmatrix} 1 & 2 \\ 0 & 4 \end{bmatrix}\right) = 4.
\end{aligned}$$

Then

$$\operatorname{adj}(\mathbf{A}) = \begin{bmatrix} C_{1,1} & C_{2,1} & C_{3,1} \\ C_{1,2} & C_{2,2} & C_{3,2} \\ C_{1,3} & C_{2,3} & C_{3,3} \end{bmatrix} = \begin{bmatrix} 24 & -12 & -2 \\ 5 & 3 & -5 \\ -4 & 2 & 4 \end{bmatrix}.$$

So,

$$\mathbf{A}^{-1} = \frac{1}{22}\begin{bmatrix} 24 & -12 & -2 \\ 5 & 3 & -5 \\ -4 & 2 & 4 \end{bmatrix}.$$

Now, consider the system of linear equations

$$\mathbf{Ax} = \mathbf{b},$$

where $\mathbf{A}$ is an $n \times n$ matrix. If $\det(\mathbf{A}) \neq 0$, then the system has a unique solution given by

$$x_j = \frac{\det\left(\mathbf{A}_j(\mathbf{b})\right)}{\det(\mathbf{A})},$$

where $\mathbf{A}_j(\mathbf{b})$ is the matrix obtained by replacing the j-th column of $\mathbf{A}$ with the column vector $\mathbf{b}$.

This is known as Cramer's rule.

Example 4.11 *Consider the system*

$$\begin{aligned} x + 2y + 3z &= 6 \\ 4y + 5z &= 7 \\ x + 6z &= 8 \end{aligned}$$

In matrix form, this is $\mathbf{Ax} = \mathbf{b}$, *where*

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 1 & 0 & 6 \end{bmatrix}, \mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 6 \\ 7 \\ 8 \end{bmatrix}.$$

We have already computed $\det(\mathbf{A}) = 22$. *Then*

$$x = \frac{\det(\mathbf{A}_1(\mathbf{b}))}{\det(\mathbf{A})} = \frac{\det\left(\begin{bmatrix} 6 & 2 & 3 \\ 7 & 4 & 5 \\ 8 & 0 & 6 \end{bmatrix}\right)}{22}.$$

Compute $\det\left(\begin{bmatrix} 6 & 2 & 3 \\ 7 & 4 & 5 \\ 8 & 0 & 6 \end{bmatrix}\right)$ *by expanding along the second column:*

$$\begin{aligned} & 2 \cdot (-1)^{1+2} \det\left(\begin{bmatrix} 7 & 5 \\ 8 & 6 \end{bmatrix}\right) + 4 \cdot (-1)^{2+2} \det\left(\begin{bmatrix} 6 & 3 \\ 8 & 6 \end{bmatrix}\right) + 0 \cdot C_{3,2} \\ = \; & 2 \cdot (-1)(42 - 40) + 4 \cdot (36 - 24) = 2 \cdot (-1) \cdot 2 + 4 \cdot 12 = -4 + 48 = 44. \end{aligned}$$

So, $x = \frac{44}{22} = 2$.

Similarly,

$$y = \frac{\det(\mathbf{A}_2(\mathbf{b}))}{\det(\mathbf{A})} = \frac{\det\left(\begin{bmatrix} 1 & 6 & 3 \\ 0 & 7 & 5 \\ 1 & 8 & 6 \end{bmatrix}\right)}{22}.$$

Compute $\det\left(\begin{bmatrix} 1 & 6 & 3 \\ 0 & 7 & 5 \\ 1 & 8 & 6 \end{bmatrix}\right)$ *by expanding along the first column:*

$$\begin{aligned} & 1 \cdot (-1)^{1+1} \det\left(\begin{bmatrix} 7 & 5 \\ 8 & 6 \end{bmatrix}\right) + 0 + 1 \cdot (-1)^{3+1} \det\left(\begin{bmatrix} 6 & 3 \\ 7 & 5 \end{bmatrix}\right) \\ = \; & 1 \cdot (42 - 40) + 1 \cdot (30 - 21) = 2 + 9 = 11. \end{aligned}$$

So, $y = \frac{11}{22} = \frac{1}{2}$.

Finally,

$$z = \frac{\det(\mathbf{A}_3(\mathbf{b}))}{\det(\mathbf{A})} = \frac{\det\left(\begin{bmatrix} 1 & 2 & 6 \\ 0 & 4 & 7 \\ 1 & 0 & 8 \end{bmatrix}\right)}{22}.$$

Compute $\det\left(\begin{bmatrix} 1 & 2 & 6 \\ 0 & 4 & 7 \\ 1 & 0 & 8 \end{bmatrix}\right)$ *by expanding along the second column:*

$$\begin{aligned} & 2 \cdot (-1)^{1+2} \det\left(\begin{bmatrix} 0 & 7 \\ 1 & 8 \end{bmatrix}\right) + 4 \cdot (-1)^{2+2} \det\left(\begin{bmatrix} 1 & 6 \\ 1 & 8 \end{bmatrix}\right) + 0 \\ = \; & 2 \cdot (-1)(0-7) + 4 \cdot (8-6) = 2 \cdot (-1) \cdot (-7) + 4 \cdot 2 = 14 + 8 = 22. \end{aligned}$$

So, $z = \frac{22}{22} = 1$.

Thus, the solution is $x = 2, y = \frac{1}{2}, z = 1$.

Exercise 4.6 *1. Use Cramer's rule to solve the system:*

$$\begin{aligned} 2x - y &= 5 \\ x + 3y &= -1 \end{aligned}$$

Solution. We have $\mathbf{A} = \begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 5 \\ -1 \end{bmatrix}$, $\det(\mathbf{A}) = 6 - (-1) = 7$.

Then

$$x = \frac{\det\left(\begin{bmatrix} 5 & -1 \\ -1 & 3 \end{bmatrix}\right)}{7} = \frac{15-1}{7} = \frac{14}{7} = 2,$$

$$y = \frac{\det\left(\begin{bmatrix} 2 & 5 \\ 1 & -1 \end{bmatrix}\right)}{7} = \frac{-2-5}{7} = \frac{-7}{7} = -1.$$

So, the solution is $x = 2, y = -1$. □

4.2 Systems of Linear Equations

4.2.1 Definitions and Interpretations

A system of m linear equations in n unknowns is a set of equations of the form:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned}$$

where a_{ij} and b_i are given numbers (or elements of a field), and $x_1, \ldots, x_n$ are the unknowns.

The system is said to be homogeneous if $b_1 = b_2 = \cdots = b_m = 0$; otherwise, it is non-homogeneous.

A solution of the system is an n-tuple $(s_1, \ldots, s_n)$ such that when we substitute $x_1 = s_1, \ldots, x_n = s_n$, all the equations are satisfied.

The system is said to be consistent if it has at least one solution; otherwise, it is inconsistent.

4.2.2 Matrix Expression and Rank of a System

The system can be written in matrix form as

$$\mathbf{A}\mathbf{x} = \mathbf{b},$$

where

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}, \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}.$$

The matrix $\mathbf{A}$ is called the coefficient matrix.

The augmented matrix of the system is the matrix $\left[\begin{array}{c|c} \mathbf{A} & \mathbf{b} \end{array}\right]$.

The rank of the system is the rank of the coefficient matrix $\mathbf{A}$.

The system is consistent if and only if the rank of $\mathbf{A}$ is equal to the rank of the augmented matrix.

If the system is consistent, then the number of free parameters in the general solution is $n - \text{rank}(\mathbf{A})$.

4.2.3 Vector Expression

The system can also be written as

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n = \mathbf{b},$$

where $\mathbf{a}_j$ is the j-th column of $\mathbf{A}$.

This expresses $\mathbf{b}$ as a linear combination of the columns of $\mathbf{A}$.

Thus, the system is consistent if and only if $\mathbf{b}$ is in the column space of $\mathbf{A}$.

4.2.4 Interpretation in Terms of Linear Maps

If we consider the linear map $T : \mathbb{R}^n \to \mathbb{R}^m$ defined by $T(\mathbf{x}) = \mathbf{A}\mathbf{x}$, then the system $\mathbf{A}\mathbf{x} = \mathbf{b}$ is consistent if and only if $\mathbf{b}$ is in the image of T.

The kernel of T is the set of solutions to the homogeneous system $\mathbf{A}\mathbf{x} = \mathbf{0}$.

4.2.5 Cramer's Systems

A system $\mathbf{Ax} = \mathbf{b}$ is called a Cramer system if $\mathbf{A}$ is a square matrix and $\det(\mathbf{A}) \neq 0$.
In this case, the system has a unique solution given by Cramer's rule.

Exercise 4.7 *1. Determine whether the following system is consistent:*

$$\begin{aligned} x + y + z &= 1 \\ 2x + 3y + 4z &= 5 \\ 4x + 5y + 6z &= 9 \end{aligned}$$

Solution. The augmented matrix is

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 2 & 3 & 4 & 5 \\ 4 & 5 & 6 & 9 \end{array}\right].$$

We reduce it to row echelon form:

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 1 & 2 & 5 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 2 \end{array}\right].$$

The last row gives $0 = 2$, which is impossible. So, the system is inconsistent. □

4.3 Exercises

Exercise 1

Without calculating them, explain why the following determinants are zero:

$$\Delta_1 = \begin{vmatrix} 0 & -1 & 10 \\ 0 & 2 & 5 \\ 0 & 1 & 1 \end{vmatrix}, \quad \Delta_2 = \begin{vmatrix} -1 & 2 & 3 \\ 1 & -2 & 5 \\ 1 & -2 & 2 \end{vmatrix}$$

$$\Delta_3 = \begin{vmatrix} 2 & -1 & 3 \\ -1 & 2 & -3 \\ 3 & 1 & 2 \end{vmatrix}, \quad \Delta_4 = \begin{vmatrix} 1 & 2 & 1 \\ 0 & 0 & 3 \\ 0 & 0 & 1 \end{vmatrix}$$

$$\Delta_5 = \begin{vmatrix} 1 & a & b+c \\ 1 & b & c+a \\ 1 & c & a+b \end{vmatrix}, \quad \Delta_6 = \begin{vmatrix} 1 & \cos 2x & 2\cos^2 x \\ 1 & -\cos 2x & 2\sin^2 x \\ \cos x \sin x & \cos x \sin x & \sin 2x \end{vmatrix}$$

Exercise 2

Show that:

$$\begin{vmatrix} 3 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 5 \end{vmatrix} = - \begin{vmatrix} 0 & 0 & 3 \\ 0 & 4 & 0 \\ 5 & 0 & 0 \end{vmatrix} = 60$$

Exercise 3

The numbers $119, 153$ and 289 are all divisible by 17. Show, without expanding it, that the determinant

$$\begin{vmatrix} 1 & 1 & 9 \\ 1 & 5 & 3 \\ 2 & 8 & 9 \end{vmatrix}$$

is divisible by 17.

Exercise 4

Let $(a, b, c) \in \mathbb{R}^3$. Consider the polynomials $P_a = (X - a)^2$, $P_b = (X - b)^2$ and $P_c = (X - c)^2$. Determine for which values of (a, b, c) the family $\mathcal{P} = (P_a, P_b, P_c)$ forms a basis of $\mathbb{R}_2[X]$?

Exercise 5

Show that

$$\begin{aligned} \Delta &= \begin{vmatrix} \cos(a-b) & \cos(b-c) & \cos(c-a) \\ \cos(a+b) & \cos(b+c) & \cos(c+a) \\ \sin(a+b) & \sin(b+c) & \sin(c+a) \end{vmatrix} \\ &= -2\sin(a-b)\sin(b-c)\sin(c-a) \end{aligned}$$

Exercise 6

Let

$$P_1 = 2X^2 - X + 1, \ P_2 = X^2 + 2X \text{ and } P_3 = X^2 - 1$$

Show that the family $\mathcal{P} = (P_1, P_2, P_3)$ is a basis of $\mathbb{R}_2[X]$.

Exercise 7

Under what condition on the real number a does the family $e = (e_1, e_2, e_3)$:

$$e_1 = (a, 1, 1) \quad e_2 = (1, a, 1) \quad e_3 = (1, 1, a)$$

form a basis of $\mathbb{R}^3$?

Exercise 8

Solve in $\mathbb{R}^3$ the systems:

$$1. \begin{cases} x - y + z = 1 \\ 3y - z = 2 \\ 2z = 8 \end{cases} \qquad 2. \begin{cases} x - y + 2z = 1 \\ 2x - 3y + z = 4 \\ x - 3y - 4z = 5 \end{cases}$$

3. $\begin{cases} x+2y+3z=1 \\ -x-3y+5z=2 \\ x+y+z=-1 \end{cases}$ 4. $\begin{cases} y+3z=0 \\ x+2y+6z=2 \\ 7x+3y+9z=14 \end{cases}$

5. $\begin{cases} x+2y+z=2 \\ 2x+y+z=-1 \\ x-3y+2z=-1 \end{cases}$ 6. $\begin{cases} 2x-y+3z=1 \\ x+y-z=2 \\ x-2y+4z=1 \end{cases}$

7. $\begin{cases} 2x-y+3z=0 \\ x+y+2z=0 \end{cases}$ 8. $\begin{cases} x+y-z=1 \\ 2x+2y-2z=2 \\ -x-y+z=-1 \end{cases}$

Exercise 9

Without attempting to solve the following systems, discuss the nature of their solution set:

$$\begin{cases} x+y-z=0 \\ x-y=0 \\ x+y+z=0 \end{cases}, \begin{cases} x+3y+2z=1 \\ 2x-2y=2 \\ x+y+z=2 \end{cases} \text{ and } \begin{cases} x+3y+2z=1 \\ 2x-2y=2 \\ x+y+z=3 \end{cases}$$

Exercise 10

Discuss, depending on the value of m, the dimension of the solution space of the following systems:

1.
$$\begin{cases} x+my+z=0 \\ mx+y+mz=0 \end{cases}$$

2.
$$\begin{cases} x+y+mz=0 \\ x+my+z=0 \\ mx+y+z=0 \end{cases}$$

4.4 Solutions

Exercise 1

1. A column of Δ_1 is zero, so $\Delta_1 = 0$.
2. The first two columns of Δ_2 are proportional, so $\Delta_2 = 0$.
3. The first column of Δ_3 is the sum of the other two, so $\Delta_3 = 0$.
4. Δ_4 being triangular, the determinant Δ_4 is equal to the product of its diagonal terms. One

of these being zero, Δ_4 is also zero.

5.

$$\Delta_5 \overset{C_3 \to C_2+C_3}{=} \begin{vmatrix} 1 & a & a+b+c \\ 1 & b & a+b+c \\ 1 & c & a+b+c \end{vmatrix}$$

and the first and third columns of Δ_5 are proportional. It follows that $\Delta_5 = 0$.

6. The last column of Δ_6 is the sum of the other two, so $\Delta_6 = 0$.

Exercise 2

Easy by column permutations.

Exercise 3

Let Δ be this determinant. We have:

$$\begin{aligned} 1000\Delta &= \begin{vmatrix} 100 & 10 & 9 \\ 100 & 50 & 3 \\ 200 & 80 & 9 \end{vmatrix} \overset{C_1 \to C_1+C_2+C_3}{=} \begin{vmatrix} 119 & 10 & 9 \\ 153 & 50 & 3 \\ 289 & 80 & 9 \end{vmatrix} \\ &= 17 \begin{vmatrix} a & 10 & 9 \\ b & 50 & 3 \\ c & 80 & 9 \end{vmatrix} \end{aligned}$$

where a, b and c denote the quotient of $119, 153$ and 289 by 17, respectively. We obtain: $\Delta = 17m$ with

$$m = \begin{vmatrix} a & 10 & 9 \\ b & 50 & 3 \\ c & 80 & 9 \end{vmatrix}$$

which is an integer. Since 17 is coprime with 1000, applying Gauss's lemma, 17 divides Δ.

Exercise 4

The matrix of the family $\mathcal{P}$ in the canonical basis $(1, X, X^2)$ of $\mathbb{R}_2[X]$ is

$$M = \begin{pmatrix} a^2 & b^2 & c^2 \\ -2a & -2b & -2c \\ 1 & 1 & 1 \end{pmatrix}$$

Using Vandermonde determinants, we find

$$\det M = -2(b-a)(c-a)(c-b).$$

The family $\mathcal{P}$ forms a basis of $\mathbb{R}_2[X]$ if and only if the scalars a, b and c are pairwise distinct.

Exercise 5

We expand along the first row and recognize the addition formulas:

$$\begin{aligned}\Delta &= \cos(a-b)[\cos(b+c)\sin(c+a) - \cos(c+a)\sin(b+c)]\\ &- \cos(b-c)[\cos(a+b)\sin(c+a) - \cos(c+a)\sin(a+b)]\\ &+ \cos(c-a)[\cos(a+b)\sin(b+c) - \cos(b+c)\sin(a+b)]\\ &= \cos(a-b)\sin(a-b) + \cos(b-c)\sin(b-c)\\ &+ \cos(c-a)\sin(c-a) = \tfrac{1}{2}(\sin 2(a-b) + \sin 2(b-c) + \sin 2(c-a))\end{aligned}$$

then we use the two formulas

$$\sin p + \sin q = 2\sin\frac{p+q}{2}\cos\frac{p-q}{2}$$

and

$$\cos p - \cos q = -2\sin\frac{p+q}{2}\sin\frac{p-q}{2}:$$

$$\begin{aligned}\Delta &= \tfrac{1}{2}(2\sin(a-c)\cos(a+c-2b) + \sin 2(c-a))\\ &= \sin(a-c)\cos(a+c-2b) + \sin(c-a)\cos(c-a)\\ &= \sin(a-c)(\cos(a+c-2b) - \cos(c-a))\\ &= -2\sin(a-b)\sin(b-c)\sin(c-a)\end{aligned}$$

Exercise 6

Letting $e = (X^2, X, 1)$ be the canonical basis of $\mathbb{R}_2[X]$, we have:

$$\underset{e}{\mathrm{Mat}}(\mathcal{P}) = \begin{pmatrix} 2 & 1 & 1\\ -1 & 2 & 0\\ 1 & 0 & -1\end{pmatrix}$$

which is invertible. The family $\mathcal{P}$ is therefore a basis of $\mathbb{R}_2[X]$.

Exercise 7

The family e forms a basis of $\mathbb{R}^3$ if and only if

$$\begin{vmatrix} a & 1 & 1\\ 1 & a & 1\\ 1 & 1 & a\end{vmatrix} \neq 0.$$

This determinant is equal to: $(a-1)^2(a+2)$.

The family e is therefore free if and only if $a \neq 1$ and $a \neq -2$.

Exercise 8

1. Working backwards, we find successively: $z = 4; y = 2; x = -1$.

2.

$$\begin{cases} x - y + 2z = 1\\ 2x - 3y + z = 4\\ x - 3y - 4z = 5\end{cases} \iff \begin{cases} x - y + 2z = 1\\ -y - 3z = 2\\ -2y - 6z = 4\end{cases}$$

The last two equations are equivalent. The system is of rank 2 and consistent. Taking z as a parameter, the solution set is $\{(-5z-1,-3z-2,z) \mid z \in \mathbb{K}\}$.

3. Cramer's system: $\left\{\left(-\frac{5}{2},1,\frac{1}{2}\right)\right\}$.

4. System of rank 2 and consistent. $\{(2,-3z,z) \mid z \in \mathbb{K}\}$.

5. Cramer's system: $\{(-2,1,2)\}$.

6. System of rank 2 but not consistent. No solution.

7. $\begin{cases} 2x-y+3z=0 \\ x+y+2z=0 \end{cases}$ i.e., $\begin{cases} 2x-y+3z=0 \\ 3x+5z=0 \end{cases}$. The system is of rank 2, so consistent. Taking z as a parameter, the solution set is $\left\{\left(-\frac{5}{3}z,-\frac{1}{3}z,z\right) \mid z \in \mathbb{K}\right\}$.

8. The system is clearly of rank 1 and consistent (we have three times the same equation). The solution set is the plane with equation $x+y-z=1$.

Exercise 9

First system: Matrix of rank 3 (invertible) so a unique solution $(0,0,0)$.

Second system: Matrix of rank 2, system not consistent, no solution.

Third system: Matrix of rank 2, system not consistent, no solution.

Exercise 10

1. If $m=1$ or $m=-1$, then the system is of rank 1. The solution space is of dimension 2. Otherwise the system is of rank 2 and the solution space is of dimension 1. In the latter case, the solution set is $\{(x,0,-x), x \in \mathbb{R}\}$.

2. If $m=1$, then the system is of rank 1. The solution space is of dimension 2. If $m=-2$, then the system is of rank 2. The solution space is of dimension 1. Otherwise the system is a Cramer system. The solution space is of dimension 0.

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