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Handout title:

Quantum Field Theory I :
Course and Exercises

For students:

Level	First Year Master
Speciality	Theoretical Physics

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Preface

This handout is intended for first-year Master students, specializing in Theoretical Physics. This educational material contains the basic elements necessary for learning the subject "Quantum Field Theory I". In addition to a detailed course, the handout contains a series of corrected exercises, capable of introducing the student to the field of relativistic quantum mechanics. This work is the result of many years of research and preparation, lasting more than eight years of hard work. Teaching the quantum field theory course since the first year of the launch of the "Theoretical Physics" specialty within the Department of Material Sciences located at Djillali Bounaama Khemis Miliana University (UDBKM), has allowed me to have a broader vision of the subject, which has given me the opportunity to choose the most effective methods for transferring my knowledge to the students.

The handout is written in accordance with the template, it offers students the opportunity to deepen their previously acquired knowledge in the field of both quantum mechanics and special relativity theories, while taking into account the contributions of electromagnetism theory and analytical mechanics theory.

Title of the Master's Degree : Theoretical Physics

Semester : 1

Title of UE : Fundamental UEF1.1

Title of the subject : Quantum Field Theory I

Credits : 6

Coefficients : 3

Educational objectives:

- Understanding the concept of global and local symmetry in quantum field theory and their implications.

Recommended prior knowledge:

- Quantum mechanics, analytical mechanics, electromagnetism, and special relativity.

Preamble

2.1 Basic principle of quantum field theory

In a general case, a scalar field is associated with each particle that possesses zero spin. To characterize N particles, one defines N scalar fields. Consequently, the system comprising these N fields will be represented by a Lagrangian density of the following form,

$$\mathcal{L} = \mathcal{L}\left(\phi_1, \partial_\mu\phi_1, \phi_2, \partial_\mu\phi_2 \ldots \phi_N, \partial_\mu\phi_N, x_\mu\right) = \mathcal{L}\left(\phi_i, \partial_\mu\phi_i, x_\mu\right) \text{ avec } i = 1 \longrightarrow N \tag{2.1}$$

The motion of these N scalar fields will be described by the following N Euler-Lagrange equations,

$$\frac{\partial\mathcal{L}}{\partial\phi_i} - \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_i)}\right) = 0 \tag{2.2}$$

It is said that the scalar field $\phi(x_\mu)$ is a system with N degrees of freedom. According to its definition, the scalar field represents the most straightforward scenario. Its transformation occurs as follows,

$$\phi(x_\mu) = \phi'(x'_\mu) \tag{2.3}$$

- The scalar field (Klein-Gordon field) is used to describe the physics of zero-spin particles with relativistic speeds c.
- The scalar field can either be real $\phi(x_\mu) = \phi^*(x_\mu)$, or complex $\phi(x_\mu) \neq \phi^*(x_\mu)$.

2.1.1 Free scalar field

One possible form of the Lagrangian density that must be chosen to obtain the free Klein-Gordon equation is given by the following equation.

$$\left(\partial_\mu\partial_\mu - m^2\right)\phi(x_\mu) = 0 \tag{2.4}$$

Response: The selection is not singular. Our choice is as follows,

$$\mathcal{L}(\phi, \partial_\mu \phi, x_\mu) = -\frac{1}{2}\left(\partial_\mu \phi\right)^2 - \frac{1}{2} m^2 \phi^2 \tag{2.5}$$

Verification: Let us replace in the Euler-Lagrange equations, where $\phi_i = \phi = \phi^*$,

$$\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) = 0 \tag{2.6}$$

with $\dfrac{\partial \mathcal{L}}{\partial \phi} = -m^2 \phi, \quad \dfrac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} = -\partial_\mu \phi.$
By substituting into equation (5.3.2), we obtain the Klein-Gordon equation

$$\left(\partial_\mu \partial_\mu - m^2\right) \phi(x_\mu) = 0 \tag{2.7}$$

2.1.2 Free complex scalar field

If $\phi = \phi^*$, what is the general form of the Lagrangian density that must be selected in order to obtain the following two equations?

$$\left(\partial_\mu \partial_\mu - m^2\right) \phi(x_\mu) = 0 \ , \ \left(\partial_\mu \partial_\mu - m^2\right) \phi^*(x_\mu) = 0 \tag{2.8}$$

Response: Our choice is the following

$$\mathcal{L}(\phi, \partial_\mu \phi, \phi^*, \partial_\mu \phi^*, x_\mu) = -\left(\partial_\mu \phi\right)\left(\partial_\mu \phi^*\right) - m^2 \phi \phi^* \tag{2.9}$$

Verification: Let's substitute in both Euler-Lagrange equations for $\phi_i = \phi, \phi^*$,

$$\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) = 0, \quad \frac{\partial \mathcal{L}}{\partial \phi^*} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^*)} \right) = 0 \tag{2.10}$$

with $\dfrac{\partial \mathcal{L}}{\partial \phi^*} = -m^2 \phi, \quad \dfrac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^*)} = -\partial_\mu \phi, \quad \dfrac{\partial \mathcal{L}}{\partial \phi} = -m^2 \phi^*, \quad \dfrac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} = -\partial_\mu \phi^*.$
By substituting into equation (5.3.2), we obtain the following two equations,

$$\left(\partial_\mu \partial_\mu - m^2\right) \phi(x_\mu) = 0 \ , \ \left(\partial_\mu \partial_\mu - m^2\right) \phi^*(x_\mu) = 0 \tag{2.11}$$

2.1.3 Complex scalar field in the presence of an external electromagnetic field

What is the general form of the Lagrangian density that must be chosen to satisfy the following two equations?

$$\left[\left(\partial_\mu - iqA_\mu\right)\left(\partial_\mu - iqA_\mu\right) - m^2\right]\phi(x_\mu) = 0 \tag{2.12}$$

$$\left[\left(\partial_\mu + iqA_\mu\right)\left(\partial_\mu + iqA_\mu\right) - m^2\right]\phi^*(x_\mu) = 0 \tag{2.13}$$

Response: Our choice is as follows,

$$\mathcal{L}(\phi, \partial_\mu\phi, \phi^*, \partial_\mu\phi^*, x_\mu) = -\left(\partial_\mu + iqA_\mu\right)\phi^*\left(\partial_\mu - iqA_\mu\right)\phi - m^2\phi\phi^* \tag{2.14}$$

A review of quantum mechanics

3.1 Introduction

Several attempts were necessary before arriving at the current formulation of quantum mechanics. Specifically, in the mid-1920s, there were two competing approaches to model quantum phenomena: that of Heisenberg, Born, Jordan, and Dirac, called the matrix mechanics, and that of Schrödinger, called wave mechanics.
Before detailing these two theories, let us recall the essential points of classical mechanics (analytical mechanics). The latter is based on the Lagrangian formalism.

3.2 Recall the formalism of Lagrange

Lagrange's formalism is an extremely powerful tool for describing the evolution of a physical problem. Initially approached in the form of the principle of least action, it allows to determine the behavior of a system as soon as the expression of a physical quantity, the Lagrangian, is known. The aim of this reminder is to review the fundamental concepts of Lagrangian theory, first in the context of studying a massive particle, and then in the field theory.

3.2.1 Principle of least action

Given an initial state, a physical system has an infinite number of ways to evolve towards a final state:

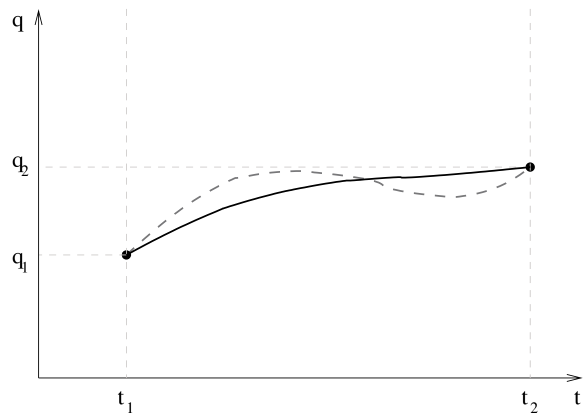


Figure 3.1: Conversion in the space of generalized coordinates

Therefore, during a real transformation, only one of these changes (evolutions) is actually carried out. How can we determine this preferred evolution and differentiate it from the others? This question is answered by the principle of least action, which can be considered as one of the postulates of physics.
According to the principle of least action, there exists a quantity called "Action" defined by,

$$S[q] = \int_{t_1}^{t_2} dt \; L(q_i(t), \dot{q}_i(t), t) \qquad , \qquad i = 1 \longrightarrow N \tag{3.1}$$

The value of the system changes during its evolution and must remain minimal throughout the actual transformation. The action S is defined as the integral of a quantity known as the "Lagrangian," which is a function of the generalized coordinates q and the generalized velocities $\dot{q}(t) = \frac{dq}{dt}$.

3.2.2 Euler-Lagrange equations

Among all the paths that connect the two fixed points ($\delta q(t_1) = \delta q(t_2) = 0$) with generalized coordinates $Q_1 = q(t_1)$ and $Q_2 = q(t_2)$, the physical trajectories are those that minimize the action S, such that $\Delta S \simeq 0$.

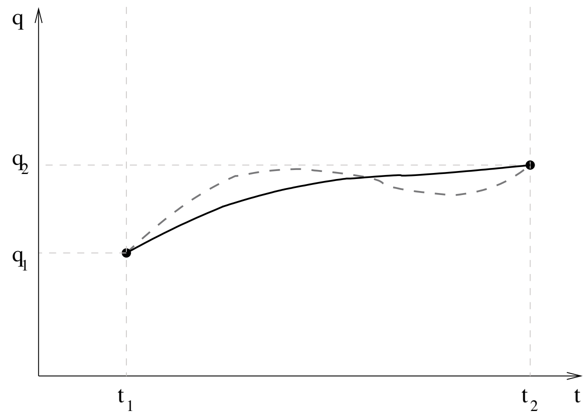


Figure 3.2: Transformation in the space of generalized coordinates

In case $\delta(q(t))$ is a infinitesimal function, then,

$$\Delta S[q] \simeq S(q+\delta q) - S(q) \tag{3.2}$$

On a

$$S[q] = \int_{t_1}^{t_2} dt\ L(q,\dot{q},t) \quad \Longrightarrow \quad \Delta S[q] = \int_{t_1}^{t_2} dt[\ L(q+\delta q, \dot{q}+\delta\dot{q}, t) -\ L(q,\dot{q},t)] \tag{3.3}$$

Or,

$$L(q+\delta q, \dot{q}+\delta\dot{q}, t) = L(q,\dot{q},t) + \frac{\partial L(q,\dot{q},t)}{\partial q}\delta q + \frac{\partial L(q,\dot{q},t)}{\partial \dot{q}}\delta\dot{q} \tag{3.4}$$

Therefore,

$$\begin{aligned} \Delta S[q] &= \int_{t_1}^{t_2} dt \left[L(q,\dot{q},t) + \frac{\partial L(q,\dot{q},t)}{\partial q}\delta q + \frac{\partial L(q,\dot{q},t)}{\partial \dot{q}}\delta\dot{q} -\ L(q,\dot{q},t) \right] \\ &= \int_{t_1}^{t_2} dt \left[\frac{\partial L(q,\dot{q},t)}{\partial q}\delta q + \frac{\partial L(q,\dot{q},t)}{\partial \dot{q}}\delta\dot{q} \right] \simeq 0 \end{aligned} \tag{3.5}$$

If we set that

$$\delta\dot{q} = \frac{d}{dt}(\delta q) \quad \Longrightarrow \quad \frac{\partial L(q,\dot{q},t)}{\partial \dot{q}}\delta\dot{q} = \frac{\partial L}{\partial \dot{q}}\frac{d}{dt}(\delta q) \tag{3.6}$$

We have also,

$$\frac{d}{dt}\left[\frac{\partial L}{\partial \dot{q}}\delta q\right] = \frac{d}{dt}\left[\frac{\partial L}{\partial \dot{q}}\right]\delta q + \frac{\partial L}{\partial \dot{q}}\frac{d}{dt}(\delta q) \quad \Longrightarrow \quad \frac{\partial L}{\partial \dot{q}}\frac{d}{dt}(\delta q) = \frac{d}{dt}\left[\frac{\partial L}{\partial \dot{q}}\delta q\right] - \frac{d}{dt}\left[\frac{\partial L}{\partial \dot{q}}\right]\delta q \tag{3.7}$$

By substituting into equation (3.5), we find,

$$\Delta S[q] = \int_{t_1}^{t_2} dt \left[\frac{\partial L}{\partial q}\delta q + \frac{d}{dt}\left[\frac{\partial L}{\partial \dot{q}}\delta q\right] - \frac{d}{dt}\left[\frac{\partial L}{\partial \dot{q}}\right]\delta q\right]$$
$$= \int_{t_1}^{t_2} dt\delta q \left[\frac{\partial L}{\partial q} - \frac{d}{dt}\left[\frac{\partial L}{\partial \dot{q}}\right]\right] + \int_{t_1}^{t_2} dt \frac{d}{dt}\left[\frac{\partial L}{\partial \dot{q}}\delta q\right] \simeq 0 \tag{3.8}$$

where

$$\int_{t_1}^{t_2} dt \frac{d}{dt}\left[\frac{\partial L}{\partial \dot{q}}\delta q\right] = \int_{t_1}^{t_2} d\left[\frac{\partial L}{\partial \dot{q}}\delta q\right] = \left[\frac{\partial L}{\partial \dot{q}}\delta q\right] = 0 \tag{3.9}$$

Finally, the Euler-Lagrange equations are expressed as

$$\frac{\partial L}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = 0 \tag{3.10}$$

3.2.3 Lagrangian selection

The choice of the Lagrangian is not unique.

- If we replace the Lagrangian L with (αL), where α is a real number, then the equations of motion remain unchanged.
- If we replace the Lagrangian L with $(\beta + L)$, where β is a constant, then the equations of motion remain unchanged.
- If we replace the Lagrangian L with $(L + \frac{dF}{dt})$, where $F = F(q, \dot{q}, t)$ is a function, then the equations of motion remain unchanged.

Exercice 1 :
Show that the variation ΔS remains invariant under the change of the Lagrangian L to $L + \frac{dF}{dt}$.

3.2.4 Hamiltonian formulation

The Hamiltonian H is given by

$$H(p, q, t) = P_i \dot{q}_i - L \tag{3.11}$$

The generalized momentum is given by

$$P_i = \frac{\partial L}{\partial \dot{q}_i} \tag{3.12}$$

Exercice 2 :
Show that if the Lagrangian L does not explicitly depend on time t, then $\frac{dH}{dt} = 0$.
Solution 3:

$$\frac{dH}{dt} = p\frac{\partial \dot{q}}{\partial t} + \dot{q}\frac{\partial p}{\partial t} - \frac{\partial L}{\partial q}\frac{\partial q}{\partial t} - \frac{\partial L}{\partial \dot{q}}\frac{\partial \dot{q}}{\partial t} \tag{3.13}$$

Or, we have

$$p = \frac{\partial L}{\partial \dot{q}} \quad \text{et} \quad \frac{\partial L}{\partial q} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}} = 0 \tag{3.14}$$

Therefore,

$$\frac{dH}{dt} = \left(\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q}\right)\frac{\partial q}{\partial t} = -\left(\frac{\partial L}{\partial q} - \frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{q}}\right)\right)\frac{\partial q}{\partial t} = 0 \tag{3.15}$$

3.3 Wave modeling

Due to the wave-like nature of matter, we need to take a closer look at what a wave is and the appropriate method to use to mathematically model its movement in spacetime. From a mathematical point of view, the dynamic of a wave can be described by solving the following wave equation:

$$\Box \phi = 0 \tag{3.16}$$

where the d'Alembertian operator is given by the expression

$$\Box := -\partial_{tt} + \Delta$$

A wave will then be modeled by a function ϕ, which is a solution of the equation (3.16). An obvious solution to the equation (3.16) is the function

$$\phi(x,t) = \phi_o e^{i(k\cdot x - \omega t)} \tag{3.17}$$

where x represents the position vector, t the time, k the wave vector (i.e., the wave propagation vector), ω is the wave frequency, and $x \cdot k$ is the dot product.

3.4 Schrödinger equation

The idea here is to model particles in the same way as waves, namely by a function ψ. The probability of finding the particle at time t is equal to

$$\int |\psi(x,t)|^2 dx. \tag{3.18}$$

This implies that

$$\int_{R^3} |\psi(x,t)|^2 dx = 1. \tag{3.19}$$

The fundamental principle of wave mechanics is stated as follows
The wave function ψ of a particle with mass m moving in vacuum and subjected to no interactions satisfies the Schrödinger equation

$$i\hbar\partial_t\psi = -\frac{\hbar^2}{2m}\Delta\psi \tag{3.20}$$

$\hbar$ is a universal constant known as the Planck constant, and Δ is a spatial Laplacian, with the following sign convention:

$$\Delta = \partial_{11} + \partial_{22} + \partial_{33}.$$

The Planck constant, denoted as $\hbar$, has dimensions of energy multiplied by time, or equivalently, momentum multiplied by length. Its value is expressed in Joule-seconds:

$$\hbar = 1,054571628 \times 10^{-34} \text{ J.s}$$

The wave function ψ of a particle placed in a potential $V(x,t)$ satisfies:

$$i\hbar\partial_t\psi = -\frac{\hbar^2}{2m}\Delta\psi + V\psi. \tag{3.21}$$

3.5 Harmonic oscillator

This section will be addressed as an exercise (see exercise 3).

3.6 Pauli equation

This section will be addressed as an exercise (see exercise 4).

3.7 Application exercises

Exercice 3 :

At time t_o, the state of the one-dimensional linear harmonic oscillator system is described by $\phi(x,0) = e^{a^+}\psi_0(x)$; where $\psi_n(x)$ are the eigenfunctions of $H_o = \hbar\omega(a^+a + \frac{1}{2})$ corresponding to the eigenvalues $E_n = \hbar\omega(n + \frac{1}{2})$, where n is an integer.

1. What is the normalized wave function at time t?
2. What is the probability of finding the energy E at time t?

Exercice 4 :

1. Using the product of Pauli matrices given by the formulae: $\sigma_i\sigma_j = \delta_{ij} + i\epsilon^{ijk}\sigma_k$, show that

$$(\overrightarrow{\sigma}\overrightarrow{A})(\overrightarrow{\sigma}\overrightarrow{B}) = \overrightarrow{A}\overrightarrow{B} + i\overrightarrow{\sigma}(\overrightarrow{A}\wedge\overrightarrow{B})$$

   when $\overrightarrow{A}$ and $\overrightarrow{B}$ commute with $\overrightarrow{\sigma}$.

2. Find the general form of the free Pauli equation.

A review of special relativity

4.1 Overview of the laws of electromagnetism

4.1.1 Maxwell equations

The laws of electromagnetism can be expressed as follows

- As a function of the electric field ($\overrightarrow{E}$) and the magnetic field ($\overrightarrow{B}$).
- As a function of the vector potential ($\overrightarrow{A}$) and scalar potential (ϕ).

Maxwell expressed the laws of electromagnetism in the form of the following four equations:

$$div \cdot \overrightarrow{D} = \rho \tag{4.1}$$

$$div \cdot \overrightarrow{B} = 0 \tag{4.2}$$

$$\overrightarrow{rot}\,\overrightarrow{H} = \overrightarrow{j} + \frac{\partial \overrightarrow{D}}{\partial t} \tag{4.3}$$

$$\overrightarrow{rot}\,\overrightarrow{E} = -\frac{\partial \overrightarrow{B}}{\partial t} \tag{4.4}$$

The equations (4.1), (4.3), (4.4) represent Gauss's law, Maxwell-Ampere's law and Lenz-Faraday's law respectively.

- $\overrightarrow{D}$ is the electrical displacement vector.
- $\overrightarrow{H}$ is the excitation field vector.
- ρ is an electrical charge density.
- $\overrightarrow{j}$ is an electric charge current.

These vectors are related to the electric field $\overrightarrow{E}$ and the magnetic field $\overrightarrow{B}$ by the next equations:

$$\overrightarrow{D} = \varepsilon \overrightarrow{E} \tag{4.5}$$

$$\overrightarrow{B} = \mu \overrightarrow{H} \tag{4.6}$$

Where ε is the dielectric permittivity and μ the magnetic permeability of the medium. In vacuum, the equations (4.5) and (4.6) become

$$\overrightarrow{D} = \varepsilon_o \overrightarrow{E} \tag{4.7}$$

$$\overrightarrow{B} = \mu_o \overrightarrow{H} \Longrightarrow \overrightarrow{H} = \frac{\overrightarrow{B}}{\mu_o} \tag{4.8}$$

Where ε_o and μ_o are two constants given respectively by: $\varepsilon_o = 8.854\,\mathrm{pF\,m^{-1}}$ and $\mu_0 = 4\pi \times 10^{-7}$ Henry/meter.
By introducing the operator $\overrightarrow{\nabla}$, the previous equations become:

$$\overrightarrow{\nabla} \cdot \overrightarrow{D} = \rho \tag{4.9}$$

$$\overrightarrow{\nabla} \cdot \overrightarrow{B} = 0 \tag{4.10}$$

$$\overrightarrow{\nabla} \wedge \overrightarrow{H} = \overrightarrow{j} + \frac{\partial \overrightarrow{D}}{\partial t} \tag{4.11}$$

$$\overrightarrow{\nabla} \wedge \overrightarrow{E} = -\frac{\partial \overrightarrow{B}}{\partial t} \tag{4.12}$$

By replacing the equations (4.7) and (4.8) in the equations (4.9), (4.10), (4.11) and (4.12), we find:

$$\overrightarrow{\nabla} \cdot (\epsilon_o \overrightarrow{E}) = \rho \Longrightarrow \overrightarrow{\nabla} \cdot \overrightarrow{E} = \frac{\rho}{\epsilon_o} \tag{4.13}$$

$$\overrightarrow{\nabla} \cdot \overrightarrow{B} = 0 \tag{4.14}$$

$$\overrightarrow{\nabla} \wedge \left(\frac{\overrightarrow{B}}{\mu_o}\right) = \overrightarrow{j} + \frac{\partial}{\partial t}\left(\overrightarrow{\epsilon_o E}\right) \Longrightarrow \overrightarrow{\nabla} \wedge \overrightarrow{B} = \mu_o \overrightarrow{j} + \mu_o \epsilon_o \frac{\partial \overrightarrow{E}}{\partial t} \tag{4.15}$$

$$\overrightarrow{\nabla} \wedge \overrightarrow{E} = -\frac{\partial \overrightarrow{B}}{\partial t} \tag{4.16}$$

The Lorentz force acting on a particle with charge q and velocity $overrightarrowV$ is given by

$$\overrightarrow{F} = q(\overrightarrow{E} + \overrightarrow{V} \wedge \overrightarrow{B}) \tag{4.17}$$

The charge conservation equation is given by,

$$\overrightarrow{\nabla} \cdot \overrightarrow{j} + \frac{\partial \rho}{\partial t} = 0 \tag{4.18}$$

Ramarque:

- Maxwell's equations and the equation for the conservation of electric charge are valid at all points in the medium and at all times. They are, therefore, local equations.

4.1.2 Vector and scalar potentials

The magnetic field $\overrightarrow{B}$ and the electric field $\overrightarrow{E}$ are derived from the Lorentz potentials $\overrightarrow{A}$ and ϕ, where

$$\overrightarrow{E} = -\overrightarrow{grad}\phi - \frac{\partial \overrightarrow{A}}{\partial t} \tag{4.19}$$

$$\overrightarrow{B} = \overrightarrow{rot}\,\overrightarrow{A} \tag{4.20}$$

The latter equations can be rewritten in terms of $\overrightarrow{\nabla}$,

$$\overrightarrow{E} = -\overrightarrow{\nabla} \cdot \phi - \frac{\partial \overrightarrow{A}}{\partial t} \tag{4.21}$$

$$\overrightarrow{B} = \overrightarrow{\nabla} \wedge \overrightarrow{A} \tag{4.22}$$

Ramarque:

In vacuum, the potential vectors $\overrightarrow{A}$ and scalar ϕ satisfy the following equation:

$$div\overrightarrow{A} + \mu_o \epsilon_o \frac{\partial \phi}{\partial t} = 0 \tag{4.23}$$

This equation is known as the Lorentz Gauge. This equation can be written as a function of $\overrightarrow{\nabla}$,

$$\overrightarrow{\nabla} \cdot \overrightarrow{A} + \mu_o \epsilon_o \frac{\partial \phi}{\partial t} = 0 \tag{4.24}$$

We have,

$$\mu_o \epsilon_o c^2 = 1 \Longrightarrow c = 1/\sqrt{\mu_o \epsilon_o} \tag{4.25}$$

Using Maxwell's equations and the Lorentz gauge, we obtain:

$$\Delta \phi - \frac{1}{c^2} \frac{\partial^2 \phi}{\partial^2 t} = 0 \tag{4.26}$$

$$\Delta \overrightarrow{A} - \frac{1}{c^2} \frac{\partial^2 \overrightarrow{A}}{\partial^2 t} = 0 \tag{4.27}$$

Solving these two equations gives the values of the potentials ϕ and $\overrightarrow{A}$.

4.2 Vector analysis in Minkowski space

The "quadi-nabla" operator is introduced into Minkowski's four-dimensional space and defined as follows:

$$\overrightarrow{\partial} = \left(\overrightarrow{\nabla}, -\frac{1}{c} \frac{\partial}{\partial t} \right) \tag{4.28}$$

of components,

$$\partial_1 = \frac{\partial}{\partial x}, \quad \partial_2 = \frac{\partial}{\partial y}, \quad \partial_3 = \frac{\partial}{\partial z}, \quad \partial_4 = -\frac{1}{c} \frac{\partial}{\partial t} \tag{4.29}$$

4.2.1 Quadri-divergence and quadri-gradient

Let be the quadri-vector $\overrightarrow{A}$, with components:

$$\overrightarrow{A} = (a_x, a_y, a_z, a_4) = (\overrightarrow{a}, a_4) \quad \text{où} \quad \overrightarrow{a} = (a_x, a_y, a_z) \tag{4.30}$$

The metric of the Minkowski space is given by $(+,+,+,-)$. So the scalar product of two quadri-vectors $\overrightarrow{A}$ and $\overrightarrow{B}$ is given by

$$\overrightarrow{A}\cdot\overrightarrow{B}=\begin{pmatrix}a_x\\a_y\\a_z\\a_4\end{pmatrix}\cdot\begin{pmatrix}b_x\\b_y\\b_z\\b_4\end{pmatrix}=+a_xb_x+a_yb_y+a_zb_z-a_4b_4 \tag{4.31}$$

The quadri-divergence of a quadri-vector $\overrightarrow{V}$ is given by

$$\overrightarrow{\partial}\cdot\overrightarrow{V}=\begin{pmatrix}\frac{\partial}{\partial x}\\\frac{\partial}{\partial y}\\\frac{\partial}{\partial z}\\-\frac{1}{c}\frac{\partial}{\partial t}\end{pmatrix}\cdot\begin{pmatrix}v_x\\v_y\\v_z\\v_4\end{pmatrix}=+\frac{\partial v_x}{\partial x}+\frac{\partial v_y}{\partial y}+\frac{\partial v_z}{\partial z}+\frac{1}{c}\frac{\partial v_4}{\partial t} \tag{4.32}$$

which can also be written as,

$$\overrightarrow{\partial}\cdot\overrightarrow{V}=\left(\overrightarrow{\partial},-\frac{1}{c}\frac{\partial}{\partial t}\right)\cdot(\overrightarrow{v},v_4)=\left(\overrightarrow{\partial}\,\overrightarrow{v},-\frac{1}{c}\frac{\partial v_4}{\partial t}\right) \tag{4.33}$$

In the same way, we define the quadri-gradient of a ϕ scalar function as,

$$\overrightarrow{\partial}\phi=\left(\overrightarrow{\partial}\phi,-\frac{1}{c}\frac{\partial\phi}{\partial t}\right) \tag{4.34}$$

4.2.2 Quad-vector current density

The equation (4.18) expresses the principle of conservation of charge. This equation can be written as

$$\frac{\partial\rho}{\partial t}+\overrightarrow{\nabla}\cdot\overrightarrow{j}=0\Longrightarrow\frac{\partial j_x}{\partial x}+\frac{\partial j_y}{\partial y}+\frac{\partial j_z}{\partial z}+\frac{1}{c}\frac{\partial}{\partial t}(c\rho)=0 \tag{4.35}$$

which can be written as:

$$\frac{\partial j_x}{\partial x}+\frac{\partial j_y}{\partial y}+\frac{\partial j_z}{\partial z}-\left(-\frac{1}{c}\frac{\partial}{\partial t}\right)(c\rho)=0\Longrightarrow\left(\overrightarrow{\partial}\,\overrightarrow{j},-\frac{1}{c}\frac{\partial}{\partial t}(\rho c)\right)=0 \tag{4.36}$$

This equation appears as the quadri-divergence of a quadri-vector

$$\left(\overrightarrow{\partial}, -\frac{1}{c}\frac{\partial}{\partial t}\right)\left(\overrightarrow{j}, \rho c\right) = 0 \Longrightarrow \overrightarrow{\partial}\,\overrightarrow{J} = 0 \tag{4.37}$$

Equation (4.37) represents the writing of the charge conservation equation in Minkowski space and the current quadri-vector is given by,

$$\overrightarrow{J} = \left(\overrightarrow{j}, \rho c\right) \tag{4.38}$$

4.2.3 Quad-vector potential

The Lorentz gauge given in the equation (**??**) can be rewritten as follows,

$$\overrightarrow{\nabla}\cdot\overrightarrow{A} + \frac{1}{c^2}\frac{\partial\phi}{\partial t} = 0 \Longrightarrow \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} + \frac{1}{c}\frac{\partial}{\partial t}\left(\frac{\phi}{c}\right) = 0 \tag{4.39}$$

which can be written as:

$$\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} - \left(-\frac{1}{c}\frac{\partial}{\partial t}\right)\left(\frac{\phi}{c}\right) = 0 \Longrightarrow \left(\overrightarrow{\partial}\,\overrightarrow{A}, -\frac{1}{c}\frac{\partial}{\partial t}\frac{\phi}{c}\right) = 0 \tag{4.40}$$

This equation appears as the quadri-divergence of a quadri-vector

$$\left(\overrightarrow{\partial}, -\frac{1}{c}\frac{\partial}{\partial t}\right)\left(\overrightarrow{A}, \frac{\phi}{c}\right) = 0 \Longrightarrow \overrightarrow{\partial}\,\overrightarrow{\mathbf{A}} = 0 \tag{4.41}$$

Equation (4.41) represents the writing of the Lorentz gauge in Minkowski space and the potential quadri-vector is given by,

$$\overrightarrow{\mathbf{A}} = \left(\overrightarrow{A}, \frac{\phi}{c}\right) \tag{4.42}$$

4.2.4 Electromagnetic field tensor

The fields $\overrightarrow{E}$ and $\overrightarrow{B}$ are given as functions of the potentials ϕ and $\overrightarrow{A}$ by the two equations (4.21) and (4.22).

Writing the equation (4.21) in three-dimensional space gives:

$$\overrightarrow{E} = -\overrightarrow{\nabla}\phi - \frac{\partial\overrightarrow{A}}{\partial t} \Longrightarrow \tag{4.43}$$

$$E_x = -\frac{\partial \phi}{\partial x} - \frac{\partial A_x}{\partial t} \tag{4.44}$$

$$E_y = -\frac{\partial \phi}{\partial y} - \frac{\partial A_y}{\partial t} \tag{4.45}$$

$$E_z = -\frac{\partial \phi}{\partial z} - \frac{\partial A_z}{\partial t} \tag{4.46}$$

These last equations can be rewritten in the following form:

$$E_x = -c\frac{\partial}{\partial x}\left(\frac{\phi}{c}\right) - c\frac{1}{c}\frac{\partial A_x}{\partial t} \tag{4.47}$$

$$E_y = -c\frac{\partial}{\partial y}\left(\frac{\phi}{c}\right) - c\frac{1}{c}\frac{\partial A_y}{\partial t} \tag{4.48}$$

$$E_z = -c\frac{\partial}{\partial z}\left(\frac{\phi}{c}\right) - c\frac{1}{c}\frac{\partial A_z}{\partial t} \tag{4.49}$$

Now, taking c as a factor, we find:

$$\frac{E_x}{c} = -\frac{\partial}{\partial x}\left(\frac{\phi}{c}\right) - \frac{1}{c}\frac{\partial A_x}{\partial t} \tag{4.50}$$

$$\frac{E_y}{c} = -\frac{\partial}{\partial y}\left(\frac{\phi}{c}\right) - \frac{1}{c}\frac{\partial A_y}{\partial t} \tag{4.51}$$

$$\frac{E_z}{c} = -\frac{\partial}{\partial z}\left(\frac{\phi}{c}\right) - \frac{1}{c}\frac{\partial A_z}{\partial t} \tag{4.52}$$

Writing the equation (4.22) in three-dimensional space gives:

$$\overrightarrow{B} = \overrightarrow{\nabla} \wedge \overrightarrow{A} \Longrightarrow \tag{4.53}$$

$$B_x = \frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} \tag{4.54}$$

$$B_y = -\left[\frac{\partial A_z}{\partial x} - \frac{\partial A_x}{\partial z}\right] \Longrightarrow B_y = \frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} \tag{4.55}$$

$$B_z = \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \tag{4.56}$$

4.2.5 Change of variable

A point in Minkowski space is represented by the quadri-vector position

$$\begin{pmatrix} x \\ y \\ z \\ -ct \end{pmatrix} \tag{4.57}$$

Let's make the following change of variables:

$$\begin{cases} x = x_1 \\ y = x_2 \\ z = x_3 \\ -ct = x_4 \end{cases} \Longrightarrow \begin{cases} \frac{\partial}{\partial x} = \frac{\partial}{\partial x_1} \\ \frac{\partial}{\partial y} = \frac{\partial}{\partial x_2} \\ \frac{\partial}{\partial z} = \frac{\partial}{\partial x_3} \\ -\frac{1}{c}\frac{\partial}{\partial t} = \frac{\partial}{\partial x_4} \end{cases} \Longrightarrow \begin{cases} \frac{\partial}{\partial x_1} = \partial_1 \\ \frac{\partial}{\partial x_2} = \partial_2 \\ \frac{\partial}{\partial x_3} = \partial_3 \\ \frac{\partial}{\partial x_4} = \partial_4 \end{cases} \tag{4.58}$$

$$\begin{cases} A_x = A_1 \\ A_y = A_2 \\ A_z = A_3 \\ \frac{\phi}{c} = A_4 \end{cases} \tag{4.59}$$

The equations (4.50), (4.51), (4.52), (4.54), (4.55) and (4.56) become,

$$\frac{E_x}{c} = -\partial_1 A_4 + \partial_4 A_1 \tag{4.60}$$

$$\frac{E_y}{c} = -\partial_2 A_4 + \partial_4 A_2 \tag{4.61}$$

$$\frac{E_z}{c} = -\partial_3 A_4 + \partial_4 A_3 \tag{4.62}$$

$$B_x = \partial_2 A_3 - \partial_3 A_2 \tag{4.63}$$

$$B_y = \partial_3 A_1 - \partial_1 A_3 \tag{4.64}$$

$$B_z = \partial_1 A_2 - \partial_2 A_1 \tag{4.65}$$

These six equations can be written in the following general form

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \qquad \mu, \nu = 1, 2, 3, 4 \tag{4.66}$$

Where the coefficients $F_{\mu\nu}$ are the matrix elements of a tensor in Minkowski space, called the "electromagnetic field tensor" and given by,

$$F^{\mu\nu} = \begin{pmatrix} F_{11} & F_{12} & F_{13} & F_{14} \\ F_{21} & F_{22} & F_{23} & F_{24} \\ F_{31} & F_{32} & F_{33} & F_{34} \\ F_{41} & F_{42} & F_{43} & F_{44} \end{pmatrix} \tag{4.67}$$

The tensor is antisymmetric $F^{\mu} = -F^{\mu}$ and $F^{\mu} = 0$. Therefore, the matrix elements of the electromagnetic field tensor are given by,

$$\frac{E_x}{c} = -\partial_1 A_4 + \partial_4 A_1 = F_{41} = -F_{14} \tag{4.68}$$

$$\frac{E_y}{c} = -\partial_2 A_4 + \partial_4 A_2 = F_{42} = -F_{24} \tag{4.69}$$

$$\frac{E_z}{c} = -\partial_3 A_4 + \partial_4 A_3 = F_{43} = -F_{34} \tag{4.70}$$

$$B_x = \partial_2 A_3 - \partial_3 A_2 = F_{23} = -F_{32} \tag{4.71}$$

$$B_y = \partial_3 A_1 - \partial_1 A_3 = F_{31} = -F_{13} \tag{4.72}$$

$$B_z = \partial_1 A_2 - \partial_2 A_1 = F_{12} = -F_{21} \tag{4.73}$$

Finally, the electromagnetic field tensor is given by,

$$F^{\mu\nu} = \begin{pmatrix} 0 & B_z & -B_y & -\frac{E_x}{c} \\ -B_z & 0 & B_x & -\frac{E_y}{c} \\ B_y & -B_x & 0 & -\frac{E_z}{c} \\ \frac{E_x}{c} & \frac{E_y}{c} & \frac{E_z}{c} & 0 \end{pmatrix} \tag{4.74}$$

4.3 Exercises

Exercice 5 :

The electric and magnetic fields $\overrightarrow{E_1}$ and $\overrightarrow{B_1}$, measured by an observer **O** linked to a Galilean reference frame **R**, are given in terms of the scalar and vector potentials $\phi_1, \overrightarrow{A_1}$ by the equations

$$\overrightarrow{E_1} = -\overrightarrow{\mathbf{grad}}\,\phi_1 - \frac{\partial \overrightarrow{A_1}}{\partial t}, \quad \overrightarrow{B_1} = \overrightarrow{\mathbf{rot}}\overrightarrow{A_1}$$

1. Give the expression for the components of the fields $\overrightarrow{E_1}$ and $\overrightarrow{B_1}$ in the reference frame **R**.
2. Find the components of the electromagnetic tensor.
3. What are the new values of the fields $\overrightarrow{E_1}'$ and $\overrightarrow{B_1}'$, measured by an observer $\mathbf{O}'$ linked to a Galilean reference frame $\mathbf{R}'$ moving at a constant speed $\overrightarrow{\mathbf{v}}$ relative to **R**?

Exercice 6 :

- Find the probability current of the Schrodinger equation $\overrightarrow{j}$ which verifies the equation

$$\frac{\partial \rho}{\partial t} + \overrightarrow{\nabla}\overrightarrow{j} = 0$$

We give : $\rho = \psi^*(\vec{r}, t)\psi(\vec{r}, t)$

Symmetry and invariance

5.1 Definition

A physical law is said to be invariant when it remains unchanged by a change of coordinates and variables.

Example:

In classical mechanics:

- The coordinates are represented by: $\overrightarrow{r}$, t,$\cdots$.
- The variables are represented by: $\overrightarrow{r}(t)$, $\overrightarrow{p}(t)$, $\cdots$.

In quantum mechanics:

- The coordinates are represented by: $\left(\overrightarrow{r}, t\right)$, $\cdots$.
- The variables are represented by: $\psi\left(\overrightarrow{r}, t\right)$, $\psi(t)$, $\cdots$.

In analytical mechanics:

- The coordinates are represented by: $q(t)$, $p(t)$ $\cdots$.
- The variables are represented by: $\dot{q}(t) = -\frac{\partial H}{\partial p_i}$, $\dot{p}(t) = -\frac{\partial H}{\partial q_i}$, $\cdots$.

5.2 Types of transformations

There are two kinds of transformation:

5.2.1 Geometric transformations

The geometric transformations that exist are:

- Moving in space.
- Moving in time.
- Rotation.
- Time reversal T.
- Inversion of the origin P.

5.2.2 Internal transformations

A particle can undergo the following internal transformations:

- Interchanging identical particles.
- Interchanging particles and anti-particles. This transformation is often called "charge conjugation", which is denoted C.

Remarque:

The three transformations C, P, T are discrete transformations.

5.2.3 Internal geometric transformations

For this type of transformation, we can cite the Galilean transformation, given by

$$\begin{cases} \overrightarrow{r} \rightarrow \overrightarrow{r}' = \overrightarrow{r} + \overrightarrow{v} t \\ t \rightarrow t' = t \end{cases} \tag{5.1}$$

5.3 Symmetries and conservation laws

In this section, it will be assumed that the Lagrangian density does not depend explicitly on (x_μ). It will also be assumed that the equations of motion (and hence the action) remain unchanged during an infinitesimal (continuous) transformation defined by,

$$\begin{cases} x_\mu \longrightarrow x'_\mu = x_\mu + \delta x_\mu \\ \phi(x_\mu) \longrightarrow \phi'(x'_\mu) = \phi(x_\mu) + \delta\phi(x_\mu) \end{cases} \tag{5.2}$$

with,

$$\begin{cases} x_\mu \longrightarrow \text{position spatio-temporelle (coordonnées)} \\ \delta x_\mu \longrightarrow \text{variation infinitisimale (deplacement l'espace et dans le temps)} \\ \phi(x_\mu) \longrightarrow \text{champ scalaire (variable)} \\ \delta\phi(x_\mu) \longrightarrow \text{variation de phase (dûe à une rotation)} \end{cases}$$

5.3.1 Example of transformation

Space-time transformation

A space-time transformation is defined by

$$\begin{cases} x_\mu \longrightarrow x'_\mu = x_\mu + a_\mu \ , \quad (a_\mu = \delta x_\mu) \\ \phi(x_\mu) \longrightarrow \phi'(x'_\mu) = \phi(x_\mu) \ , \quad (\delta\phi(x_\mu) = 0) \end{cases} \tag{5.3}$$

Where a_μ represents the quadri-vector displacement in space-time.
According to the infinitesimal transformation given in equation (5.3),

$$\phi'(x'_\mu) = \phi'(x_\mu + a_\mu) = \phi(x_\mu) \tag{5.4}$$

therefore;

$$\phi'(x_\mu + a_\mu) = \phi(x_\mu) \tag{5.5}$$

Global phase transformation $(\phi(x_\mu) \neq \phi^*(x_\mu))$

This transformation is given by,

$$\begin{cases} x_\mu \longrightarrow x'_\mu = x_\mu \ , \quad (\delta x_\mu = 0) \\ \phi(x_\mu) \longrightarrow \phi'(x'_\mu) = \phi(x_\mu) + \delta\phi(x_\mu) = e^{-iq\theta(x_\mu)}\phi(x_\mu) \end{cases} \tag{5.6}$$

Where $theta(x_\mu)$ is a real scalar.
According to the infinitesimal transformation given in equation (5.6),

$$\phi'(x'_\mu) = \phi'(x_\mu) = \phi(x_\mu) + \delta\phi(x_\mu) = e^{-iq\theta(x_\mu)}\phi(x_\mu) \tag{5.7}$$

therefor,

$$\phi'^*(x_\mu) = e^{+iq\theta(x_\mu)}\phi^*(x_\mu) \tag{5.8}$$

Local phase transformation $(\phi(x_\mu) = \phi^*(x_\mu))$

This transformation is given by,

$$\begin{cases} x_\mu \longrightarrow x'_\mu = x_\mu \ , \quad (\delta x_\mu = 0) \\ \phi(x_\mu) \longrightarrow \phi'(x'_\mu) = \phi(x_\mu) + \delta\phi(x_\mu) = e^{-iq\theta(x_\mu)}\phi(x_\mu) \end{cases} \tag{5.9}$$

Where $theta(x_\mu)$ is a real scalar.
According to the infinitesimal transformation given in equation (5.9),

$$\phi'(x'_\mu) = \phi'(x_\mu) = \phi(x_\mu) + \delta\phi(x_\mu) = e^{-iq\theta(x_\mu)}\phi(x_\mu) \tag{5.10}$$

therefor,

$$\phi'^*(x_\mu) = e^{+iq\theta(x_\mu)}\phi(x_\mu) \tag{5.11}$$

5.3.2 Noether's theorem

Statement

For any continuous transformation of the action S, there is a current J_μ satisfying the equation

$$\partial_\mu J_\mu = 0 \tag{5.12}$$

This implies that there is a self-preserving charge, defined by

$$Q = \int \rho \, d^3x \tag{5.13}$$

Demonstration

The equations of motion are said to be invariant if the action S is stationary.

$$\delta S = S' - S \simeq 0 \tag{5.14}$$

We have

$$S = \int d^4x \, \mathcal{L}(\phi, \partial_\mu\phi) \Rightarrow S' = \int d^4x' \, \mathcal{L}(\phi', \partial'_\mu\phi') \tag{5.15}$$

Given $\mathcal{L} = \mathcal{L}(\phi, \partial_\mu \phi)$ (where the Lagrangian density does not have explicit dependence on x_μ). Let us consider infinitesimal transformations of the form,

$$\begin{cases} x_\mu \longrightarrow x'_\mu = x_\mu + \delta x_\mu \\ \phi(x_\mu) \longrightarrow \phi'(x'_\mu) = \phi(x_\mu) + \delta\phi(x_\mu) \end{cases} \tag{5.16}$$

where

$$\delta\phi(x) = \phi'(x') - \phi(x) \tag{5.17}$$

The symbol $\delta\phi(x_\mu)$ represents the variation of the field due to both the transformation of the field (variable) and the transformation of the coordinates (x_μ).
Thus, the change at a specific point in 4-dimensional space is determined by

$$\delta_o\phi(x) = \phi'(x) - \phi(x) \ , \quad \text{pour } x' = x \tag{5.18}$$

The relationship between the spacetime derivatives is expressed by

$$d^4x' = [1 + \partial_\mu(\delta x_\mu)]d^4x \tag{5.19}$$

Let's now examine the relationship between the field variation at two different points $\delta\phi$ and the field variation at a fixed point $\delta_o\phi$.
The variation of the field at two different points is given by

$$\delta\phi(x) = \phi'(x') - \phi(x) = \phi'(x') - \phi'(x) + \phi'(x) - \phi(x) \tag{5.20}$$

$$\delta\phi(x) = \phi'(x) + (\partial_\nu\phi)\delta x_\nu - \phi'(x) + \delta_o\phi(x) \tag{5.21}$$

with

$$\phi'(x') = \phi'(x_\mu + \delta x_\mu) = \phi'(x_\mu) + (\partial_\nu\phi)\delta x_\nu = \phi'(x) + (\partial_\nu\phi)\delta x_\nu \tag{5.22}$$

Therefor,

$$\delta\phi(x) = \delta_o\phi(x) + (\partial_\nu\phi)\delta x_\nu \tag{5.23}$$

Let us calculate the term $\partial'_\mu\phi'$
We have

$$\partial'_\mu\phi'(x') = \partial'_\mu(\phi + \delta\phi) = \frac{\partial}{\partial x'_\mu}(\phi + \delta\phi) \tag{5.24}$$

$$= \frac{\partial}{\partial x_\nu}\frac{\partial x_\nu}{\partial x'_\mu}(\phi+\delta\phi) = \frac{\partial}{\partial x_\nu}(\phi+\delta\phi)\frac{\partial x_\nu}{\partial x'_\mu} \tag{5.25}$$

We have also

$$x'_\nu = x_\nu + \delta x_\nu \Rightarrow x_\nu = x'_\nu - \delta x_\nu \tag{5.26}$$

Therefor

$$\frac{\partial x_\nu}{\partial x'_\mu} = \frac{\partial x'_\nu}{\partial x'_\mu} + \frac{\partial}{\partial x'_\mu}(\delta x_\nu) \tag{5.27}$$

Finally, we get

$$\frac{\partial x_\nu}{\partial x'_\mu} = \delta_{\mu\nu} - \partial_\mu(\delta x_\nu) \tag{5.28}$$

By substituting the equation (**??**) into equation (5.24), we obtain

$$\partial'_\mu\phi'(x') = \frac{\partial}{\partial x_\nu}(\phi+\delta\phi)\frac{\partial x_\nu}{\partial x'_\mu} \tag{5.29}$$

$$= \left(\frac{\partial\phi}{\partial x_\nu} + \frac{\partial}{\partial x_\nu}(\delta\phi)\right)(\delta_{\mu\nu} - \partial_\mu(\delta x_\nu)) \tag{5.30}$$

$$= (\partial_\nu\phi + \partial_\nu(\delta\phi))(\delta_{\mu\nu} - \partial_\mu(\delta x_\nu)) \tag{5.31}$$

$$= (\partial_\nu\phi)\delta_{\mu\nu} - (\partial_\nu\phi)\partial_\mu(\delta x_\nu) + \partial_\nu(\delta\phi)\delta_{\mu\nu} - \partial_\nu(\delta\phi)\partial_\mu(\delta x_\nu) \tag{5.32}$$

$$\partial'_\mu\phi'(x') = (\partial_\mu\phi) - (\partial_\nu\phi)\partial_\mu(\delta x_\nu) + \partial_\mu(\delta\phi) \tag{5.33}$$

The term $\partial_\nu(\delta\phi)\partial_\mu(\delta x_\nu)$ is neglected, as it is a higher-order term.
The Lagrangian density does not explicitly depend on x_μ, which implies that $\mathcal{L} = \mathcal{L}(\phi, \partial_\mu\phi)$.
Therefor,

$$\mathcal{L}(\phi', \partial'_\mu\phi') = \mathcal{L}(\phi+\delta\phi, (\partial_\mu\phi) - (\partial_\nu\phi)\partial_\mu(\delta x_\nu) + \partial_\mu(\delta\phi)) \tag{5.34}$$

$$= \mathcal{L}(\phi, \partial_\mu\phi) + \frac{\partial\mathcal{L}}{\partial\phi}\delta\phi + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}[\partial_\mu(\delta\phi) - (\partial_\nu\phi)\partial_\mu(\delta x_\nu)] \tag{5.35}$$

we get

$$\mathcal{L}(\phi', \partial'_\mu\phi') = \mathcal{L}(\phi, \partial_\mu\phi) + \frac{\partial\mathcal{L}}{\partial\phi}\delta\phi + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\partial_\mu(\delta\phi) - \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}(\partial_\nu\phi)\partial_\mu(\delta x_\nu) \tag{5.36}$$

By substituting the equation (5.19) into the equation (5.14), one arrives at the following result

$$\delta S = \int d^4x' \,\mathcal{L}(\phi', \partial'_\mu \phi') - \int d^4x \,\mathcal{L}(\phi, \partial_\mu \phi) \simeq 0 \tag{5.37}$$

$$= \int [1 + \partial_\mu (\delta x_\mu)] d^4x \,\mathcal{L}(\phi', \partial'_\mu \phi') - \int d^4x \,\mathcal{L}(\phi, \partial_\mu \phi) \simeq 0 \tag{5.38}$$

$$\delta S = \int [\mathcal{L}(\phi', \partial'_\mu \phi') - \mathcal{L}(\phi, \partial_\mu \phi) + \partial_\mu (\delta x_\mu) \mathcal{L}] d^4x \simeq 0 \tag{5.39}$$

Let us calculate the following term: $\mathcal{L}(\phi', \partial'_\mu \phi') - \mathcal{L}(\phi, \partial_\mu \phi)$

$$\mathcal{L}(\phi', \partial'_\mu \phi') - \mathcal{L}(\phi, \partial_\mu \phi) = \mathcal{L}(\phi, \partial_\mu \phi) + \frac{\partial \mathcal{L}}{\partial \phi} \delta\phi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial_\mu (\delta\phi) - \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} (\partial_\nu \phi) \partial_\mu (\delta x_\nu) - \mathcal{L}(\phi, \partial_\mu \phi)$$

$$\mathcal{L}(\phi', \partial'_\mu \phi') - \mathcal{L}(\phi, \partial_\mu \phi) = \frac{\partial \mathcal{L}}{\partial \phi} \delta\phi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial_\mu (\delta\phi) - \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} (\partial_\nu \phi) \partial_\mu (\delta x_\nu) \tag{5.40}$$

According to the Euler-Lagrange equations,

$$\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) = 0$$

then

$$\frac{\partial \mathcal{L}}{\partial \phi} = \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) \tag{5.41}$$

We have also

$$\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \delta\phi \right) = \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) \delta\phi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial_\mu (\delta\phi)$$

Therefor

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial_\mu (\delta\phi) = \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \delta\phi \right) - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) \delta\phi \tag{5.42}$$

By replacing equations (5.41) and (5.42) in equation (5.40), we obtain

$$\mathcal{L}(\phi', \partial'_\mu \phi') - \mathcal{L}(\phi, \partial_\mu \phi) = \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) \delta\phi + \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \delta\phi \right) - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) \delta\phi - \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} (\partial_\nu \phi) \partial_\mu (\delta x_\nu)$$

$$\mathcal{L}(\phi', \partial'_\mu \phi') - \mathcal{L}(\phi, \partial_\mu \phi) = \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \delta\phi \right) - \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} (\partial_\nu \phi) \partial_\mu (\delta x_\nu) \tag{5.43}$$

We have

$$\delta\phi = \delta_o\phi + (\partial_\nu\phi)\delta x_\nu$$

Then

$$\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\delta\phi\right) = \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}(\delta_o\phi + (\partial_\nu\phi)(\delta x_\nu))\right)$$

$$\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\delta\phi\right) = \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\delta_o\phi\right) + \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}(\partial_\nu\phi)(\delta x_\nu)\right) \tag{5.44}$$

Let us calculate the term $\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}(\partial_\nu\phi)(\delta x_\nu)\right)$:

$$\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}(\partial_\nu\phi)(\delta x_\nu)\right) = \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\right)(\partial_\nu\phi)(\delta x_\nu) + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\partial_\mu\left((\partial_\nu\phi)\right)(\delta x_\nu) + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}(\partial_\nu\phi)\partial_\mu\left(\delta x_\nu\right)$$

By neglecting higher order terms, one can find

$$\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}(\partial_\nu\phi)(\delta x_\nu)\right) = \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\right)(\partial_\nu\phi)(\delta x_\nu) + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}(\partial_\nu\phi)\partial_\mu\left(\delta x_\nu\right) \tag{5.45}$$

Therefor

$$\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\delta\phi\right) = \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\delta_o\phi\right) + \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\right)(\partial_\nu\phi)(\delta x_\nu) + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}(\partial_\nu\phi)\partial_\mu\left(\delta x_\nu\right) \tag{5.46}$$

By inserting equation (5.46) into equation (5.43), we get

$$\mathcal{L}(\phi', \partial'_\mu\phi') - \mathcal{L}(\phi, \partial_\mu\phi) = \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\delta\phi\right) - \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}(\partial_\nu\phi)\partial_\mu(\delta x_\nu)$$

$$= \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\delta_o\phi\right) + \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\right)(\partial_\nu\phi)(\delta x_\nu) + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}(\partial_\nu\phi)\partial_\mu\left(\delta x_\nu\right) - \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}(\partial_\nu\phi)\partial_\mu\left(\delta x_\nu\right)$$

So,

$$\mathcal{L}(\phi', \partial'_\mu\phi') - \mathcal{L}(\phi, \partial_\mu\phi) = \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\delta_o\phi\right) + \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\right)(\partial_\nu\phi)(\delta x_\nu)$$

Calculating the term $\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\right)(\partial_\nu\phi)(\delta x_\nu)$:

$$\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\right)(\partial_\nu\phi)(\delta x_\nu) = \frac{\partial\mathcal{L}}{\partial\phi}(\partial_\nu\phi)(\delta x_\nu) = \frac{\partial\mathcal{L}}{\partial x_\mu}\frac{\partial x_\mu}{\partial\phi}\frac{\partial\phi}{\partial x_\nu}\delta x_\nu$$

$$= \frac{\partial \mathcal{L}}{\partial x_\mu}\frac{\partial x_\mu}{\partial \partial x_\nu}\delta x_\nu = \frac{\partial \mathcal{L}}{\partial x_\mu}\delta_{\mu\nu}\delta x_\nu = \partial_\mu \mathcal{L}\ \delta x_\mu$$

Finally, we get

$$\mathcal{L}(\phi', \partial'_\mu \phi') - \mathcal{L}(\phi, \partial_\mu \phi) = \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\delta_o \phi\right) + \partial_\mu \mathcal{L}\ \delta x_\mu \tag{5.47}$$

The variation of the action in the equation (5.39) becomes

$$\delta S = \int \left[\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\delta_o \phi\right) + \partial_\mu \mathcal{L}\ \delta x_\mu + \partial_\mu(\delta x_\mu)\mathcal{L}\right] d^4x \simeq 0$$

We have

$$\partial_\mu \mathcal{L}\ \delta x_\mu + \partial_\mu(\delta x_\mu)\mathcal{L} = \partial_\mu(\mathcal{L}\,\delta x_\mu)$$

Then,

$$\delta S = \int \left[\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\delta_o \phi\right) + \partial_\mu(\mathcal{L}\,\delta x_\mu)\right] d^4x \simeq 0$$

$$\delta S = \int \partial_\mu \left[\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\delta_o \phi\right) + \mathcal{L}\,\delta x_\mu\right] d^4x \simeq 0$$

$$\Rightarrow \partial_\mu \left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\delta_o \phi + \mathcal{L}\,\delta x_\mu\right] = 0$$

The final equation can be expressed in the following form

$$\partial_\mu J_\mu = 0$$

with

$$J_\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\delta_o \phi + \mathcal{L}\,\delta x_\mu \longrightarrow \text{ Courant de Noether}$$

<u>Exercice</u> 7 :

1. Demonstrate that the Lagrangian density of the free complex scalar field remains invariant under the following global phase transformation

$$\begin{cases} \phi(x) \longrightarrow \phi'(x) = e^{i\theta}\phi(x) \\ \phi^*(x) \longrightarrow \phi'^*(x) = e^{-i\theta}\phi^*(x) \end{cases}$$

   θ is a real constant that does not depend on x_μ.

2. What are the currents and charges that are conserved?

Exercice 8 :

The dynamics of a system consisting of a real scalar field ϕ_1 and two complex scalar fields ϕ_2 and ϕ_3 is described by the Lagrangian density.

$$\mathcal{L} = -\frac{1}{2}(\partial_\mu\phi_1)^2 - \frac{1}{2}m_1^2\phi_1^2 - (\partial_\mu\phi_2^*)(\partial_\mu\phi_2) - m_2^2\phi_2^*\phi_2 - (\partial_\mu + iqA_\mu)\phi_3^*(\partial_\mu - iqA_\mu)\phi_3 - m_3^2\phi_3^*\phi_3$$

In which m_1, m_2, and m_3 represent constants.

1. Find the equations of motion?
2. It is known that the Lagrangian density remains invariant under the following two global phase transformations.

$$\begin{cases} \phi_1(x) \longrightarrow \phi_1'(x) = e^{-i\alpha_1}\phi_1(x) \\ \phi_1^*(x) \longrightarrow \phi_1'^*(x) = e^{i\alpha_1}\phi_1^*(x) \end{cases}, \begin{cases} \phi_2(x) \longrightarrow \phi_2'(x) = e^{+i\alpha_2}\phi_2(x) \\ \phi_2^*(x) \longrightarrow \phi_2'^*(x) = e^{-i\alpha_2}\phi_2^*(x) \end{cases}$$

   α_1 and α_2 are real constants with no dependence on x.
   What are the currents and charges that are conserved in these transformations?

Solution 9:

1^o/ The dynamics of a system are characterized by the Lagrangian density,

$$\mathcal{L} = -\frac{1}{2}\left(\partial_\mu\phi_1\right)^2 - \frac{1}{2}m_1^2\phi_1^2 - \left(\partial_\mu\phi_2\right)\left(\partial_\mu\phi_2^*\right) - m_2^2\phi_2\phi_2^* - \left(\partial_\mu + iqA_\mu\right)\phi_3^*\left(\partial_\mu - iqA_\mu\right)\phi_3 - m^2\phi_3\phi_3^*$$

The Lagrangian density can be expressed in the following form:

$$\mathcal{L} = \mathcal{L}_1 + \mathcal{L}_2 + \mathcal{L}_3$$

a°/: The real scalar field is determined by the following Lagrangian density,

$$\mathcal{L}_1(\phi_1, \partial_\mu\phi_1, x_\mu) = -\frac{1}{2}\left(\partial_\mu\phi_1\right)^2 - \frac{1}{2}m_1^2\phi_1^2$$

Equations of motion: Let us replace in the Euler-Lagrange equations, where $\phi_i = \phi_1 = \phi_1^*$

$$\frac{\partial\mathcal{L}_1}{\partial\phi_1} - \partial_\mu\left(\frac{\partial\mathcal{L}_1}{\partial(\partial_\mu\phi_1)}\right) = 0$$

with $\dfrac{\partial\mathcal{L}_1}{\partial\phi_1} = -m_1^2\phi_1, \quad \dfrac{\partial\mathcal{L}_1}{\partial(\partial_\mu\phi_1)} = -\partial_\mu\phi_1.$

The Klein-Gordon equation is obtained by replacing

$$\left(\partial_\mu \partial_\mu - m_1^2\right) \phi_1(x_\mu) = 0$$

b°/: The complex scalar field is defined by the following Lagrangian density:

$$\mathcal{L}_2(\phi_2, \partial_\mu \phi_2, \phi_2^*, \partial_\mu \phi_2^*, x_\mu) = -\left(\partial_\mu \phi_2\right)\left(\partial_\mu \phi_2^*\right) - m_2^2 \phi_2 \phi_2^*$$

Equations of motion: Let's substitute $\phi_i = \phi_2,\ \phi_2^*$ in both Euler-Lagrange equations,

$$\frac{\partial \mathcal{L}_2}{\partial \phi_2} - \partial_\mu \left(\frac{\partial \mathcal{L}_2}{\partial(\partial_\mu \phi_2)}\right) = 0, \quad \frac{\partial \mathcal{L}_2}{\partial \phi_2^*} - \partial_\mu \left(\frac{\partial \mathcal{L}_2}{\partial(\partial_\mu \phi_2^*)}\right) = 0$$

with $\dfrac{\partial \mathcal{L}_2}{\partial \phi_2^*} = -m_2^2 \phi_2, \quad \dfrac{\partial \mathcal{L}_2}{\partial(\partial_\mu \phi_2^*)} = -\partial_\mu \phi_2, \quad \dfrac{\partial \mathcal{L}_2}{\partial \phi_2} = -m_2^2 \phi_2^*, \quad \dfrac{\partial \mathcal{L}_2}{\partial(\partial_\mu \phi_2)} = -\partial_\mu \phi_2^*.$
By substituting into equation (5.3.2), we obtain the following two equations,

$$\left(\partial_\mu \partial_\mu - m_2^2\right) \phi_2(x_\mu) = 0 \ \ , \ \left(\partial_\mu \partial_\mu - m_2^2\right) \phi_2^*(x_\mu) = 0$$

c°/: The Lagrangian density for the complex scalar field in the presence of an external electromagnetic field is given by the following expression,

$$\mathcal{L}_3(\phi_3, \partial_\mu \phi_3, \phi_3^*, \partial_\mu \phi_3^*, x_\mu) = -\left(\partial_\mu + iqA_\mu\right) \phi_3^* \left(\partial_\mu - iqA_\mu\right) \phi_3 - m^2 \phi_3 \phi_3^*$$

Equations of motion for the field ϕ_3: Let's substitute in the Euler-Lagrange equation for $\phi_i = \phi_3^*$,

$$\frac{\partial \mathcal{L}_3}{\partial \phi_3^*} - \partial_\mu \left(\frac{\partial \mathcal{L}_3}{\partial(\partial_\mu \phi_3^*)}\right) = 0$$

we have

$$\mathcal{L}_3 = -\left(\partial_\mu \phi_3^*\right)\left(\partial_\mu - iqA_\mu\right) \phi_3 - iqA_\mu \phi_3^* \left(\partial_\mu - iqA_\mu\right) \phi_3 - m^2 \phi_3 \phi_3^*$$

with $\dfrac{\partial \mathcal{L}_3}{\partial \phi_3^*} = -iqA_\mu \left(\partial_\mu - iqA_\mu\right) \phi_3 - m_3^2 \phi_3, \quad \dfrac{\partial \mathcal{L}_3}{\partial(\partial_\mu \phi_3^*)} = -\left(\partial_\mu - iqA_\mu\right) \phi_3.$
By substituting into equation (5.3.2), we obtain the following two equations,

$$-iqA_\mu \left(\partial_\mu - iqA_\mu\right) \phi_3 - m_3^2 \phi_3 + \partial_\mu \left(\partial_\mu - iqA_\mu\right) \phi_3 = 0$$

$$\left[\left(\partial_\mu - iqA_\mu\right)\left(\partial_\mu - iqA_\mu\right) - m_3^2\right] \phi_3(x_\mu) = 0$$

Equations of motion for the field ϕ_3^*: Let's substitute in the Euler-Lagrange equation for $\phi_i = \phi_3$,

$$\frac{\partial \mathcal{L}_3}{\partial \phi_3} - \partial_\mu \left(\frac{\partial \mathcal{L}_3}{\partial(\partial_\mu \phi_3)} \right) = 0$$

$$\mathcal{L}_3 = -\left(\partial_\mu \phi_3\right)\left(\partial_\mu + iqA_\mu\right)\phi_3^* + iqA_\mu \phi_3 \left(\partial_\mu + iqA_\mu\right)\phi_3^* - m^2 \phi_3 \phi_3^*$$

with $\dfrac{\partial \mathcal{L}_3}{\partial \phi_3} = iqA_\mu \left(\partial_\mu + iqA_\mu\right)\phi_3^* - m_3^2 \phi_3^*, \quad \dfrac{\partial \mathcal{L}_3}{\partial(\partial_\mu \phi_3)} = -\left(\partial_\mu + iqA_\mu\right)\phi_3^*.$

By substituting into equation (5.3.2), we obtain the following two equations,

$$iqA_\mu \left(\partial_\mu + iqA_\mu\right)\phi_3^* - m_3^2 \phi_3^* + \partial_\mu \left(\partial_\mu + iqA_\mu\right)\phi_3^* = 0$$

$$\left[\left(\partial_\mu + iqA_\mu\right)\left(\partial_\mu + iqA_\mu\right) - m_3^2\right]\phi_3^*(x_\mu) = 0$$

2^o/ Find the currents and charges associated with the two global phase transformations:
According th Noether's Theorem

$$\delta S = \int \partial_\mu \left[\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)} \delta_o \phi_i \right) + \mathcal{L}\, \delta x_\mu \right] d^4x \simeq 0$$

a°/ Current and charge associated with the real scalar field in the transformation:

$$\begin{cases} \phi_1(x) \longrightarrow \phi_1^{'}(x) = e^{-i\alpha_1} \phi_1(x) \\ \phi_1^*(x) \longrightarrow \phi_1^{'*}(x) = e^{i\alpha_1} \phi_1^*(x) \end{cases}$$

α_1 is a real constant that does not depend on x.
Through the application of Noether's theorem:

$$\delta S = \int \partial_\mu \left(\frac{\partial \mathcal{L}_1}{\partial(\partial_\mu \phi_1)} \delta_o \phi_1 + \frac{\partial \mathcal{L}_1}{\partial(\partial_\mu \phi_1^*)} \delta_o \phi_1^* \right) d^4x \simeq 0$$

with

$$\frac{\partial \mathcal{L}_1}{\partial(\partial_\mu \phi_1^*)} = 0, \quad \frac{\partial \mathcal{L}_1}{\partial(\partial_\mu \phi_1)} = -\partial_\mu \phi_1$$

$$\delta_o \phi_1 = \phi_1^{'} - \phi_1 = e^{-i\alpha_1}\phi_1 - \phi_1 = (e^{-i\alpha_1} - 1)\phi_1 = (1 - i\alpha_1 - 1)\phi_1 = -i\alpha_1 \phi_1 \ \text{ pour } \ \alpha_1 << 1$$

Therefor,

$$\delta S = \int \partial_\mu \left(-i(\partial_\mu \phi_1)\phi_1\right) \alpha_1 d^4x \simeq 0$$

$$\alpha_1 << 1 \Rightarrow \partial_\mu \left(-i(\partial_\mu \phi_1)\phi_1\right) = 0 \Rightarrow \partial_\mu J^1_\mu = 0$$

Therefore, the current of the free real scalar field is given by,

$$J^1_\mu = -i(\partial_\mu \phi_1)\phi_1 = (j_i, \rho_1), \text{ avec } \rho_1 = \frac{j_t}{i} = (-\partial_t \phi_1)\phi_1$$

The charge Q_1 related to this transformation is determined by,

$$Q_1 = \int d^3x \rho_1 = \int d^3x \, (-\partial_t \phi_1)\phi_1$$

<u>b°/</u> Current and charge associated with the complex scalar field in the transformation:

$$\begin{cases} \phi_2(x) \longrightarrow \phi_2'(x) = e^{+i\alpha_2}\phi_2(x) \\ \phi_2^*(x) \longrightarrow \phi_2'^*(x) = e^{-i\alpha_2}\phi_2^*(x) \end{cases}$$

The real number α_2 is independent of x.
By applying Noether's theorem:

$$\delta S = \int \partial_\mu \left[\left(\frac{\partial \mathcal{L}_2}{\partial(\partial_\mu \phi_2)} \delta_o \phi_2 \right) + \left(\frac{\partial \mathcal{L}_2}{\partial(\partial_\mu \phi_2^*)} \delta_o \phi_2^* \right) \right] d^4x \simeq 0$$

With

$$\frac{\partial \mathcal{L}_2}{\partial(\partial_\mu \phi_2)} = -\partial_\mu \phi_2^*, \; \frac{\partial \mathcal{L}_2}{\partial(\partial_\mu \phi_2^*)} = -\partial_\mu \phi_2$$

$$\delta_o \phi_2 = \phi_2' - \phi_2 = e^{i\alpha_2}\phi_2 - \phi_2 = (e^{i\alpha_2} - 1)\phi_2 = (1 + i\alpha_2 - 1)\phi_2 = i\alpha_2 \phi_2 \text{ pour } \alpha_2 << 1$$

$$\delta_o \phi_2^* = \phi_2^{*'} - \phi_2^* = e^{-i\alpha_2}\phi_2^* - \phi_2^* = (e^{-i\alpha_2} - 1)\phi_2^* = (1 - i\alpha_2 - 1)\phi_2^* = -i\alpha_2 \phi_2^* \text{ pour } \alpha_2 << 1$$

Therefor,

$$\delta S = \int \partial_\mu \left(-i(\partial_\mu \phi_2^*)\phi_2 + i(\partial_\mu \phi_2)\phi_2^*\right) \alpha_2 d^4x \simeq 0$$

$$\alpha_2 << 1 \Rightarrow \partial_\mu \left(-i(\partial_\mu \phi_2^*)\phi_2 + i(\partial_\mu \phi_2)\phi_2^*\right) = 0 \Rightarrow \partial_\mu J^2_\mu = 0$$

The current of the real free scalar field is therefore given by,

$$J^2_\mu = -i(\partial_\mu \phi_2^*)\phi_2 + i(\partial_\mu \phi_2)\phi_2^* = (j_i, \rho_2), \text{ avec } \rho_2 = \frac{j_t}{i} = -(\partial_t \phi_2^*)\phi_2 + (\partial_t \phi_2)\phi_2^*$$

The charge Q_1 linked to this transformation is expressed as:

$$Q_2 = \int d^3x\rho_2 = \int d^3x\,((\partial_t\phi_2)\phi_2^* - (\partial_t\phi_2^*)\phi_2)$$

5.4 Energy-Momentum Tensor of the scalar field

Since the Lagrangian density $\mathcal{L}$ does not explicitly depend on the four-position vector x_μ, its derivative with respect to x_μ is as follows

$$\partial_\mu\mathcal{L} = \partial_\mu\mathcal{L}(\phi, \partial_\mu\phi) \quad \text{où} \quad \partial_\mu = \frac{\partial}{\partial x_\mu} \tag{5.48}$$

Therefor

$$\partial_\mu\mathcal{L} = \frac{\partial\mathcal{L}}{\partial x_\mu} \tag{5.49}$$

We have,

$$\partial_\mu\mathcal{L} = \frac{\partial\mathcal{L}}{\partial x_\mu} = \frac{\partial\mathcal{L}}{\partial\phi}\frac{\partial\phi}{\partial x_\mu} + \frac{\partial\mathcal{L}}{\partial(\partial_\nu\phi)}\frac{\partial(\partial_\nu\phi)}{\partial x_\mu} \tag{5.50}$$

According to the Euler-Lagrange equation, we have

$$\frac{\partial\mathcal{L}}{\partial\phi} - \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\nu\phi)}\right) = 0 \quad \Rightarrow \frac{\partial\mathcal{L}}{\partial\phi} = \partial_\nu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\nu\phi)}\right) \quad \text{pour} \qquad \mu = \nu \tag{5.51}$$

Therefor,

$$\partial_\mu\mathcal{L} = \frac{\partial\mathcal{L}}{\partial x_\mu} = \partial_\nu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\nu\phi)}\right)\partial_\mu\phi + \frac{\partial\mathcal{L}}{\partial(\partial_\nu\phi)}\partial_\mu\left(\partial_\nu\phi\right) \tag{5.52}$$

By setting,

$$\partial_\mu\left(\partial_\nu\phi\right) = \partial_\nu\left(\partial_\mu\phi\right) \tag{5.53}$$

we found that,

$$\partial_\mu\mathcal{L} = \partial_\nu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\nu\phi)}\right)\partial_\mu\phi + \frac{\partial\mathcal{L}}{\partial(\partial_\nu\phi)}\partial_\nu\left(\partial_\mu\phi\right) = \partial_\nu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\nu\phi)}\partial_\mu\phi\right) \tag{5.54}$$

The expression $\partial_\mu\mathcal{L}$ can also be represented in the following way:

$$\partial_\mu\mathcal{L} = \frac{\partial\mathcal{L}}{\partial x_\mu} = \frac{\partial\mathcal{L}}{\partial x_\nu}\frac{\partial x_\nu}{\partial x_\mu} = \left(\partial_\nu\mathcal{L}\right)\delta_{\mu\nu} = \partial_\nu\left(\mathcal{L}\delta_{\mu\nu}\right) \tag{5.55}$$

Comparing equations (5.54) and (5.55), we can see that

$$\partial_\mu \mathcal{L} = \partial_\nu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\nu \phi)} \partial_\mu \phi \right) = \partial_\nu \left(\mathcal{L} \delta_{\mu\nu} \right) \tag{5.56}$$

Therefor,

$$\partial_\nu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\nu \phi)} \partial_\mu \phi - \mathcal{L} \delta_{\mu\nu} \right) = 0 \tag{5.57}$$

Now, if we replace ν by μ

$$\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial_\nu \phi - \mathcal{L} \delta_{\mu\nu} \right) = 0 \tag{5.58}$$

The letter can be rewritten in the following form,

$$\partial_{\mu\nu} T_{\mu\nu} = 0 \text{ avec } T_{\mu\nu} = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial_\nu \phi - \mathcal{L} \delta_{\mu\nu} \tag{5.59}$$

The tensor $T_{\mu\nu}$ denotes the energy-momentum tensor of the scalar field.

5.5 Exercises

Exercice 9 :

In the position space $\{|\overrightarrow{r}\rangle\}$, the geometric transformation origin inversion is defined as:

$$\Pi\,|\overrightarrow{r}\rangle = |-\overrightarrow{r}\rangle\,,$$

Π represents the parity operator.

1. Calculate $\Pi\,|\overrightarrow{p}\rangle$
2. Calculate $\Pi\,|\psi(t)\rangle$
3. The transformed $\overrightarrow{A}'$ of an operator $\overrightarrow{A}$ is defined by $\overrightarrow{A}' \equiv \Pi\,\overrightarrow{A}\,\Pi^{-1}$. Calculate the transforms of the position, momentum, and angular momentum operators given respectively by $\overrightarrow{R}' \equiv \Pi\,\overrightarrow{R}\,\Pi^{-1}$, $\overrightarrow{P}' \equiv \Pi\,\overrightarrow{P}\,\Pi^{-1}$ and $\overrightarrow{L}' \equiv \Pi\,\overrightarrow{L}\,\Pi^{-1}$

Klein-Gordon equation

6.1 Introduction

The construction of quantum mechanics, which considers time as decoupled from space variables, is not compatible with the principles of special relativity. Additionally, experimental observations show that quantum mechanics is only accurate when the observed phenomena involve particles at low speeds. For example, it is not a suitable model for describing experiments involving interaction between light and matter.
In this chapter, we introduce the initial efforts to modify quantum mechanics to incorporate relativistic principles. Our first objective will be to derive a relativistic equation. In other words, we will begin our exploration with a particle that possesses zero spin. Within this context, it is logical to operate within the framework of Minkowski space, which is fundamental to special relativity, in order to develop a relativistic theory.
In order to describe quantum particles with zero spin and relativistic speeds, the Klein-Gordon equation is introduced. This equation is the relativistic equivalent of the Schrödinger equation given by,

$$H\psi = E\psi \tag{6.1}$$

By applying the principle of equivalence, we can write

$$i\hbar\frac{\partial}{\partial t}\psi = \frac{\overrightarrow{P}^2}{2m}\psi = -\frac{\hbar^2}{2m}\nabla^2\psi, \qquad \overrightarrow{p} = -i\hbar\overrightarrow{\nabla} \tag{6.2}$$

It is known that in the case of plane waves, the functions $\psi(\overrightarrow{r}, t)$ which are solutions of the Schrödinger equation are given by.

$$\psi(\overrightarrow{r}, t) = e^{i(\frac{\overrightarrow{p}\cdot\overrightarrow{r}}{\hbar} - \frac{E\cdot t}{\hbar})} \tag{6.3}$$

Let's attempt to find the general form of the Klein-Gordon equation, which allows us to describe the motion of free particles with zero spin and relativistic velocities, starting from the Schrödinger equation.

6.2 Quadri-vectors in field theory.

It is important to remember that the relativistic energy of a free particle is determined by

$$E = \sqrt{\overrightarrow{p}^2 c^2 + m^2 c^4} \tag{6.4}$$

- $\overrightarrow{p}$: impulsion
- c : velocity of light
- m : mass of the particle

The energy-momentum quadri-vector $\overrightarrow{P}$ is defined by.

$$\overrightarrow{\mathbf{P}} = \left(\overrightarrow{p}, \frac{E}{c}\right) \tag{6.5}$$

In field theory, the Einstein convention is used. If $\overrightarrow{A}$ is a quadri-vector, it is denoted as A_μ with $\mu = 1, 2, 3, 4$. The quadri-vector A_μ has the following components:

$$A_\mu = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ ia_4 \end{pmatrix} \tag{6.6}$$

When calculating the dot product of two quadri-vectors A_μ and B_ν, the result is obtained

$$A_\mu B_\nu = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ ia_4 \end{pmatrix} \cdot \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ ib_4 \end{pmatrix} = +a_1 b_1 + a_2 b_2 + a_3 b_3 - a_4 b_4 \tag{6.7}$$

The scalar product satisfies the metric of Minkowski space $(+, +, +, -)$.
In the field theory, the energy-momentum quadri-vector is written as:

$$P_\mu = \left(\overrightarrow{p}, i\frac{E}{c}\right) \tag{6.8}$$

It should be emphasized that in quantum mechanics, E and $\overrightarrow{p}$ are defined as:

$$E \rightarrow i\hbar\frac{\partial}{\partial t} \qquad\qquad \overrightarrow{p} \rightarrow -i\hbar\overrightarrow{\nabla} \tag{6.9}$$

By substituting (6.9) into (6.8), we obtain

$$P_\mu = \left(-i\hbar\overrightarrow{\nabla}, i\frac{i\hbar}{c}\frac{\partial}{\partial t}\right) = -i\hbar\left(\overrightarrow{\nabla}, -\frac{i}{c}\frac{\partial}{\partial t}\right) \tag{6.10}$$

If we set,

$$\partial_\mu = \left(\overrightarrow{\nabla}, -\frac{i}{c}\frac{\partial}{\partial t}\right) \tag{6.11}$$

The quadri-vector spatio-temporal derivative is represented by ∂_μ, where we find

$$P_\mu = -i\hbar\partial_\mu \tag{6.12}$$

6.3 Free Klein-Gordon equation

Let's now find the equation of the free Klein-Gordon describing the motion (displacement) of a quantum particle, with zero spin and relativistic speed
In quantum mechanics, a free particle is described by the Schrödinger's evolution equation.

$$i\hbar\frac{\partial}{\partial t}\phi(\overrightarrow{r}, t) = E\phi(\overrightarrow{r}, t) \quad \text{où} \quad E = H = E_c + V = E_c + 0 = \frac{1}{2}mv^2 \quad \text{avec} \quad v << c \tag{6.13}$$

For a free relativistic particle

$$E_R = \sqrt{\overrightarrow{p}^2c^2 + m^2c^4} \tag{6.14}$$

The dynamics of these relativistic particles will be described by the following equation

$$i\hbar\frac{\partial}{\partial t}\phi(\overrightarrow{r}, t) = E_R \quad \phi(\overrightarrow{r}, t) = \sqrt{\overrightarrow{p}^2c^2 + m^2c^4} \quad \phi(\overrightarrow{r}, t) \tag{6.15}$$

$$\left(i\hbar\frac{\partial}{\partial t}\right)^2 \phi(\overrightarrow{r}, t) = \left(\sqrt{\overrightarrow{p}^2c^2 + m^2c^4}\right)^2 \quad \phi(\overrightarrow{r}, t) \tag{6.16}$$

$$-\hbar^2\frac{\partial^2}{\partial^2 t}\phi(\overrightarrow{r}, t) = \left(\overrightarrow{p}^2c^2 + m^2c^4\right) \quad \phi(\overrightarrow{r}, t) \tag{6.17}$$

$$-\hbar^2 \frac{\partial^2}{\partial^2 t} \phi(\overrightarrow{r}, t) = \left(\left(-i\hbar \overrightarrow{\nabla} \right)^2 c^2 + m^2 c^4 \right) \quad \phi(\overrightarrow{r}, t) \tag{6.18}$$

$$-\hbar^2 \frac{\partial^2}{\partial^2 t} \phi(\overrightarrow{r}, t) = \left(\left(-i\hbar \overrightarrow{\nabla} \right)^2 c^2 + m^2 c^4 \right) \quad \phi(\overrightarrow{r}, t) \tag{6.19}$$

$$\frac{-\hbar^2}{\hbar^2 c^2} \frac{\partial^2}{\partial^2 t} \phi(\overrightarrow{r}, t) + \frac{\hbar^2 \overrightarrow{\nabla}^2 c^2}{\hbar^2 c^2} \quad \phi(\overrightarrow{r}, t) - \frac{m^2 c^4}{\hbar^2 c^2} \quad \phi(\overrightarrow{r}, t) = 0 \tag{6.20}$$

By setting $\overrightarrow{\nabla}^2 = \Delta$, we obtain the following equation

$$\left(\Delta - \frac{1}{c^2} \frac{\partial^2}{\partial^2 t} - \frac{m^2 c^2}{\hbar^2} \right) \quad \phi(\overrightarrow{r}, t) = 0 \tag{6.21}$$

The final equation represents the free Klein-Gordon equation written in real space. Let us now seek the form of this equation in Minkowski space.
We have

$$\partial_\mu = \left(\overrightarrow{\nabla}, -\frac{i}{c} \frac{\partial}{\partial t} \right) \Longrightarrow \partial_\mu^2 = \partial_\mu \cdot \partial_\mu = \left(\overrightarrow{\nabla}, -\frac{i}{c} \frac{\partial}{\partial t} \right) \cdot \left(\overrightarrow{\nabla}, -\frac{i}{c} \frac{\partial}{\partial t} \right) \tag{6.22}$$

$$\partial_\mu^2 = \left(\Delta, -\frac{1}{c^2} \frac{\partial^2}{\partial^2 t} \right) = \Delta - \frac{1}{c^2} \frac{\partial^2}{\partial^2 t} \tag{6.23}$$

By replacing (6.23) in (6.21), we get

$$\left(\partial_\mu^2 - \frac{m^2 c^2}{\hbar^2} \right) \quad \phi(\overrightarrow{r}, t) = 0 \tag{6.24}$$

By setting $\hbar = c = 1$ and defining $(\overrightarrow{r}, t) = x_\mu$, where x_μ denotes a point in Minkowski space and $\mu = 1, 2, 3, 4$, the equation (6.24) is transformed

$$\left(\partial_\mu^2 - m^2 \right) \phi(x_\mu) = 0 \tag{6.25}$$

This equation represents the free Klein-Gordon equation expressed in Minkowski space.

6.4 Invariance of the free Klein-Gordon equation under gauge transformation

Exercice 10 :

The motion of a particle with mass m, zero spin, and relativistic speed c is governed by the following free Klein-Gordon equation

$$\left(\partial_\mu^2 - m^2\right)\phi(x_\mu) = 0$$

- Demonstrate the invariance of this equation under the following gauge transformation

$$\phi(x_\mu) \longrightarrow \phi^{'}(x_\mu) = e^{-iq\alpha(x_\mu)}\phi(x_\mu) \quad , \quad \phi(x_\mu), \alpha(x_\mu) \quad \text{sont deux réels arbitraires.}$$

6.5 Solutions to the free Klein-Gordon equation

The free for Klein-Gordon equation is given by

$$\left(\partial_\mu^2 - m^2\right)\phi(x_\mu) = 0 \quad \text{qu'on peut écrire} \quad \left(\Delta - \frac{1}{c^2}\frac{\partial^2}{\partial^2 t} - \frac{m^2c^2}{\hbar^2}\right) \quad \phi(\overrightarrow{r}, t) = 0 \tag{6.26}$$

This equation has a solution in steady states. Its general form is given by,

$$\phi(\overrightarrow{r}, t) = f(t) \cdot \psi(\overrightarrow{r}) \tag{6.27}$$

It is said that a steady-state solution is a solution with separable variables. Substituting (6.27) into (6.26), we find

$$\left(\Delta - \frac{1}{c^2}\frac{\partial^2}{\partial^2 t} - \frac{m^2c^2}{\hbar^2}\right) \quad f(t) \cdot \psi(\overrightarrow{r}) = 0 \tag{6.28}$$

$$f(t)\Delta\psi(\overrightarrow{r}) - \psi(\overrightarrow{r})\frac{1}{c^2}\frac{\partial^2}{\partial^2 t}f(t) - \frac{m^2c^2}{\hbar^2}f(t)\psi(\overrightarrow{r}) = 0 \tag{6.29}$$

Dividing the entire equation by $f(t)\psi(\overrightarrow{r})$ yields

$$\frac{f(t)\Delta\psi(\overrightarrow{r})}{f(t)\psi(\overrightarrow{r})} - \frac{1}{f(t)\psi(\overrightarrow{r})}\psi(\overrightarrow{r})\frac{1}{c^2}\frac{\partial^2}{\partial^2 t}f(t) - \frac{1}{f(t)\psi(\overrightarrow{r})}\frac{m^2c^2}{\hbar^2}f(t)\psi(\overrightarrow{r}) = 0 \tag{6.30}$$

$$\frac{\Delta\psi(\overrightarrow{r})}{\psi(\overrightarrow{r})} - \frac{1}{f(t)}\frac{1}{c^2}\frac{\partial^2}{\partial^2 t}f(t) - \frac{m^2c^2}{\hbar^2} = 0 \tag{6.31}$$

This equation represents a second-order equation with two independent variables.

$$\frac{\Delta\psi(\overrightarrow{r})}{\psi(\overrightarrow{r})} - \frac{m^2c^2}{\hbar^2} = \frac{1}{c^2}\frac{f''(t)}{f(t)} = \text{constante}, \quad \text{avec} \quad f'' = \frac{\partial^2}{\partial^2 t}f(t) \tag{6.32}$$

If we define $constant = \omega^2$, we can deduce

$$\frac{\Delta\psi(\overrightarrow{r})}{\psi(\overrightarrow{r})} - \frac{m^2c^2}{\hbar^2} = \frac{1}{c^2}\frac{f''(t)}{f(t)} = \omega^2 \tag{6.33}$$

From this equation, we derive the two following equations:

$$\frac{\Delta\psi(\overrightarrow{r})}{\psi(\overrightarrow{r})} - \frac{m^2c^2}{\hbar^2} = \omega^2 \Longrightarrow \frac{\Delta\psi(\overrightarrow{r})}{\psi(\overrightarrow{r})} = \omega^2 + \frac{m^2c^2}{\hbar^2} \Longrightarrow \Delta\psi(\overrightarrow{r}) - \left(\omega^2 + \frac{m^2c^2}{\hbar^2}\right)\psi(\overrightarrow{r}) = 0 \tag{6.34}$$

$$\frac{1}{c^2}\frac{f''(t)}{f(t)} = \omega^2 \Longrightarrow \frac{f''(t)}{f(t)} = c^2\omega^2 \Longrightarrow f''(t) = c^2\omega^2 f(t) \Longrightarrow f''(t) - c^2\omega^2 f(t) = 0 \tag{6.35}$$

Equation (6.35) can be expressed in the following general form

$$f''(t) \pm (c\omega)^2 f(t) = 0 \tag{6.36}$$

Equation (6.35) then has solutions of the form

$$f(t) = A\, e^{c\omega t} + B\, e^{-c\omega t} \tag{6.37}$$

In order to have continuous solutions everywhere, we set

$$c\,\omega = \frac{i\,E}{\hbar}, \quad E \text{ est un réel.} \tag{6.38}$$

By substituting (6.37) into (6.38), we obtain

$$f(t) = A\,e^{\frac{i\,E}{\hbar}t} + B\,e^{-\frac{i\,E}{\hbar}t} \tag{6.39}$$

We have

$$c\,\omega = \frac{i\,E}{\hbar} \Longrightarrow c^2\,\omega^2 = -\frac{E^2}{\hbar^2} \Longrightarrow \omega^2 = -\frac{E^2}{c^2\hbar^2} \tag{6.40}$$

Let us now substitute into equation (6.34)

$$\Delta\psi(\overrightarrow{r}) - \left(-\frac{E^2}{c^2\hbar^2} + \frac{m^2c^2}{\hbar^2}\right)\psi(\overrightarrow{r}) = 0 \Longrightarrow \tag{6.41}$$

By finding a common denominator, one can determine

$$\Delta\psi(\overrightarrow{r}) - \left(-\frac{E^2}{c^2\hbar^2} + \frac{m^2c^4}{c^2\hbar^2}\right)\psi(\overrightarrow{r}) = 0 \Longrightarrow \Delta\psi(\overrightarrow{r}) - \left(\frac{-E^2 + m^2c^4}{c^2\hbar^2}\right)\psi(\overrightarrow{r}) = 0 \tag{6.42}$$

Or,

$$E^2 = \overrightarrow{p}^2c^2 + m^2c^4 \Longrightarrow -\overrightarrow{p}^2c^2 = -E^2 + m^2c^4 \tag{6.43}$$

By substituting into the previous equation, we find

$$\Delta\psi(\overrightarrow{r}) - \left(\frac{-\overrightarrow{p}^2c^2}{c^2\hbar^2}\right)\psi(\overrightarrow{r}) = 0 \Longrightarrow \Delta\psi(\overrightarrow{r}) - \left(\frac{-\overrightarrow{p}^2}{\hbar^2}\right)\psi(\overrightarrow{r}) = 0 \Longrightarrow \tag{6.44}$$

$$\Delta\psi(\overrightarrow{r}) - \left(\frac{i\,\overrightarrow{p}}{\hbar}\right)^2\psi(\overrightarrow{r}) = 0 \tag{6.45}$$

This equation has solutions of the following form

$$\psi(\overrightarrow{r}) = C\,e^{\frac{i\,\overrightarrow{p}\,\overrightarrow{r}}{\hbar}} + D\,e^{-\frac{i\,\overrightarrow{p}\,\overrightarrow{r}}{\hbar}} \tag{6.46}$$

6.6 Physical interpretation of solutions to the free Klein-Gordon equation

In order to give a physical meaning to the solutions, we assume

- $e^{-\frac{i\,E}{\hbar}t}$ Represents a particle that was created in the past $(-\infty)$ and is traveling towards the future $(+\infty)$.

- $e^{\frac{iE}{\hbar}t}$ Represents a particle created in the future $(+\infty)$ and travels towards the past $(-\infty)$.
- A Represents the probability that the particle being created in the future $(+\infty)$ and traveling towards the past $(-\infty)$.
- B Represents the probability that the particle was created in the past, extending from negative infinity $(-\infty)$, and is now moving towards the future, represented by positive infinity $(+\infty)$.

Therefor, the physical solution is given by

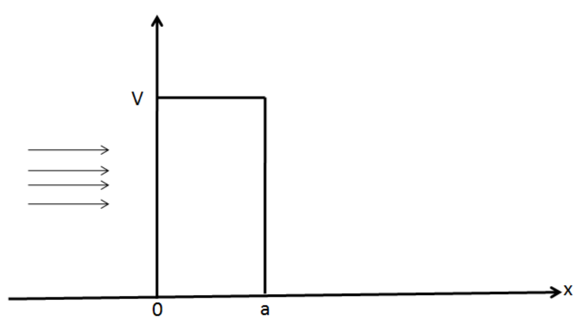
$$f(t) = e^{-\frac{iE}{\hbar}t} \tag{6.47}$$

It signifies the probability that the particle was created in the past, extending from negative infinity, and is now moving towards the future, represented by positive infinity

$$\phi(\overrightarrow{r},t) = f(t)\cdot\psi(\overrightarrow{r}) = e^{-\frac{iE}{\hbar}t}\left(C\,e^{\frac{i\overrightarrow{p}\,\overrightarrow{r}}{\hbar}} + D\,e^{-\frac{i\overrightarrow{p}\,\overrightarrow{r}}{\hbar}}\right) \tag{6.48}$$

Exercice 11 :

The particles with spin 0, charge q, and mass m are approaching from $(+\infty)$ to $(-\infty)$ on a potential barrier of height V and width a. Given that the energy of these particles is given by $E = qV/2$, where $qV > 2mc^2$,



1. Calculate the transmission coefficients T and reflection coefficients R.
2. Calculate the current density J_x in each region.

Indication: Working on one dimension.

6.7 Klein-Gordon equation in the presence of an external electromagnetic field

This equation describes the interaction between a particle with charge q and the external electromagnetic field, which is represented by the four-vector potential $A_\mu = \left(\overrightarrow{A}, i\frac{\phi}{c}\right)$.
To obtain the Klein-Gordon equation in the presence of an external electromagnetic field, the minimal coupling method is employed, which involves substituting the momentum and energy $(\overrightarrow{p}, E)$ with.

$$E \to E - q\phi \qquad \overrightarrow{p} \to \overrightarrow{p} - q\overrightarrow{A} \tag{6.49}$$

In the free Klein-Gordon equation, the transformation presented in equation (6.49) can be reformulated using four-vectors. Its expression is provided by:

$$P_\mu \to P_\mu - qA_\mu \tag{6.50}$$

Exercice 12 :

- Demonstrate the equivalence of the two transformations provided in equations (6.49) and (6.50).

We have

$$P_\mu = -i\hbar\,\partial_\mu \Longrightarrow P_\mu = -i\,\partial_\mu \quad \text{pour } \hbar = 1 \tag{6.51}$$

The transformation (6.50) becomes,

$$-i\,\partial_\mu \to -i\,\partial_\mu - qA_\mu \Longrightarrow \partial_\mu \to \partial_\mu - iqA_\mu \Longrightarrow \partial_\mu \cdot \partial_\mu \to \left(\partial_\mu - iqA_\mu\right)\left(\partial_\mu - iqA_\mu\right) \tag{6.52}$$

If we replace in the free Klein-Gordon equation, we get

$$\left[\left(\partial_\mu - iqA_\mu\right)\left(\partial_\mu - iqA_\mu\right) - m^2\right]\phi(x_\mu) = 0 \tag{6.53}$$

This equation is known as the Klein-Gordon equation in the presence of an external electromagnetic field A_μ. Introducing $D_\mu = \left(\partial_\mu - iqA_\mu\right)$, equation (6.53) can be expressed as

$$\left[D_\mu D_\mu - m^2\right]\phi(x_\mu) = 0 \tag{6.54}$$

The conjugate of the latter equation is provided by

$$\left[D^*_\mu D^*_\mu - m^2\right]\phi^*(x_\mu) = 0 \Longrightarrow \left[\left(\partial_\mu + iqA_\mu\right)\left(\partial_\mu + iqA_\mu\right) - m^2\right]\phi^*(x_\mu) = 0 \qquad (6.55)$$

6.8 invariance of the Klein-Gordon equation under the presence of an external electromagnetic field through gauge transformation

Exercice 13 :

In the presence of an external electromagnetic field $A_\mu(\overrightarrow{A}, iV)$, the motion of a particle with mass m, zero spin, and relativistic speed c is characterized by the following Klein-Gordon equation

$$\left[(\partial_\mu - iqA_\mu)(\partial_\mu - iqA_\mu) - m^2\right]\phi(x_\mu) = 0$$

- Demonstrate the invariance of this equation under the following gauge transformation

$$\begin{cases} A_\mu \longrightarrow A'_\mu = A_\mu - \partial_\mu\alpha(x_\mu) \\ \phi(x_\mu) \longrightarrow \phi'(x_\mu) = e^{-iq\alpha(x_\mu)}\phi(x_\mu) \quad , \quad \phi(x_\mu), \alpha(x_\mu) \text{sont deux réels arbitraires.} \end{cases}$$

6.9 Klein-Gordon equation current in the presence of an external electromagnetic field

Exercice 14 :

The Klein-Gordon equation, which governs the dynamics of a relativistic particle with mass m, charge q, and subject to an external electromagnetic-magnetic field $A_\mu(\overrightarrow{A}, i\phi)$, is presented

$$\left[(\partial_\mu - iqA_\mu)(\partial_\mu - iqA_\mu) - m^2\right]\psi(x) = 0$$

Determine the quadri-vector current expression of Klein-Gordon J_μ that solves the equation

$$\partial_\mu J_\mu = 0$$

We give : $(\partial_\mu^* - iqA_\mu^*)(\partial_\mu^* - iqA_\mu^*) = (\partial_\mu + iqA_\mu)(\partial_\mu + iqA_\mu)$

6.10 Exercises

Exercice 15 :

Particles with spin 0, charge q, and mass m are approaching from $(-\infty)$ towards $(+\infty)$ a potential barrier of height V and width a.
Given that the energy of these particles is determined by $E = qV/2$, where $qV > 2mc^2$,

1. Recover the general form of the wave function outside the potential barrier.
2. Calculate the current density J_x outside the potential barrier when the wave function is provided
$$\phi(x) = e^{ipx}$$
3. Demonstrate the expression for the transmission coefficient T
$$T = \frac{4pp'}{(p+p')\, e^{ia(p-p')} - (p-p')\, e^{ia(p+p')}}$$
when the momentum p of particles outside the potential barrier is different from the momentum p' of particles inside the potential barrier.
We give: $p = \sqrt{E^2 - m^2}$ et $p' = \sqrt{(E-qV)^2 - m^2}$ avec $c = \hbar = 1$. Indication: Work in one dimension

Exercice 16 :

1. Reconstruct the general form of the free Klein-Gordon equation from the Schrödinger equation.
2. Derive the general form of the Klein-Gordon equation in the presence of an external electromagnetic field by employing the method of minimal coupling.
3. Find the solutions of the free Klein-Gordon equation.

Exercice 17 :

The Klein-Gordon equation (Adjoint), in the presence of an external electromagnetic-magnetic field $A_\mu(\overrightarrow{A}, \frac{i\phi}{c})$, is provided by

$$\left[(\partial_\mu + iqA_\mu)(\partial_\mu + iqA_\mu) - m^2\right]\phi^*(x_\mu) = 0$$

1. Demonstrate the invariance of this equation under the following gauge transformation:

$$\begin{cases} A_\mu \longrightarrow A'_\mu = A_\mu + \partial_\mu \alpha(x_\mu) \\ \phi^*(x) \longrightarrow \phi^*(x)' = e^{-iq\alpha(x_\mu)}\phi^*(x) \;\;,\;\; \alpha(x_\mu) \text{ est un réel arbitraire} \end{cases}$$

Exercice 18 :

1. Obtain the quadri-current density vector expression from the continuity equation.
2. Derive the expression for the quadri-current potential from the Lorentz gauge equation.

Dirac equation

7.1 Introduction

We will now attempt to develop a relativistic theory for particles with non-zero spin. Initially, we will consider a scenario in which the electromagnetic field is not taken into account.
To achieve a satisfactory model, it is necessary for the state vector ψ to be governed by an equation that generalizes both the Schrödinger equation (which does not account for relativistic phenomena) and the Klein-Gordon equation (which does not consider spin). This equation must possess two primary properties.

1. It must remain invariant under the action of the Lorentz group.
2. It must be of the first order in t and, more specifically, take the following form.

$$i\hbar\partial_t\psi = H_D\psi \tag{7.1}$$

   where H_D represents an operator. The proof follows the same methodology as that employed to derive the Klein-Gordon equation.

7.2 The shortcomings of the Klein-Gordon equation

The Klein-Gordon equation is deemed unsatisfactory due to the presence of solutions with negative energy. This issue ultimately led Dirac to propose the existence of the "positron," a particle analogous to the electron but possessing a positive charge.
Before examining the physical consequences of negative energies, it is essential to first establish the underlying theory. Let us proceed with the standard approach when analyzing a second-order ordinary differential equation that we wish to reduce to first order (in terms of t only).
Let us define the following vector

$$\phi = \begin{pmatrix} \psi \\ \partial_t\psi \end{pmatrix}.$$

We are led to the following first-order equation:

$$\partial_t \phi = \begin{pmatrix} 0 & Id \\ \Delta + \frac{m^2}{\hbar^2} & 0 \end{pmatrix} \phi.$$

In fact, it will be more convenient to place.

$$\phi_1 = \psi + \frac{i\hbar}{m}\partial_t \psi \;\text{ et }\; \phi_2 = \psi - \frac{i\hbar}{m}\partial_t \psi \tag{7.2}$$

It is important to note that the wave function defined by $\Phi = (\phi_1, \phi_2)$ satisfies the following equation

$$\partial_t \Phi = \frac{1}{2} \begin{pmatrix} \frac{i\hbar}{m}\Delta & \frac{i\hbar}{m}\Delta + \frac{2im}{\hbar} \\ -\frac{i\hbar}{m}\Delta - \frac{2im}{\hbar} & -\frac{i\hbar}{m}\Delta \end{pmatrix} \Phi. \tag{7.3}$$

If the velocity of the particle is small compared to the speed of light, we can disregard its kinetic energy in relation to its internal energy, leading to the conclusion that the total energy is approximately equal to $E \simeq mc^2 = m^2$. This relationship is expressed in terms of observables as $i\hbar\partial_t \psi = m\psi$, which implies that in non-relativistic scenarios, $\phi_2 \simeq 0$. By setting $\phi_2 = 0$ and examining the first coordinate in (7.3), we derive the equation

$$\partial_t \phi_1 = \frac{i\hbar}{2m}\Delta \phi_1$$

. In other words, we arrive at the non-relativistic Schrödinger equation.

7.3 Dirac's Hamiltonian

To prevent the use of particles with negative energies, as was the case with the Hamiltonian (the total energy) from which the Klein-Gordon equation for a free particle was derived, Paul Dirac suggested in 1928 that the general form of the Hamiltonian be expressed as follows:

$$H_{Dirac} = \vec{\alpha}\,.\,\vec{p}\,c + \beta\, mc^2 = \sum_{i=1}^{3} \alpha_i.p_i c + \beta\, mc^2 = \alpha_i.p_i c + \beta\, mc^2 \tag{7.4}$$

where the coefficients β and α_i are constants that do not commute.
- We are seeking the values of these two constants.

By calculating the square of the Dirac Hamiltonian H^2_{Dirac}, one arrives at the following expression

$$H^2 = \left(\alpha_i.p_i c + \beta\, mc^2\right)\left(\alpha_j.p_j c + \beta\, mc^2\right) = \overrightarrow{p}^2 c^2 + m^2 c^4 \tag{7.5}$$

$$H^2 = p_i p_j \alpha_i \alpha_j c^2 + \beta^2 mc^2 c^4 + mc^3 p_i \left(\beta\alpha_j + \alpha_j \beta\right) = \overrightarrow{p}^2 c^2 + m^2 c^4 \tag{7.6}$$

- It is observed through comparison that

$$\beta^2 = 1 \Longrightarrow \beta\beta^{-1} = 1 \Longrightarrow \beta = \beta^{-1} \tag{7.7}$$

$$\beta\alpha_j + \alpha_j\beta = 0 \tag{7.8}$$

$$p_i p_j \alpha_i \alpha_j = p^2 \tag{7.9}$$

for $i = j = 1, 2, 3$ we can get:

$$p_i p_j \alpha_i \alpha_j = p_1^2 \alpha_1^2 + p_2^2 \alpha_2^2 + p_1 p_2 \left(\alpha_1\alpha_2 + \alpha_2\alpha_1\right) + p_1 p_3 \left(\alpha_1\alpha_3 + \alpha_3\alpha_1\right) + p_2 p_3 \left(\alpha_2\alpha_3 + \alpha_3\alpha_2\right) \tag{7.10}$$

$$p_i p_j \alpha_i \alpha_j = p_1^2 + p_2^2 + p_3^2 \tag{7.11}$$

For (7.10) to be equal to (7.11), it is necessary that

$$\alpha_1^2 = \alpha_2^2 = \alpha_3^2 = 1 \tag{7.12}$$

$$\alpha_1\alpha_2 + \alpha_2\alpha_1 = \alpha_1\alpha_3 + \alpha_3\alpha_1 = \alpha_2\alpha_3 + \alpha_3\alpha_2 = 0 \tag{7.13}$$

Therefor, if we suppose that $\alpha_i^2 = 1$ où $i = 1, 2, 3$ then

$$\{\alpha_i, \alpha_j\} = 2\delta_{ij} \tag{7.14}$$

In this context, $\{A, B\} = AB + BA$ represents the anti-commutator of the two quantities A and B. Finally, the dimensionless constants α_i and β satisfy the following anti-commutation relations

$$\beta^2 = 1 \tag{7.15}$$
$$\{\beta, \alpha_i\} = 0 \tag{7.16}$$
$$\{\alpha_i, \alpha_j\} = 2\delta_{ij} \tag{7.17}$$
$$\alpha_i^2 = 1 \tag{7.18}$$
$$\tag{7.19}$$

Therefor,

$$\alpha_1^2 = \alpha_2^2 = \alpha_3^2 = \beta^2 = 1, \tag{7.20}$$
$$\{\alpha_1, \alpha_2\} = \{\alpha_1, \alpha_3\} = \{\alpha_2, \alpha_3\} = \{\beta, \alpha_1\} = \{\beta, \alpha_2\} = \{\beta, \alpha_3\} = 0, \tag{7.21}$$

7.4 The characteristics of Dirac matrices

Prior to formulating the Dirac equation that describes particles with non-zero spin, it is essential to ascertain the order of the matrices present in the expression of the Dirac Hamiltonian. Establishing the order of the matrices β and α_i will facilitate the determination of the number of components in the spinor that characterizes the state of such a particle in the relativistic context. To achieve this:

1. The eigenvalues of matrices are determined. $\beta\,, \alpha_i : i = 1, 2, 3$.
   The eigenvalue equation, pertaining to β (and similarly to the α_i), is expressed in the following form.
   $$\beta\,\overrightarrow{X} = \lambda\,\overrightarrow{X}.$$
   A second application of β (or the α_i) yields, taking into account (**??**):
   $$\beta^2\,\overrightarrow{X} = \lambda\,\beta\,\overrightarrow{X} \quad \Rightarrow \quad 1.\overrightarrow{X} = \lambda^2\,\overrightarrow{X}$$
   $$\lambda^2 = 1 \quad \Rightarrow \quad \lambda = \pm 1.$$
   Therefore, the eigenvalues of the matrices β and α_i are either $+1$ or -1.
2. It is subsequently demonstrated that the traces $Tr(\beta) = Tr(\alpha_i) = 0$. To achieve this, we will utilize, on one hand, the anti-commutation of the matrices in question, and on the other

hand, the well-known properties.

$$
\begin{aligned}
Tr(A\,B) &= Tr(B\,A),\\
Tr(\lambda\,A) &= \lambda\,Tr(A).
\end{aligned}
\tag{7.22}
$$

Indeed,

$$
\begin{aligned}
Tr(\alpha_i) &= Tr(1.\alpha_i) = Tr(\beta^2\,\alpha_i) = Tr\big[\beta(\beta\,\alpha_i)\big] = Tr\big[\beta(-\alpha_i\,\beta)\big]\\
&= -Tr\big[\beta(\alpha_i\,\beta)\big] = -Tr\big[(\alpha_i\,\beta)\beta\big] = -Tr\big[\alpha_i\,\beta^2\big]\\
&= -Tr(\alpha_i)\\
&\Rightarrow \qquad Tr(\alpha_i) = 0.
\end{aligned}
\tag{7.23}
$$

A similar demonstration can be conducted to illustrate that $Tr(\beta) = 0$.

3. We will utilize the property that Hermitian matrices M are diagonalizable, meaning there exists an invertible matrix S such that.

$$
S\,M\,S^{-1} = M_D = \begin{pmatrix} \lambda_1 & ... & 0 & ... & 0\\ \vdots & \ddots & \vdots & \ddots & \vdots\\ 0 & ... & \lambda_i & ... & 0\\ \vdots & \ddots & \vdots & \ddots & \vdots\\ 0 & ... & 0 & ... & \lambda_n \end{pmatrix},
\tag{7.24}
$$

where the λ_i represent the eigenvalues of M. Additionally, the equality of the traces of the two matrices M and M_D is also utilized. Indeed,

$$
Tr(M) = Tr\big[S^{-1}(M_D S)\big] = Tr\big[(M_D S)S^{-1}\big] = Tr(M_D)
\tag{7.25}
$$

Since the matrices β and α_i are Hermitian, it is possible to apply the aforementioned properties, which can be expressed in the context of β and α_i as follows

$$
\begin{aligned}
Tr(\beta) = Tr(\alpha_i) = 0 \qquad &\Rightarrow \qquad Tr(\beta_D) = Tr\big[(\alpha_i)_D\big] = 0\\
&\Rightarrow \qquad \sum_{i=1}^{n} \lambda_i = 0\\
&\Rightarrow \qquad \underbrace{(1+1-1+..-1+1)}_{\text{n termes}} = 0.
\end{aligned}
$$

In order to achieve a sum of zero, it is necessary for the +1 and -1 values to completely offset each other. This condition is met only when the dimensions of the matrices β_D, $(\alpha_i)_D$, or alternatively, β and α_i, are even, specifically when $n = 2p$.
For $n = 2$, A basis for the complex matrices $M_{2\times 2}$ consists of the set of Pauli matrices, along with the identity matrix $\{\sigma_1, \sigma_2, \sigma_3, 1\}$. In this scenario, there is no solution, as equating the α_i with the σ_i necessitates that $\beta = 1$. However, β has a trace that differs from 1 ($Tr(1) = 2$), which is contradictory.
for $n = 4$, Solutions do exist. They can be expressed in standard representation in the following form.

$$\overrightarrow{\alpha} = \begin{pmatrix} 0 & \overrightarrow{\sigma} \\ \overrightarrow{\sigma} & 0 \end{pmatrix} \quad , \quad \beta = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix}, \tag{7.26}$$

where $\mathbf{1}$ represents the identity matrix of size (2×2) and $\overrightarrow{\sigma} = \overrightarrow{e_1}\,\sigma_1 + \overrightarrow{e_2}\,\sigma_2 + \overrightarrow{e_3}\,\sigma_3$. The three Pauli matrices are defined as follows

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad , \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad , \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{7.27}$$

In conclusion, it can be stated that the matrices β and α_i present in the Dirac Hamiltonian are of order 4×4. Consequently, the wave function that characterizes the state of a particle with non-zero spin is a four-component spinor. This spinor is capable of describing both the particle and its non-zero spin antiparticle. In standard representation, it is customary to employ the following condensed notation.

$$\psi = \begin{pmatrix} \varphi \\ \chi \end{pmatrix}, \tag{7.28}$$

In this context, φ and χ represent two-component spinors, which correspond to the particle and its antiparticle, respectively.

7.5 Standard representation

The representation of Dirac matrices in the standard form is provided by

$$\gamma^k = \begin{pmatrix} \mathbb{O} & -i\sigma_k \\ i\sigma_k & \mathbb{O} \end{pmatrix} \tag{7.29}$$

$$\gamma^4 = \begin{pmatrix} \mathbb{I} & \mathbb{O} \\ \mathbb{O} & -\mathbb{I} \end{pmatrix} \tag{7.30}$$

where σ_k represents the Pauli matrices (which are 2×2 matrices), defined as follows.

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{7.31}$$

and

$$\mathbb{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \longrightarrow \text{matrice unitaire}, \qquad \mathbb{O} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \tag{7.32}$$

Exercice 19 :

1. Provide the explicit forms of the following Dirac matrices: γ^1, γ^2, γ^3, and γ^4.
2. Demonstrate that

$$(\gamma^\mu)^+ = \gamma^\mu, \quad (\gamma^\mu)^1 = 1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \{\gamma^\mu, \gamma^\nu\} = 2\delta_{\mu\nu} \tag{7.33}$$

Solution 20:

1^o/ The four Dirac matrices are provided by,

$$\gamma^1 = \begin{pmatrix} \mathbb{O} & -i\sigma_1 \\ i\sigma_1 & \mathbb{O} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix} \tag{7.34}$$

$$\gamma^2 = \begin{pmatrix} \mathbb{O} & -i\sigma_2 \\ i\sigma_2 & \mathbb{O} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} \tag{7.35}$$

$$\gamma^3 = \begin{pmatrix} \mathbb{O} & -i\sigma_3 \\ i\sigma_3 & \mathbb{O} \end{pmatrix} = \begin{pmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & i \\ i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix} \tag{7.36}$$

$$\gamma^4 = \begin{pmatrix} \mathbb{I} & \mathbb{O} \\ \mathbb{O} & -\mathbb{I} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{7.37}$$

$2^o/a/$ Demonstrate that:

$$(\gamma^\mu)^+ = \gamma^\mu, \quad \text{où} \quad \text{si} \quad A = a_{ij} \quad \text{alors} \quad A^+ = a^*_{ji} \tag{7.38}$$

- For $\mu = 1$

$$\left(\gamma^1\right)^+ = \left[\begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}\right]^+ = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix} = \gamma^1 \tag{7.39}$$

- For $\mu = 2$

$$\left(\gamma^2\right)^+ = \left[\begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}\right]^+ = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} = \gamma^2 \tag{7.40}$$

- For $\mu = 3$

$$\left(\gamma^3\right)^+ = \left[\begin{pmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & i \\ i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix}\right]^+ = \begin{pmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & i \\ i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix} = \gamma^3 \tag{7.41}$$

- For $\mu = 4$

$$\left(\gamma^4\right)^+ = \left[\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}\right]^+ = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} = \gamma^4 \tag{7.42}$$

Therefor, $(\gamma^\mu)^+ = \gamma^\mu$.
$2^o/b/$ Demonstrate that:

$$(\gamma^\mu)^2 = 1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{7.43}$$

- For $\mu = 1$

$$\left(\gamma^1\right)^2 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = 1 \tag{7.44}$$

- For $\mu = 2$

$$\left(\gamma^2\right)^2 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = 1 \tag{7.45}$$

- For $\mu = 3$

$$\left(\gamma^3\right)^2 = \begin{pmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & i \\ i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & i \\ i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = 1 \tag{7.46}$$

- For $\mu = 4$

$$\left(\gamma^4\right)^2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = 1 \tag{7.47}$$

Therefor $(\gamma^\mu)^2 = 1$.
$2^o/c/$ Demonstrate that:

$$\{\gamma^\mu, \gamma^\nu\} = 2\delta_{\mu\nu} \tag{7.48}$$

- Pour $\mu = \nu = 1$

$$\{\gamma^1, \gamma^1\} = \gamma^1\gamma^1 + \gamma^1\gamma^1 = 2\left(\gamma^1\right)^2 = 2\delta_{11} = 2 \tag{7.49}$$

- Pour $\mu = 1, \nu = 2$

$$\{\gamma^1, \gamma^2\} = \gamma^1\gamma^2 + \gamma^2\gamma^1 = 2\delta_{12} = 0 \tag{7.50}$$

Verification:

$$\gamma^1\gamma^2 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & 0 & 0 & -i \end{pmatrix} \tag{7.51}$$

$$\gamma^2\gamma^1 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} -i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & 0 & 0 & i \end{pmatrix} \tag{7.52}$$

Therefor,

$$\{\gamma^1, \gamma^2\} = \gamma^1\gamma^2 + \gamma^2\gamma^1 = \begin{pmatrix} i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & 0 & 0 & -i \end{pmatrix} + \begin{pmatrix} -i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & 0 & 0 & i \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{7.53}$$

Therefor,

$$\{\gamma^\mu, \gamma^\nu\} = 2 \quad \text{lorsque} \quad \mu = \nu, \quad \{\gamma^\mu, \gamma^\nu\} = 0 \quad \text{lorsque} \quad \mu \neq \nu \tag{7.54}$$

7.6 Free Dirac equation

In the following discussion, we will attempt to derive the Dirac equation from the Schrödinger evolution equation,

$$i\hbar\frac{\partial\psi}{\partial t} = H_{Shrdinger}\psi, \quad \text{avec} \qquad H_{Shrodinger} = -\frac{\hbar^2}{2m}\overrightarrow{\nabla}^2 \tag{7.55}$$

The Dirac's Hamiltonian is given by,

$$H_{Dirac} = \alpha_i.p_i c + \beta\, mc^2 \tag{7.56}$$

and

$$\overrightarrow{p} = -i\hbar\overrightarrow{\nabla} = -i\hbar\,\overrightarrow{\partial} = -i\,\hbar\,\partial_i \tag{7.57}$$

We have also,

$$\partial_4 = \frac{-i}{c}\frac{\partial}{\partial t} \Longrightarrow i\frac{\partial}{\partial t} = -c\,\partial_4 \tag{7.58}$$

By substituting into (7.55), one obtains.

$$i\hbar\frac{\partial\psi}{\partial t} = \left(\alpha_i.p_i c + \beta\, mc^2\right)\psi \Longrightarrow -c\,\hbar\,\partial_4\psi = \left(-i\,\hbar\,\alpha_i\,\partial_i\,c + \beta\, mc^2\right)\psi \tag{7.59}$$

If we divide both sides of the equation (7.59) by c, we obtain

$$-\hbar\,\partial_4\psi = \left(-i\,\hbar\,\alpha_i\,\partial_i + \beta\, m\, c\right)\psi \tag{7.60}$$

At this point, if we divide both sides of the equation (7.60) by β, we obtain

$$-\beta\,\hbar\,\partial_4\psi = \left(-i\,\beta\,\hbar\,\alpha_i\,\partial_i + m\,c\right)\psi\text{avec} \quad \beta = \beta^{-1} \tag{7.61}$$

$$\left(\partial_4\,\beta + \partial_i\left(-i\,\beta\,\alpha_i\right) + \frac{m\,c}{\hbar}\right)\psi = 0 \tag{7.62}$$

$$\left(\partial_4\,\gamma^4 + \partial_i\,\gamma^i + \frac{m\,c}{\hbar}\right)\psi = 0 \tag{7.63}$$

with

$$\gamma^4 = \beta \tag{7.64}$$
$$\gamma^i = -i\,\beta\,\alpha_i \tag{7.65}$$

Finally, we found,

$$\left(\partial_4\,\gamma^4 + \partial_i\,\gamma^i + m\right)\psi = 0 \quad \text{avec} \quad \hbar = c = 1 \tag{7.66}$$

By employing the representation of the two quadri-vectors.

$$\partial_\mu = (\partial_i, \partial_4) \tag{7.67}$$
$$\gamma^\mu = \left(\gamma^i, \gamma^4\right) \tag{7.68}$$

where,

$$(\partial_i, \partial_4) \cdot \left(\gamma^i, \gamma^4\right) = \partial_4\,\gamma^4 + \partial_i\,\gamma^i \tag{7.69}$$

This equation can be rewritten as follow,

$$\left(\partial_\mu\,\gamma^\mu + m\right)\psi = 0 \tag{7.70}$$

The last equation represents the Dirac equation for a free particle.
If we make the following assumption,

$$\not{\partial} = \partial_\mu\,\gamma^\mu \tag{7.71}$$

We get,

$$(\not{\partial} + m)\,\psi(x) = 0 \quad \text{avec} \quad \psi(x) = \begin{pmatrix} \psi_1(x) \\ \psi_2(x) \\ \psi_3(x) \\ \psi_4(x) \end{pmatrix} \longrightarrow \quad \text{spineur de dirac} \tag{7.72}$$

Therefore, the Dirac matrices exhibit the following properties for subscripts $\mu, \nu = 1, 2, 3, 4$

$$(\gamma^\mu)^+ = \gamma^\mu \tag{7.73}$$
$$(\gamma^\mu)^2 = 1 \tag{7.74}$$
$$\{\gamma^\mu, \gamma^\nu\} = 2\delta_{\mu\nu} \tag{7.75}$$

7.7 Physical interpretation of the negatives energies

The advantage of utilizing component vectors (spinors) lies in their ability to represent fermions (such as electrons). Specifically, two components of the Dirac spinor are employed to characterize the two spin states ($\pm\frac{1}{2}$) of the particle, which possesses an energy of ($\sqrt{p^2c^2 + m^2c^4}$). The remaining two components of the spinor are used to describe the spin state of the antiparticle, which has an energy of ($-\sqrt{p^2c^2 + m^2c^4}$).
The antiparticle simply represents the absence of matter (a void).
For instance, when a particle transitions from a lower energy level to a higher energy level, the vacancy created by the particle, known as a hole, is regarded as the antiparticle of energy ($E = -\sqrt{p^2c^2 + m^2c^4}$), commonly referred to as a positron. A positron has the same mass as an electron but carries a positive charge ($+q$).

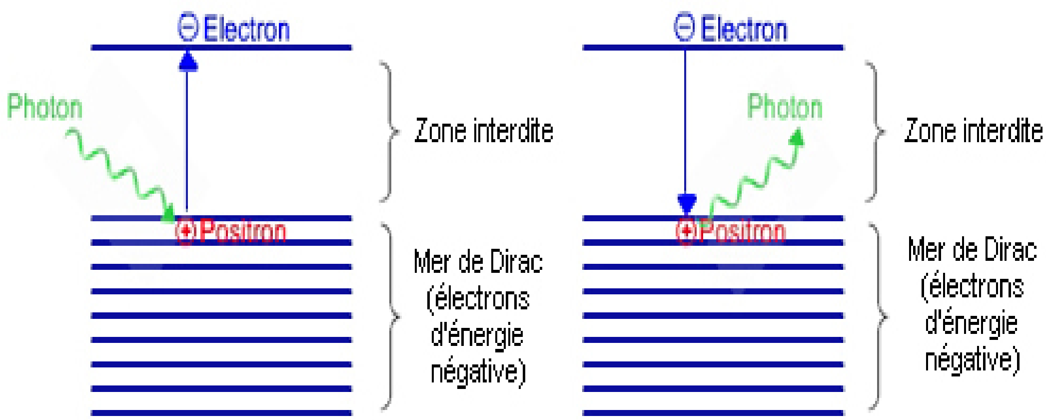


Figure 7.1: Diagram of the Dirac Sea.

When an electron returns to its initial state, it emits a photon of energy ($h\nu$)

$$e^- + e^+ \longrightarrow \gamma \tag{7.76}$$

This process is referred to as the annihilation phenomenon. It can be observed in particle accelerators, where electrons and positrons are accelerated to speeds approaching that of light, subsequently colliding to produce new particles (such as pions, mesons, etc.) that possess extremely short lifetimes.

7.8 Current of free Dirac equation

We seek the expression of the current associated with the Dirac equation, which satisfies the given continuity equation.

$$\frac{\partial \rho}{\partial t} + \overrightarrow{\nabla} \cdot \overrightarrow{j} = 0 \Longrightarrow \partial_\mu J_\mu = 0 \quad \text{avec} \quad \mu = 1,2,3,4 \tag{7.77}$$

The free Dirac equation is given by,

$$(\not{\partial} + m)\,\psi(x) = 0 \Longrightarrow (\partial_\mu\, \gamma^\mu + m)\,\psi(x) = 0 \tag{7.78}$$

- By calculating the conjugate of the Dirac equation, we arrive at the following result

$$\left[(\partial_\mu\, \gamma^\mu + m)\,\psi(x)\right]^* = 0 \Longrightarrow \psi^+(x)\left(\partial_\mu^*\, (\gamma^\mu)^+ + m\right) = 0 \tag{7.79}$$

We have

$$\partial_\mu = (\partial_i, \partial_4) \Longrightarrow \partial_\mu^* = (\partial_i^*, \partial_4^*) \tag{7.80}$$

with

$$\partial_i^* = \partial_i, \quad \partial_4^* = -\partial_4 \tag{7.81}$$

Therefor,

$$\partial_\mu^* = (\partial_i, -\partial_4) \tag{7.82}$$

and

$$\gamma^\mu = \left(\gamma^i, \gamma^4\right) \Longrightarrow (\gamma^\mu)^+ = \gamma^\mu = \left(\gamma^i, \gamma^4\right) \tag{7.83}$$

Therefor,

$$\partial_\mu^*\, (\gamma^\mu)^+ = \partial_i\, \gamma^i - \partial_4\, \gamma^4 \tag{7.84}$$

Substituting (7.84) in (7.79) we get,

$$\psi^+(x)\left(\partial_i\, \gamma^i - \partial_4\, \gamma^4 + m\right) = 0 \tag{7.85}$$

By multiplying both sides of the equation (7.85) by (γ^4), one arrives at the following result.

$$\left[\psi^+(x)\left(\partial_\mu^*\, (\gamma^\mu)^+ + m\right) = 0\right] \times \gamma^4 \tag{7.86}$$

$$\psi^{+}(x)\left(\partial_i\,\gamma^i\gamma^4-\partial_4\,\gamma^4\gamma^4+m\,\gamma^4\right)=0 \tag{7.87}$$

Or

$$\{\gamma^{\mu},\gamma^{\nu}\}=2\,\delta_{\mu\nu}\Longrightarrow\{\gamma^1,\gamma^4\}=\gamma^1\gamma^4+\gamma^4\gamma^1=0\Longrightarrow\gamma^1\gamma^4=-\gamma^4\gamma^1 \tag{7.88}$$

Therefor,

$$\psi^{+}\left(-\gamma^4\,\partial_i\,\gamma^i-\gamma^4\,\partial_4\,\gamma^4+\gamma^4\,m\right)=0\Longrightarrow \tag{7.89}$$

$$\psi^{+}\,\gamma^4\left(-\partial_i\,\gamma^i-\partial_4\,\gamma^4+m\right)=0\Longrightarrow\psi^{+}\,\gamma^4\left(-\partial_{\mu}\,\gamma^{\mu}+m\right)=0 \tag{7.90}$$

If we define $\overline{\psi}=\psi^{+}\,\gamma^4$, the adjoint equation of the free Dirac equation is transformed

$$\overline{\psi}\left(-\partial_{\mu}\,\gamma^{\mu}+m\right)=0\Longrightarrow\overline{\psi}\left(\partial_{\mu}\,\gamma^{\mu}-m\right)=0 \tag{7.91}$$

It can be expressed in the following final form,

$$\overline{\psi}\left(\overleftarrow{\not\partial}-m\right)=0 \tag{7.92}$$

By multiplying equation (7.78) by $\overline{\psi}$ and equation (7.92) by ψ, we obtain the following results

$$\overline{\psi}\left(\partial_{\mu}\,\gamma^{\mu}+m\,\right)\psi=0 \tag{7.93}$$

$$\overline{\psi}\left(\partial_{\mu}\,\gamma^{\mu}-m\right)\psi=0 \tag{7.94}$$

By calculating the sum of the two equations (7.93) and (7.94), one finds that

$$\overline{\psi}\left(\partial_{\mu}\,\gamma^{\mu}+m\,\right)\psi+\overline{\psi}\left(\partial_{\mu}\,\gamma^{\mu}-m\right)\psi=0\Longrightarrow \tag{7.95}$$

$$\overline{\psi}\,\overleftarrow{\partial_{\mu}}\,\gamma^{\mu}\psi+m\overline{\psi}\,\psi+\overline{\psi}\,\overrightarrow{\partial_{\mu}}\,\gamma^{\mu}\psi-m\overline{\psi}\psi=0\Longrightarrow \tag{7.96}$$

$$\partial_{\mu}\left(\overline{\psi}\,\gamma^{\mu}\,\psi\right)=0\Longrightarrow\partial_{\mu}J_{\mu}^{Dirac}=0 \tag{7.97}$$

with

$$J^{Dirac}=k\,\overline{\psi}\,\gamma^{\mu}\,\psi=i\,\overline{\psi}\,\gamma^{\mu}\,\psi\quad avec\quad k=i \tag{7.98}$$

7.8.1 vector current and total charge

Let us compute the expressions for the components of the momentum vector of j_4 and j_i

$$J_4 = i\,\overline{\psi}\,\gamma^4\,\psi = i\,\psi^+\,\gamma^4\,\gamma^4\,\psi = i\,\psi^+\,\psi = \rho \tag{7.99}$$

$$J_i = i\,\overline{\psi}\,\gamma^i\,\psi = i\,\psi^+\,\gamma^4\,\gamma^i\,\psi \tag{7.100}$$

Or

$$\gamma^i = -i\,\beta\,\alpha_i, \quad \beta = \gamma^4 \Longrightarrow \gamma^i = -i\,\gamma^4\,\alpha_i \Longrightarrow \tag{7.101}$$

$$\gamma^4\,\gamma^i = -i\,\gamma^4\,\gamma^4\,\alpha_i \Longrightarrow \alpha_i = i\,\gamma^4\,\gamma^i \tag{7.102}$$

Therefor,

$$J_i = \psi^+\,\alpha_i\,\psi \Longrightarrow \overrightarrow{J} = \psi^+\,\overrightarrow{\alpha}\,\psi \tag{7.103}$$

Finally, the total charge is given by,

$$Q = \int d^3x\,\rho = i\int d^3x\,\psi^+\,\psi \tag{7.104}$$

7.9 Dirac equation in the presence of an external electromagnetic field

To recover the Dirac equation in the presence of an external electromagnetic field A_μ, the method of minimal coupling is employed

$$\partial_\mu \rightarrow \partial_\mu - iqA_\mu \tag{7.105}$$

$$(\,\not\partial + m\,)\,\psi(x) = 0 \Longrightarrow \left(\partial_\mu\,\gamma^\mu + m\,\right)\psi(x) = 0 \tag{7.106}$$

Substituting (7.105) in (7.106) we get,

$$\left(\left(\partial_\mu - iqA_\mu\right)\,\gamma^\mu + m\,\right)\psi(x) = 0 \Longrightarrow \left(\partial_\mu\,\gamma^\mu - iqA_\mu\,\gamma^\mu + m\,\right)\psi(x) = 0 \tag{7.107}$$

$$(\,\not\partial - iq\,\not A + m\,)\,\psi(x) = 0 \tag{7.108}$$

This is the Dirac equation in the presence of an external electromagnetic field A_μ.

7.10 Lagrangian of the complex spinor field

It is possible to derive the Dirac equation and the adjoint Dirac equation by employing the Lagrangian formulation. Our selection of the Lagrangian is as follows

$$\mathcal{L}(\psi, \partial_\mu \psi, \overline{\psi}, \partial_\mu \overline{\psi}, x_\mu) = -\overline{\psi}\,(\not{\partial} + m\,)\,\psi \tag{7.109}$$

Verification: Let us verify that this Lagrangian density enables us to obtain the equations of motion for the free complex spinor field $(\psi, \overline{\psi})$. To conduct this verification, it is necessary to substitute the expression of the Lagrangian density into the Euler-Lagrange equations for a field,

$$\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) = 0 \quad \text{avec} \quad \phi_i = \psi = \overline{\psi} \tag{7.110}$$

Therefore, each value of ϕ_i corresponds to a motion equation

$$\frac{\partial \mathcal{L}}{\partial \psi} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)} \right) = 0 \longrightarrow \text{This equation allows for the derivation of the adjoint Dirac equation.} \tag{7.111}$$

$$\frac{\partial \mathcal{L}}{\partial \overline{\psi}} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \overline{\psi})} \right) = 0 \longrightarrow \text{This equation allows for the derivation of the Dirac equation,} \tag{7.112}$$

1- Let us revisit the Dirac adjoint equation

$$\frac{\partial \mathcal{L}}{\partial \overline{\psi}} = -\partial_\mu \gamma^\mu - m\,\psi, \quad \frac{\partial \mathcal{L}}{\partial(\partial_\mu \overline{\psi})} = 0 \tag{7.113}$$

So,

$$\frac{\partial \mathcal{L}}{\partial \overline{\psi}} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \overline{\psi})} \right) = 0 \Longrightarrow -\left(\partial_\mu\, \gamma^\mu + m\,\right)\psi = 0 \Longrightarrow (\not{\partial} + m\,)\,\psi = 0 \tag{7.114}$$

2- Let us revisit the Dirac equation

$$\frac{\partial \mathcal{L}}{\partial \psi} = -m\,\overline{\psi}, \quad \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)} = -\overline{\psi}\gamma^\mu \tag{7.115}$$

Therefor

$$\frac{\partial \mathcal{L}}{\partial \psi} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)} \right) = 0 \Longrightarrow -m\,\overline{\psi} + \overline{\psi}\gamma^\mu \Longrightarrow \overline{\psi}\left(\overleftarrow{\partial_\mu}\,\gamma^\mu - m \right) = 0 \Longrightarrow \overline{\psi}\left(\overleftarrow{\not\partial} + m \right) = 0 \quad (7.116)$$

Therefore, the Lagrangian density of the free complex spinor field is expressed as

$$\mathcal{L} = -\overline{\psi}\,(\not\partial + m\,)\,\psi = -\overline{\psi}\,(\partial_\mu\,\gamma^\mu + m\,)\,\psi = -\overline{\psi}\overrightarrow{\partial_\mu}\,\gamma^\mu\psi - m\overline{\psi}\psi \quad (7.117)$$

7.11 Lagrangian of the complex spinor field in the presence of an external electromagnetic field.

To derive the two equations of motion for the spinor fields ψ and $\overline{\psi}$ in the presence of an external electromagnetic field A_μ, the following Lagrangian density is employed

$$\mathcal{L} = -\overline{\psi}\,(\not\partial - iq\,\not A + m\,)\,\psi = -\overline{\psi}\,(\partial_\mu\,\gamma^\mu - iq\,A_\mu\gamma^\mu + m\,)\,\psi \quad (7.118)$$

That we can write in the following form,

$$\mathcal{L} = -\overline{\psi}\overrightarrow{\partial_\mu}\,\gamma^\mu\psi - iq\,A_\mu\,\gamma^\mu\,\overline{\psi}\psi + m\overline{\psi}\psi \quad (7.119)$$

It is important to recall that the Dirac equations and the adjoint Dirac equation in the presence of an external electromagnetic field are expressed as follows,

$$(\not\partial - iq\,\not A + m\,)\,\psi(x) = 0 \quad (7.120)$$

$$\overline{\psi}\left(\overleftarrow{\not\partial} + iq\,\not A + m \right)\psi(x) = 0 \quad (7.121)$$

Verification: Let us verify that this Lagrangian density enables us to derive the equations of motion for the complex spinor field in the presence of an electromagnetic field. To conduct this verification, it is necessary to substitute the expression of the Lagrangian density into the Euler-Lagrange equations for a field,

$$\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) = 0 \quad \text{avec} \quad \phi_i = \psi = \overline{\psi} \quad (7.122)$$

Therefore, each value of ϕ_i corresponds to a motion equation

$$\frac{\partial \mathcal{L}}{\partial \psi} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)} \right) = 0 \tag{7.123}$$

$$\frac{\partial \mathcal{L}}{\partial \overline{\psi}} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \overline{\psi})} \right) = 0 \tag{7.124}$$

1- Let us revisit the adjoint equation

$$\frac{\partial \mathcal{L}}{\partial \overline{\psi}} = -\left(\partial\!\!\!/ - iq \ A\!\!\!/ + m \right) \psi, \quad \frac{\partial \mathcal{L}}{\partial(\partial_\mu \overline{\psi})} = 0 \tag{7.125}$$

So

$$\frac{\partial \mathcal{L}}{\partial \overline{\psi}} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \overline{\psi})} \right) = 0 \Longrightarrow -\left(\partial\!\!\!/ - iq \ A\!\!\!/ + m \right) \psi = 0 \Longrightarrow \left(\partial\!\!\!/ - iq \ A\!\!\!/ + m \right) \psi = 0 \tag{7.126}$$

2- Let us revisit the Dirac equation

$$\frac{\partial \mathcal{L}}{\partial \psi} = iq \, A_\mu \, \gamma^\mu \, \overline{\psi} - m \, \overline{\psi}, \quad \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)} = -\overline{\psi}\gamma^\mu \tag{7.127}$$

alors

$$\frac{\partial \mathcal{L}}{\partial \psi} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)} \right) = 0 \Longrightarrow iq \, A_\mu \, \gamma^\mu \, \overline{\psi} - m \, \overline{\psi} + \overline{\psi}\gamma^\mu = 0 \Longrightarrow \overline{\psi} \left(\overleftarrow{\partial\!\!\!/} + iq \, A_\mu \, \gamma^\mu - m \right) = 0 \tag{7.128}$$

7.12 Exercises

<u>Exercice</u> 20 :

1^o/ In the presence of an external electromagnetic field A_μ, the dynamics of a relativistic particle with charge q, mass m, and non-zero spin can be described by the following Lagrangian density of the spinor field.

$$\mathcal{L}_2 = -\bar{\psi}(\partial\!\!\!/ - iq \ A\!\!\!/ + m)\psi = -\psi^\dagger \gamma^4 (\partial_\mu \gamma^\mu - iq A_\mu \gamma^\mu + m)\psi$$

1. Derive the equations of motion by utilizing the Euler-Lagrange equations.

2^o/ In the absence of the electromagnetic field, the dynamics of a free particle can be described by

$$\mathcal{L}_3 = -\bar{\psi}\ \partial\!\!\!/ \psi = -\psi^\dagger \gamma^4 \partial_\mu \gamma^\mu \psi$$

1. Demonstrate that this Lagrangian density remains invariant under the following phase transformation:

$$\begin{cases} \psi(x) \longrightarrow \psi^{'}(x) = e^{-i\theta\gamma^5}\psi(x) \\ \bar{\psi}(x) \longrightarrow \bar{\psi}^{'}(x) = \bar{\psi}(x) e^{-i\theta\gamma^5} \end{cases} , \quad \theta \text{ is a constant.}$$

   where $\gamma^5 = \gamma^1\gamma^2\gamma^3\gamma^4$ and verify the following relations: $\{\gamma^5, \gamma^\mu\} = 0$, $(\gamma^5)^\dagger = \gamma^5$ et $(\gamma^5)^2 = 1$.

2. Employ the Noether Theorem to identify the conserved quantities associated with this transformation.

3^o/ If we set:

$$\begin{cases} \psi_L(x) = \left(\frac{1+\gamma^5}{2}\right)\psi(x) \\ \psi_R(x) = \left(\frac{1-\gamma^5}{2}\right)\psi(x) \end{cases}$$

1. Rewrite the expression for the Lagrangian density $\mathcal{L}_3$ in terms of ψ_L and ψ_R.

2. Examine the invariance of the Lagrangian density $\mathcal{L}_3$ under the following phase transformation.

$$\begin{cases} \psi_L(x) \longrightarrow \psi^{'}_L(x) = \psi_L(x) e^{-i\alpha} \\ \psi_R(x) \longrightarrow \psi^{'}_R(x) = \psi_R(x) \end{cases} , \quad \alpha \text{ is a constant.}$$

Somme References

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