*République Algérienne Démocratique et Populaire*

*Ministère de L'enseignement Supérieur et de la Recherche Scientifique*

*Université de Djilali Bounâama-Khemis Miliana*

# *Mémoire de fin d'étude*

* * * * *

*Département des Mathématiques et Informatiques*

*Master 2*

*Analyse Mathématique et Application*

**Présenté par :**

*BELKACEMI Hadjer*

## Inégalités fractionnaires pondérées au sens de Hadamard .

**soutenu devant le jury composé de**

Examinateur 1 : Mr Mohamed HOUAS..........................................Univ. Djilali Bounâama.

Examinateur 2 : Mr Abdelkader YACHE.........................................Univ. Djilali Bounâama.

Encadreur : Mr Mohamed BEZZIOU..............................................Univ. Djilali Bounâama.

**Année Universitaire : 2019/2020**

# Dédicace

*Tous les mots ne sauraient exprimer la gratitude, l'amour, le respect, la reconnaissance, c'est tous simplement que : Je dédie cette mémoire de Master à :*

*À Ma chère Mère :* ***Mereim*** *.*

*À Mon cher Père :* ***Ahemad*** *.*

*À mes chers frères .*

*À mes très amis surtout :* ***Naima***, ***Nadjeya*** *.*

*À mon cher mari* : **Hamid** .

# Remerciement

*En premier lieu, je remercie **"ALLAH"** le tout puissant qui m'a donné la force, la volonté et le courage pour accomplir ce modeste travail.*

*Je remercie **Dr : Mohamed BEZZIOU**, ma encadrant de mémoire de fin d'étude, pour ses précieux conseils et son orientation ficelée tout au long de notre recherche.*

*Je veux exprimer mes remerciements les plus dévoués aux membres de jury qui m'ont honorées en acceptant d'évaluer ce travail.*

*mes remerciements s'étendent également à tous mes enseignants durant les années des études.*

*A mes familles et mes amis qui par leurs prières et leur encouragements, on a pu surmonter tous les obstacles.*

Merci

#  Résumé

Dans ce mémoire, on travail sur la généralisation des inégalités fractionnaires de type de Grüss d'une part et de l'autre par les inégalités de Tchebychev, par l'utilisation de la proche des fonctions synchrones au sens de l'intégrale fractionnaire $(s, \varphi)$ - Hadamard, sachant que $\varphi$ est une fonction positive, croissante de classe $\in C^1([a, b])$.

**Mots clés :** Intégrale fractionnaire au sens de Hadamard, intégrale fractionnaire $(s, \varphi)$ Hadamard, inégalités fractionnaires de Chebyshev, inégalités fractionnaires de Chebyshev.

# Abstract

In this thesis, we work on the generalization of Grüss -type fractional inequalities on the one hand and on the other by Chebyshev inequalities, by the use of the close of synchronous functions in the sense of the fractional integral $(s, \varphi)$ - Hadamard, konwing that $\varphi$ is a posiytive, increasing function of class $\in C^1([a, b])$.

# Table des matières

# Introduction

En mathématiques, le calcul fractionnaire est une branche de l'analyse qui étudie des intégrales et des opérateurs différentiels ayant un ordre non entier . L'objet ce travail est d'étudier et généraliser quelque inégalités intégrales classiques en utilisant l'approche fractionnaire au sens de $(s, \varphi)$- Hadamard .

Dans ce mémoire, nous allons proposer une introduction et trois chapitres .

Dans le premier chapitre on introduit des notions de base de l'intégrale fractionnaire au sens de Hadamard .

Le deuxième chapitre est consacré à établir quelque inégalités intégrables de type Grüss .

Enfin, notre dernier chapitre sera consacré à l'étude de quelque inégalité intégrales de type Chebyshev à un et deux poids positif et généraliser des résultat classique .

# Chapitre 1

# Préliminaires sur le calcule fractionnaire

Dans ce chapitre on donne quelques rappelles et définitions sur la théorie de calcule fractionnaire, on commence par les définitions des fonctions Gamma et Bêta d'Euler et après on présente quelques généralités sur les intégrales fractionnaires.

## 1 Fonctions spéciales

### 1.1 Fonction Gamma

Parmi les fonctions fondamentales qui interviennent dans la définition de l'intégrale fractionnaire, la fonction Gamma d' Euler qui généralise le factoriel d'un entie.

### 1.2 Définition

La fonction Gamma est définie par l'intégrale suivante :

$$\Gamma(\alpha) = \int_0^{+\infty} x^{\alpha-1} e^{-x} dx, \quad \alpha > 0$$

### 1.3 proposition

La fonction Gamma est bien définie pour tout $\alpha > 0$.

*Démonstration.* On va étudier la convergence de $\Gamma(\alpha) = \int_0^{+\infty} x^{\alpha-1}e^{-x}dx$.
Si $\alpha = 1$, on a

$$\Gamma(1) = \int_0^{+\infty} e^{-x}dx = 1.$$

Si $\alpha > 1$, on a

$$\Gamma(\alpha) = \int_0^{+\infty} x^{\alpha-1}e^{-x}dx = \int_0^{A} x^{\alpha-1}e^{-x}dx + \int_A^{+\infty} x^{\alpha-1}e^{-x}dx,$$

la première intégrale existe puisque la fonction $x^{\alpha-1}e^{-x}$ est continue sur $[0, A]$,
et pour la deuxième intégrale on montre qu'elle est bien définie .

On a $\lim_{x\to+\infty} x^2x^{\alpha-1}e^{-x} = 0 \Rightarrow \forall\varepsilon > 0,\ \exists B(\varepsilon) > 0,\ x > B(\varepsilon) \Rightarrow x^2x^{\alpha-1}e^{-x} < \varepsilon,$

si $\varepsilon = 1,\ \exists B(1) > 0$ tq : $\forall x > B(1),$ on a $x^{\alpha-1}e^{-x} < \frac{1}{x^2},$

on voit que $\int_A^{+\infty} \frac{1}{x^2}dx$ existe, il en résulte que l'intégrale $\int_A^{+\infty} x^{\alpha-1}e^{-x}dx$ existe.

donc $\Gamma(\alpha)$ est bien définie.

pour $0 < \alpha < 1$ , on trouve

$$\Gamma(\alpha) = \int_0^{+\infty} x^{\alpha-1}e^{-x}dx = \int_0^{A} x^{\alpha-1}e^{-x}dx + \int_A^{+\infty} x^{\alpha-1}e^{-x}dx,$$

et $\int_A^{+\infty} x^{\alpha-1}e^{-x}dx$ existe ( même preuve que $\alpha > 1$ ).

si $\int_0^{A} x^{\alpha-1}e^{-x}dx$ , on a $x^{\alpha-1}e^{-x} \simeq x^{\alpha-1}$ ,

implique $\int_0^{A} x^{\alpha-1}e^{-x}dx \simeq \int_0^{A} x^{\alpha-1}dx = \frac{A^\alpha}{\alpha},$

on résulte : $\int_0^{A} x^{\alpha-1}e^{-x}dx$ existe.

donc, $\Gamma(\alpha)$ est définie pour tout $\alpha > 0$. □

**• La relation Récursive**

on a

$$\Gamma(\alpha+1) = \alpha\Gamma(\alpha)\ .$$

*Démonstration.* en utilisant l'intégration par partie

$$\begin{aligned}\Gamma(\alpha+1) =& \int_0^{+\infty} x^{\alpha}e^{-x}dx = \left[-x^{\alpha}e^{-x}\right]_0^{+\infty} + \alpha\int_0^{+\infty} x^{\alpha-1}e^{-x}dx \\ =& 0+\alpha\Gamma(\alpha).\end{aligned}$$

□

• la fonction Gamma généralise le factoriel d'un entier car $\Gamma(n+1) = n!, \quad \forall n \in \mathbb{N}$.
•$0! = \Gamma(1) = \int_0^{+\infty} x^0 e^{-x}dx = \int_0^{+\infty} e^{-x}dx = 1.$
•$1! = \Gamma(2) = \int_0^{+\infty} xe^{-x}dx = [-xe^{-x}]_0^{+\infty} + \int_0^{+\infty} e^{-x}dx = 1.$

## 1.4 La fonction Bêta :

on présente la définition de la fonction Bêta

**Définition 1.** *La fonction Bêta d'Euler est définie par la formule d'intégration suivante :*

$$\beta(p,q) = \int_0^1 t^{p-1}(1-t)^{q-1}dt, \qquad Re(p) > 0, Re(q) > 0.$$

## Propriétés de la fonction Bêta

**1) La forme trigonométrique de Bêta**

On a

$$\beta(p,q) = \int_0^1 t^{p-1}(1-t)^{q-1}dt,$$

On pose

$$t = \sin^2\theta \Longrightarrow dt = 2\sin\theta\cos\theta d\theta,$$

$$si\ t = 1\ alors\ \theta = \frac{\pi}{2},$$

$$si\ t = 0\ alors\ \theta = 0.$$

d'où

$$\begin{aligned}
B(p,q) &= \int_0^{\frac{\pi}{2}} \left(\sin^2\theta\right)^{p-1} \left(\cos^2\theta\right)^{q-1} (2\sin\theta\cos\theta) d\theta \\
&= 2\int_0^{\frac{\pi}{2}} (\sin\theta)^{2(p-1)} (\cos\theta)^{2(q-1)} (\sin\theta\cos\theta) d\theta \\
&= 2\int_0^{\frac{\pi}{2}} (\sin\theta)^{2p-1} (\cos\theta)^{2q-1} d\theta.
\end{aligned}$$

**• Liens entre la fonction Gamma et la fonction Bêta**

la relation est donnée par :

$$\beta(p,q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}$$

*Démonstration.* On a

$$\Gamma(p)\Gamma(q) = \int_0^{+\infty} x^{p-1}e^{-x}dx \int_0^{+\infty} y^{q-1}e^{-y}dy = \int_0^{+\infty}\int_0^{+\infty} x^{p-1}y^{q-1}e^{-(x+y)}dxdy,$$

prend : $\int_0^{+\infty} x^{p-1}e^{-x}dx,$ on et met $x = t^2,$

et pour :$\int_0^{+\infty} y^{q-1}e^{-y}dy,$ on pose $y = s^2,$

on trouve

$$\Gamma(p)\Gamma(q) = 4\int_0^{+\infty}\int_0^{+\infty} t^{2p-1}s^{2q-1}e^{-\left(s^2+t^2\right)}dsdt,$$

si $t = r\cos\theta, s = r\sin\theta,$

on obtient

$$\begin{aligned}
\Gamma(p)\Gamma(q) &= 4\int_0^{+\infty}\int_0^{\frac{\Pi}{2}} r^{2p-1}(\cos\theta)^{2p-1}r^{2q-1}(\sin\theta)^{2q-1}e^{-r^2}rdrd\theta \\
&= \left(2\int_0^{+\infty}(r^2)^{p+q-1}re^{-r^2}dr\right)\left(2\int_0^{+\infty}(\cos\theta)^{2p-1}(\sin\theta)^{2q-1}d\theta\right) \\
&= 2\left(\int_0^{+\infty}(r^2)^{p+q-1}re^{-r^2}dr\right)B(p,q),
\end{aligned}$$

pour $r^2 = R,$

donc

$$\Gamma(p)\Gamma(q) = \left( \int_0^{+\infty} R^{(p+q)-1} e^{-R} dR \right) B(p,q)$$

$$\Gamma(p)\Gamma(q) = \Gamma(p+q)B(p,q).$$

D'où le résultat.

□

## 2 Intégrale fractionnaire au sens de Hadamard

**Définition 2.** *Soit* $f : [a,b] \to \mathbb{R}$ *une fonction continue. On appelle intégrale fractionnaire au sens de Hadamard de* $f$ *d'ordre* $\alpha \geq 0$ *l'intégrale définie par :*

$$I_a^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (log\frac{t}{s})^{\alpha-1} \ \frac{f(s)}{s} \ ds, \ \ \alpha > 0$$

## 3 Intégrale fractionnaire (s,$\varphi$) au sens de Hadamard

**Définition 3.** *Soit* $f \in L^1[a,b]$ *une fonction monotone positive et* $h \in C^1([a,b])$ *une fonction mesurable ,croissante .L'intégrale fractionnaire* $(s,\varphi)$ *Hadamard est définit par :*

$${}^sI_{a,\varphi}^\alpha(f(t)) = \frac{(s+1)^{1-\alpha}}{\Gamma(\alpha)} \int_a^t \left( log^{s+1}\frac{\varphi(t)}{\varphi(\tau)} \right)^{\alpha-1} \times log^s\varphi(\tau)\frac{\varphi'(\tau)}{\varphi(\tau)} f(\tau) \ \frac{d\tau}{\tau}, \quad \alpha > 0$$

*telle que* $0 < a < t \leqslant b, s \in \mathbb{R} \setminus \{-1\}$.

# Chapitre 2

# Inégalités fractionnaires de type Grüss

## 1 Introduction

On applique l'opérateur ${}^{s}I^{\alpha}_{a,\varphi}$ pour étudier certaines inégalités de type Grüss .

## 2 Principaux résultats

### 2.1 Inégalité fractionnaire avec un seul paramètre ($\alpha$)

on présente notre premier résultat comme suit

**Théorème 1.** *Si $f$ et $g$ sont deux fonctions intégrables sur $[0,\infty[$ vérifiant les conditions suivantes*

$$r \leq f(x) \leq R, n \leq g(x) \leq N, \quad r, R, n, N \in \mathbb{R}, x \in [a,b]. \tag{2.1}$$

*Alors* $\forall \ t > 0, \alpha > 0,$
*on a*

$$\begin{aligned} &\left|\tfrac{(s+1)^{1-\alpha}}{\Gamma(\alpha+1)}(log^{s+1}\varphi(t) - log^{s+1}\varphi(a))^{\alpha} \ {}^{s}I^{\alpha}_{a,\varphi}fg(t) - {}^{s}I^{\alpha}_{a,\varphi}f(t){}^{s}I^{\alpha}_{a,\varphi}g(t)\right| \\ &\leq \left(\tfrac{(s+1)^{1-\alpha}log^{s+1}\varphi(t) - log^{s+1}\varphi(a))^{\alpha}}{2\Gamma(\alpha+1)}\right)^{2}(R-r)(N-n). \end{aligned} \tag{2.2}$$

avant de montrer ce théorème on a besoin de montrer le lemme suivant

**Lemme 1.** *Si $f$ une fonction intégrable sur $[0,\infty[$ vérifie la condition* (2.1) *sur* $[0,\infty[$ *. Alors* $\forall \ t > 0,$ $\alpha > 0,$
*on a*

$$\begin{aligned}
&\tfrac{(s+1)^{1-\alpha}}{\Gamma(\alpha+1)}(log^{s+1}\varphi(t)-log^{s+1}\varphi(a)^{\alpha}\ {}^{s}I^{\alpha}_{a,\varphi}\ f^{2}(t)-\left(\ {}^{s}I^{\alpha}_{a,\varphi}\ f(t)\right)^{2}\\
&=\left(R\tfrac{(s+1)^{1-\alpha}}{\Gamma(\alpha+1)}(log^{s+1}\varphi(t)-log^{s+1}\varphi(a))^{\alpha})-\ {}^{s}I^{\alpha}_{a,\varphi}\ f(t)\right)\\
&\left(\ {}^{s}I^{\alpha}_{a,\varphi}\ f(t)-r\frac{(s+1)^{1-\alpha}}{\Gamma(\alpha+1)}(log^{s+1}\varphi(t)-log^{s+1}\varphi(a))^{\alpha}\right)\\
&-\tfrac{(s+1)^{1-\alpha}}{\Gamma(\alpha+1)}(log^{s+1}\varphi(t)-log^{s+1}\varphi(a))^{\alpha}\ {}^{s}I^{\alpha}_{a,\varphi}(R-f(t))(f(t)-r).
\end{aligned} \tag{2.3}$$

*Démonstration.* Si $f$ intégrable sur $[0,\infty[$ vérifié la condition (2.1) sur $[0,\infty[$. $\forall\ \tau,\rho\in[0,\infty[$, on a

$$\begin{aligned}
&(R-f(\rho))(f(\tau)-r)+(R-f(\tau))(f(\rho)-r)\\
&-(R-f(\tau))(f(\tau)-r)-(R-f(\rho))(f(\rho)-r)\\
&=f^{2}(\tau)+f^{2}(\rho)-2f(\tau)f(\rho),
\end{aligned} \tag{2.4}$$

En multipliant (2.4) par la quantité suivant $\frac{(s+1)^{1-\alpha}}{\Gamma(\alpha)}\ (log^{s+1}\varphi(t)-log^{s+1}\ \varphi(\tau))^{\alpha-1}log^{s}\ \ \varphi(\tau)\ \frac{\varphi'(\tau)}{\varphi(\tau)}$, $\tau\in(a,t),t>a$
on trouve

$$\begin{aligned}
&\tfrac{(s+1)^{1-\alpha}}{\Gamma(\alpha)}\ (log^{s+1}\varphi(t)-log^{s+1}\ \varphi(\tau))^{\alpha-1}log^{s}\ \ \varphi(\tau)\ \tfrac{\varphi'(\tau)}{\varphi(\tau)}\Big\{(R-f(\rho))(f(\tau)-r)+(R-f(\tau))(f(\rho)-r)\\
&-(R-f(\tau))(f(\tau)-r)-(R-f(\rho))(f(\rho)-r)\Big\}\\
&=\tfrac{(s+1)^{1-\alpha}}{\Gamma(\alpha)}\ (log^{s+1}\varphi(t)-log^{s+1}\ \varphi(\tau))^{\alpha-1}log^{s}\ \ \varphi(\tau)\ \tfrac{\varphi'(\tau)}{\varphi(\tau)}\Big\{f^{2}(\tau)+f^{2}(\rho)-2f(\tau)f(\rho)\Big\},
\end{aligned} \tag{2.5}$$

alors

$$\begin{aligned}
&(R-f(\rho))(\tfrac{(s+1)^{1-\alpha}}{\Gamma(\alpha)}\ (log^{s+1}\varphi(t)-log^{s+1}\ \varphi(\tau))^{\alpha-1}log^{s}\ \ \varphi(\tau)\ \tfrac{\varphi'(\tau)}{\varphi(\tau)}f(\tau)\\
&-\tfrac{(s+1)^{1-\alpha}}{\Gamma(\alpha)}\ (log^{s+1}\varphi(t)-log^{s+1}\ \varphi(\tau))^{\alpha-1}log^{s}\ \ \varphi(\tau)\ \tfrac{\varphi'(\tau)}{\varphi(\tau)}r)
\end{aligned}$$

$$+(f(\rho)-r)(\frac{(s+1)^{1-\alpha}}{\Gamma(\alpha)}\ \ (log^{s+1}\varphi(t)-log^{s+1}\ \varphi(\tau))^{\alpha-1}log^{s}\ \ \varphi(\tau)\ \frac{\varphi'(\tau)}{\varphi(\tau)}R$$

$$-(\frac{(s+1)^{1-\alpha}}{\Gamma(\alpha)}\ \ (log^{s+1}\varphi(t)-log^{s+1}\ \varphi(\tau))^{\alpha-1}log^{s}\ \ \varphi(\tau)\ \frac{\varphi'(\tau)}{\varphi(\tau)}f(\tau))$$

$$-\frac{(s+1)^{1-\alpha}}{\Gamma(\alpha)}\ \ (log^{s+1}\varphi(t)-log^{s+1}\ \varphi(\tau))^{\alpha-1}log^{s}\ \ \varphi(\tau)\ \frac{\varphi'(\tau)}{\varphi(\tau)}(R-f(\tau))(f(\tau)-r)$$

$$-(R-f(\rho))(f(\rho)-r)\frac{(s+1)^{1-\alpha}}{\Gamma(\alpha)}\ \ (log^{s+1}\varphi(t)-log^{s+1}\ \varphi(\tau))^{\alpha-1}log^{s}\ \ \varphi(\tau)\ \frac{\varphi'(\tau)}{\varphi(\tau)} \tag{2.6}$$

$$=\frac{(s+1)^{1-\alpha}}{\Gamma(\alpha)}\ \ (log^{s+1}\varphi(t)-log^{s+1}\ \varphi(\tau))^{\alpha-1}log^{s}\ \ \varphi(\tau)\ \frac{\varphi'(\tau)}{\varphi(\tau)}f^{2}(\tau)$$

$$+f^{2}(\rho)\frac{(s+1)^{1-\alpha}}{\Gamma(\alpha)}\ \ (log^{s+1}\varphi(t)-log^{s+1}\ \varphi(\tau))^{\alpha-1}log^{s}\ \ \varphi(\tau)\ \frac{\varphi'(\tau)}{\varphi(\tau)}$$

$$-2f(\rho)\frac{(s+1)^{1-\alpha}}{\Gamma(\alpha)}\ \ (log^{s+1}\varphi(t)-log^{s+1}\ \varphi(\tau))^{\alpha-1}log^{s}\ \ \varphi(\tau)\ \frac{\varphi'(\tau)}{\varphi(\tau)}f(\tau)$$

on intègre (2.6) par rapport à $\tau$ sur $(a,t)$,
on obtient

$$(R-f(\rho))(\frac{(s+1)^{1-\alpha}}{\Gamma(\alpha)}\ \ \int_a^t(log^{s+1}\varphi(t)-log^{s+1}\ \varphi(\tau))^{\alpha-1}log^{s}\ \ \varphi(\tau)\ \frac{\varphi'(\tau)}{\varphi(\tau)}f(\tau)\ \ \frac{d\tau}{\tau}$$

$$-r\frac{(s+1)^{1-\alpha}}{\Gamma(\alpha)}\ \ \int_a^t(log^{s+1}\varphi(t)-log^{s+1}\ \varphi(\tau))^{\alpha-1}log^{s}\ \ \varphi(\tau)\ \frac{\varphi'(\tau)}{\varphi(\tau)})\ \ \frac{d\tau}{\tau}$$

$$+(f(\rho)-r)(R\frac{(s+1)^{1-\alpha}}{\Gamma(\alpha)}\ \ \int_a^t(log^{s+1}\varphi(t)-log^{s+1}\ \varphi(\tau))^{\alpha-1}log^{s}\ \ \varphi(\tau)\ \frac{\varphi'(\tau)}{\varphi(\tau)}\ \ \frac{d\tau}{\tau}$$

$$-(\frac{(s+1)^{1-\alpha}}{\Gamma(\alpha)}\ \ \int_a^t(log^{s+1}\varphi(t)-log^{s+1}\ \varphi(\tau))^{\alpha-1}log^{s}\ \ \varphi(\tau)\ \frac{\varphi'(\tau)}{\varphi(\tau)}f(\tau)\ \ \frac{d\tau}{\tau}$$

$$-\frac{(s+1)^{1-\alpha}}{\Gamma(\alpha)}\ \ \int_a^t(log^{s+1}\varphi(t)-log^{s+1}\ \varphi(\tau))^{\alpha-1}log^{s}\ \ \varphi(\tau)\ \frac{\varphi'(\tau)}{\varphi(\tau)}(R-f(\tau))(f(\tau)-r)\ \ \frac{d\tau}{\tau} \tag{2.7}$$

$$-(R-f(\rho))(f(\rho)-r)\frac{(s+1)^{1-\alpha}}{\Gamma(\alpha)}\ \ \int_a^t(log^{s+1}\varphi(t)-log^{s+1}\ \varphi(\tau))^{\alpha-1}log^{s}\ \ \varphi(\tau)\ \frac{\varphi'(\tau)}{\varphi(\tau)}\ \ \frac{d\tau}{\tau}$$

$$=\frac{(s+1)^{1-\alpha}}{\Gamma(\alpha)}\ \ \int_a^t(log^{s+1}\varphi(t)-log^{s+1}\ \varphi(\tau))^{\alpha-1}log^{s}\ \ \varphi(\tau)\ \frac{\varphi'(\tau)}{\varphi(\tau)}f^{2}(\tau)\ \ \frac{d\tau}{\tau}$$

$$+f^{2}(\rho)\frac{(s+1)^{1-\alpha}}{\Gamma(\alpha)}\ \ \int_a^t(log^{s+1}\varphi(t)-log^{s+1}\ \varphi(\tau))^{\alpha-1}log^{s}\ \ \varphi(\tau)\ \frac{\varphi'(\tau)}{\varphi(\tau)}\ \ \frac{d\tau}{\tau}$$

$$-2f(\rho)\frac{(s+1)^{1-\alpha}}{\Gamma(\alpha)}\ \ \int_a^t(log^{s+1}\varphi(t)-log^{s+1}\ \varphi(\tau))^{\alpha-1}log^{s}\ \ \varphi(\tau)\ \frac{\varphi'(\tau)}{\varphi(\tau)}f(\tau)\ \ \frac{d\tau}{\tau}$$

alors

$$(R-f(\rho))({}^sI^\alpha_{a,\varphi}f(t))-r\frac{(s+1)^{1-\alpha}}{\Gamma(\alpha)}\int_a^t(log^{s+1}\varphi(t)-log^{s+1}\varphi(\tau))^{\alpha-1}log^s\ \varphi(\tau)\ \frac{\varphi'(\tau)}{\varphi(\tau)})\ \frac{d\tau}{\tau}$$

$$+(f(\rho)-r)(R\frac{(s+1)^{1-\alpha}}{\Gamma(\alpha)}\int_a^t(log^{s+1}\varphi(t)-log^{s+1}\varphi(\tau))^{\alpha-1}log^s\ \varphi(\tau)\ \frac{\varphi'(\tau)}{\varphi(\tau)}\ d\tau-\ {}^sI^\alpha_{a,\varphi}f(t))\ \frac{d\tau}{\tau}$$

$$-\ {}^sI^\alpha_{a,\varphi}((R-f(t))(f(t)-r)) \tag{2.8}$$

$$-(R-f(\rho))(f(\rho)-r)\frac{(s+1)^{1-\alpha}}{\Gamma(\alpha)}\int_a^t(log^{s+1}\varphi(t)-log^{s+1}\varphi(\tau))^{\alpha-1}log^s\ \varphi(\tau)\ \frac{\varphi'(\tau)}{\varphi(\tau)}\ \frac{d\tau}{\tau}$$

$$=\ {}^sI^\alpha_{a,\varphi}f^2(t)+f^2(\rho)\frac{(s+1)^{1-\alpha}}{\Gamma(\alpha)}\int_a^t(log^{s+1}\varphi(t)-log^{s+1}\varphi(\tau))^{\alpha-1}log^s\ \varphi(\tau)\ \frac{\varphi'(\tau)}{\varphi(\tau)}\ \frac{d\tau}{\tau}-2f(\rho){}^sI^\alpha_{a,\varphi}f(t)$$

Donc

$$(R-f(\rho))({}^sI^\alpha_{a,\varphi}f(t))-r\frac{(s+1)^{1-\alpha}}{\Gamma(\alpha+1)}\ (log^{s+1}\varphi(t)-log^{s+1}\varphi(a))^\alpha$$

$$+(f(\rho)-r)(R\frac{(s+1)^{1-\alpha}}{\Gamma(\alpha+1)}\ (log^{s+1}\varphi(t)-log^{s+1}\varphi(a))^\alpha-\ {}^sI^\alpha_{a,\varphi}f(t))$$

$$-\ {}^sI^\alpha_{a,\varphi}((R-f(t))(f(t)-r)) \tag{2.9}$$

$$-(R-f(\rho))(f(\rho)-r)\frac{(s+1)^{1-\alpha}}{\Gamma(\alpha+1)}\ (log^{s+1}\varphi(t)-log^{s+1}\varphi(a))^\alpha$$

$$=\ {}^sI^\alpha_{a,\varphi}f^2(t)+f^2(\rho)\frac{(s+1)^{1-\alpha}}{\Gamma(\alpha+1)}\ (log^{s+1}\varphi(t)-log^{s+1}\varphi(a))^\alpha-2f(\rho){}^sI^\alpha_{a,\varphi}f(t)$$

en multipliant (2.9) par la quantité $\frac{(s+1)^{1-\alpha}}{\Gamma(\alpha+1)}\ (log^{s+1}\varphi(t)-log^{s+1}\varphi(\rho))^{\alpha-1}log^s\ \varphi(\rho)\ \frac{\varphi'(\rho)}{\varphi(\rho)}, \rho\in(a,t)$, on a

$$\frac{(s+1)^{1-\alpha}}{\Gamma(\alpha)}\ (log^{s+1}\varphi(t)-log^{s+1}\varphi(\rho))^{\alpha-1}log^s\ \varphi(\rho)\ \frac{\varphi'(\rho)}{\varphi(\rho)}\Big\{(R-f(\rho))({}^sI^\alpha_{a,\varphi}f(t))$$

$$-r\frac{(s+1)^{1-\alpha}}{\Gamma(\alpha+1)}\ (log^{s+1}\varphi(t)-log^{s+1}\varphi(a))^\alpha$$

$$+(f(\rho)-r)(R\frac{(s+1)^{1-\alpha}}{\Gamma(\alpha+1)}\ (log^{s+1}\varphi(t)-log^{s+1}\varphi(a))^\alpha-\ {}^sI^\alpha_{a,\varphi}f(t))$$

$$-\ {}^sI^\alpha_{a,\varphi}((R-f(t))(f(t)-r))$$

$$-(R-f(\rho))(f(\rho)-r)\tfrac{(s+1)^{1-\alpha}}{\Gamma(\alpha+1)}\ \ (log^{s+1}\varphi(t)-log^{s+1}\ \varphi(a))^{\alpha}\Big\}$$

$$= \ \tfrac{(s+1)^{1-\alpha}}{\Gamma(\alpha)}\ \ (log^{s+1}\varphi(t)-log^{s+1}\ \varphi(\rho))^{\alpha-1}log^{s}\ \ \varphi(\rho)\ \tfrac{\varphi'(\rho)}{\varphi(\rho)}\Big\{{}^{s}I^{\alpha}_{a,\varphi}f^{2}(t) \tag{2.10}$$

$$+f^{2}(\rho)\tfrac{(s+1)^{1-\alpha}}{\Gamma(\alpha+1)}\ \ (log^{s+1}\varphi(t)-log^{s+1}\ \varphi(a))^{\alpha}-2f(\rho){}^{s}I^{\alpha}_{a,\varphi}f(t)\Big\}$$

ce qui donne

$$({}^{s}I^{\alpha}_{a,\varphi}f(t))-r\tfrac{(s+1)^{1-\alpha}}{\Gamma(\alpha+1)}(log^{s+1}\varphi(t)-log^{s+1}\ \varphi(a))^{\alpha})$$

$$\tfrac{(s+1)^{1-\alpha}}{\Gamma(\alpha)}\ \ (log^{s+1}\varphi(t)-log^{s+1}\ \varphi(\rho))^{\alpha-1}log^{s}\ \ \varphi(\rho)\ \tfrac{\varphi'(\rho)}{\varphi(\rho)}(R-f(\rho))$$

$$+((R\tfrac{(s+1)^{1-\alpha}}{\Gamma(\alpha+1)}(log^{s+1}\varphi(t)-log^{s+1}\ \varphi(a))^{\alpha})-\ \ {}^{s}I^{\alpha}_{a,\varphi}f(t))$$

$$\tfrac{(s+1)^{1-\alpha}}{\Gamma(\alpha)}\ \ (log^{s+1}\varphi(t)-log^{s+1}\ \varphi(\rho))^{\alpha-1}log^{s}\ \ \varphi(\rho)\ \tfrac{\varphi'(\rho)}{\varphi(\rho)}(f(\rho)-r)$$

$$-\ \ {}^{s}I^{\alpha}_{a,\varphi}(R-f(t))(f(t)-r)\varphi(t)-log^{s+1}\ \varphi(\rho))^{\alpha-1}log^{s}\ \ \varphi(\rho)\ \tfrac{\varphi'(\rho)}{\varphi(\rho)}$$

$$-\tfrac{(s+1)^{1-\alpha}}{\Gamma(\alpha+1)}(log^{s+1}\varphi(t)-log^{s+1}\ \varphi(a))^{\alpha}) \tag{2.11}$$

$$\tfrac{(s+1)^{1-\alpha}}{\Gamma(\alpha)}\ \ (log^{s+1}\varphi(t)-log^{s+1}\ \varphi(\rho))^{\alpha-1}log^{s}\ \ \varphi(\rho)\ \tfrac{\varphi'(\rho)}{\varphi(\rho)}(R-f(\rho))(f(\rho)-r)$$

$$= \ \ {}^{s}I^{\alpha}_{a,\varphi}f^{2}(t)\ \ \tfrac{(s+1)^{1-\alpha}}{\Gamma(\alpha)}\ \ (log^{s+1}\varphi(t)-log^{s+1}\ \varphi(\rho))^{\alpha-1}log^{s}\ \ \varphi(\rho)\ \tfrac{\varphi'(\rho)}{\varphi(\rho)}$$

$$+\tfrac{(s+1)^{1-\alpha}}{\Gamma(\alpha+1)}\ \ (log^{s+1}\varphi(t)-log^{s+1}\ \varphi(a))^{\alpha}$$

$$\tfrac{(s+1)^{1-\alpha}}{\Gamma(\alpha)}(log^{s+1}\varphi(t)-log^{s+1}\ \varphi(\rho))^{\alpha-1}log^{s}\ \ \varphi(\rho)\ \tfrac{\varphi'(\rho)}{\varphi(\rho)}f^{2}(\rho)$$

$$-2f(\rho){}^{s}I^{\alpha}_{a,\varphi}f(t)\ \ \tfrac{(s+1)^{1-\alpha}}{\Gamma(\alpha)}\ \ (log^{s+1}\varphi(t)-log^{s+1}\ \varphi(\rho))^{\alpha-1}log^{s}\ \ \varphi(\rho)\ \tfrac{\varphi'(\rho)}{\varphi(\rho)}$$

par l'intégration de (2.11)par rapport à $\rho$ sur $(a,t)$, on trouve

$$
\begin{aligned}
&({}^{s}I^{\alpha}_{a,\varphi}f(t)) - r\tfrac{(s+1)^{1-\alpha}}{\Gamma(\alpha+1)}(log^{s+1}\varphi(t) - log^{s+1}\ \varphi(a))^{\alpha})\ \tfrac{(s+1)^{1-\alpha}}{\Gamma(\alpha)} \\
&\textstyle\int_a^t (log^{s+1}\varphi(t) - log^{s+1}\ \varphi(\rho))^{\alpha-1}log^{s}\ \ \varphi(\rho)\ \frac{\varphi'(\rho)}{\varphi(\rho)}(R - f(\rho))\frac{d\rho}{\rho} \\
&+((R\tfrac{(s+1)^{1-\alpha}}{\Gamma(\alpha+1)}(log^{s+1}\varphi(t) - log^{s+1}\ \varphi(a))^{\alpha}) -\ {}^{s}I^{\alpha}_{a,\varphi}f(t))\ \tfrac{(s+1)^{1-\alpha}}{\Gamma(\alpha)} \\
&\textstyle\int_a^t (log^{s+1}\varphi(t) - log^{s+1}\ \varphi(\rho))^{\alpha-1}log^{s}\ \ \varphi(\rho)\ \frac{\varphi'(\rho)}{\varphi(\rho)}(f(\rho) - r)\ \ \frac{d\rho}{\rho} \\
&-\ {}^{s}I^{\alpha}_{a,\varphi}(R - f(t))(f(t) - r)\textstyle\int_a^t\ \varphi(t) - log^{s+1}\ \varphi(\rho))^{\alpha-1}log^{s}\ \ \varphi(\rho)\ \frac{\varphi'(\rho)}{\varphi(\rho)}\ \ \frac{d\rho}{\rho} \\
&-\tfrac{(s+1)^{1-\alpha}}{\Gamma(\alpha+1)}(log^{s+1}\varphi(t) - log^{s+1}\ \varphi(a))^{\alpha})\ \tfrac{(s+1)^{1-\alpha}}{\Gamma(\alpha)} \\
&\textstyle\int_a^t (log^{s+1}\varphi(t) - log^{s+1}\ \varphi(\rho))^{\alpha-1}log^{s}\ \ \varphi(\rho)\ \frac{\varphi'(\rho)}{\varphi(\rho)}(R - f(\rho))(f(\rho) - r)\ \ \frac{d\rho}{\rho} \\
&=\ {}^{s}I^{\alpha}_{a,\varphi}f^2(t)\ \ \tfrac{(s+1)^{1-\alpha}}{\Gamma(\alpha)}\ \textstyle\int_a^t\ (log^{s+1}\varphi(t) - log^{s+1}\ \varphi(\rho))^{\alpha-1}log^{s}\ \ \varphi(\rho)\ \frac{\varphi'(\rho)}{\varphi(\rho)}\ \ \frac{d\rho}{\rho} \\
&+\tfrac{(s+1)^{1-\alpha}}{\Gamma(\alpha+1)}\ \ (log^{s+1}\varphi(t) - log^{s+1}\ \varphi(a))^{\alpha}\tfrac{(s+1)^{1-\alpha}}{\Gamma(\alpha)} \\
&\textstyle\int_a^t (log^{s+1}\varphi(t) - log^{s+1}\ \varphi(\rho))^{\alpha-1}log^{s}\ \ \varphi(\rho)\ \frac{\varphi'(\rho)}{\varphi(\rho)}f^2(\rho)\ \ \frac{d\rho}{\rho} \\
&-2{}^{s}I^{\alpha}_{a,\varphi}f(t)\ \ \tfrac{(s+1)^{1-\alpha}}{\Gamma(\alpha)} \\
&\textstyle\int_a^t (log^{s+1}\varphi(t) - log^{s+1}\ \varphi(\rho))^{\alpha-1}log^{s}\ \ \varphi(\rho)\ \frac{\varphi'(\rho)}{\varphi(\rho)}f(\rho)\ \ \frac{d\rho}{\rho} \\
&\left({}^{s}I^{\alpha}_{a,\varphi}f(t) - r\tfrac{(s+1)^{1-\alpha}}{\Gamma(\alpha+1)}(log^{s+1}\varphi(t) - log^{s+1}\ \varphi(a))^{\alpha}\right) \\
&\left(R\tfrac{(s+1)^{1-\alpha}}{\Gamma(\alpha+1)}(log^{s+1}\varphi(t) - log^{s+1}\ \varphi(a))^{\alpha} -\ {}^{s}I^{\alpha}_{a,\varphi}f(t)\right) \\
&+\left(R\tfrac{(s+1)^{1-\alpha}}{\Gamma(\alpha+1)}(log^{s+1}\varphi(t) - log^{s+1}\ \varphi(a))^{\alpha} -\ {}^{s}I^{\alpha}_{a,\varphi}f(t)\right)
\end{aligned}
\tag{2.12}
$$

Alors

$$\left({}^sI^{\alpha}_{a,\varphi}f(t) - r\tfrac{(s+1)^{1-\alpha}}{\Gamma(\alpha+1)}(log^{s+1}\varphi(t) - log^{s+1}\ \varphi(a))^{\alpha}\right)$$

$$-\tfrac{(s+1)^{1-\alpha}}{\Gamma(\alpha+1)}(log^{s+1}\varphi(t) - log^{s+1}\ \varphi(a))^{\alpha})\ {}^sI_{a,\varphi}\left((R - f(t))\left(f(t) - r\right)\right)$$

$$-{}^sI_{a,\varphi}\left((R - f(t))\left(f(t) - r\right)\right)\tfrac{(s+1)^{1-\alpha}}{\Gamma(\alpha+1)}(log^{s+1}\varphi(t) - log^{s+1}\ \varphi(a))^{\alpha}) \tag{2.13}$$

$$= \tfrac{(s+1)^{1-\alpha}}{\Gamma(\alpha+1)}(log^{s+1}\varphi(t) - log^{s+1}\ \varphi(a))^{\alpha}\ \ {}^sI^{\alpha}_{a,\varphi}f^2(t)$$

$$+\tfrac{(s+1)^{1-\alpha}}{\Gamma(\alpha+1)}(log^{s+1}\varphi(t) - log^{s+1}\ \varphi(a))^{\alpha}\ \ {}^sI^{\alpha}_{a,\varphi}f^2(t) - 2\left({}^sI^{\alpha}_{a,\varphi}f(t)\right)^2$$

d'où la résultat . □

maintenant on peut montrer le théorème

**prouve de théorème**

*Démonstration.* Si $f$ et $g$ sont deux fonctions intégrables sur $[a, b]$, vérifié les conditions (2.1) .On a

$$H(\tau, \rho) := (f(\tau) - f(\rho))(g(\tau) - g(\rho)), \quad \tau, \rho \in (a, t),\ \ t > a, \tag{2.14}$$

alors

$$H(\tau, \rho) = f(\tau)g(\tau) + f(\rho)g(\rho) - f(\tau)g(\rho) - f(\rho)g(\tau), \tag{2.15}$$

on multiplie (2.15) par $\frac{(s+1)^{1-\alpha}}{\Gamma(\alpha)}\ \ (log^{s+1}\varphi(t) - log^{s+1}\ \varphi(\tau))^{\alpha-1}log^{s}\ \ \varphi(\tau)\ \frac{\varphi'(\tau)}{\varphi(\tau)}, \tau \in (a, t)$,
on obtient

$$\tfrac{(s+1)^{1-\alpha}}{\Gamma(\alpha)}\ \ (log^{s+1}\varphi(t) - log^{s+1}\ \varphi(\tau))^{\alpha-1}log^{s}\ \ \varphi(\tau)\ \tfrac{\varphi'(\tau)}{\varphi(\tau)}H(\tau, \rho)$$

$$= \tfrac{(s+1)^{1-\alpha}}{\Gamma(\alpha)}\ \ (log^{s+1}\varphi(t) - log^{s+1}\ \varphi(\tau))^{\alpha-1}log^{s}\ \ \varphi(\tau)\ \tfrac{\varphi'(\tau)}{\varphi(\tau)}f(\tau)g(\tau)$$

$$+\tfrac{(s+1)^{1-\alpha}}{\Gamma(\alpha)}\ \ (log^{s+1}\varphi(t) - log^{s+1}\ \varphi(\tau))^{\alpha-1}log^{s}\ \ \varphi(\tau)\ \tfrac{\varphi'(\tau)}{\varphi(\tau)}f(\rho)g(\rho) \tag{2.16}$$

$$-\tfrac{(s+1)^{1-\alpha}}{\Gamma(\alpha)}\ \ (log^{s+1}\varphi(t) - log^{s+1}\ \varphi(\tau))^{\alpha-1}log^{s}\ \ \varphi(\tau)\ \tfrac{\varphi'(\tau)}{h(\tau)}f(\tau)g(\rho)$$

$$-\tfrac{(s+1)^{1-\alpha}}{\Gamma(\alpha)}\ \ (log^{s+1}\varphi(t) - log^{s+1}\ \varphi(\tau))^{\alpha-1}log^{s}\ \ \varphi(\tau)\ \tfrac{\varphi'(\tau)}{\varphi(\tau)}f(\rho)g(\tau)$$

par l'intégration de (2.16) par rapport à $\tau$ sur $(a, t)$, on obtient

$$\begin{aligned}
&\tfrac{(s+1)^{1-\alpha}}{\Gamma(\alpha)} \int_a^t (log^{s+1}\varphi(t) - log^{s+1}\varphi(\tau))^{\alpha-1} log^s\ \varphi(\tau)\ \tfrac{\varphi'(\tau)}{\varphi(\tau)} H(\tau,\rho)\ \tfrac{d\tau}{\tau} \\
&= \tfrac{(s+1)^{1-\alpha}}{\Gamma(\alpha)} \int_a^t (log^{s+1}\varphi(t) - log^{s+1}\varphi(\tau))^{\alpha-1} log^s\ \varphi(\tau)\ \tfrac{\varphi'(\tau)}{\varphi(\tau)} f(\tau)g(\tau)\ \tfrac{d\tau}{\tau} \\
&+ \tfrac{(s+1)^{1-\alpha}}{\Gamma(\alpha)} \int_a^t (log^{s+1}\varphi(t) - log^{s+1}\varphi(\tau))^{\alpha-1} log^s\ \varphi(\tau)\ \tfrac{\varphi'(\tau)}{\varphi(\tau)} f(\rho)g(\rho)\ \tfrac{d\tau}{\tau} \\
&- \tfrac{(s+1)^{1-\alpha}}{\Gamma(\alpha)} \int_a^t (log^{s+1}\varphi(t) - log^{s+1}\varphi(\tau))^{\alpha-1} log^s\ \varphi(\tau)\ \tfrac{\varphi'(\tau)}{\varphi(\tau)} f(\tau)g(\rho)\ \tfrac{d\tau}{\tau} \\
&- \tfrac{(s+1)^{1-\alpha}}{\Gamma(\alpha)} \int_a^t (log^{s+1}\varphi(t) - log^{s+1}\varphi(\tau))^{\alpha-1} log^s\ \varphi(\tau)\ \tfrac{\varphi'(\tau)}{\varphi(\tau)} f(\rho)g(\tau)\ \tfrac{d\tau}{\tau}
\end{aligned} \tag{2.17}$$

Alors

$$\begin{aligned}
&\tfrac{(s+1)^{1-\alpha}}{\Gamma(\alpha)} \int_a^t (log^{s+1}\varphi(t) - log^{s+1}\varphi(\tau))^{\alpha-1} log^s\ \varphi(\tau)\ \tfrac{\varphi'(\tau)}{\varphi(\tau)} H(\tau,\rho)\ \tfrac{d\tau}{\tau} \\
&= {}^sI_{a,\varphi} f\ g(t) + \tfrac{(s+1)^{1-\alpha}}{\Gamma(\alpha+1)} (log^{s+1}\varphi(t) - log^{s+1}\varphi(a))^{\alpha}\ f(\rho)g(\rho) \\
&\quad - g(\rho)\ {}^sI_{a,\varphi} f(t) - f(\rho)\ {}^sI_{a,\varphi} g(t)
\end{aligned} \tag{2.18}$$

maintenant on multiplie (2.18) par $\frac{(s+1)^{1-\alpha}}{\Gamma(\alpha)}\ (log^{s+1}\varphi(t) - log^{s+1}\varphi(\rho))^{\alpha-1} log^s\ \varphi(\rho)\ \frac{\varphi'(\rho)}{\varphi(\rho)}, \rho \in (a,t)$, on trouve

$$\begin{aligned}
&\tfrac{(s+1)^{1-\alpha}}{\Gamma(\alpha)} (log^{s+1}\varphi(t) - log^{s+1}\varphi(\rho))^{\alpha-1} log^s\ \varphi(\rho)\ \tfrac{\varphi'(\rho)}{\varphi(\rho)} \tfrac{(s+1)^{1-\alpha}}{\Gamma(\alpha)} \\
&\int_a^t (log^{s+1}\varphi(t) - log^{s+1}\varphi(\tau))^{\alpha-1} log^s\ \varphi(\tau)\ \tfrac{\varphi'(\tau)}{\varphi(\tau)} H(\tau,\rho)\ d\tau \\
&= \tfrac{(s+1)^{1-\alpha}}{\Gamma(\alpha)} (log^{s+1}\varphi(t) - log^{s+1}\varphi(\rho))^{\alpha-1} log^s\ \varphi(\rho)\ \tfrac{\varphi'(\rho)}{\varphi(\rho)}\ {}^sI_{a,\varphi} f\ g(t) \\
&+ \tfrac{(s+1)^{1-\alpha}}{\Gamma(\alpha)} (log^{s+1}\varphi(t) - log^{s+1}\varphi(\rho))^{\alpha-1} log^s\ \varphi(\rho)\ \tfrac{\varphi'(\rho)}{\varphi(\rho)} \\
&\tfrac{(s+1)^{1-\alpha}}{\Gamma(\alpha+1)} (log^{s+1}\varphi(t) - log^{s+1}\varphi(a))^{\alpha}\ f(\rho)g(\rho) \\
&- \tfrac{(s+1)^{1-\alpha}}{\Gamma(\alpha)} (log^{s+1}\varphi(t) - log^{s+1}\varphi(\rho))^{\alpha-1} log^s\ \varphi(\rho)\ \tfrac{\varphi'(\rho)}{\varphi(\rho)} g(\rho)\ {}^sI_{a,\varphi} f(t) \\
&- \tfrac{(s+1)^{1-\alpha}}{\Gamma(\alpha)} (log^{s+1}\varphi(t) - log^{s+1}\varphi(\rho))^{\alpha-1} log^s\ \varphi(\rho)\ \tfrac{\varphi'(\rho)}{\varphi(\rho)} f(\rho)\ {}^sI_{a,\varphi} g(t)
\end{aligned} \tag{2.19}$$

on intégre (2.19) par rapport à $\rho$ sur $(a,t)$,on peut écrire

$$\frac{(s+1)^{1-\alpha}}{\Gamma(\alpha)} \int_a^t (log^{s+1}\varphi(t) - log^{s+1}\varphi(\rho))^{\alpha-1} log^s \varphi(\rho) \frac{\varphi'(\rho)}{\varphi(\rho)} \frac{d\rho}{\rho} \frac{(s+1)^{1-\alpha}}{\Gamma(\alpha)}$$

$$\int_a^t (log^{s+1}\varphi(t) - log^{s+1}\varphi(\tau))^{\alpha-1} log^s \varphi(\tau) \frac{\varphi'(\tau)}{\varphi(\tau)} H(\tau,\rho) \frac{d\tau}{\tau}$$

$$= \frac{(s+1)^{1-\alpha}}{\Gamma(\alpha)} \int_a^t (log^{s+1}\varphi(t) - log^{s+1}\varphi(\rho))^{\alpha-1} log^s \varphi(\rho) \frac{\varphi'(\rho)}{\varphi(\rho)} \frac{d\rho}{\rho} \, {}^s I_{a,\varphi} f\, g(t)$$

$$+ \frac{(s+1)^{1-\alpha}}{\Gamma(\alpha)} \int_a^t (log^{s+1}\varphi(t) - log^{s+1}\varphi(\rho))^{\alpha-1} log^s \varphi(\rho) \frac{\varphi'(\rho)}{\varphi(\rho)} \tag{2.20}$$

$$\frac{(s+1)^{1-\alpha}}{\Gamma(\alpha+1)} (log^{s+1}\varphi(t) - log^{s+1}\varphi(a))^{\alpha} f(\rho)g(\rho) \frac{d\rho}{\rho}$$

$$- \frac{(s+1)^{1-\alpha}}{\Gamma(\alpha)} \int_a^t (log^{s+1}\varphi(t) - log^{s+1}\varphi(\rho))^{\alpha-1} log^s \varphi(\rho) \frac{\varphi'(\rho)}{\varphi(\rho)} g(\rho) \, {}^s I_{a,\varphi} f(t) \frac{d\rho}{\rho}$$

$$- \frac{(s+1)^{1-\alpha}}{\Gamma(\alpha)} \int_a^t (log^{s+1}\varphi(t) - log^{s+1}\varphi(\rho))^{\alpha-1} log^s \varphi(\rho) \frac{\varphi'(\rho)}{\varphi(\rho)} f(\rho) \, {}^s I_{a,\varphi} g(t) \frac{d\rho}{\rho}$$

Alors

$$\frac{(s+1)^{1-\alpha}}{\Gamma(\alpha)} \int_a^t (log^{s+1}\varphi(t) - log^{s+1}\varphi(\rho))^{\alpha-1} log^s \varphi(\rho) \frac{\varphi'(\rho)}{\varphi(\rho)} \frac{d\rho}{\rho}$$

$$\frac{(s+1)^{1-\alpha}}{\Gamma(\alpha)} \int_a^t (log^{s+1}\varphi(t) - log^{s+1}\varphi(\tau))^{\alpha-1} log^s \varphi(\tau) \frac{\varphi'(\tau)}{\varphi(\tau)} H(\tau,\rho) \frac{d\tau}{\tau} \tag{2.21}$$

$$= 2 \frac{(s+1)^{1-\alpha}}{\Gamma(\alpha+1)} (log^{s+1}\varphi(t) - (log^{s+1}\varphi(a)^{\alpha} \, {}^s I_{a,\varphi} f g(t)$$

$$- 2 \, {}^s I_{a,\varphi} f(t) \, {}^s I_{a,\varphi} g(t)$$

maintenant on applique l'inégalité de Cauchy Schwarz, on a

$$\left( \frac{(s+1)^{1-\alpha}}{\Gamma(\alpha+1)} (log^{s+1}\varphi(t) - (log^{s+1}\varphi(a)^{\alpha} \, {}^s I_{a,\varphi} f g(t) - {}^s I_{a,\varphi} f(t) \, {}^s I_{a,\varphi} g(t) \right)^2$$

$$\leq \left( \frac{(s+1)^{1-\alpha}}{\Gamma(\alpha+1)} (log^{s+1}\varphi(t) - (log^{s+1}\varphi(a)^{\alpha} \, {}^s I_{a,\varphi} f(t)^2 - ({}^s I_{a,\varphi} f(t))^2 \right) \tag{2.22}$$

$$\left( \frac{(s+1)^{1-\alpha}}{\Gamma(\alpha+1)} (log^{s+1}\varphi(t) - (log^{s+1}\varphi(a)^{\alpha s} I^{\alpha}_{a,\varphi} g^2(t) - \left({}^s I^{\alpha}_{a,\varphi} g(t)\right)^2 \right).$$

comme $(R - f(x))(f(x) - r) \geq 0$ et $(N - g(x))(g(x) - n) \geq 0, alors$

$$\frac{(s+1)^{1-\alpha}}{\Gamma(\alpha+1)} (log^{s+1}\varphi(t) - (log^{s+1}\varphi(a))^{\alpha} \, {}^s I^{\alpha}_{a,\varphi} (R - f(t))(f(t) - r) \geq 0, \tag{2.23}$$

et on a
$$\frac{(s+1)^{1-\alpha}}{\Gamma(\alpha+1)}\ (log^{s+1}\varphi(t)-(log^{s+1}\varphi(a))^{\alpha}\ \ {}^{s}I^{\alpha}_{a,\varphi}(N-g(t))(g(t)-n)\geq 0, \tag{2.24}$$

Alors
$$\begin{aligned}
&\left(R\tfrac{(s+1)^{1-\alpha}}{\Gamma(\alpha+1)}\ (log^{s+1}\varphi(t)-(log^{s+1}\varphi(a))^{\alpha}-\ {}^{s}I^{\alpha}_{a,\varphi}f(t)\right)\\
&\left({}^{s}I^{\alpha}_{a,\varphi}f(t)-r\frac{(s+1)^{1-\alpha}}{\Gamma(\alpha+1)}\ (log^{s+1}\varphi(t)-(log^{s+1}\varphi(a))^{\alpha}\right)\\
&-\tfrac{(s+1)^{1-\alpha}}{\Gamma(\alpha+1)}\ (log^{s+1}\varphi(t)-(log^{s+1}\varphi(a))^{\alpha}\ \ {}^{s}I^{\alpha}_{a,\varphi}(R-f(t))(f(t)-r)\\
&\leq\left(R\tfrac{(s+1)^{1-\alpha}}{\Gamma(\alpha+1)}\ (log^{s+1}\varphi(t)-(log^{s+1}\varphi(a))^{\alpha}-{}^{s}I^{\alpha}_{a,\varphi}f(t)\right)\\
&\left({}^{s}I^{\alpha}_{a,\varphi}f(t)-\frac{(s+1)^{1-\alpha}}{\Gamma(\alpha+1)}\ (log^{s+1}\varphi(t)-(log^{s+1}\varphi(a))^{\alpha}\right),
\end{aligned} \tag{2.25}$$

et
$$\begin{aligned}
&\left(R-{}^{s}I^{\alpha}_{a,\varphi}g(t)\right)\left({}^{s}I^{\alpha}_{a,\varphi}g(t)-r\frac{(s+1)^{1-\alpha}}{\Gamma(\alpha+1)}\ (log^{s+1}\varphi(t)-(log^{s+1}\varphi(a))^{\alpha}\right)\\
&-\tfrac{(s+1)^{1-\alpha}}{\Gamma(\alpha+1)}\ (log^{s+1}\varphi(t)-(log^{s+1}\varphi(a))^{\alpha s}I^{\alpha}_{a,\varphi}(N-g(t))(g(t)-n)\\
&\leq\left(N\tfrac{(s+1)^{1-\alpha}}{\Gamma(\alpha+1)}\ (log^{s+1}\varphi(t)-(log^{s+1}\varphi(a))^{\alpha}-{}^{s}I^{\alpha}_{a,\varphi}g(t)\right)\\
&\left({}^{s}I^{\alpha}_{a,\varphi}g(t)-n\frac{(s+1)^{1-\alpha}}{\Gamma(\alpha+1)}\ (log^{s+1}\varphi(t)-(log^{s+1}\varphi(a))^{\alpha}\right),
\end{aligned} \tag{2.26}$$

à partir du lemme (1) , on a
$$\begin{aligned}
&\tfrac{(s+1)^{1-\alpha}}{\Gamma(\alpha+1)}\ \ (log^{s+1}\varphi(t)-(log^{s+1}\varphi(a))^{\alpha s}I^{\alpha}_{a,\varphi}f^{2}(t)-\left({}^{s}I^{\alpha}_{a,\varphi}f(t)\right)^{2}\\
&\leq\left(R\tfrac{(s+1)^{1-\alpha}}{\Gamma(\alpha+1)}\ (log^{s+1}\varphi(t)-(log^{s+1}\varphi(a))^{\alpha}-{}^{s}I^{\alpha}_{a,\varphi}f(t)\right)\\
&\left({}^{s}I^{\alpha}_{a,\varphi}f(t)-r\frac{(s+1)^{1-\alpha}}{\Gamma(\alpha+1)}\ (log^{s+1}\varphi(t)-(log^{s+1}\varphi(a))^{\alpha}\right),
\end{aligned} \tag{2.27}$$

et

$$\frac{(s+1)^{1-\alpha}}{\Gamma(\alpha+1)} \ (log^{s+1}\varphi(t)-(log^{s+1}\varphi(a))^{\alpha} \ {}^{s}I^{\alpha}_{a,\varphi}g^{2}(t)-\left({}^{s}I^{\alpha}_{a,\varphi}g(t)\right)^{2}$$

$$\leq \left(N\frac{(s+1)^{1-\alpha}}{\Gamma(\alpha+1)} \ (log^{s+1}\varphi(t)-(log^{s+1}\varphi(a))^{\alpha}-{}^{s}I^{\alpha}_{a,\varphi}g(t)\right) \tag{2.28}$$

$$\left({}^{s}I^{\alpha}_{a,\varphi}g(t)-n\frac{(s+1)^{1-\alpha}}{\Gamma(\alpha+1)} \ (log^{s+1}\varphi(t)-(log^{s+1}\varphi(a))^{\alpha}\right),$$

D'après l'inégalité de Cauchy Schwarz on peut écrire

$$\left(\frac{(s+1)^{1-\alpha}}{\Gamma(\alpha+1)} \ (log^{s+1}\varphi(t)-(log^{s+1}\varphi(a))^{\alpha} \ {}^{s}I^{\alpha}_{a,\varphi}fg(t)- \ {}^{s}I^{\alpha}_{a,\varphi}f(t) \ {}^{s}I^{\alpha}_{a,\varphi}g(t)\right)^{2}$$

$$\leq \left(\frac{(s+1)^{1-\alpha}}{\Gamma(\alpha+1)} \ (log^{s+1}\varphi(t)-(log^{s+1}\varphi(a))^{\alpha} \ {}^{s}I^{\alpha}_{a,\varphi}f^{2}(t)-\left( \ {}^{s}I^{\alpha}_{a,\varphi}f(t)\right)^{2}\right)$$

$$\left(\frac{(s+1)^{1-\alpha}}{\Gamma(\alpha+1)} \ (log^{s+1}\varphi(t)-(log^{s+1}\varphi(a))^{\alpha} \ {}^{s}I^{\alpha}_{a,\varphi}g^{2}(t)-\left( \ {}^{s}I^{\alpha}_{a,\varphi}g(t)\right)^{2}\right),$$

à partir de (2.27) , (2.28) , on trouve

$$\left(\frac{(s+1)^{1-\alpha}}{\Gamma(\alpha+1)} \ (log^{s+1}\varphi(t)-(log^{s+1}\varphi(a))^{\alpha} \ {}^{s}I^{\alpha}_{a,\varphi}fg(t)-{}^{s}I^{\alpha}_{a,\varphi}f(t){}^{s}I^{\alpha}_{a,\varphi}g(t)\right)^{2}$$

$$\leq \left(R\frac{(s+1)^{1-\alpha}}{\Gamma(\alpha+1)} \ (log^{s+1}\varphi(t)-(log^{s+1}\varphi(a))^{\alpha}-{}^{s}I^{\alpha}_{a,\varphi}f(t)\right)$$

$$\left({}^{s}I^{\alpha}_{a,\varphi}f(t)-r\frac{(s+1)^{1-\alpha}}{\Gamma(\alpha+1)} \ (log^{s+1}\varphi(t)-(log^{s+1}\varphi(a))^{\alpha}\right) \tag{2.29}$$

$$\times\left(N\frac{(s+1)^{1-\alpha}}{\Gamma(\alpha+1)} \ (log^{s+1}\varphi(t)-(log^{s+1}\varphi(a))^{\alpha}-{}^{s}I^{\alpha}_{a,\varphi}g(t)\right)$$

$$\left({}^{s}I^{\alpha}_{a,\varphi}g(t)-n\frac{(s+1)^{1-\alpha}}{\Gamma(\alpha+1)} \ (log^{s+1}\varphi(t)-(log^{s+1}\varphi(a))^{\alpha}\right).$$

par l'utilisation de l'inégalité élémentaire $4rs \leq (r+s)^2, r, s \in \mathbb{R}$ , on a

$$4\left(R\frac{(s+1)^{1-\alpha}}{\Gamma(\alpha+1)}\ (log^{s+1}\varphi(t)-(log^{s+1}\varphi(a))^{\alpha}-{}^{s}I^{\alpha}_{a,\varphi}f(t)\right)$$

$$\left({}^{s}I^{\alpha}_{a,\varphi}f(t)-r\frac{(s+1)^{1-\alpha}}{\Gamma(\alpha+1)}\ (log^{s+1}\varphi(t)-(log^{s+1}\varphi(a))^{\alpha}\right) \tag{2.30}$$

$$\leq\left(\frac{(s+1)^{1-\alpha}}{\Gamma(\alpha+1)}\ (log^{s+1}\varphi(t)-(log^{s+1}\varphi(a))^{\alpha}(R-r)\right)^2,$$

et

$$4\left(N\frac{(s+1)^{1-\alpha}}{\Gamma(\alpha+1)}\ (log^{s+1}\varphi(t)-(log^{s+1}\varphi(a))^{\alpha}-{}^{s}I^{\alpha}_{a,\varphi}g(t)\right)$$

$$\left({}^{s}I^{\alpha}_{a,\varphi}g(t)-n\frac{(s+1)^{1-\alpha}}{\Gamma(\alpha+1)}\ (log^{s+1}\varphi(t)-(log^{s+1}\varphi(a))^{\alpha}\right) \tag{2.31}$$

$$\leq\left(\frac{(s+1)^{1-\alpha}}{\Gamma(\alpha+1)}\ (log^{s+1}\varphi(t)-(log^{s+1}\varphi(a))^{\alpha}(N-n)\right)^2,$$

on remplaçant (2.30) et (2.31) dans (2.29) ,
nous trouvons

$$\left(\frac{(s+1)^{1-\alpha}}{\Gamma(\alpha+1)}(log^{s+1}\varphi(t)-log^{s+1}\ \varphi(a))^{\alpha}\ \ {}^{s}I^{\alpha}_{a,\varphi}fg(t)-{}^{s}I^{\alpha}_{a,\varphi}f(t){}^{s}I^{\alpha}_{a,\varphi}g(t)\right)^2$$

$$\leq\left(\frac{(s+1)^{1-\alpha}}{2\Gamma(\alpha+1)}(log^{s+1}\varphi(t)-log^{s+1}\ \varphi(a))^{\alpha}(R-r)\right)^2 .$$

$$\left(\frac{(s+1)^{1-\alpha}}{2\Gamma(\alpha+1)}(log^{s+1}\varphi(t)-log^{s+1}\ \varphi(a))^{\alpha}(N-n)\right)^2$$

d'où

$$\left|\frac{(s+1)^{1-\alpha}}{\Gamma(\alpha+1)}\ (log^{s+1}\varphi(t)-(log^{s+1}\varphi(a))^{\alpha}\ \ {}^{s}I^{\alpha}_{a,\varphi}fg(t)-{}^{s}I^{\alpha}_{a,\varphi}f(t){}^{s}I^{\alpha}_{a,\varphi}g(t)\right| \tag{2.32}$$

$$\leq\left(\frac{(s+1)^{1-\alpha}}{\Gamma(\alpha+1)}\ (log^{s+1}\varphi(t)-(log^{s+1}\varphi(a))^{\alpha}\right)^2(R-r)(N-n).$$

□

## 2.2 Inégalités fractionnaires avec deux paramètres $(\alpha, \beta)$

voici un autre résultat qu'est donné par le théorème suivant

**Théorème 2.** *Si $f$ et $g$ sont deux fonctions intégrables sur $[0, \infty[$ vérifié la condition* (2.1) *sur $[0, \infty[$ alors $\forall\ t > 0, \alpha > 0, \beta > 0,$ on a :*

$$
\begin{aligned}
&[\tfrac{(s+1)^{1-\alpha}}{\Gamma(\alpha+1)}\ (log^{s+1}\varphi(t) - (log^{s+1}\varphi(a))^{\alpha}\ \ {}^{s}I^{\beta}_{a,\varphi}fg(t) + \\
&\tfrac{(s+1)^{1-\beta}}{\Gamma(\beta+1)}\ (log^{s+1}\varphi(t) - (log^{s+1}\varphi(a))^{\beta s}I^{\alpha}_{a,\varphi}fg(t) \\
&- {}^{s}I^{\alpha}_{a,\varphi}f(t)^{s}I^{\beta}_{a,\varphi}g(t) - {}^{s}I^{\beta}_{a,\varphi}f(t)^{s}I^{\alpha}_{a,\varphi}g(t)]^{2} \\
&\leq \left((R\tfrac{(s+1)^{1-\alpha}}{\Gamma(\alpha+1)}\ (log^{s+1}\varphi(t) - (log^{s+1}\varphi(a))^{\alpha} - {}^{s}I^{\alpha}_{a,\varphi}f(t)\right) \\
&\left({}^{s}I^{\beta}_{a,\varphi}f(t) - r\frac{(s+1)^{1-\beta}}{\Gamma(\beta+1)}\ (log^{s+1}\varphi(t) - (log^{s+1}\varphi(a))^{\beta}\right) \\
&+ \left({}^{s}I^{\alpha}_{a,\varphi}f(t) - r\frac{(s+1)^{1-\alpha}}{\Gamma(\alpha+1)}\ (log^{s+1}\varphi(t) - (log^{s+1}\varphi(a))^{\alpha}\right) \\
&\left(R\tfrac{(s+1)^{1-\beta}}{\Gamma(\beta+1)}\ (log^{s+1}\varphi(t) - (log^{s+1}\varphi(a))^{\beta} - {}^{s}I^{\beta}_{a,\varphi}f(t)\right) \\
&\times \left(N\tfrac{(s+1)^{1-\alpha}}{\Gamma(\alpha+1)}\ (log^{s+1}\varphi(t) - (log^{s+1}\varphi(a))^{\alpha} - {}^{s}I^{\alpha}_{a,\varphi}g(t)\right) \\
&\left({}^{s}I^{\beta}_{a,\varphi}g(t) - n\frac{(s+1)^{1-\beta}}{\Gamma(\beta+1)}\ (log^{s+1}\varphi(t) - (log^{s+1}\varphi(a))^{\beta}\right) \\
&+ \left({}^{s}I^{\alpha}_{a,\varphi}g(t) - n\frac{(s+1)^{1-\alpha}}{\Gamma(\alpha+1)}\ (log^{s+1}\varphi(t) - (log^{s+1}\varphi(a))^{\alpha}\right) \\
&\left(N\tfrac{(s+1)^{1-\beta}}{\Gamma(\beta+1)}\ (log^{s+1}\varphi(t) - (log^{s+1}\varphi(a))^{\beta} - {}^{s}I^{\beta}_{a,\varphi}g(t)\right).
\end{aligned}
\tag{2.33}
$$

Pour montrer le théorème (2), nous avons besoin des lemmes suivants :

**Lemme 2.** *Si $f$ et $g$ sont deux fonctions intégrables sur $[0, \infty[$ alors $\forall\ t > 0, \alpha > 0, \beta > 0,$ on a :*

$$[\tfrac{(s+1)^{1-\alpha}}{\Gamma(\alpha+1)} \ (log^{s+1}\varphi(t) - (log^{s+1}\varphi(a))^{\alpha s} I^{\beta}_{a,\varphi} fg(t) +$$

$$\tfrac{(s+1)^{1-\beta}}{\Gamma(\beta+1)} \ (log^{s+1}\varphi(t) - (log^{s+1}\varphi(a))^{\beta s} I^{\alpha}_{a,\varphi} fg(t)$$

$$- \ {}^{s}I^{\alpha}_{a,\varphi} f(t) {}^{s}I^{\beta}_{a,\varphi} g(t) - {}^{s}I^{\beta}_{a,\varphi} f(t) {}^{s}I^{\alpha}_{a,\varphi} g(t)]^2$$

$$\leq [\tfrac{(s+1)^{1-\alpha}}{\Gamma(\alpha+1)} \ (log^{s+1}\varphi(t) - (log^{s+1}\varphi(a))^{\alpha s} I^{\beta}_{a,\varphi} f^2(t) + \tag{2.34}$$

$$\tfrac{(s+1)^{1-\beta}}{\Gamma(\beta+1)} \ (log^{s+1}\varphi(t) - (log^{s+1}\varphi(a))^{\beta s} I^{\alpha}_{a,\varphi} f^2(t) - 2^{s} I^{\alpha}_{a,\varphi} f(t) {}^{s}I^{\beta}_{a,\varphi} f(t)]$$

$$\times [\tfrac{(s+1)^{1-\alpha}}{\Gamma(\alpha+1)} \ (log^{s+1}\varphi(t) - (log^{s+1}\varphi(a))^{\alpha s} I^{\beta}_{a,\varphi} g^2(t) +$$

$$\tfrac{(s+1)^{1-\beta}}{\Gamma(\beta+1)} \ (log^{s+1}\varphi(t) - (log^{s+1}\varphi(a))^{\beta s} I^{\alpha}_{a,\varphi} g^2(t) - 2^{s} I^{\alpha}_{a,\varphi} g(t) {}^{s}I^{\beta}_{a,\varphi} g(t)].$$

*Démonstration.* On multiplie (2.18) par la quantité suivant $\tfrac{(s+1)^{1-\beta}}{\Gamma(\beta)} \ (log^{s+1}\varphi(t) - (log^{s+1}\varphi(\rho))^{\beta-1} \, log^{s}\varphi(\rho) \tfrac{\varphi'(\rho)}{\varphi(\rho)}$, $\rho \in (0,t)$, on trouve

$$\tfrac{(s+1)^{1-\beta}}{\Gamma(\beta)} \ (log^{s+1}\varphi(t) - (log^{s+1}\varphi(\rho))^{\beta-1} \, log^{s}\varphi(\rho) \tfrac{\varphi'(\rho)}{\varphi(\rho)}$$

$$\tfrac{(s+1)^{1-\alpha}}{\Gamma(\alpha)} \ \int_a^t (log^{s+1}\varphi(t) - log^{s+1} \ h(\tau))^{\alpha-1} log^{s} \ \varphi(\tau) \ \tfrac{\varphi'(\tau)}{\varphi(\tau)} H(\tau,\rho) \ \tfrac{d\tau}{\tau}$$

$$= \tfrac{(s+1)^{1-\beta}}{\Gamma(\beta)} \ (log^{s+1}\varphi(t) - (log^{s+1}\varphi(\rho))^{\beta-1} \, log^{s}\varphi(\rho) \tfrac{\varphi'(\rho)}{\varphi(\rho)} {}^{s}I^{\alpha}_{a,\varphi} fg(t)$$

$$+ \ \tfrac{(s+1)^{1-\beta}}{\Gamma(\beta)} \ (log^{s+1}\varphi(t) - (log^{s+1}\varphi(\rho))^{\beta-1} \ \ log^{s}\varphi(\rho) \ \tfrac{\varphi'(\rho)}{\varphi(\rho)} \ \ \tfrac{(s+1)^{1-\alpha}((log^{s+1}\varphi(t) - (log^{s+1}\varphi(a))^{\alpha}}{\Gamma(\alpha+1)} f(\rho) g(\rho)$$

$$\tfrac{(s+1)^{1-\beta}}{\Gamma(\beta)} \ (log^{s+1}\varphi(t) - (log^{s+1}\varphi(\rho))^{\beta-1} \, log^{s}\varphi(\rho) \ \tfrac{\varphi'(\rho)}{\varphi(\rho)} g(\rho) \ \ {}^{s}I^{\alpha}_{a,\varphi} f(t)$$

$$- \ \tfrac{(s+1)^{1-\beta}}{\Gamma(\beta)} \ (log^{s+1}\varphi(t) - (log^{s+1}\varphi(\rho))^{\beta-1} \, log^{s}\varphi(\rho) \ \tfrac{\varphi'(\rho)}{\varphi(\rho)} f(\rho) {}^{s}I^{\alpha}_{a,\varphi} g(t), \tag{2.35}$$

en intégrant (2.35) par rapport à $\rho$ sur $(a,t)$, on obtient

$$\frac{(s+1)^{1-\beta}}{\Gamma(\beta)}\int_a^t (log^{s+1}\varphi(t)-(log^{s+1}\varphi(\rho))^{\beta-1})\ log^s\varphi(\rho)\frac{\varphi'(\rho)}{\varphi(\rho)}\frac{d\rho}{\rho}$$

$$\frac{(s+1)^{1-\alpha}}{\Gamma(\alpha)}\int_a^t (log^{s+1}\varphi(t)-log^{s+1}\ h(\tau))^{\alpha-1}log^s\ \ \varphi(\tau)\ \frac{\varphi'(\tau)}{\varphi(\tau)}(H(\tau,\rho)\frac{d\tau}{\tau}$$

$$= \frac{(s+1)^{1-\beta}}{\Gamma(\beta)}\ \int_a^t (log^{s+1}\varphi(t)-(log^{s+1}\varphi(\rho))^{\beta-1}\ log^s\varphi(\rho)\frac{\varphi'(\rho)}{\varphi(\rho)}\ \ \frac{d\rho}{\rho}{}^sI^{\alpha}_{a,\varphi}fg(t)$$

$$+\ \frac{(s+1)^{1-\beta}}{\Gamma(\beta)}\ \int_a^t (log^{s+1}\varphi(t)-(log^{s+1}\varphi(\rho))^{\beta-1}\ log^s\varphi(\rho)\frac{\varphi'(\rho)}{\varphi(\rho)} \tag{2.36}$$

$$\frac{(s+1)^{1-\alpha}((log^{s+1}\varphi(t)-(log^{s+1}\varphi(a))^{\alpha}}{\Gamma(\alpha+1)}f(\rho)g(\rho)\frac{d\rho}{\rho}$$

$$\frac{(s+1)^{1-\beta}}{\Gamma(\beta)}\ \int_a^t (log^{s+1}\varphi(t)-(log^{s+1}\varphi(\rho))^{\beta-1}log^s\varphi(\rho)\ \frac{\varphi'(\rho)}{\varphi(\rho)}g(\rho)\frac{d\rho}{\rho}\ \ {}^sI^{\alpha}_{a,\varphi}f(t)$$

$$-\ \frac{(s+1)^{1-\beta}}{\Gamma(\beta)}\ \ \int_a^t (log^{s+1}\varphi(t)-(log^{s+1}\varphi(\rho))^{\beta-1}log^s\varphi(\rho)\ \frac{\varphi'(\rho)}{\varphi(\rho)}f(\rho)\frac{d\rho}{\rho}{}^sI^{\alpha}_{a,\varphi}g(t).$$

Par conséquent

$$\frac{(s+1)^{1-\beta}}{\Gamma(\beta)}\int_a^t (log^{s+1}\varphi(t)-(log^{s+1}\varphi(\rho))^{\beta-1}\ log^s\varphi(\rho)\frac{\varphi'(\rho)}{\varphi(\rho)}\frac{d\rho}{\rho}$$

$$\frac{(s+1)^{1-\alpha}}{\Gamma(\alpha)}(log^{s+1}\varphi(t)-log^{s+1}\ h(\tau))^{\alpha-1}log^s\ \ \varphi(\tau)\ \frac{\varphi'(\tau)}{\varphi(\tau)}(H(\tau,\rho)\frac{d\rho}{\rho} \tag{2.37}$$

$$=^s\ I^{\alpha}_{a,\varphi}(\varphi(t))^sI^{\alpha}_{a,\varphi}fg(t)+\frac{(s+1)^{1-\beta}}{\Gamma(\beta+1)}(log^{s+1}\varphi(t)-(log^{s+1}\varphi(a))^{\beta s}I^{\beta}_{a,\varphi}fg(t)$$

$$-{}^sI^{\alpha}_{a,\varphi}f(t)^sI^{\beta}_{a,\varphi}g(t)-{}^sI^{\beta}_{a,\varphi}f(t)^sI^{\alpha}_{a,\varphi}g(t).$$

En appliquant l'inégalité de Cauchy-Schwarz , nous obtenons

$$[(\frac{(s+1)^{1-\alpha}((log^{s+1}\varphi(t)-(log^{s+1}\varphi(a))^{\alpha}}{\Gamma(\alpha+1)}{}^{s}I^{\beta}_{a,\varphi}fg(t)+\frac{(s+1)^{1-\beta}((log^{s+1}\varphi(t)-(log^{s+1}\varphi(a))^{\beta}}{\Gamma(\beta+1)}{}^{s}I^{\alpha}_{a,\varphi}fg(t)$$

$$-\ {}^{s}I^{\alpha}_{a,\varphi}f(t){}^{s}I^{\beta}_{a,\varphi}g(t)-{}^{s}I^{\beta}_{a,\varphi}f(t){}^{s}I^{\alpha}_{a,\varphi}g(t)]^{2}$$

$$\leq[\frac{(s+1)^{1-\alpha}((log^{s+1}\varphi(t)-(log^{s+1}\varphi(a))^{\alpha}}{\Gamma(\alpha+1)}{}^{s}I^{\beta}_{a,\varphi}f^{2}(t)+\frac{(s+1)^{1-\beta}((log^{s+1}\varphi(t)-(log^{s+1}\varphi(a))^{\beta}}{\Gamma(\beta+1)}$$

$${}^{s}I^{\alpha}_{a,\varphi}f^{2}(t)-2{}^{s}I^{\alpha}_{a,\varphi}f(t){}^{s}I^{\beta}_{a,\varphi}f(t)]$$

$$\times[(\frac{(s+1)^{1-\alpha}((log^{s+1}\varphi(t)-(log^{s+1}\varphi(a))^{\alpha}}{\Gamma(\alpha+1)}{}^{s}I^{\beta}_{a,\varphi}g^{2}(t)+\frac{(s+1)^{1-\beta}((log^{s+1}\varphi(t)-(log^{s+1}\varphi(a))^{\beta}}{\Gamma(\beta+1)}$$

$${}^{s}I^{\alpha}_{a,\varphi}g^{2}(t)-2{}^{s}I^{\alpha}_{a,\varphi}g(t){}^{s}I^{\beta}_{a,\varphi}g(t)].$$

□

**Lemme 3.** *Soit $f$ une fonction intégrable sur $[a,b]$ vérifié la condition* (2.1) *sur $[a,b]$ . Alors $\forall\ \ t>a$, $\alpha>0$, $\beta>0$, on a :*

$$\frac{(s+1)^{1-\alpha}}{\Gamma(\alpha+1)}(log^{s+1}\varphi(t)-log^{s+1}\varphi(a)^{\alpha s}I^{\beta}_{a,\varphi}f^{2}(t)$$

$$+\frac{(s+1)^{1-\beta}}{\Gamma(\beta+1)}(log^{s+1}\varphi(t)-log^{s+1}\varphi(a)^{\beta s}I^{\alpha}_{a,\varphi}f^{2}(t)$$

$$-2^{s}I^{\alpha}_{a,\varphi}f(t){}^{s}I^{\beta}_{a,\varphi}f(t)$$

$$=\left(R\frac{(s+1)^{1-\alpha}}{\Gamma(\alpha+1)}(log^{s+1}\varphi(t)-log^{s+1}\varphi(a)^{\alpha}-{}^{s}I^{\alpha}_{a,\varphi}f(t)\right)$$

$$\left({}^{s}I^{\beta}_{a,\varphi}f(t)-r\frac{(s+1)^{1-\beta}}{\Gamma(\beta+1)}(log^{s+1}\varphi(t)-log^{s+1}\varphi(a)^{\beta}\right)$$

$$+\left(R\tfrac{(s+1)^{1-\beta}}{\Gamma(\beta+1)}(log^{s+1}\varphi(t)-log^{s+1}\varphi(a)^{\beta}-{}^{s}I^{\beta}_{a,\varphi}f(t)\right)$$

$$\left({}^{s}I^{\alpha}_{a,\varphi}f(t)-r\frac{(s+1)^{1-\alpha}}{\Gamma(\alpha+1)}(log^{s+1}\varphi(t)-log^{s+1}\varphi(a)^{\alpha}\right)$$

$$-\tfrac{(s+1)^{1-\alpha}}{\Gamma(\alpha+1)}(log^{s+1}\varphi(t)-log^{s+1}\varphi(a)^{\alpha s}I^{\beta}_{a,\varphi}(R-f(t))(f(t)-r) \tag{2.38}$$

$$-\tfrac{(s+1)^{1-\beta}}{\Gamma(\beta+1)}(log^{s+1}\varphi(t)-log^{s+1}\varphi(a)^{\beta s}I^{\alpha}_{a,\varphi}(R-f(t))(f(t)-r).$$

*Démonstration.* On multiplie (2.9) par la quantité suivant $\frac{(s+1)^{1-\beta}}{\Gamma(\beta)}\ (log^{s+1}\varphi(t)-log^{s+1}\ \varphi(\rho))^{\beta-1}log^{s}\varphi(\rho)\frac{\varphi'(\rho)}{\varphi(\rho)}$, $\rho\in(a,t)$ , on obtient

$$\tfrac{(s+1)^{1-\beta}}{\Gamma(\beta)}\ (log^{s+1}\varphi(t)-log^{s+1}\ \varphi(\rho))^{\beta-1}log^{s}\ \ \varphi(\rho)\ \tfrac{\varphi'(\rho)}{\varphi(\rho)}$$

$$\Big\{(R-f(\rho))({}^{s}I^{\alpha}_{a,\varphi}f(t))-r\tfrac{(s+1)^{1-\alpha}}{\Gamma(\alpha+1)}\ (log^{s+1}\varphi(t)-log^{s+1}\ \varphi(a))^{\alpha}$$

$$+(f(\rho)-r)(R\tfrac{(s+1)^{1-\alpha}}{\Gamma(\alpha+1)}\ (log^{s+1}\varphi(t)-log^{s+1}\ \varphi(a))^{\alpha}-\ {}^{s}I^{\alpha}_{a,\varphi}f(t))$$

$${}^{s}I^{\alpha}_{a,\varphi}((R-f(t))(f(t)-r)) \tag{2.39}$$

$$-(R-f(\rho))(f(\rho)-r)\tfrac{(s+1)^{1-\alpha}}{\Gamma(\alpha+1)}\ (log^{s+1}\varphi(t)-log^{s+1}\ \varphi(a))^{\alpha}\Big\}$$

$$=\ \tfrac{(s+1)^{1-\beta}}{\Gamma(\beta)}\ (log^{s+1}\varphi(t)-log^{s+1}\ \varphi(\rho))^{\beta-1}log^{s}\ \ \varphi(\rho)\ \tfrac{\varphi'(\rho)}{\varphi(\rho)}\Big\{{}^{s}I^{\alpha}_{a,\varphi}f^{2}((t))$$

$$+f^{2}(\rho)\tfrac{(s+1)^{1-\alpha}}{\Gamma(\alpha+1)}\ (log^{s+1}\varphi(t)-log^{s+1}\ \varphi(a))^{\alpha}-2f(\rho)^{s}I^{\alpha}_{a,\varphi}f(t)\Big\}$$

ce que donne

$$\{^{s}I^{\alpha}_{a,\varphi}f(t) - r\tfrac{(s+1)^{1-\alpha}}{\Gamma(\alpha+1)}\ (log^{s+1}\varphi(t) - log^{s+1}\ \varphi(a))^{\alpha}\}$$

$$\tfrac{(s+1)^{1-\beta}}{\Gamma(\beta)}\ \ (log^{s+1}\varphi(t) - log^{s+1}\ \varphi(\rho))^{\beta-1}log^{s}\ \ \varphi(\rho)\ \tfrac{\varphi'(\rho)}{\varphi(\rho)}(R - f(\rho))$$

$$+\{(R\tfrac{(s+1)^{1-\alpha}}{\Gamma(\alpha+1)}(log^{s+1}\varphi(t) - log^{s+1}\ \varphi(a))^{\alpha} - \ ^{s}I^{\alpha}_{a,\varphi}f(t))\}$$

$$\tfrac{(s+1)^{1-\beta}}{\Gamma(\beta)}\ \ (log^{s+1}\varphi(t) - log^{s+1}\ \varphi(\rho))^{\beta-1}log^{s}\ \ \varphi(\rho)\ \tfrac{\varphi'(\rho)}{\varphi(\rho)}$$

$$-\{^{s}I^{\alpha}_{a,\varphi}(R - f(\rho))(f(\rho) - r)\}\tfrac{(s+1)^{1-\alpha}}{\Gamma(\alpha+1)}(log^{s+1}\varphi(t) - log^{s+1}\ \varphi(a))^{\alpha}\tfrac{(s+1)^{1-\beta}}{\Gamma(\beta)}$$

$$(log^{s+1}\varphi(t) - log^{s+1}\ \varphi(\rho))^{\beta-1}log^{s}\ \ \varphi(\rho)\ \tfrac{\varphi'(\rho)}{\varphi(\rho)}$$

$$-\tfrac{(s+1)^{1-\alpha}}{\Gamma(\alpha+1)}(log^{s+1}\varphi(t) - log^{s+1}\ \varphi(a))^{\alpha} \tag{2.40}$$

$$\tfrac{(s+1)^{1-\beta}}{\Gamma(\beta)}\ \ (log^{s+1}\varphi(t) - log^{s+1}\ \varphi(\rho))^{\beta-1}log^{s}\ \ \varphi(\rho)\ \tfrac{\varphi'(\rho)}{\varphi(\rho)}(R - f(\rho))(f(\rho) - r)$$

$$=\ \ ^{s}I^{\alpha}_{a,\varphi}f^{2}((t))\tfrac{(s+1)^{1-\alpha}}{\Gamma(\alpha+1)}(log^{s+1}\varphi(t) - log^{s+1}\ \varphi(a))^{\alpha}$$

$$\tfrac{(s+1)^{1-\beta}}{\Gamma(\beta)}\ \ (log^{s+1}\varphi(t) - log^{s+1}\ \varphi(\rho))^{\beta-1}log^{s}\ \ \varphi(\rho)\ \tfrac{\varphi'(\rho)}{\varphi(\rho)}$$

$$+\tfrac{(s+1)^{1-\alpha}}{\Gamma(\alpha+1)}(log^{s+1}\varphi(t) - log^{s+1}\ \varphi(a))^{\alpha}$$

$$\tfrac{(s+1)^{1-\beta}}{\Gamma(\beta)}\ \ (log^{s+1}\varphi(t) - log^{s+1}\ \varphi(\rho))^{\beta-1}log^{s}\ \ \varphi(\rho)\ \tfrac{\varphi'(\rho)}{\varphi(\rho)}f^{2}(\rho)$$

$$-2f(\rho)^{s}I^{\alpha}_{a,\varphi}f(t)\tfrac{(s+1)^{1-\beta}}{\Gamma(\beta)}(log^{s+1}\varphi(t) - log^{s+1}\ \varphi(\rho))^{\beta-1}log^{s}\ \ \varphi(\rho)\ \tfrac{\varphi'(\rho)}{\varphi(\rho)}$$

on intègre (2.41) par rapport à $\rho$ sur $(a,t)$, on trouve

$$\{{}^sI^{\alpha}_{a,\varphi}f(t)-r\tfrac{(s+1)^{1-\alpha}}{\Gamma(\alpha+1)}\}\ \ (log^{s+1}\varphi(t)-log^{s+1}\ \varphi(a))^{\alpha}\tfrac{(s+1)^{1-\beta}}{\Gamma(\beta)}$$

$$\int_a^t\ (log^{s+1}\varphi(t)-log^{s+1}\ \varphi(\rho))^{\beta-1}log^s\ \ \varphi(\rho)\ \tfrac{\varphi'(\rho)}{\varphi(\rho)}(R-f(\rho))\tfrac{d\rho}{\rho}$$

$$+\{(R\tfrac{(s+1)^{1-\alpha}}{\Gamma(\alpha+1)}-\ \ {}^sI^{\alpha}_{a,\varphi}f(t))\}(log^{s+1}\varphi(t)-log^{s+1}\ \varphi(a))^{\alpha}\tfrac{(s+1)^{1-\beta}}{\Gamma(\beta)}$$

$$\int_a^t(log^{s+1}\varphi(t)-log^{s+1}\ \varphi(\rho))^{\beta-1}log^s\ \ \varphi(\rho)\ \tfrac{\varphi'(\rho)}{\varphi(\rho)}\ \ \tfrac{d\rho}{\rho}$$

$$-\{{}^sI^{\alpha}_{a,\varphi}(log^{s+1}\varphi(t)-log^{s+1}\ \varphi(a))^{\alpha}\tfrac{(s+1)^{1-\beta}}{\Gamma(\beta)}\}$$
$$\int_a^t(R-f(\rho))(f(\rho)-r)\ \ (log^{s+1}\varphi(t)-log^{s+1}\ \varphi(\rho))^{\beta-1}log^s\ \ \varphi(\rho)\ \tfrac{\varphi'(\rho)}{\varphi(\rho)}\tfrac{d\rho}{\rho}$$

$$-(log^{s+1}\varphi(t)-log^{s+1}\ \varphi(a))^{\alpha}\tfrac{(s+1)^{1-\beta}}{\Gamma(\beta)}$$

$$\int_a^t\ (log^{s+1}\varphi(t)-log^{s+1}\ \varphi(\rho))^{\beta-1}log^s\ \ \varphi(\rho)\ \tfrac{\varphi'(\rho)}{\varphi(\rho)}(R-f(\rho))(f(\rho)-r)\ \ \tfrac{d\rho}{\rho}$$

$$=\ \ {}^sI^{\alpha}_{a,\varphi}f^2((t))(log^{s+1}\varphi(t)-log^{s+1}\ \varphi(a))^{\alpha}\tfrac{(s+1)^{1-\beta}}{\Gamma(\beta)}\ \ \int_a^t\ (log^{s+1}\varphi(t)-log^{s+1}\ \varphi(\rho))^{\beta-1}log^s\ \ \varphi(\rho)\ \tfrac{\varphi'(\rho)}{\varphi(\rho)}\ \tfrac{d\rho}{\rho}$$

$$+(log^{s+1}\varphi(t)-log^{s+1}\ \varphi(a))^{\alpha}\tfrac{(s+1)^{1-\beta}}{\Gamma(\beta)}\ \ \int_a^t\ (log^{s+1}\varphi(t)-log^{s+1}\ \varphi(\rho))^{\beta-1}log^s\ \ \varphi(\rho)\ \tfrac{\varphi'(\rho)}{\varphi(\rho)}f^2(\rho)\ \ \tfrac{d\rho}{\rho}$$

$$-2^sI^{\alpha}_{a,\varphi}f(t)\ \int_a^t(log^{s+1}\varphi(t)-log^{s+1}\ \varphi(\rho))^{\beta-1}log^s\ \ \varphi(\rho)\ \tfrac{\varphi'(\rho)}{\varphi(\rho)}f(\rho)\ \ \tfrac{d\rho}{\rho} \tag{2.41}$$

C'est- à - dire

$$\left({}^{s}I^{\alpha}_{a,\varphi}f(t) - r\tfrac{(s+1)^{1-\alpha}}{\Gamma(\alpha+1)}\ \left(log^{s+1}\varphi(t) - log^{s+1}\ \varphi(a)\right)^{\alpha}\right)$$
$$\left(R\tfrac{(s+1)^{1-\beta}}{\Gamma(\beta+1)}(log^{s+1}\varphi(t) - log^{s+1}\ \varphi(a))^{\beta} -^{s} I^{\beta}_{a,\varphi}f(t)\right)$$
$$+\left(R\tfrac{(s+1)^{1-\alpha}}{\Gamma(\alpha+1)}(log^{s+1}\varphi(t) - log^{s+1}\ \varphi(a))^{\alpha} -^{s} I^{\alpha}_{a,\varphi}f(t)\right)$$
$$\left({}^{s}I^{\beta}_{a,\varphi}f(t) - r\tfrac{(s+1)^{1-\beta}}{\Gamma(\beta+1)}\ \left(log^{s+1}\varphi(t) - log^{s+1}\ \varphi(a)\right)^{\beta}\right)$$
$$-\ {}^{s}I^{\alpha}_{a,\varphi}((R-f(t))(f(t)-r))\tfrac{(s+1)^{1-\beta}}{\Gamma(\beta+1)}\ (log^{s+1}\varphi(t) - log^{s+1}\ \varphi(a))^{\beta} \tag{2.42}$$
$$-\ \tfrac{(s+1)^{1-\alpha}}{\Gamma(\alpha+1)}\ (log^{s+1}\varphi(t) - log^{s+1}\ \varphi(a))^{\alpha}\ \ {}^{s}I^{\beta}_{a,\varphi}((R-f(t))(f(t)-r))$$
$$= \tfrac{(s+1)^{1-\beta}}{\Gamma(\beta+1)}\ (log^{s+1}\varphi(t) - log^{s+1}\ \varphi(a))^{\beta}\ {}^{s}I^{\alpha}_{a,\varphi}f^{2}(t)$$
$$+\tfrac{(s+1)^{1-\alpha}}{\Gamma(\alpha+1)}\ (log^{s+1}\varphi(t) - log^{s+1}\ \varphi(a))^{\alpha}\ {}^{s}I^{\beta}_{a,\varphi}f^{2}(t) -\ \ 2\ {}^{s}I^{\alpha}_{a,\varphi}f(t)\ \ {}^{s}I^{\beta}_{a,\varphi}f(t)$$

d'où la résultat .

## Preuve du théorème 2

*Démonstration.* Si $f$ et $g$ sont deux fonctions intégrables sur $[0,\infty[$ vérifié la condition (2.1) sur $[0,\infty[$

comme $(R-f(x))(f(x)-r) \geq 0$ et $(N-g(x))(g(x)-n) \geq 0$ . Alors

$$\tfrac{(s+1)^{1-\alpha}}{\Gamma(\alpha+1)}\ (log^{s+1}\varphi(t) - log^{s+1}\ \varphi(a))^{\alpha}\ {}^{s}I^{\beta}_{a,\varphi}(R-f(t))(f(t)-r)$$
$$+\tfrac{(s+1)^{1-\alpha}}{\Gamma(\alpha+1)}\ (log^{s+1}\varphi(t) - log^{s+1}\ \varphi(a))^{\alpha}\ {}^{s}I^{\alpha}_{a,\varphi}(R-f(t))(f(t)-r) \geq 0 \tag{2.43}$$

et

$$\tfrac{(s+1)^{1-\alpha}}{\Gamma(\alpha+1)}\ (log^{s+1}\varphi(t) - log^{s+1}\ \varphi(a))^{\alpha}\ {}^{s}I^{\beta}_{a,\varphi}(N-g(t))(g(t)-n)$$
$$+\tfrac{(s+1)^{1-\alpha}}{\Gamma(\alpha+1)}\ (log^{s+1}\varphi(t) - log^{s+1}\ \varphi(a))^{\alpha}\ {}^{s}I^{\alpha}_{a,\varphi}(N-g(t))(g(t)-n) \geq 0 \tag{2.44}$$

Nous avons

$$\left(R\tfrac{(s+1)^{1-\alpha}}{\Gamma(\alpha+1)}\ (log^{s+1}\varphi(t)-log^{s+1}\ \varphi(a))^{\alpha}\ -\ {}^{s}I^{\alpha}_{a,\varphi}f(t)\right)$$

$$\left({}^{s}I^{\beta}_{a,\varphi}f(t)-r\tfrac{(s+1)^{1-\beta}}{\Gamma(\beta+1)}\ (log^{s+1}\varphi(t)-log^{s+1}\ \varphi(a))^{\beta}\right)$$

$$\left(R\tfrac{(s+1)^{1-\beta}}{\Gamma(\beta+1)}\ (log^{s+1}\varphi(t)-log^{s+1}\ \varphi(a))^{\beta}\ -\ {}^{s}I^{\beta}_{a,\varphi}f(t)\right)$$

$$\left({}^{s}I^{\alpha}_{a,\varphi}f(t)-r\tfrac{(s+1)^{1-\alpha}}{\Gamma(\alpha+1)}\ (log^{s+1}\varphi(t)-log^{s+1}\ \varphi(a))^{\alpha}\right)$$

$$-\ \tfrac{(s+1)^{1-\alpha}}{\Gamma(\alpha+1)}\ (log^{s+1}\varphi(t)-log^{s+1}\ \varphi(a))^{\alpha}\,{}^{s}I^{\alpha}_{a,\varphi}(R-f(t))(f(t)-r)$$

$$-\ \tfrac{(s+1)^{1-\beta}}{\Gamma(\beta+1)}\ (log^{s+1}\varphi(t)-log^{s+1}\ \varphi(a))^{\beta}\,{}^{s}I^{\beta}_{a,\varphi}(R-f(t))(f(t)-r)\leqslant$$

$$\left(R\tfrac{(s+1)^{1-\alpha}}{\Gamma(\alpha+1)}\ (log^{s+1}\varphi(t)-log^{s+1}\ \varphi(a))^{\alpha}\ -\ {}^{s}I^{\alpha}_{a,\varphi}f(t)\right)$$

$$\left({}^{s}I^{\beta}_{a,\varphi}f(t)-r\tfrac{(s+1)^{1-\beta}}{\Gamma(\beta+1)}\ (log^{s+1}\varphi(t)-log^{s+1}\ \varphi(a))^{\beta}\right)$$

$$\left(R\tfrac{(s+1)^{1-\beta}}{\Gamma(\beta+1)}\ (log^{s+1}\varphi(t)-log^{s+1}\ \varphi(a))^{\beta}\ -\ {}^{s}I^{\beta}_{a,\varphi}f(t)\right)$$

$$\left({}^{s}I^{\alpha}_{a,\varphi}f(t)-r\tfrac{(s+1)^{1-\alpha}}{\Gamma(\alpha+1)}\ (log^{s+1}\varphi(t)-log^{s+1}\ \varphi(a))^{\alpha}\right)$$

et

$$\left(N\tfrac{(s+1)^{1-\alpha}}{\Gamma(\alpha+1)}\ (log^{s+1}\varphi(t)-log^{s+1}\ \varphi(a))^{\alpha}\ -\ {}^{s}I^{\alpha}_{a,\varphi}g(t)\right)$$

$$\left({}^{s}I^{\beta}_{a,\varphi}g(t)-n\tfrac{(s+1)^{1-\beta}}{\Gamma(\beta+1)}\ (log^{s+1}\varphi(t)-log^{s+1}\ \varphi(a))^{\beta}\right)$$

$$\left(N\tfrac{(s+1)^{1-\beta}}{\Gamma(\beta+1)}\ (log^{s+1}\varphi(t)-log^{s+1}\ \varphi(a))^{\beta}\ -\ {}^{s}I^{\beta}_{a,\varphi}g(t)\right)$$

$$\left({}^{s}I^{\alpha}_{a,\varphi}g(t)-n\tfrac{(s+1)^{1-\alpha}}{\Gamma(\alpha+1)}\ (log^{s+1}\varphi(t)-log^{s+1}\ \varphi(a))^{\alpha}\right)$$

$$- \ \frac{(s+1)^{1-\alpha}}{\Gamma(\alpha+1)} \ (log^{s+1}\varphi(t) - log^{s+1}\ \varphi(a))^{\alpha}\ {}^{s}I^{\alpha}_{a,\varphi}(N-g(t))(g(t)-n)$$

$$- \ \frac{(s+1)^{1-\beta}}{\Gamma(\beta+1)} \ (log^{s+1}\varphi(t) - log^{s+1}\ \varphi(a))^{\beta}\ {}^{s}I^{\beta}_{a,\varphi}(N-g(t))(g(t)-n) \leqslant$$

$$\left(N\frac{(s+1)^{1-\alpha}}{\Gamma(\alpha+1)} \ (log^{s+1}\varphi(t) - log^{s+1}\ \varphi(a))^{\alpha} \ - \ {}^{s}I^{\alpha}_{a,\varphi}g(t)\right)$$

$$\left({}^{s}I^{\beta}_{a,\varphi}g(t) - n\frac{(s+1)^{1-\beta}}{\Gamma(\beta+1)} \ (log^{s+1}\varphi(t) - log^{s+1}\ \varphi(a))^{\beta}\right)$$

$$\left(N\frac{(s+1)^{1-\beta}}{\Gamma(\beta+1)} \ (log^{s+1}\varphi(t) - log^{s+1}\ \varphi(a))^{\beta} \ - \ {}^{s}I^{\beta}_{a,\varphi}g(t)\right)$$

$$\left({}^{s}I^{\alpha}_{a,\varphi}g(t) - n\frac{(s+1)^{1-\alpha}}{\Gamma(\alpha+1)} \ (log^{s+1}\varphi(t) - log^{s+1}\ \varphi(a))^{\alpha}\right)$$

D'après le lemme 3 , on trouve

$$\begin{aligned}
&\frac{(s+1)^{1-\beta}}{\Gamma(\beta+1)} \ (log^{s+1}\varphi(t) - log^{s+1}\ \varphi(a))^{\beta}\ {}^{s}I^{\alpha}_{a,\varphi}f^{2}(t) \\
&+\frac{(s+1)^{1-\alpha}}{\Gamma(\alpha+1)} \ (log^{s+1}\varphi(t) - log^{s+1}\ \varphi(a))^{\alpha}\ {}^{s}I^{\beta}_{a,\varphi}f^{2}(t) - \ 2\ {}^{s}I^{\alpha}_{a,\varphi}f(t)\ \ {}^{s}I^{\beta}_{a,\varphi}f(t) \leqslant \\
&\left(R\frac{(s+1)^{1-\alpha}}{\Gamma(\alpha+1)} \ (log^{s+1}\varphi(t) - log^{s+1}\ \varphi(a))^{\alpha} \ - \ {}^{s}I^{\alpha}_{a,\varphi}f(t)\right) \\
&\left({}^{s}I^{\beta}_{a,\varphi}f(t) - r\frac{(s+1)^{1-\beta}}{\Gamma(\beta+1)} \ (log^{s+1}\varphi(t) - log^{s+1}\ \varphi(a))^{\beta}\right) \\
&\left(R\frac{(s+1)^{1-\beta}}{\Gamma(\beta+1)} \ (log^{s+1}\varphi(t) - log^{s+1}\ \varphi(a))^{\beta} \ - \ {}^{s}I^{\beta}_{a,\varphi}f(t)\right) \\
&\left({}^{s}I^{\alpha}_{a,\varphi}f(t) - r\frac{(s+1)^{1-\alpha}}{\Gamma(\alpha+1)} \ (log^{s+1}\varphi(t) - log^{s+1}\ \varphi(a))^{\alpha}\right)
\end{aligned} \tag{2.45}$$

et

$$\frac{(s+1)^{1-\beta}}{\Gamma(\beta+1)} \ (log^{s+1}\varphi(t) - log^{s+1}\ \varphi(a))^{\beta}\ {}^{s}I^{\alpha}_{a,\varphi}g^2(t)$$

$$+\frac{(s+1)^{1-\alpha}}{\Gamma(\alpha+1)} \ (log^{s+1}\varphi(t) - log^{s+1}\ \varphi(a))^{\alpha}\ {}^{s}I^{\beta}_{a,\varphi}g^2(t) - \ 2\ {}^{s}I^{\alpha}_{a,\varphi}g(t)\ \ {}^{s}I^{\beta}_{a,\varphi}g(t) \leqslant$$

$$\left(P\frac{(s+1)^{1-\alpha}}{\Gamma(\alpha+1)} \ (log^{s+1}\varphi(t) - log^{s+1}\ \varphi(a))^{\alpha} \ - \ \ {}^{s}I^{\alpha}_{a,\varphi}g(t)\right)$$

$$\left(\ {}^{s}I^{\beta}_{a,\varphi}g(t) - p\frac{(s+1)^{1-\beta}}{\Gamma(\beta+1)} \ (log^{s+1}\varphi(t) - log^{s+1}\ \varphi(a))^{\beta}\right) \tag{2.46}$$

$$\left(P\frac{(s+1)^{1-\beta}}{\Gamma(\beta+1)} \ (log^{s+1}\varphi(t) - log^{s+1}\ \varphi(a))^{\beta} \ - \ \ {}^{s}I^{\beta}_{a,\varphi}g(t)\right)$$

$$\left(\ {}^{s}I^{\alpha}_{a,\varphi}g(t) - p\frac{(s+1)^{1-\alpha}}{\Gamma(\alpha+1)} \ (log^{s+1}\varphi(t) - log^{s+1}\ \varphi(a))^{\alpha}\right)$$

et d'après le lemme 2 on a

$$[(\frac{(s+1)^{1-\alpha}((log^{s+1}\varphi(t)-(log^{s+1}\varphi(a))^{\alpha}}{\Gamma(\alpha+1)}{}^{s}I^{\beta}_{a,\varphi}fg(t) + \frac{(s+1)^{1-\beta}((log^{s+1}\varphi(t)-(log^{s+1}\varphi(a))^{\beta}}{\Gamma(\beta+1)}{}^{s}I^{\alpha}_{a,\varphi}fg(t)$$

$$- \ {}^{s}I^{\alpha}_{a,\varphi}f(t){}^{s}I^{\beta}_{a,\varphi}g(t) - {}^{s}I^{\beta}_{a,\varphi}f(t){}^{s}I^{\alpha}_{a,\varphi}g(t)]^2$$

$$\leq [\frac{(s+1)^{1-\alpha}((log^{s+1}\varphi(t)-(log^{s+1}\varphi(a))^{\alpha}}{\Gamma(\alpha+1)}{}^{s}I^{\beta}_{a,\varphi}f^2(t) + \frac{(s+1)^{1-\beta}((log^{s+1}\varphi(t)-(log^{s+1}\varphi(a))^{\beta}}{\Gamma(\beta+1)}$$

$${}^{s}I^{\alpha}_{a,\varphi}f^2(t) - 2{}^{s}I^{\alpha}_{a,\varphi}f(t){}^{s}I^{\beta}_{a,\varphi}f(t)]$$

$$\times[(\frac{(s+1)^{1-\alpha}((log^{s+1}\varphi(t)-(log^{s+1}\varphi(a))^{\alpha}}{\Gamma(\alpha+1)}{}^{s}I^{\beta}_{a,\varphi}g^2(t) + \frac{(s+1)^{1-\beta}((log^{s+1}\varphi(t)-(log^{s+1}\varphi(a))^{\beta}}{\Gamma(\beta+1)}$$

$${}^{s}I^{\alpha}_{a,\varphi}g^2(t) - 2{}^{s}I^{\alpha}_{a,\varphi}g(t){}^{s}I^{\beta}_{a,\varphi}g(t)].$$

Donc

$$[(\frac{(s+1)^{1-\alpha}((log^{s+1}\varphi(t)-(log^{s+1}\varphi(a))^{\alpha}}{\Gamma(\alpha+1)}{}^{s}I^{\beta}_{a,\varphi}fg(t)+\frac{(s+1)^{1-\beta}((log^{s+1}\varphi(t)-(log^{s+1}\varphi(a))^{\beta}}{\Gamma(\beta+1)}{}^{s}I^{\alpha}_{a,\varphi}fg(t)$$

$$-\ {}^{s}I^{\alpha}_{a,\varphi}f(t){}^{s}I^{\beta}_{a,\varphi}g(t)-{}^{s}I^{\beta}_{a,\varphi}f(t){}^{s}I^{\alpha}_{a,\varphi}g(t)]^{2}\leq$$

$$\left(R\frac{(s+1)^{1-\alpha}}{\Gamma(\alpha+1)}\ (log^{s+1}\varphi(t)-log^{s+1}\ \varphi(a))^{\alpha}\ -\ {}^{s}I^{\alpha}_{a,\varphi}f(t)\right)\left({}^{s}I^{\beta}_{a,\varphi}f(t)-r\frac{(s+1)^{1-\beta}}{\Gamma(\beta+1)}\ (log^{s+1}\varphi(t)-log^{s+1}\ \varphi(a))^{\beta}\right)$$

$$+\left({}^{s}I^{\alpha}_{a,\varphi}f(t)-r\frac{(s+1)^{1-\alpha}}{\Gamma(\alpha+1)}\ (log^{s+1}\varphi(t)-log^{s+1}\ \varphi(a))^{\alpha}\right)\left(R\frac{(s+1)^{1-\beta}}{\Gamma(\beta+1)}\ (log^{s+1}\varphi(t)-log^{s+1}\ \varphi(a))^{\beta}\ -\ {}^{s}I^{\alpha}_{a,\varphi}f(t)\right)$$

$$\left(N\frac{(s+1)^{1-\alpha}}{\Gamma(\alpha+1)}\ (log^{s+1}\varphi(t)-log^{s+1}\ \varphi(a))^{\alpha}\ -\ {}^{s}I^{\alpha}_{a,\varphi}g(t)\right)\left({}^{s}I^{\beta}_{a,\varphi}g(t)-n\frac{(s+1)^{1-\beta}}{\Gamma(\beta+1)}\ (log^{s+1}\varphi(t)-log^{s+1}\ \varphi(a))^{\beta}\right)$$

$$+\left({}^{s}I^{\alpha}_{a,\varphi}g(t)-n\frac{(s+1)^{1-\alpha}}{\Gamma(\alpha+1)}\ (log^{s+1}\varphi(t)-log^{s+1}\ \varphi(a))^{\alpha}\right)\left(N\frac{(s+1)^{1-\beta}}{\Gamma(\beta+1)}\ (log^{s+1}\varphi(t)-log^{s+1}\ \varphi(a))^{\beta}\ -\ {}^{s}I^{\alpha}_{a,\varphi}g(t)\right)$$

**Remarque 1.** *Si en prend $\alpha=\beta$ dons le théorème 2 , on trouve le théorème 1 .*

□

□

# Chapitre 3

# Inégalités fractionnaires de type Chebyshev

## 1 Introduction

Dons ce chapitre on présent quelques inégalités de type de Chebyshev .

## 2 Les fonctions synchrones

**Définition 4.** *Soient $f$ et $g$ sont deux fonctions définies sur $[a,b]$ .Les fonctions $f$ et $g$ sont dites synchrones sur $[a,b]$ si*

$$(f(\tau)-f(\rho))(g(\tau)-g(\rho))\geq 0, \qquad \tau,\rho\in[a,b].$$

## 3 Principaux résultats

### 3.1 Inégalité pondérées avec un seul paramètre ($\alpha$)

on donne le premier résultat

**Théorème 3.** *Si $f$ et $g$ sont deux fonctions synchrones sur $[0,\infty[$ et soit $l,p,q:[0,\infty[\to[0,\infty[$. Alors* *pour tout* $t > 0$, $\alpha > 0$, *on a :*

$$\begin{aligned}
&{}^sI^{\alpha}_{a,\varphi}l(t)\Big[\ {}^sI^{\alpha}_{a,\varphi}p(t)\ {}^sI^{\alpha}_{a,\varphi}(qfg)(t) + \ {}^sI^{\alpha}_{a,\varphi}q(t)\ {}^sI^{\alpha}_{a,\varphi}(pfg)(t)\Big] \\
&+ \ {}^sI^{\alpha}_{a,\varphi}p(t)\ {}^sI^{\alpha}_{a,\varphi}(q)(t))\ {}^sI^{\alpha}_{a,\varphi}(lfg)(t) \geq \\
&\tfrac{1}{2}\Big\{\ {}^sI^{\alpha}_{a,\varphi}r(t)\Big[\ {}^sI^{\alpha}_{a,\varphi}(pf)(t)\ {}^sI^{\alpha}_{a,\varphi}(qg)(t) + \ {}^sI^{\alpha}_{a,\varphi}(qf)(t)\ {}^sI^{\alpha}_{a,\varphi}(pg)(t)\Big] \\
&+ \ {}^sI^{\alpha}_{a,\varphi}p(t)\Big[\ {}^sI^{\alpha}_{a,\varphi}(rf)(t)\ {}^sI^{\alpha}_{a,\varphi}(qg)(t) + \ {}^sI^{\alpha}_{a,\varphi}(qf)(t)\ {}^sI^{\alpha}_{a,\varphi}(rg)(t)\Big] \\
&+ \ {}^sI^{\alpha}_{a,\varphi}q(t)\Big[\ {}^sI^{\alpha}_{a,\varphi}(rf)(t)\ {}^sI^{\alpha}_{a,\varphi}(pg)(t) + \ {}^sI^{\alpha}_{a,\varphi}(pf)(t)\ {}^sI^{\alpha}_{a,\varphi}(rg)(t)\Big]\Big\}.
\end{aligned} \tag{3.1}$$

avant de montrer ce théorème on a besoin le lemme suivant

**Lemme 4.** *Si* $f$ *et* $g$*sont deux fonctions synchrones sur* $]0,\infty[$ *et soit* $u, v :]0,\infty[\to]0,\infty[$. *Alors* $\forall t > 0, \alpha > 0$, *on a :*

$$\begin{aligned}
&{}^sI^{\alpha}_{a,\varphi}u(t)\ {}^sI^{\alpha}_{a,\varphi}(vfg)(t) + \ {}^sI^{\alpha}_{a,\varphi}v(t)\ {}^sI^{\alpha}_{a,\varphi}(ufg)(t) \\
\geq\ &{}^sI^{\alpha}_{a,\varphi}(uf)(t)\ {}^sI^{\alpha}_{a,\varphi}(vg)(t) + \ {}^sI^{\alpha}_{a,\varphi}(vf)(t)\ {}^sI^{\alpha}_{a,\varphi}(ug)(t)
\end{aligned}. \tag{3.2}$$

*Démonstration.* comme $f$ et $g$ sont des fonctions synchrones sur $[0,\infty[$, alors $\forall \tau \geq 0$, $\rho \geq 0$, on peut écrire :

$$(f(\tau) - f(\rho))(g(\tau) - g(\rho)) \geq 0, \tag{3.3}$$

implique que

$$f(\tau)g(\tau) + f(\rho)g(\rho) - f(\tau)g(\rho) - f(\rho)g(\tau) \geq 0, \tag{3.4}$$

c'est -à - dire

$$f(\tau)g(\tau) + f(\rho)g(\rho) \geq f(\tau)g(\rho) + f(\rho)g(\tau). \tag{3.5}$$

Multipliant (3.5) par la quantité suivante $\frac{(s+1)^{1-\alpha}}{\Gamma(\alpha)}\ (log^{s+1}\varphi(t) - log^{s+1}\varphi(\tau))^{\alpha-1}log^s\ \ \varphi(\tau)\frac{\varphi'(\tau)}{\varphi(\tau)}u(\tau)$ on trouve :

$$\begin{aligned}
&\frac{(s+1)^{1-\alpha}}{\Gamma(\alpha)} \ (log^{s+1}\varphi(t)-log^{s+1}\varphi(\tau))^{\alpha-1} \ log^s \ \varphi(\tau)\frac{\varphi'(\tau)}{\varphi(\tau)}u(\tau)f(\tau)g(\tau)\\
&+\frac{(s+1)^{1-\alpha}}{\Gamma(\alpha)} \ (log^{s+1}\varphi(t)-log^{s+1}\varphi(\tau))^{\alpha-1} \ log^s \ \varphi(\tau) \ \frac{\varphi'(\tau)}{\varphi(\tau)}u(\tau)f(\rho)g(\rho)\\
&\geq \frac{(s+1)^{1-\alpha}}{\Gamma(\alpha)} \ (log^{s+1}\varphi(t)-log^{s+1}\varphi(\tau))^{\alpha-1} \ log^s \ \varphi(\tau)\frac{\varphi'(\tau)}{\varphi(\tau)} \ u(\tau) \ f(\tau) \ g(\rho)\\
&+\frac{(s+1)^{1-\alpha}}{\Gamma(\alpha)} \ (log^{s+1}\varphi(t)-log^{s+1}\varphi(\tau))^{\alpha-1} \ log^s \ \varphi(\tau)\frac{\varphi'(\tau)}{\varphi(\tau)} \ u(\tau) \ f(\rho) \ g(\tau)
\end{aligned} \tag{3.6}$$

On intégrant (3.6) par rapport à $\tau$ sur $(a,t)$, on trouve :

$$\begin{aligned}
&\frac{(s+1)^{1-\alpha}}{\Gamma(\alpha)}\int_a^t (log^{s+1}\varphi(t)-log^{s+1}\varphi(\tau))^{\alpha-1} \ log^s \ \varphi(\tau)\frac{\varphi'(\tau)}{\varphi(\tau)} \ u(\tau) \ f(\tau) \ g(\tau)\frac{d\tau}{\tau}\\
&+\frac{(s+1)^{1-\alpha}}{\Gamma(\alpha)}\int_a^t(log^{s+1}\varphi(t)-log^{s+1}\varphi(\tau))^{\alpha-1} \ log^s \ \varphi(\tau)\frac{\varphi'(\tau)}{\varphi(\tau)} \ u(\tau) \ f(\rho) \ g(\rho)\frac{d\tau}{\tau}\\
&\geq \frac{(s+1)^{1-\alpha}}{\Gamma(\alpha)}\int_a^t(log^{s+1}\varphi(t)-log^{s+1}\varphi(\tau))^{\alpha-1} \ log^s \ \varphi(\tau)\frac{\varphi'(\tau)}{\varphi(\tau)} \ u(\tau) \ f(\tau) \ g(\rho)\frac{d\tau}{\tau}\\
&+\frac{(s+1)^{1-\alpha}}{\Gamma(\alpha)}\int_a^t(log^{s+1}\varphi(t)-log^{s+1}\varphi(\tau))^{\alpha-1} \ log^s \ \varphi(\tau)\frac{\varphi'(\tau)}{\varphi(\tau)} \ u(\tau) \ f(\rho) \ g(\tau) \ \frac{d\tau}{\tau}
\end{aligned} \tag{3.7}$$

□

d'où :

$$\begin{aligned}
&{}^sI^{\alpha}_{a,\varphi}(ufg)(t)+f(\rho)g(\rho)\frac{(s+1)^{1-\alpha}}{\Gamma(\alpha)}\int_a^t(log^{s+1}\varphi(t)-log^{s+1}\varphi(\tau))^{\alpha-1} \ log^s \ \varphi(\tau)\frac{\varphi'(\tau)}{\varphi(\tau)} \ u(\tau) \ \frac{d\tau}{\tau}\\
&\geq g(\rho)\frac{(s+1)^{1-\alpha}}{\Gamma(\alpha)}\int_a^t(log^{s+1}\varphi(t)-log^{s+1}\varphi(\tau))^{\alpha-1} \ log^s \ \varphi(\tau)\frac{\varphi'(\tau)}{\varphi(\tau)} \ u(\tau) \ f(\tau) \ \frac{d\tau}{\tau}\\
&+f(\rho)\frac{(s+1)^{1-\alpha}}{\Gamma(\alpha)}\int_a^t(log^{s+1}\varphi(t)-log^{s+1}\varphi(\tau))^{\alpha-1} \ log^s \ \varphi(\tau)\frac{\varphi'(\tau)}{\varphi(\tau)} \ u(\tau) \ g(\tau) \ \frac{d\tau}{\tau}
\end{aligned} \tag{3.8}$$

Alors :

$${}^sI^{\alpha}_{a,\varphi}(ufg)(t)+f(\rho)g(\rho) \ {}^sI^{\alpha}_{a,\varphi}u(t)\geq g(\rho) \ {}^sI^{\alpha}_{a,\varphi}(uf)(t)+f(\rho) \ {}^sI^{\alpha}_{a,\varphi}(ug)(t) \tag{3.9}$$

maintenant on multiplie (3.9) par $\frac{(s+1)^{1-\alpha}}{\Gamma(\alpha)} \ (log^{s+1}\varphi(t)-log^{s+1} \ \varphi(\tau))^{\alpha-1}log^s \ \varphi(\rho) \ \frac{\varphi'(\rho)}{\varphi(\rho)} \ v(\rho)$ ;

$\rho \in (a,t)$ On trouve

$$\begin{aligned}
&\tfrac{(s+1)^{1-\alpha}}{\Gamma(\alpha)} \ (log^{s+1}\varphi(t) - log^{s+1} \ \varphi(\rho))^{\alpha-1} \ log^{s} \ \varphi(\rho) \ \tfrac{\varphi'(\rho)}{\varphi(\rho)} \ v(\rho)^{s}I^{\alpha}_{a,\varphi}(ufg)(t)\\
&+\tfrac{(s+1)^{1-\alpha}}{\Gamma(\alpha)} \ (log^{s+1}\varphi(t) - log^{s+1} \ \varphi(\rho))^{\alpha-1} \ log^{s} \ \varphi(\rho) \ \tfrac{\varphi'(\rho)}{\varphi(\rho)} \ v(\rho)f(\rho)g(\rho)^{s}I^{\alpha}_{a,\varphi}u(t)\\
&\geq \tfrac{(s+1)^{1-\alpha}}{\Gamma(\alpha)} \ (log^{s+1}\varphi(t) - log^{s+1} \ \varphi(\rho))^{\alpha-1} \ log^{s} \ \varphi(\rho) \ \tfrac{\varphi'(\rho)}{\varphi(\rho)} \ v(\rho)g(\rho)^{s}I^{\alpha}_{a,\varphi}(uf)(t)\\
&+\tfrac{(s+1)^{1-\alpha}}{\Gamma(\alpha)} \ (log^{s+1}\varphi(t) - log^{s+1} \ \varphi(\rho))^{\alpha-1} \ log^{s} \ \varphi(\rho) \ \tfrac{\varphi'(\rho)}{\varphi(\rho)} \ v(\rho)f(\rho)^{s}I^{\alpha}_{a,\varphi}(ug)(t)
\end{aligned} \tag{3.10}$$

de la même manière on intègre (3.10) par rapport à $\rho$ sur (a,t) et on peut écrire

$$\begin{aligned}
&\tfrac{(s+1)^{1-\alpha}}{\Gamma(\alpha)}\int_a^t \ (log^{s+1}\varphi(t) - log^{s+1} \ \varphi(\rho))^{\alpha-1} \ log^{s} \ \varphi(\rho) \ \tfrac{\varphi'(\rho)}{\varphi(\rho)} \ v(\rho)^{s}I^{\alpha}_{a,h}(ufg)(t)\tfrac{d\rho}{\rho}\\
&+\tfrac{(s+1)^{1-\alpha}}{\Gamma(\alpha)}\int_a^t \ (log^{s+1}\varphi(t) - log^{s+1} \ \varphi(\rho))^{\alpha-1} \ log^{s} \ \varphi(\rho) \ \tfrac{\varphi'(\rho)}{\varphi(\rho)} \ v(\rho)f(\rho)g(\rho)^{s}I^{\alpha}_{a,\varphi}u(t)\tfrac{d\rho}{\rho}\\
&\geq \tfrac{(s+1)^{1-\alpha}}{\Gamma(\alpha)}\int_a^t \ (log^{s+1}\varphi(t) - log^{s+1} \ \varphi(\rho))^{\alpha-1} \ log^{s} \ \varphi(\rho) \ \tfrac{\varphi'(\rho)}{\varphi(\rho)} \ v(\rho)g(\rho)^{s}I^{\alpha}_{a,\varphi}(uf)(t)\tfrac{d\rho}{\rho}\\
&+\tfrac{(s+1)^{1-\alpha}}{\Gamma(\alpha)}\int_a^t \ (log^{s+1}\varphi(t) - log^{s+1} \ \varphi(\rho))^{\alpha-1} \ log^{s} \ \varphi(\rho) \ \tfrac{\varphi'(\rho)}{\varphi(\rho)} \ v(\rho)f(\rho)^{s}I^{\alpha}_{a,\varphi}(ug)(t)\tfrac{d\rho}{\rho}
\end{aligned} \tag{3.11}$$

ce qui donne

$$\begin{aligned}
&\tfrac{(s+1)^{1-\alpha}}{\Gamma(\alpha)} \ {}^{s}I^{\alpha}_{a,\varphi}(ufg)(t)\int_a^t \ (log^{s+1}\varphi(t) - log^{s+1} \ \varphi(\rho))^{\alpha-1} \, log^{s} \ \varphi(\rho) \ \tfrac{\varphi'(\rho)}{\varphi(\rho)} \, v(\rho)\tfrac{d\rho}{\rho}\\
&+\tfrac{(s+1)^{1-\alpha}}{\Gamma(\alpha)} \ {}^{s}I^{\alpha}_{a,\varphi}(u)(t)\int_a^t \ (log^{s+1}\varphi(t) - log^{s+1} \ \varphi(\rho))^{\alpha-1}log^{s} \ \varphi(\rho) \ \tfrac{\varphi'(\rho)}{\varphi(\rho)} \ v(\rho) \ f(\rho) \ g(\rho)\tfrac{d\rho}{\rho}\\
&\geq \tfrac{(s+1)^{1-\alpha}}{\Gamma(\alpha)} \ {}^{s}I^{\alpha}_{a,\varphi}(uf)(t)\int_a^t \ (log^{s+1}\varphi(t) - log^{s+1} \ \varphi(\rho))^{\alpha-1}log^{s} \ \varphi(\rho) \ \tfrac{\varphi'(\rho)}{\varphi(\rho)} \ v(\rho) \ g(\rho)\tfrac{d\rho}{\rho}\\
&\tfrac{(s+1)^{1-\alpha}}{\Gamma(\alpha)} \ {}^{s}I^{\alpha}_{a,\varphi}(ug)(t)\int_a^t \ (log^{s+1}\varphi(t) - log^{s+1} \ \varphi(\rho))^{\alpha-1} \, log^{s} \ \varphi(\rho) \ \tfrac{\varphi'(\rho)}{\varphi(\rho)} \, v(\rho) \ f(\rho)\tfrac{d\rho}{\rho}
\end{aligned} \tag{3.12}$$

Alors :

$$\begin{aligned}
&{}^{s}I^{\alpha}_{a,\varphi}(v)(t) \ {}^{s}I^{\alpha}_{a,\varphi}(ufg)(t) + \ {}^{s}I^{\alpha}_{a,\varphi}(u)(t) \ {}^{s}I^{\alpha}_{a,\varphi}(vfg)(t)\\
&\geq \ {}^{s}I^{\alpha}_{a,\varphi}(uf)(t) \ {}^{s}I^{\alpha}_{a,\varphi}(vg)(t) + \ {}^{s}I^{\alpha}_{a,\varphi}(vf)(t) \ {}^{s}I^{\alpha}_{a,\varphi}(ug)(t)
\end{aligned} \tag{3.13}$$

**Remarque 2.**

1. *en appliquant le lemme* (4) *pour* $u=1$*,on trouve :*

$$\begin{aligned} & {}^{s}I^{\alpha}_{a,\varphi}(1)\ {}^{s}I^{\alpha}_{a,\varphi}(vfg)(t)+\ {}^{s}I^{\alpha}_{a,\varphi}v(t)\ {}^{s}I^{\alpha}_{a,\varphi}(fg)(t) \\ \geq\ & {}^{s}I^{\alpha}_{a,\varphi}(f)(t)\ {}^{s}I^{\alpha}_{a,\varphi}(vg)(t)+\ {}^{s}I^{\alpha}_{a,\varphi}(vf)(t)\ {}^{s}I^{\alpha}_{a,\varphi}(g)(t) \end{aligned}.$$

2. *la même chose si* $v=1$

$$\begin{aligned} & {}^{s}I^{\alpha}_{a,\varphi}u(t)\ {}^{s}I^{\alpha}_{a,\varphi}(fg)(t)+\ {}^{s}I^{\alpha}_{a,\varphi}(1)\ {}^{s}I^{\alpha}_{a,\varphi}(ufg)(t) \\ \geq\ & {}^{s}I^{\alpha}_{a,\varphi}(uf)(t)\ {}^{s}I^{\alpha}_{a,\varphi}(g)(t)+\ {}^{s}I^{\alpha}_{a,\varphi}(f)(t)\ {}^{s}I^{\alpha}_{a,\varphi}(ug)(t) \end{aligned}.$$

3. *pour* $u=v=1, alors$

$$\begin{aligned} & {}^{s}I^{\alpha}_{a,\varphi}(1)\ {}^{s}I^{\alpha}_{a,\varphi}(fg)(t)+\ {}^{s}I^{\alpha}_{a,\varphi}(1)\ {}^{s}I^{\alpha}_{a,\varphi}(fg)(t) \\ \geq\ & {}^{s}I^{\alpha}_{a,\varphi}(f)(t)\ {}^{s}I^{\alpha}_{a,\varphi}(g)(t)+\ {}^{s}I^{\alpha}_{a,\varphi}(f)(t)\ {}^{s}I^{\alpha}_{a,\varphi}(g)(t) \end{aligned}$$

$$2\ \ {}^{s}I^{\alpha}_{a,\varphi}(1)\ {}^{s}I^{\alpha}_{a,\varphi}(fg)(t)\geq 2\ \ {}^{s}I^{\alpha}_{a,\varphi}(f)(t)\ {}^{s}I^{\alpha}_{a,\varphi}(g)(t)$$

$$ {}^{s}I^{\alpha}_{a,\varphi}(fg)(t)\geq \left(\ {}^{s}I^{\alpha}_{a,\varphi}(1)\right)^{-1}\ {}^{s}I^{\alpha}_{a,\varphi}(f)(t)\ {}^{s}I^{\alpha}_{a,\varphi}(g)(t) \tag{3.14}$$

maintenant on peut montrer le théorème 1

## Preuve du théorème 1

*Démonstration.* par l'utilisation de le lemme (4) avec $u=p, v=q$, on a

$$\begin{aligned} & {}^{s}I^{\alpha}_{a,\varphi}p(t)\ {}^{s}I^{\alpha}_{a,\varphi}(qfg)(t)+\ {}^{s}I^{\alpha}_{a,h\varphi}q(t)\ {}^{s}I^{\alpha}_{a,\varphi}(pfg)(t) \\ \geq\ & {}^{s}I^{\alpha}_{a,\varphi}(pf)(t)\ {}^{s}I^{\alpha}_{a,\varphi}(qg)(t)+\ {}^{s}I^{\alpha}_{a,\varphi}(qf)(t)\ {}^{s}I^{\alpha}_{a,\varphi}(pg)(t) \end{aligned} \tag{3.15}$$

On multiplie (3.15) par ${}^{s}I_{(a,\varphi)}(l)(t)$ on obtient :

$$\begin{aligned} &{}^{s}I_{a,\varphi}(l)(t)\left[\ {}^{s}I^{\alpha}_{a,\varphi}p(t)\ \ {}^{s}I^{\alpha}_{a,\varphi}(qfg)(t)\right] +\ {}^{s}I_{a,h}(l)(t)\left[\ {}^{s}I^{\alpha}_{a,\varphi}q(t)\ \ {}^{s}I^{\alpha}_{a,\varphi}(pfg)(t)\right] \\ \geq\ &{}^{s}I_{a,h}(l)(t)\left[\ {}^{s}I^{\alpha}_{a,\varphi}(pf)(t)\ \ {}^{s}I^{\alpha}_{a,\varphi}(qg)(t)\right] +\ {}^{s}I_{(a,\varphi)}(r)(t)\left[\ {}^{s}I^{\alpha}_{a,\varphi}(qf)(t)\ \ {}^{s}I^{\alpha}_{a,\varphi}(pg)(t)\right] \end{aligned} \tag{3.16}$$

On remplaçe $u\ par\ l\ \ et\ v\ par\ q$ dans le lemme (4) on obtient :

$$\begin{aligned} &{}^{s}I^{\alpha}_{a,\varphi}l(t)\ \ {}^{s}I^{\alpha}_{a,\varphi}(qfg)(t) +\ {}^{s}I^{\alpha}_{a,\varphi}q(t)\ \ {}^{s}I^{\alpha}_{a,\varphi}(lfg)(t) \\ \geq\ &{}^{s}I^{\alpha}_{a,\varphi}(lf)(t)\ \ {}^{s}I^{\alpha}_{a,\varphi}(qg)(t) +\ {}^{s}I^{\alpha}_{a,\varphi}(qf)(t)\ \ {}^{s}I^{\alpha}_{a,\varphi}(lg)(t). \end{aligned} \tag{3.17}$$

la même chose On multiplie (3.17)par ${}^{s}I^{\alpha}_{a,\varphi}p(t)$, alors :

$$\begin{aligned} &{}^{s}I^{\alpha}_{a,\varphi}p(t)\left[\ {}^{s}I^{\alpha}_{a,\varphi}l(t)\ \ {}^{s}I^{\alpha}_{a,\varphi}(qfg)(t) +\ {}^{s}I^{\alpha}_{a,\varphi}q(t)\ \ {}^{s}I^{\alpha}_{a,\varphi}(lfg)(t)\right] \\ \geq\ &{}^{s}I^{\alpha}_{a,\varphi}p(t)\left[\ {}^{s}I^{\alpha}_{a,\varphi}(lf)(t)\ \ {}^{s}I^{\alpha}_{a,\varphi}(qg)(t) +\ {}^{s}I^{\alpha}_{a,\varphi}(qf)(t)\ \ {}^{s}I^{\alpha}_{a,\varphi}(lg)(t)\right]. \end{aligned} \tag{3.18}$$

On pose $u = l, v\ =\ p$ et en utilisant à nouveau le lemme (4) , on à

$$\begin{aligned} &{}^{s}I^{\alpha}_{a,\varphi}l(t)\ \ {}^{s}I^{\alpha}_{a,\varphi}(qfg)(t) +\ {}^{s}I^{\alpha}_{a,\varphi}q(t)\ \ {}^{s}I^{\alpha}_{a,\varphi}(lfg)(t) \\ \geq\ &{}^{s}I^{\alpha}_{a,\varphi}(lf)(t)\ \ {}^{s}I^{\alpha}_{a,\varphi}(qg)(t) +\ {}^{s}I^{\alpha}_{a,\varphi}(qf)(t)\ \ {}^{s}I^{\alpha}_{a,\varphi}(lg)(t). \end{aligned} \tag{3.19}$$

On multiplie (3.19) par ${}^{s}I^{\alpha}_{a,h}q(t)$ , donc

$$\begin{aligned} &{}^{s}I^{\alpha}_{a,\varphi}q(t)\left[\ {}^{s}I^{\alpha}_{a,\varphi}l(t)\ \ {}^{s}I^{\alpha}_{a,\varphi}(qfg)(t) +\ {}^{s}I^{\alpha}_{a,\varphi}q(t)\ \ {}^{s}I^{\alpha}_{a,\varphi}(lfg)(t)\right] \\ \geq^{s}\ &I^{\alpha}_{a,\varphi}q(t)\left[\ {}^{s}I^{\alpha}_{a,\varphi}(lf)(t)\ \ {}^{s}I^{\alpha}_{a,\varphi}(qg)(t) +\ {}^{s}I^{\alpha}_{a,\varphi}(qf)(t)\ \ {}^{s}I^{\alpha}_{a,\varphi}(lg)(t)\right]. \end{aligned} \tag{3.20}$$

On sommant les intégralités (3.15),(3.18),(3.20) on obtient l'intégrale (3.1). □

**Remarque 3.**

1. *On utilise le théorème* (3) *si* $l = 1$,*on trouve :*

$$
\begin{aligned}
&{}^{s}I^{\alpha}_{a,\varphi}(1)\Big[\ {}^{s}I^{\alpha}_{a,\varphi}p(t)\ \ {}^{s}I^{\alpha}_{a,\varphi}(qfg)(t) +\ {}^{s}I^{\alpha}_{a,\varphi}q(t)\ \ {}^{s}I^{\alpha}_{a,\varphi}(pfg)(t)\Big] \\
&+\ {}^{s}I^{\alpha}_{a,\varphi}p(t)\ \ {}^{s}I^{\alpha}_{a,\varphi}(q)(t))\ \ {}^{s}I^{\alpha}_{a,\varphi}(fg)(t) \geq \\
&\tfrac{1}{2}\Big\{\ {}^{s}I^{\alpha}_{a,\varphi}(1)\Big[\ {}^{s}I^{\alpha}_{a,\varphi}(pf)(t)\ \ {}^{s}I^{\alpha}_{a,\varphi}(qg)(t) +\ {}^{s}I^{\alpha}_{a,\varphi}(qf)(t)\ \ {}^{s}I^{\alpha}_{a,\varphi}(pg)(t)\Big] \\
&+\ {}^{s}I^{\alpha}_{a,\varphi}p(t)\Big[\ {}^{s}I^{\alpha}_{a,\varphi}(f)(t)\ \ {}^{s}I^{\alpha}_{a,h}(qg)(t) +\ {}^{s}I^{\alpha}_{a,\varphi}(qf)(t)\ \ {}^{s}I^{\alpha}_{a,\varphi}(g)(t)\Big] \\
&+\ {}^{s}I^{\alpha}_{a,\varphi}q(t)\Big[\ {}^{s}I^{\alpha}_{a,\varphi}(f)(t)\ \ {}^{s}I^{\alpha}_{a,\varphi}(pg)(t) +\ {}^{s}I^{\alpha}_{a,\varphi}(pf)(t)\ \ {}^{s}I^{\alpha}_{a,\varphi}(g)(t)\Big]\Big\}.
\end{aligned}
$$

2. *On utilise le théorème* (3) *si* $p = 1$,*on a :*

$$
\begin{aligned}
&{}^{s}I^{\alpha}_{a,\varphi}l(t)\Big[\ {}^{s}I^{\alpha}_{a,\varphi}(1)\ \ {}^{s}I^{\alpha}_{a,\varphi}(qfg)(t) +\ {}^{s}I^{\alpha}_{a,\varphi}q(t)\ \ {}^{s}I^{\alpha}_{a,\varphi}(fg)(t)\Big] \\
&+\ {}^{s}I^{\alpha}_{a,\varphi}(1)\ \ {}^{s}I^{\alpha}_{a,\varphi}(q)(t))\ \ {}^{s}I^{\alpha}_{a,\varphi}(lfg)(t) \geq \\
&\tfrac{1}{2}\Big\{\ {}^{s}I^{\alpha}_{a,\varphi}l(t)\Big[\ {}^{s}I^{\alpha}_{a,\varphi}(f)(t)\ \ {}^{s}I^{\alpha}_{a,\varphi}(qg)(t) +\ {}^{s}I^{\alpha}_{a,\varphi}(qf)(t)\ \ {}^{s}I^{\alpha}_{a,\varphi}(g)(t)\Big] \\
&+\ {}^{s}I^{\alpha}_{a,\varphi}(1)\Big[\ {}^{s}I^{\alpha}_{a,\varphi}(lf)(t)\ \ {}^{s}I^{\alpha}_{a,\varphi}(qg)(t) +\ {}^{s}I^{\alpha}_{a,\varphi}(qf)(t)\ \ {}^{s}I^{\alpha}_{a,\varphi}(lg)(t)\Big] \\
&+\ {}^{s}I^{\alpha}_{a,\varphi}q(t)\Big[\ {}^{s}I^{\alpha}_{a,\varphi}(lf)(t)\ \ {}^{s}I^{\alpha}_{a,\varphi}(g)(t) +\ {}^{s}I^{\alpha}_{a,\varphi}(f)(t)\ \ {}^{s}I^{\alpha}_{a,\varphi}(lg)(t)\Big]\Big\}.
\end{aligned}
$$

3. *de la même façon on appliquant le théorème* (3) *si* $q = 1$ *on a :*

$$
\begin{aligned}
&{}^{s}I^{\alpha}_{a,\varphi}l(t)\Big[\ {}^{s}I^{\alpha}_{a,\varphi}p(t)\ {}^{s}I^{\alpha}_{a,\varphi}(fg)(t) +\ {}^{s}I^{\alpha}_{a,\varphi}(1)\ {}^{s}I^{\alpha}_{a,\varphi}(pfg)(t)\Big] \\
&+\ {}^{s}I^{\alpha}_{a,\varphi}p(t)\ {}^{s}I^{\alpha}_{a,\varphi}(1)\ {}^{s}I^{\alpha}_{a,\varphi}(lfg)(t) \geq \\
&\tfrac{1}{2}\Big\{\ {}^{s}I^{\alpha}_{a,\varphi}l(t)\Big[\ {}^{s}I^{\alpha}_{a,\varphi}(pf)(t)\ {}^{s}I^{\alpha}_{a,\varphi}(g)(t) +\ {}^{s}I^{\alpha}_{a,\varphi}(f)(t)\ {}^{s}I^{\alpha}_{a,\varphi}(pg)(t)\Big] \\
&+\ {}^{s}I^{\alpha}_{a,\varphi}p(t)\Big[\ {}^{s}I^{\alpha}_{a,\varphi}(lf)(t)\ {}^{s}I^{\alpha}_{a,\varphi}(g)(t) +\ {}^{s}I^{\alpha}_{a,\varphi}(f)(t)\ {}^{s}I^{\alpha}_{a,\varphi}(lg)(t)] \\
&+\ {}^{s}I^{\alpha}_{a,\varphi}(1)\Big[\ {}^{s}I^{\alpha}_{a,\varphi}(lf)(t)\ {}^{s}I^{\alpha}_{a,\varphi}(pg)(t) +\ {}^{s}I^{\alpha}_{a,\varphi}(pf)(t)\ {}^{s}I^{\alpha}_{a,\varphi}(lg)(t)\Big]\Big\}.
\end{aligned}
$$

4. *d'après le théorème* (3) *et si* $p = q = 1$ *on obtient :*

$$
\begin{aligned}
&{}^{s}I^{\alpha}_{a,\varphi}l(t)\ {}^{s}I^{\alpha}_{a,\varphi}(1)\ {}^{s}I^{\alpha}_{a,\varphi}(fg)(t) +\ \left({}^{s}I^{\alpha}_{a,\varphi}(1)\right)^2\ \ {}^{s}I^{\alpha}_{a,\varphi}(lfg)(t) \geq \\
&\tfrac{1}{2}\Big\{\ {}^{s}I^{\alpha}_{a,\varphi}(1)\Big[\ {}^{s}I^{\alpha}_{a,\varphi}(lf)(t)\ {}^{s}I^{\alpha}_{a,\varphi}(g)(t) +\ {}^{s}I^{\alpha}_{a,\varphi}(f)(t)\ {}^{s}I^{\alpha}_{a,\varphi}(lg)(t)\Big] \\
&+\ {}^{s}I^{\alpha}_{a,\varphi}(l)(t)\ {}^{s}I^{\alpha}_{a,\varphi}(f)(t)\ {}^{s}I^{\alpha}_{a,\varphi}(g)(t)\Big\}.
\end{aligned}
$$

5. *la même chose on applique le théorème* (3) *si* $p = r = 1$ *on trouve :*

$$
\begin{aligned}
&{}^{s}I^{\alpha}_{a,\varphi}q(t)\ {}^{s}I^{\alpha}_{a,\varphi}(1)\ {}^{s}I^{\alpha}_{a,\varphi}(fg)(t) +\ \left({}^{s}I^{\alpha}_{a,\varphi}(1)\right)^2\ \ {}^{s}I^{\alpha}_{a,\varphi}(qfg)(t) \geq \\
&\tfrac{1}{2}\Big\{\ {}^{s}I^{\alpha}_{a,\varphi}(1)\Big[\ {}^{s}I^{\alpha}_{a,\varphi}(qf)(t)\ {}^{s}I^{\alpha}_{a,\varphi}(g)(t) +\ {}^{s}I^{\alpha}_{a,\varphi}(f)(t)\ {}^{s}I^{\alpha}_{a,\varphi}(qg)(t)\Big] \\
&+\ {}^{s}I^{\alpha}_{a,\varphi}(q)(t)\ {}^{s}I^{\alpha}_{a,\varphi}(f)(t)\ {}^{s}I^{\alpha}_{a,\varphi}(g)(t)\Big\}.
\end{aligned}
$$

## 3.2 Inégalités fractionnaires pondérées avec deux paramètres $(\alpha, \beta)$

voici un autre résultat qu'est donné par le théorème suivant

**Théorème 4.** *Si* $f$ *et* $g$ *sont deux fonctions synchrones sur* $[a, b]$ *et soit* $r, p, q : [a, b] \to [a, b]$ *sont des* *fonctions intégrables . Alors* $\forall\ t > a, \alpha > 0\ \beta > 0$, *on a :*

$$\begin{aligned}&{}^sI^{\alpha}_{a,\varphi}l(t)\Big[\ {}^sI^{\alpha}_{a,\varphi}p(t)\ {}^sI^{\alpha}_{a,\varphi}(qfg)(t) +\ {}^sI^{\alpha}_{a,\varphi}q(t)\ {}^sI^{\alpha}_{a,\varphi}(pfg)(t)\Big]\\ &+\ {}^sI^{\alpha}_{a,\varphi}p(t)\ {}^sI^{\alpha}_{a,\varphi}(q)(t))\ {}^sI^{\alpha}_{a,\varphi}(lfg)(t) \geq\\ &\tfrac{1}{2}\Big\{\ {}^sI^{\alpha}_{a,\varphi}l(t)\Big[\ {}^sI^{\alpha}_{a,\varphi}(pf)(t)\ {}^sI^{\alpha}_{a,\varphi}(qg)(t) +\ {}^sI^{\alpha}_{a,\varphi}(qf)(t)\ {}^sI^{\alpha}_{a,\varphi}(pg)(t)\Big]\\ &+\ {}^sI^{\alpha}_{a,\varphi}p(t)\Big[\ {}^sI^{\alpha}_{a,\varphi}(lf)(t)\ {}^sI^{\alpha}_{a,\varphi}(qg)(t) +\ {}^sI^{\alpha}_{a,\varphi}(qf)(t)\ {}^sI^{\alpha}_{a,\varphi}(lg)(t)\Big]\\ &+\ {}^sI^{\alpha}_{a,\varphi}q(t)\Big[\ {}^sI^{\alpha}_{a,\varphi}(rf)(t)\ {}^sI^{\alpha}_{a,\varphi}(pg)(t) +\ {}^sI^{\alpha}_{a,\varphi}(pf)(t)\ {}^sI^{\alpha}_{a,\varphi}(lg)(t)\Big]\Big\}.\end{aligned} \tag{3.21}$$

avant de montrer ce théorème on a besoin le lemme suivant

**Lemme 5.** *Si* $f$ *et* $g$ *sont deux fonctions synchrones sur* $[0, \infty[$ *et* $u, v : [0, \infty[\longrightarrow [0, \infty[$. *Alors* $\forall\ t > 0$, $\alpha > 0$, *on a :*

$$\begin{aligned}&{}^sI^{\alpha}_{a,\varphi}u(t){}^sI^{\beta}_{a,\varphi}(vfg)(t) + {}^sI^{\beta}_{a,\varphi}v(t){}^sI^{\alpha}_{a,\varphi}(ufg)(t) \geq\\ &{}^sI^{\alpha}_{a,\varphi}(uf)(t){}^sI^{\beta}_{a,\varphi}(vg)(t) + {}^sI^{\beta}_{a,h}(vf)(t){}^sI^{\alpha}_{a,h}(ug)(t)\end{aligned}. \tag{3.22}$$

on donne la preuve du lemme 02

## Preuve de Lemme 2

*Démonstration.* Multipliant (3.9) par la quantité $\frac{(s+1)^{1-\beta}}{\Gamma(\beta)}\ (log^{s+1}\varphi(t) - log^{s+1}\ \varphi(\rho))^{\beta-1}log^s\ \varphi(\rho)\ \frac{\varphi'(\rho)}{\varphi(\rho)}\ v(\rho)$ ; $\rho \in (a, t)$ on obtient

$$\begin{aligned}&\tfrac{(s+1)^{1-\beta}}{\Gamma(\beta)}\ (log^{s+1}\varphi(t) - log^{s+1}\ \varphi(\rho))^{\beta-1}log^s\ \varphi(\rho)\ \tfrac{\varphi'(\rho)}{\varphi(\rho)}\ v(\rho){}^sI^{\alpha}_{a,\varphi}(ufg)(t)\\ &+\tfrac{(s+1)^{1-\beta}}{\Gamma(\beta)}\ (log^{s+1}\varphi(t) - log^{s+1}\ \varphi(\rho))^{\beta-1}log^s\ \varphi(\rho)\ \tfrac{\varphi'(\rho)}{\varphi(\rho)}\ v(\rho)f(\rho)g(\rho)\ {}^sI^{\alpha}_{a,\varphi}u(t)\\ &\geq \tfrac{(s+1)^{1-\beta}}{\Gamma(\beta)}\ (log^{s+1}\varphi(t) - log^{s+1}\ \varphi(\rho))^{\beta-1}log^s\ \varphi(\rho)\ \tfrac{\varphi'(\rho)}{\varphi(\rho)}\ v(\rho)g(\rho)\ {}^sI^{\alpha}_{a,\varphi}(uf)(t)\\ &+\tfrac{(s+1)^{1-\beta}}{\Gamma(\beta)}\ (log^{s+1}\varphi(t) - log^{s+1}\ \varphi(\rho))^{\beta-1}log^s\ \varphi(\rho)\ \tfrac{\varphi'(\rho)}{\varphi(\rho)}\ v(\rho)f(\rho)\ {}^sI^{\alpha}_{a,\varphi}(ug)(t)\end{aligned} \tag{3.23}$$

on intègre (3.23) par rapport à $\rho$ sur (a, t), on trouve l'inégalité suivante

$$\begin{aligned}
&\tfrac{(s+1)^{1-\beta}}{\Gamma(\beta)} \int_a^t (log^{s+1}\varphi(t) - log^{s+1}\ \varphi(\rho))^{\beta-1} log^s\ \varphi(\rho)\ \tfrac{\varphi'(\rho)}{\varphi(\rho)}\ v(\rho)^s I^{\alpha}_{a,\varphi}(ufg)(t)\ \tfrac{d\rho}{\rho} \\
&+\tfrac{(s+1)^{1-\beta}}{\Gamma(\beta)} \int_a^t (log^{s+1}\varphi(t) - log^{s+1}\ \varphi(\rho))^{\beta-1} log^s\ \varphi(\rho)\ \tfrac{\varphi'(\rho)}{\varphi(\rho)}\ v(\rho)f(\rho)g(\rho)\ {}^sI^{\alpha}_{a,\varphi}u(t)\ \tfrac{d\rho}{\rho} \\
&\geq \tfrac{(s+1)^{1-\beta}}{\Gamma(\beta)} \int_a^t (log^{s+1}\varphi(t) - log^{s+1}\ \varphi(\rho))^{\beta-1} log^s\ \varphi(\rho)\ \tfrac{\varphi'(\rho)}{\varphi(\rho)}\ v(\rho)g(\rho)\ {}^sI^{\alpha}_{a,\varphi}(uf)(t)\ \tfrac{d\rho}{\rho} \\
&+\tfrac{(s+1)^{1-\beta}}{\Gamma(\beta)} \int_a^t (log^{s+1}\varphi(t) - log^{s+1}\ \varphi(\rho))^{\beta-1} log^s\ \varphi(\rho)\ \tfrac{\varphi'(\rho)}{\varphi(\rho)}\ v(\rho)f(\rho)\ {}^sI^{\alpha}_{a,\varphi}(ug)(t)\ \tfrac{d\rho}{\rho}
\end{aligned} \tag{3.24}$$

alors

$$\begin{aligned}
&{}^sI^{\alpha}_{a,\varphi}(ufg)(t)\ \tfrac{(s+1)^{1-\beta}}{\Gamma(\beta)} \int_a^t (log^{s+1}\varphi(t) - log^{s+1}\ \varphi(\rho))^{\beta-1} log^s\ \varphi(\rho)\ \tfrac{\varphi'(\rho)}{\varphi(\rho)}\ \varphi(\rho)\ \tfrac{d\rho}{\rho} \\
&+\ {}^sI^{\alpha}_{a,\varphi}u(t)\ \tfrac{(s+1)^{1-\beta}}{\Gamma(\beta)} \int_a^t (log^{s+1}\varphi(t) - log^{s+1}\ \varphi(\rho))^{\beta-1} log^s\ \varphi(\rho)\ \tfrac{\varphi'(\rho)}{\varphi(\rho)}\ v(\rho)f(\rho)g(\rho)\ \tfrac{d\rho}{\rho} \\
&\geq\ {}^sI^{\alpha}_{a,\varphi}(uf)(t)\ \tfrac{(s+1)^{1-\beta}}{\Gamma(\beta)} \int_a^t (log^{s+1}\varphi(t) - log^{s+1}\ \varphi(\rho))^{\beta-1} log^s\ \varphi(\rho)\ \tfrac{\varphi'(\rho)}{\varphi(\rho)}\ v(\rho)g(\rho)\ \tfrac{d\rho}{\rho} \\
&+\ {}^sI^{\alpha}_{a,\varphi}(ug)(t)\ \tfrac{(s+1)^{1-\beta}}{\Gamma(\beta)} \int_a^t (log^{s+1}\varphi(t) - log^{s+1}\ \varphi(\rho))^{\beta-1} log^s\ \varphi(\rho)\ \tfrac{\varphi'(\rho)}{\varphi(\rho)}\ v(\rho)f(\rho)\ \tfrac{d\rho}{\rho}
\end{aligned} \tag{3.25}$$

c'est-à-dire

$$\begin{aligned}
&{}^sI^{\beta}_{a,\varphi}v(t){}^sI^{\alpha}_{a,\varphi}(ufg)(t) + {}^sI^{\alpha}_{a,\varphi}u(t){}^sI^{\beta}_{a,\varphi}(vfg)(t) \geq \\
&{}^sI^{\alpha}_{a,\varphi}(uf)(t){}^sI^{\beta}_{a,\varphi}(vg)(t) + {}^sI^{\beta}_{a,\varphi}(vf)(t){}^sI^{\alpha}_{a,\varphi}(ug)(t)
\end{aligned} \tag{3.26}$$

□

maintenant on démontré le théorème 02

## Preuve du théorème 02

*Démonstration.* D'après le lemme (5) et $u = p, v = q$ , on écrit

$$\begin{aligned}
&{}^sI^{\alpha}_{a,\varphi}p(t)\ {}^sI^{\beta}_{a,\varphi}(q\ f\ g)(t) +\ {}^sI^{\beta}_{a,\varphi}q(t)\ \ {}^sI^{\alpha}_{a,\varphi}(u\ f\ g)(t)\ \geq \\
&{}^sI^{\alpha}_{a,\varphi}(p\ f)(t)\ {}^sI^{\beta}_{a,\varphi}(q\ g)(t)\ +\ {}^sI^{\beta}_{a,\varphi}(q\ f)(t){}^sI^{\alpha}_{a,\varphi}(p\ g)(t)
\end{aligned}\ . \tag{3.27}$$

et en multipliant (3.27) par ${}^sI^{\alpha}_{a,h}(r)(t)$, on trouve le résultat suivant

$$\begin{array}{c} {}^sI^{\alpha}_{a,\varphi}(l)(t)\left[{}^sI^{\alpha}_{a,\varphi}p(t)\ \ {}^sI^{\beta}_{a,h}(q\ f\ g)(t)+\ \ {}^sI^{\beta}_{a,\varphi}q(t)\ \ \ {}^sI^{\alpha}_{a,\varphi}(u\ f\ g)(t)\right]\ \geq \\ {}^sI^{\alpha}_{a,\varphi}(l)(t)\left[{}^sI^{\alpha}_{a,\varphi}(p\ f)(t)\ \ {}^sI^{\beta}_{a,\varphi}(q\ g)(t)\ \ +\ \ {}^sI^{\beta}_{a,\varphi}(q\ f)(t){}^sI^{\alpha}_{a,\varphi}(p\ g)(t)\right] \end{array}. \tag{3.28}$$

On remplaçe $u$ par $l$ et $v$ par $q$ dans le lemme (5) , on obtient l'inégalité

$$\begin{array}{c} {}^sI^{\alpha}_{a,\varphi}l(t)\ \ {}^sI^{\beta}_{a,\varphi}(q\ f\ g)(t)+\ \ {}^sI^{\beta}_{a,\varphi}q(t)\ \ {}^sI^{\alpha}_{a,\varphi}(l\ f\ g)(t)\geq \\ {}^sI^{\alpha}_{a,\varphi}(l\ f)(t)\ \ {}^sI^{\beta}_{a,\varphi}(q\ \ g)(t)+\ \ {}^sI^{\beta}_{a,\varphi}(q\ f)(t)\ \ {}^sI^{\alpha}_{a,\varphi}(l\ g)(t) \end{array}. \tag{3.29}$$

la même chose en multipliant les deux côtés de (3.29) par ${}^sI^{\alpha}_{a,\varphi}(p)(t)$, on a

$$\begin{array}{c} {}^sI^{\alpha}_{a,\varphi}(p)(t)\left[\ {}^sI^{\alpha}_{a,\varphi}l(t)\ {}^sI^{\beta}_{a,\varphi}(q\ f\ g)(t)+\ \ {}^sI^{\beta}_{a,\varphi}q(t){}^sI^{\alpha}_{a,\varphi}(lfg)(t)\right]\geq \\ {}^sI^{\alpha}_{a,\varphi}(p)(t)\left[\ {}^sI^{\alpha}_{a,\varphi}(l\ f)(t)\ \ {}^sI^{\beta}_{a,\varphi}(qg)(t)+\ \ {}^sI^{\beta}_{a,\varphi}(q\ f)(t)\ \ {}^sI^{\alpha}_{a,\varphi}(l\ g)(t)\right] \end{array}. \tag{3.30}$$

d'après le lemme (5) et $u=l, v=p$, on peut écrire

$$\begin{array}{c} {}^sI^{\alpha}_{a,\varphi}l(t){}^sI^{\beta}_{a,\varphi}(p\ f\ g)(t)+{}^sI^{\beta}_{a,\varphi}p(t){}^sI^{\alpha}_{a,\varphi}(l\ f\ g)(t)\geq \\ {}^sI^{\alpha}_{a,\varphi}(l\ f)(t){}^sI^{\beta}_{a,\varphi}(p\ g)(t)+{}^sI^{\beta}_{a,\varphi}(p\ f)(t){}^sI^{\alpha}_{a,\varphi}(l\ g)(t) \end{array}. \tag{3.31}$$

En multipliant (3.31) par ${}^sI^{\alpha}_{a,h}(q)(t)$, on obtient

$$\begin{array}{c} {}^sI^{\alpha}_{a,\varphi}(q)(t)\left[{}^sI^{\alpha}_{a,\varphi}l(t){}^sI^{\beta}_{a,\varphi}(p\ f\ g)(t)+{}^sI^{\beta}_{a,\varphi}p(t){}^sI^{\alpha}_{a,\varphi}(l\ f\ g)(t)\right]\geq \\ {}^sI^{\alpha}_{a,\varphi}(q)(t)\left[{}^sI^{\alpha}_{a,\varphi}(l\ f)(t){}^sI^{\beta}_{a,\varphi}(p\ g)(t)+{}^sI^{\beta}_{a,\varphi}(p\ f)(t){}^sI^{\alpha}_{a,\varphi}(l\ g)(t)\right] \end{array}. \tag{3.32}$$

En sommant les inégalités (3.28), (3.30) et (3.32), on trouve le résultat (3.21)

□

## 3.3 Inégalité fractionnaire de n fonctions

**Théorème 5.** *Si $(f_i)_{i=1,\ldots,n}$ sont des fonctions croissantes positives sur $[0,\infty[$. Alors $\forall\ t>0\ ,\alpha>0$, on a*

$$ {}^sI^{\alpha}_{a,\varphi}\left(\prod_{i=1}^{n}f_i\right)(t)\geq\left({}^sI^{\alpha}_{a,\varphi}(1)\right)^{1-n}\prod_{i=1}^{n}{}^sI^{\alpha}_{a,\varphi}f_i(t). \tag{3.33}$$

*Démonstration.* on peut démontrer ce théorème par récurrence .

si $n = 1$ , on a

$$^{s}I^{\alpha}_{a,\varphi}\left(f_1\right)(t) \geq {}^{s}I^{\alpha}_{a,\varphi}\left(f_1\right)(t), \quad t>0, \alpha>0,$$

et si $n = 2$ , en appliquant (3.14), on trouve

$$^{s}I^{\alpha}_{a,\varphi}(f_1f_2)(t) \geq \left({}^{s}I^{\alpha}_{a,\varphi}(1)\right)^{-1} {}^{s}I^{\alpha}_{a,\varphi}f_1(t) {}^{s}I^{\alpha}_{a,\varphi}f_2(t), \quad t>0, \alpha>0,$$

et par l'hypothèse de récurrence on trouve

$$^{s}I^{\alpha}_{a,\varphi}\left(\prod_{i=1}^{n-1} f_i\right)(t) \geq \left({}^{s}I^{\alpha}_{a,\varphi}(1)\right)^{2-n} \prod_{i=1}^{n-1} {}^{s}I^{\alpha}_{a,\varphi}f_i(t). \tag{3.34}$$

comme $(f_i)_{i=1,\ldots,n}$ sont des fonctions croissantes positives, alors $\left(\prod_{i=1}^{n-1} f_i\right)(t)$ est une fonction croissante.

par l'inégalité (3.14) les fonctions $\prod_{i=1}^{n-1} f_i = g, f_n = f$, on a

$$^{s}I^{\alpha}_{a,\varphi}\left(\prod_{i=1}^{n} f_i\right)(t) = {}^{s}I^{\alpha}_{a,\varphi}(fg)(t) \geq \left({}^{s}I^{\alpha}_{a,\varphi}(1)\right)^{-1} {}^{s}I^{\alpha}_{a,\varphi}\left(\prod_{i=1}^{n-1} f_i\right)(t) {}^{s}I^{\alpha}_{a,\varphi}\left(f_n\right)(t). \tag{3.35}$$

et par l'inégalité (3.34) , on obtient

$$^{s}I^{\alpha}_{a,\varphi}\left(\prod_{i=1}^{n} f_i\right)(t) \geq \left({}^{s}I^{\alpha}_{a,\varphi}(1)\right)^{-1}\left(\left({}^{s}I^{\alpha}_{a,\varphi}(1)\right)^{2-n}\left(\prod_{i=1}^{n-1} {}^{s}I^{\alpha}_{a,\varphi}f_i\right)(t)\right) {}^{s}I^{\alpha}_{a,\varphi}\left(f_n\right)(t), \tag{3.36}$$

ce qui implique

$$^{s}I^{\alpha}_{a,\varphi}\left(\prod_{i=1}^{n} f_i\right)(t) \geq \left({}^{s}I^{\alpha}_{a,\varphi}(1)\right)^{1-n} \prod_{i=1}^{n} {}^{s}I^{\alpha}_{a,\varphi}f_i(t).$$

□

**Remarque 4.** *Si $f$ et $g$ sont deux fonctions définies sur $[a,b]$.*

***(A)*** *Suppose que $f$ est une fonction croissante, $g$ est une fonction différentiable et il existe un nombre réel $M := \sup_{t\geq 0} g'(t)$ . Alors $\forall\ t>0, \alpha>0$, on a*

$$\begin{aligned} &{}^{s}I^{\alpha}_{a,\varphi}(fg)(t) \geq \left({}^{s}I^{\alpha}_{a,\varphi}(1)\right)^{-1} {}^{s}I^{\alpha}_{a,\varphi}f(t) {}^{s}I^{\alpha}_{a,\varphi}g(t) - Mt {}^{s}I^{\alpha}_{a,\varphi}f(t) + M {}^{s}I^{\alpha}_{a,\varphi}(tf(t)) \\ &pour\ \ toute\ \ t>a, \alpha>0. \end{aligned} \tag{3.37}$$

***(B)*** *Suppose que $f$ et $g$ sont différentiables et il existe $m_1 := \inf_{t\geq 0} f'(x)$*

$m_2 := \inf_{t\geq 0} g'(t)$, *alors on a*

$$\begin{aligned} &{}^sI^{\alpha}_{a,\varphi}(fg)(t) - m_1{}^sI^{\alpha}_{a,\varphi}tg(t) - m_2{}^sI^{\alpha}_{a,\varphi}tf(t) + m_1m_2{}^sI^{\alpha}_{a,\varphi}t^2 \\ &\quad\geq \left({}^sI^{\alpha}_{a,\varphi}(1)\right)^{-1}\left({}^sI^{\alpha}_{a,\varphi}f(t){}^sI^{\alpha}_{a,\varphi}g(t) - m_1{}^sI^{\alpha}_{a,\varphi}t{}^sI^{\alpha}_{a,\varphi}g(t) - m_2{}^sI^{\alpha}_{a,\varphi}t{}^sI^{\alpha}_{a,\varphi}f(t) + m_1m_2\left({}^sI^{\alpha}_{a,\varphi}t\right)^2\right). \end{aligned} \tag{3.38}$$

***(C)*** *On supposons que f et g sont différentiables et qu'il existe* $M_1 := \sup_{t\geq 0} f'(t), M_2 := \sup_{t\geq 0} g'(t)$. *Alors l'inégalité*

$$\begin{aligned} &I^{\alpha}_{a,\varphi}(fg)(t) - M_1I^{\alpha}_{a,\varphi}\,\mathrm{tg}(t) - M_2I^{\alpha}_{a,\varphi}tf(t) + M_1M_2I^{\alpha}_{a,\varphi}t^2 \\ &\quad\geq \left(I^{\alpha}_{a,\varphi}(1)\right)^{-1}\left(I^{\alpha}_{a,\varphi}f(t)I^{\alpha}_{a,\varphi}g(t) - M_1I^{\alpha}_{a,\varphi}tI^{\alpha}_{a,\varphi}g(t) - M_2I^{\alpha}_{a,\varphi}tI^{\alpha}_{a,\varphi}f(t) + M_1M_2\left(I^{\alpha}_{a,\varphi}t\right)^2\right) \end{aligned} \tag{3.39}$$

*est vérifié.*

*Démonstration.*
**(A)** d'après l'inégalité (3.14) les fonctions $f$ et $G(t) := g(t) - m_2t$. on trouve le résultat
**(B)** le même chose que (A)on applique l'inégalité (3.14) aux fonctions $F$ et $G$, où : $F(t) := f(t) - m_1t, G(t) := g(t) - m_2t$.
**(C)** aussi on peut appliquer l'inégalité (3.14) on obtient ce qu'on veut .

$$F(t) := f(t) - M_1t, \quad G(t) := g(t) - M_2t.$$

□

# Bibliographie


[1] M. BEZZIOU, Z. Dahmani, M. Z. Sarikaya, New operators for fractional integration theory with some applications, Journal of Mathematical Extension, 12(4), 2018, 87-100

[2] Z. Dahmani, New inequalities in fractional integrals, International Journal of Nonlinear Science, 4, 9, 2010, 493-497.

[3] Z. Dahmani, L. Tabharit and S. Taf, New generalisations of Grüss inequality using Riemann-Liouville fractional integrals, Bulletin of Mathematical Analysis and Applications, 2, 3(2010), 93-99.

[4] S. Anber, Z. Dahmani and B. Bendoukha, Some new results using integration of arbitrary order, Int. J. Nonlinear Anal. Appl. , 4(2), 2013, 45-52.

[5] S. Belarbi, Z. Dahmani, On some new fractional integral inequalities,journal of inequalities in pure and applied mathematics, 10, 2009, 3.

[6] S. M. Malamuda, Some complements to the Jenson and Chebyshev inequalities and a problem of W. Walter, Proc. Amer. Math. Soc., 129(9), 2001, 2671–2678.

[7] S. Mubeen, and G. M. Habibullah, k-fractional integrals and application, Int. J. Contemp. Math. Sciences, 7(2), 2012, 89-94.

[8] B. G. PACHPATTE, A note on Chebyshev-Grüss type inequalities for differential functions, Tamsui Oxford Journal of Mathematical Sciences, 22(1), 2006, 29–36.

[9] A. Rehman, S. Mubeen, N. Sadiq and F. Shaheen, Some inequalities involving k-gamma and k-beta functions with applications, Journal of Inequalities and Applications , 224(1) :16,2014

[10] A. Rehman, R. Munir, S. S. Mubeen, A note on some k-functions and their probability distributions. Sci. Int. (Lahore), 29(1), 2017, 35-40

[11] S. G. Samko, A. A. Kilbas and O. I.Marichev, Fractional Integrals and Derivatives Theory and Application, Gordan and Breach Science, New York, 1993.

[12] D. Grüss, Uber das maximum des absoluten Betrages von $\frac{1}{b-a}\int_a^b f(x)g(x) - \frac{1}{(b-a)^2}\int_a^b f(x)dx \int_a^b g(x)dx$, Math. Z. 39, 1935, 215-226.