People's Democratic Republic of Algeria
Ministry of Higher Education and Scientific Research

Djilali Bounaama Khemis Miliana University
Faculty of Science and Technology
Department of Mathematics and Computer Science

Master Thesis

presented for the purpose of obtaining

MASTER DEGREE IN MATHEMATICS

Specialty : Mathematical Analysis and Applications

By

Khassani Assia

Solvabilité d'un problème aux limites fractionnaires à deux points dans le cas de résonance

(Solvability of two-points fractional boundary value problem at resonance)

Defended on July 2019 in front of the jury:

Jury President :	Mr Bezziou Mohamed	*Khemis-Miliana University*
Supervisor :	Mr Benbachir Maamar	*Khemis-Miliana University*
Examiner :	Mr Hachama Mohammed	*Khemis-Miliana University*
Examiner :	Mr Houas Mohamed	*Khemis-Miliana University*

COLLEGE YEAR : 2018-2019

Dedication

I dedicate this thesis to my parents Youcef and Fatiha for their limitless love, encouragement, and support.

Acknowledgements

First, I would like to express my sincere gratitude to my supervisor Professor Benbachir Maamar for the continuous support of my Master study and related research, for his patience, motivation, and immense knowledge.

Second, I would like to express my deep thanks to my thesis reviewers for having accepted to discuss it, and for reading this manuscript carefully and for their support and helpful comments.

I also would like to thank all my colleagues and friends for creating the friendly, pleasant and fertile atmosphere to work.

Finally, I would like to express my deepest gratitude to my family for the love, support, and encouragement.

Abstract

This thesis deals with the BVP multi-point existence of nonlinear fractional differential equations at resonance, where the kernel's dimension of the fractional differential operator is equal to one. The main results are based on Mawhin's theory of coincidence.

Résumé

Ce mémoire traite l'existence de solution d'un problème aux limites à point multiple d'une équation différentielle fractionnaire non linéaire à la résonance, où la dimension du noyau de l'opérateur différentiel fractionnaire est égale à un. L'outil principal utilisé dans dans ce mémoire est basé principalement sur la théorie de la coïncidence du degré dû à J.L. Mawhin.

Contents

Notations

$X :=$ Banach space.

$\Gamma(\cdot) :=$ Euler's Gamma function.

$B(\cdot,\cdot) :=$ Euler's Beta function.

$I_a^\alpha :=$ Riemann–Liouville Fractional Integral of Order α where $\mathrm{Re}(\alpha) > 0$.

$D_a^\alpha :=$ Riemann–Liouville Fractional Derivative of order α where $\mathrm{Re}(\alpha) > 0$.

$D = \dfrac{d}{dt}$.

$L^1[0,1] := \{u : [0,1] \to \mathbb{R}$ is measurable on $[0,1]$ and $\displaystyle\int_0^1 |u(t)|dt < \infty\}$.

$\mathcal{C}^k[a,b] := \{u : [0,1] \to \mathbb{R};\ u$ has a continuous kth derivative, $k \in \mathbb{N}\}$.

$\mathcal{C}[a,b] := \mathcal{C}^0[a,b]$.

$\mathcal{C}(\overline{\Omega}) := \{u : \overline{\Omega} \subset X \to X;\ u$ is a continuous in $\overline{\Omega}\}$.

$\mathcal{C}^{\alpha-1}[0,1] := \{u(t) | u(t) = I_{0+}^{\alpha-1}x(t), x \in \mathcal{C}[0,1], t \in [0,1], \mathrm{Re}(\alpha) > 0\}$.

$u'(t_0) = \left(\dfrac{du_i}{dt_j}\right)_{1\le i,j\le n}(t_0); u \in \mathcal{C}^1(\overline{\Omega})$.

$S_u :=$ The set of critical points of $u \in \Omega$.

Introduction

Fractional calculus developed since 17th century through the pioneering works of Leibniz, Euler, Lagrange, Able, Liouville and many others deals with the generalization of differentiation and integration to fractional order. In recent years the term **fractional calculus** refers to integration and differentiation to an arbitrary order. Fractional differential equations have been of great interest recently. It is caused both by the intensive development of the theory of fractional calculus itself and by the applications of such constructions in various sciences such as physics, mechanics, chemistry, engineering, etc.

Gaines and Mawhin introduced **coincidence degree theory** in 1970s in analyzing functional and differential equations [4]. Mawhin has continued studies on this theory later and has made so important contributions on this subject since then this theory is also known as Mahwin's coincidence degree theory. Coincidence theory is a very powerful technique especially in existence of solutions problems in nonlinear equations. It has especially so broad applications in the existence of periodic solutions of nonlinear differential equations so that many researchers have used it for their investigations [7].

The main goal in the coincidence degree theory is to search the existence of a solutions of the operator equation

$$Lu = Nu \tag{1}$$

in some bounded and open set Ω in some Banach space for L being a linear operator and N nonlinear operator. In finite dimensional case, for $\Omega \subset \mathbb{R}^n$, $u \in C(\overline{\Omega})$, and $p \in \mathbb{R}^n \backslash u(\partial\Omega)$, the degree of u on Ω with respect to $p, d(u, \Omega, p)$ is well defined. But unfortunately this is not the case in infinite dimension for $u \in C(\overline{\Omega})$.

Luckily, in an arbitrary Banach space X, Leray and Schauder proved that for $\Omega \in X$ open, bounded set, $M : \overline{\Omega} \to X$ compact operator and for $p \in X\backslash(I - M)(\partial\Omega)$ the degree of compact perturbation of identity $I - M$ in Ω with respect to $p, \deg(I - M, \Omega, p)$ is well defined. One of the main useful properties of degree theory is that if $\deg(I - M, \Omega, p) \neq 0$ then $(I - M)x = p$ has at least one solution in Ω In particular if we take $p = 0$ and $\deg(I - M, \Omega, p) \neq 0$ then the compact operator M has at least one fixed point in Ω.

In [4] Gaines and Mawhin studied existence of a solution of an operator equation (1) defined on a Banach space X in an open bounded set Ω using the Leray-Schauder degree theory but since the operator $I-(L-N)$ is not compact in general the need to define a compact operator M such that its set of fixed points in Ω would be equal to a solution set of (1) in Ω.
In [4] the compact operator M is given and the coincidence degree for the couple (L,N) in Ω is defined by $\deg[(L,N),\Omega]=\deg(I-M,\Omega,0)$.

The purpose of these thesis is to study the existence of solution to fractional boundary value problems at resonance in Banach spaces. Our study is based upon the coincidence degree theory of Mawhin.

Chapter 1

Theoretical and Basic Tools

Contents

1.1 Preliminaries on the Fractional Calculation

1.1.1 Gamma Function

One of the basic functions of the fractional calculus is Eulcr[1]'s gamma function Γ, which generalizes the factorial, $n!$.

Definition 1.1.1 *Gamma function* (Γ) *is defined as:*

$$\Gamma(\alpha) = \int_0^{\infty} e^{-t} t^{\alpha-1} dt. \tag{1.1}$$

Theorem 1.1.1 *Function* $\Gamma(\alpha)$ *is convergent for* $\mathrm{Re}(\alpha) > 0$.

Example 1.1.1 *Evaluate* $\Gamma\left(\frac{1}{2}\right)$ *by definition* 1.1.1

$$\Gamma\left(\frac{1}{2}\right) = \int_0^{+\infty} \frac{1}{\sqrt{t}} e^{-t} dt.$$

Since we have by using the substitution $t = s^2$,

$$dt = 2sds,$$

where

$$\begin{aligned} \Gamma\left(\frac{1}{2}\right) &= \int_0^{+\infty} \frac{1}{s} e^{-s^2} 2sds \\ &= 2\int_0^{+\infty} e^{-s^2} ds. \end{aligned}$$

Compute $\left[\Gamma\left(\frac{1}{2}\right)\right]^2$, *we find*

$$\begin{aligned} \left[\Gamma\left(\frac{1}{2}\right)\right]^2 &= \left[2\int_0^{+\infty} e^{-s^2} ds\right]^2 \\ &= \left[2\int_0^{+\infty} e^{-s^2} ds\right]\left[2\int_0^{+\infty} e^{-z^2} dz\right] \\ &= 4\int_0^{+\infty}\int_0^{+\infty} e^{-\left(z^2+s^2\right)} dzds, \end{aligned}$$

for

$$\begin{cases} z = r\cos\theta \\ s = r\sin\theta \end{cases}, r \in \left[0, +\infty\right[, \theta \in \left[0, \frac{\pi}{2}\right]$$

[1] Leonhard Euler (1707-1783) was a Swiss mathematician, physicist, astronomer, logician and engineer who made important and influential discoveries in many branches of mathematics, such as infinitesimal calculus and graph theory, while also making pioneering contributions to several branches such as topology and analytic number theory.

we will have

$$\begin{aligned}\left[\Gamma\left(\frac{1}{2}\right)\right]^2 &= 4\int_{\theta=0}^{\frac{\pi}{2}}\int_{r=0}^{+\infty} e^{-r^2} r dr d\theta \\ &= 4\int_0^{\frac{\pi}{2}}\left[-\frac{e^{-r^2}}{2}\right]_0^{+\infty} d\theta \\ &= 4\left[-\frac{e^{-r^2}}{2}\right]_0^{+\infty}\int_0^{\frac{\pi}{2}} d\theta \\ &= \pi \\ &\Rightarrow \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}.\end{aligned}$$

Some Properties of the Gamma Function

The basic properties of the Gamma function are:

1. The function $\Gamma(\alpha)$ obeys the property:

$$\Gamma(\alpha+1) = \alpha\Gamma(\alpha). \tag{1.2}$$

Since we have

$$\Gamma(\alpha+1) = \int_0^{\infty} e^{-t}t^{\alpha}dt = -\left[e^{-t}t^{\alpha}\right]_0^{\infty} + \alpha\int_0^{\infty} e^{-t}t^{\alpha-1}dt = \alpha\Gamma(\alpha). \tag{1.3}$$

2. The following particular values for Γ function can be useful for calculation purposes:

$$\begin{aligned}\Gamma(n+1) &= n!, \\ \frac{1}{\Gamma(0)} &= 0, \\ \Gamma\left(-\frac{1}{2}\right) &= -2\sqrt{\pi}, \\ \Gamma\left(\frac{-3}{2}\right) &= \frac{4}{3}\sqrt{\pi}, \\ \Gamma\left(\frac{3}{2}\right) &= \frac{\sqrt{\pi}}{2}, \\ \Gamma\left(\frac{5}{2}\right) &= \frac{3\sqrt{\pi}}{4}.\end{aligned}$$

1.1.2 Beta Function

Definition 1.1.2 *The Beta function, or the first order Euler function, can be defined as:*

$$B(p,q) = \int_0^1 t^{p-1}(1-t)^{q-1}dt,$$

where

$$\mathrm{Re}(p) > 0 \text{ and } \mathrm{Re}(q) > 0.$$

Some Properties of the Beta Function

The basic properties of the Beta function are:

1. For every $\mathrm{Re}(p) > 0$ and $\mathrm{Re}(q) > 0$, we have:

$$B(p,q) = B(q,p).$$

since we have by using the substitution $t = 1 - s$

$$B(p,q) = \int_0^1 t^{p-1}(1-t)^{q-1}dt = -\int_1^0 (1-s)^{p-1}s^{q-1}ds = \int_0^1 s^{q-1}(1-s)^{p-1}ds = B(q,p).$$

2. For every $\mathrm{Re}(p) > 0$ and $\mathrm{Re}(q) > 0$, it is valid the identity:

$$B(p,q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}.$$

Since we use the definition 1.1.1 to obtain

$$\Gamma(p)\Gamma(q) = \int_0^\infty e^{-t}t^{p-1}dt \int_0^\infty e^{-s}s^{q-1}ds = \int_0^\infty \int_0^\infty e^{-(t+s)}t^{p-1}s^{q-1}dtds = \int_0^\infty \int_0^\infty F(t,s)dtds.$$

Now we apply the change of variables $t = xy = \phi(x,y)$ and $s = x(1-y) = \psi(x,y)$ to this double integral.

Note that $t + s = x$ and that $0 < t < \infty$ and $0 < s < \infty$ imply that $0 < x < \infty$ and $0 < y < 1$.

The Jacobian of this transformation is

$$Jac = \begin{vmatrix} \frac{\partial\phi}{\partial x} & \frac{\partial\phi}{\partial y} \\ \frac{\partial\psi}{\partial x} & \frac{\partial\psi}{\partial y} \end{vmatrix} = \begin{vmatrix} y & x \\ 1-y & -x \end{vmatrix} = -xy - x + xy = -x.$$

Since $x > 0$ we conclude that

$$dtds = |Jac|dxdy = xdxdy.$$

Hence we have

$$\begin{aligned}
\Gamma(p)\Gamma(q) &= \int_0^1 \int_0^\infty F(\phi(x,y),\psi(x,y))|Jac|dxdy \\
&= \int_0^1 \int_0^\infty e^{-x}x^{p-1}y^{p-1}x^{q-1}(1-y)^{q-1}xdxdy \\
&= \int_0^\infty e^{-x}x^{p+q-1}dx \int_0^1 y^{p-1}(1-y)^{q-1}dy \\
&= \Gamma(p+q)B(p,q).
\end{aligned}$$

1.1.3 Riemann–Liouville Differential and Integral Operators

Definition 1.1.3 *Let u be a continuous function, for every $\mathrm{Re}(\alpha) > 0$, the Riemann*[2]*–Liouville*[3] *fractional integral of order α is defined:*

$$I_a^{\alpha} u(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} u(s) ds, \quad -\infty \leq a < t < \infty. \tag{1.4}$$

Example 1.1.2 *We consider the function u defined by $u(t) = (t-a)^{\beta}$ and we will calculate their integral of order α where $\mathrm{Re}(\alpha) > 0$,*

$$I_a^{\alpha} (t-a)^{\beta} = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} (s-a)^{\beta} ds.$$

We assume the change $s = a + (t-a)\tau$ we obtain:

$$ds = (t-a)d\tau,$$

$$s = a \Longrightarrow \tau = 0,$$

$$s = t \Longrightarrow \tau = 1.$$

So

$$\begin{aligned} I_a^{\alpha} (t-a)^{\beta} &= \frac{1}{\Gamma(\alpha)} \int_a^t (t-a-(t-a)\tau)^{\alpha-1} (a+(t-a)\tau-a)^{\beta} (t-a) d\tau \\ &= \frac{1}{\Gamma(\alpha)} \int_0^1 (t-a)^{\alpha-1} (1-\tau)^{\alpha-1} (t-a)^{\beta} \tau^{\beta} (t-a) d\tau \\ &= \frac{(t-a)^{\alpha+\beta}}{\Gamma(\alpha)} \int_0^1 (1-\tau)^{\alpha-1} \tau^{\beta} d\tau \\ &= \frac{\Gamma(\alpha)\Gamma(\beta+1)}{\Gamma(\alpha)\Gamma(\alpha+\beta+1)} (t-a)^{\alpha+\beta} \\ &= \frac{\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)} (t-a)^{\alpha+\beta}. \end{aligned} \tag{1.5}$$

Corollary 1.1.1 *Let u be a continuous function of two variables (s, z), we have the next Dirichlet*[4] *equality*

$$\int_a^t \int_a^s u(s,z)\, \mathrm{d}z\mathrm{d}s = \int_a^t \int_z^t u(s,z)\, \mathrm{d}s\mathrm{d}z.$$

If u does not depend on s, but on z alone (see the integration area in Figure 1.1):

$$\begin{aligned} \int_a^t \int_a^s u(z)\, \mathrm{d}z\mathrm{d}s = \int_a^t \int_z^t u(z)\, \mathrm{d}s\mathrm{d}z = \int_a^t u(z) \int_z^t \mathrm{d}s\mathrm{d}z \\ = \int_a^t u(z)(t-z)\, \mathrm{d}z. \end{aligned} \tag{1.6}$$

[2]B. Riemann (1826-1866) was a German mathematician who made contributions to analysis, number theory, and differential geometry.

[3]J. Liouville (1809-1882) was a French mathematician known for his work in analysis, differential geometry, and number theory.

[4]J.P.G.L. Dirichlet (1805–1859) was a German mathematician who made deep contributions to number theory, and to the theory of Fourier series and other topics in mathematical analysis.

When the indefinite limit of integration comes first, (1.6) *becomes (see the integration area in Figure 1.1)*

$$\int_t^a \int_s^a u(z)\,\mathrm{d}z\mathrm{d}s = \int_t^a \int_t^z u(z)\,\mathrm{d}s\mathrm{d}z = \int_t^a u(z) \int_t^z \mathrm{d}s\mathrm{d}z$$
$$= \int_t^a u(z)(z-t)\,\mathrm{d}z. \tag{1.7}$$

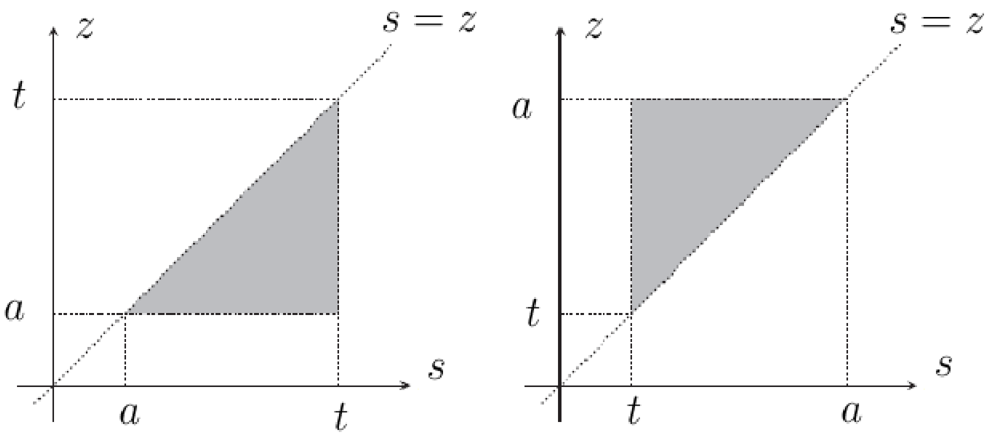

Figure 1.1: Left: integration area of (1.6); right: integration area of (1.7)

Theorem 1.1.2 *The exponents property:*

$$I_a^\alpha I_a^\beta u(t) = I_a^{\alpha+\beta} u(t).$$

Proof. For $\mathrm{Re}(\alpha) > 0$, $\mathrm{Re}(\beta) > 0$, it results:

$$I_a^\alpha I_a^\beta u(t) = \frac{1}{\Gamma(\alpha)} \frac{1}{\Gamma(\beta)} \int_a^t (t-s)^{\alpha-1} \int_a^s (s-z)^{\beta-1} u(z) dz ds.$$

If we apply the Dirichlet equality

$$\int_a^t \int_a^s u(z) dz ds = \int_a^t \int_z^t u(z) ds dz,$$

we obtain:

$$I_a^\alpha I_a^\beta u(t) = \frac{1}{\Gamma(\alpha)} \frac{1}{\Gamma(\beta)} \int_a^t \int_z^t (t-s)^{\alpha-1} (s-z)^{\beta-1} u(z) ds dz,$$
$$s = z + k(t-z),$$

$$ds = (t-z)dk, \quad t-s = (1-k)(t-z),$$
$$I_a^\alpha I_a^\beta u(t) = \frac{1}{\Gamma(\alpha)} \frac{1}{\Gamma(\beta)} \int_a^t (t-z)^{\alpha+\beta-1} u(z) \int_0^1 (1-k)^{\alpha-1} k^{\beta-1} dz dk,$$

but:

$$\int_0^1 (1-k)^{\alpha-1} k^{\beta-1} ds = B(\alpha,\beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}.$$

Finally, it results:

$$I_a^{\alpha} I_a^{\beta} u(t) = \frac{1}{\Gamma(\alpha+\beta)} \int_a^t (t-z)^{\alpha+\beta-1} u(z) dz = I_a^{\alpha+\beta} u(t).$$

■

Definition 1.1.4 *[2] Let* $I_{0+}^{\alpha}\left(L^1(0,1)\right)$, $\mathrm{Re}(\alpha) > 0$, *denote the space of functions* u, *represented by fractional integral of order* α *of a summable function:* $u = I_{0+}^{\alpha} v, v \in L^1(0,1)$.

In the following Lemma, we use the unified notation of both for fractional integrals and fractional derivatives assuming that $I_{0+}^{\alpha} = D_{0+}^{\alpha}$ for $\mathrm{Re}(\alpha) < 0$.

Lemma 1.1.1 *[2] The relation*

$$I_{0+}^{\alpha} I_{0+}^{\beta} \varphi = I_{0+}^{\alpha+\beta} \varphi,$$

is valid in any of the following cases:

1. $\beta \geq 0, \alpha+\beta \geq 0, \quad \varphi(t) \in L^1(0,1).$
2. $\beta \leq 0, \alpha \geq 0, \quad \varphi(t) \in I_{0+}^{-\beta}\left(L^1(0,1)\right).$
3. $\alpha \leq 0, \alpha+\beta \leq 0, \quad \varphi(t) \in I_{0+}^{-\alpha-\beta}\left(L^1(0,1)\right).$

Theorem 1.1.3 *Linearity property:*

$$I_a^{\alpha}\left[C_1 f(t) + C_2 g(t)\right] = C_1 \; I_a^{\alpha} f(t) + C_2 \; I_a^{\alpha} g(t), \tag{1.8}$$

where C_1 *and* C_2 *are constants and* $f(t)$ *and* $g(t)$ *are functions.*

Proof.

$$\begin{aligned} I_a^{\alpha}\left[C_1 f(t) + C_2 g(t)\right] &= \frac{1}{\Gamma(\alpha)} \int_a^t (t-y)^{\alpha-1} \left[C_1 f(y) + C_2 g(y)\right] dy \\ &= C_1 \frac{1}{\Gamma(\alpha)} \int_a^t (t-y)^{\alpha-1} f(y) dy \\ &\quad + C_2 \frac{1}{\Gamma(\alpha)} \int_a^t (t-y)^{\alpha-1} g(y) dy \\ &= C_1 \; I_a^{\alpha} f(t) + C_2 \; I_a^{\alpha} g(t). \end{aligned}$$

■

Definition 1.1.5 *For every* α, *and* $n = [\alpha]+1$ *the Riemann–Liouville derivative of order* α *can be defined as:*

$$D_a^{\alpha} u(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_a^t (t-s)^{n-\alpha-1} u(s) ds. \tag{1.9}$$

Example 1.1.3 *We consider the function u defined by $u(t) = (t-a)^\beta$ and we will calculate their fractional derivative of order α such that $n = [\alpha] + 1$ by example (1.5) we get:*

$$D_a^\alpha (t-a)^\beta = \left(\frac{d}{dt}\right)^n \left[I^{n-\alpha}(t-a)^\beta\right]$$

so

$$\begin{aligned} D_a^\alpha (t-a)^\beta &= \left(\frac{d}{dt}\right)^n \left[\frac{\Gamma(\beta+1)}{\Gamma(\beta+1+n-\alpha)}(t-a)^{\beta+n-\alpha}\right] \\ &= \frac{\Gamma(\beta+1)}{\Gamma(\beta+1+n-\alpha)}\left(\frac{d}{dt}\right)^n (t-a)^{\beta+n-\alpha} \\ &= \frac{\Gamma(\beta+1)}{\Gamma(\beta+1+n-\alpha)}\frac{\Gamma(\beta+n-\alpha+1)}{\Gamma(\beta-\alpha+1)}(t-a)^{\beta-\alpha} \\ &= \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}(t-a)^{\beta-\alpha}. \end{aligned}$$

Because

$$\begin{aligned} \left(\frac{d}{dt}\right)^n (t-a)^{\beta+n-\alpha} &= (\beta+n-\alpha)(\beta+n-\alpha-1)\dots(\beta+n-\alpha-(n-1))(t-a)^{\beta-\alpha} \\ &= \frac{\Gamma(\beta+n-\alpha+1)}{\Gamma(\beta-\alpha+1)}(x-a)^{\beta-\alpha}. \end{aligned}$$

Remark 1.1.1 *The Riemann-Liouvelle fractional integration and fractional differentiation operators of the power functions t^β yield power functions of the same form. For* $\mathrm{Re}(\alpha) \geq 0$, $\mathrm{Re}(\beta) > -1$, *there are*

$$I_{0+}^\alpha t^\beta = \frac{\Gamma(\beta+1)}{\Gamma(\beta+\alpha+1)}t^{\beta+\alpha}, \quad D_{0+}^\alpha t^\beta = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}t^{\beta-\alpha}.$$

Theorem 1.1.4 *Linearity property:*

$$D_a^\alpha \left[C_1 f(t) + C_2 g(t)\right] = C_1\ D_a^\alpha f(t) + C_2\ D_a^\alpha g(t),$$

where C_1 and C_2 are constants and $f(t)$ and $g(t)$ are functions.

Proof. Since we have by using (1.8)

$$\begin{aligned} D_a^\alpha \left[C_1 f(t) + C_2 e(t)\right] &= \left(\frac{d}{dt}\right)^n \left(I_a^{n-\alpha}[C_1 f(t) + C_2 g(t)]\right) \\ &= \left(\frac{d}{dt}\right)^n \left(C_1 I_a^{n-\alpha} f(t) + C_2 I_a^{n-\alpha} g(t)\right) \\ &= C_1 \left(\frac{d}{dt}\right)^n I_a^{n-\alpha} f(t) + C_2 \left(\frac{d}{dt}\right)^n I_a^{n-\alpha} g(t) \\ &= C_1 D_a^\alpha f(t) + C_2 D_a^\alpha g(t). \end{aligned}$$

■

Theorem 1.1.5 *The following integration and derivation rules are valid:*

(*a*) $\dfrac{d}{dt}(I_a^\alpha u)(t) = (I_a^{\alpha-1}u)(t)$, *where* $\mathrm{Re}(\alpha) > 1$.

(*b*) $D_a^\alpha(I_a^\alpha u)(t) = u(t)$, *where* $\mathrm{Re}(\alpha) > 0$.

(*c*) $I_a^\alpha(D_a^\alpha u(t)) = u(t) - \sum_{j=1}^{n}\left[D_a^{\alpha-j}u(t)\right]_{t=a}\dfrac{(t-a)^{\alpha-j}}{\Gamma(\alpha-j+1)}$, *where* $\mathrm{Re}(\alpha) > 0$, $n \in \mathbb{N}$.

Proof.

(*a*) Since we have by using (1.2)

$$\begin{aligned}
\left(\frac{d}{dt}\right)(I_a^\alpha u)(t) &= \left(\frac{d}{dt}\right)\frac{1}{(\alpha-1)\Gamma(\alpha-1)}\int_a^t (t-s)^{\alpha-1}u(s)ds \\
&= \left(\frac{d}{dt}\right)\frac{1}{(\alpha-1)\Gamma(\alpha-1)}\int_0^{+\infty} (t-s)^{\alpha-1}u(s)\mathbb{1}_{[a,t]}(s)ds \\
&= \frac{1}{(\alpha-1)\Gamma(\alpha-1)}\int_0^{+\infty} (\alpha-1)(t-s)^{\alpha-2}u(s)\mathbb{1}_{[a,t]}(s)ds \\
&= \frac{1}{\Gamma(\alpha-1)}\int_a^t (t-s)^{(\alpha-1)-1}u(s)ds \\
&= (I_a^{\alpha-1}u)(t).
\end{aligned}$$

(*b*) We need to consider the case of an integer $\alpha = n \geq 1$

$$\begin{aligned}
\left[\left({}^{RL}D_a^n \circ I_a^n\right)u\right](t) &= \left(\frac{d}{dx}\right)^n\left[\left(I_a^n u\right)(t)\right] \\
&= \left(\frac{d}{dt}\right)^n\left[\frac{1}{(n-1)!}\int_a^t (t-s)^{n-1}u(s)ds\right] \\
&= \frac{d}{dt}\left[\frac{1}{(n-1)!}\int_a^t \left(\frac{d}{dt}\right)^{n-1}(t-s)^{n-1}u(s)ds\right] \\
&= \frac{d}{dt}\left[\frac{1}{(n-1)!}\int_a^t (n-1)!u(s)ds\right] \\
&= \frac{d}{dt}\int_a^t u(s)ds \\
&= u(t).
\end{aligned}$$

We take now $\alpha \in]n-1, n[$ we will have

$$\begin{aligned}
\left[D_a^\alpha\left(I_a^\alpha u\right)\right](t) &= \left(\frac{d}{dt}\right)^n\left[\left(I_a^{n-\alpha}\left(I_a^\alpha u\right)\right)(t)\right] \\
&= \left(\frac{d}{dt}\right)^n\left[\left(I_a^n f\right)(t)\right] \\
&= u(t).
\end{aligned}$$

(*c*) We have

$$\begin{aligned}
I_a^\alpha(D_a^\alpha u(t)) &= \frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}D_a^\alpha u(s)ds \\
&= \frac{d}{dt}\left\{\frac{1}{\Gamma(\alpha+1)}\int_a^t (t-s)^\alpha D_a^\alpha u(s)ds\right\}.
\end{aligned}$$

Where $\alpha \in]n-1, n[$ and there is $k \in \mathbb{N}$ such that $n = m + k$, so

$$\begin{aligned}
\frac{1}{\Gamma(\alpha+1)}\int_a^t (t-s)^\alpha D_a^\alpha u(s)ds &= \frac{1}{\Gamma(\alpha+1)}\int_a^t (t-s)^\alpha \left(\frac{d}{ds}\right)^{m+k} I^{m+k-\alpha}u(s)ds \\
&= \frac{1}{\Gamma(\alpha+1)}\int_a^t (t-s)^\alpha \left(\frac{d}{ds}\right)^{k} D_a^{-(k-\alpha)}u(s)ds \\
&= \frac{1}{\Gamma(\alpha-k+1)}\int_a^t (t-s)^{\alpha-k}\{D_a^{-(k-\alpha)}u(s)\}ds \\
&\quad -\sum_{j=1}^{k}\left[\left(\frac{d}{dt}\right)^{k-j}(D_a^{-(k-\alpha)}u(t))\right]_{t=a}\frac{(t-a)^{\alpha-j+1}}{\Gamma(2+\alpha-j)} \\
&= \frac{1}{\Gamma(\alpha-k+1)}\int_a^t (t-s)^{\alpha-k}\{D_a^{-(k-\alpha)}u(s)\}ds \\
&\quad -\sum_{j=1}^{k}\left[\left(\frac{d}{dt}\right)^{k-j}\left(\frac{d}{dt}\right)^{m} I^{m+k-\alpha}u(t)\right]_{t=a}\frac{(t-a)^{\alpha-j+1}}{\Gamma(2+\alpha-j)} \\
&= \left(\frac{d}{dt}\right)^{n}\frac{1}{\Gamma(n+\alpha-k+1)}\int_a^t (t-s)^{m+\alpha-k}\{D_a^{-(k-\alpha)}u(s)\}ds \\
&\quad -\sum_{j=1}^{k}\left[\left(\frac{d}{dt}\right)^{m+k-j} I^{m+k-j-(\alpha-j)}u(t)\right]_{t=a}\frac{(t-a)^{\alpha-j+1}}{\Gamma(2+\alpha-j)} \\
&= D_a^{-(\alpha-k+1)}\{D_a^{-(k-\alpha)}u(t)\} - \sum_{j=1}^{k}\left[D^{\alpha-j}u(t)\right]_{t=a}\frac{(t-a)^{\alpha-j+1}}{\Gamma(2+\alpha-j)} \\
&= D_a^{-1}u(t) - \sum_{j=1}^{k}\left[D^{\alpha-j}u(s)\right]_{t=a}\frac{(t-a)^{\alpha-j+1}}{\Gamma(2+\alpha-j)}.
\end{aligned}$$

Hence

$$\begin{aligned}
I_a^\alpha(D_a^\alpha u(t)) &= \frac{d}{dt}\left\{D_a^{-1}u(t) - \sum_{j=1}^{k}\left[D^{\alpha-j}u(t)\right]_{t=a}\frac{(t-a)^{\alpha-j+1}}{\Gamma(2+\alpha-j)}\right\} \\
&= u(t) - \sum_{j=1}^{k}\left[D_a^{\alpha-j}u(t)\right]_{t=a}\frac{(t-a)^{\alpha-j}}{\Gamma(\alpha-j+1)}.
\end{aligned}$$

■

Lemma 1.1.2 *For $n = [\alpha] + 1$ and $m \in \mathbb{N}$ we have:*

$$(D^m D_{0+}^\alpha u)(t) = (D_{0+}^{\alpha+m}u)(t).$$

Proof.

$$\begin{aligned}
(D^m D_{0+}^\alpha u)(t) &= \left(\frac{d}{dt}\right)^{m}\left(\frac{d}{dt}\right)^{n} I^{n-\alpha}u(t) \\
&= \left(\frac{d}{dt}\right)^{m+n} I^{n-m+m-\alpha}u(t) \\
&= (D_{0+}^{\alpha+m}u)(t).
\end{aligned}$$

■

1.2 Some functional Spaces

Definition 1.2.1 *A real Hilbert space X is a real Banach space associated with an inner product $\langle\cdot,\cdot\rangle : X\times X\to\mathbb{R}$ with the following two properties:*

- *$\langle u,v\rangle$ is a bilinear form on X, that is, it is linear in u and in v;*
- *$\|u\| = \sqrt{\langle u,u\rangle}$.*

Definition 1.2.2 *A function $u:[a,b]\to\mathbb{R}$ is said to be absolutely continuous on $[a,b]$ if, given $\varepsilon>0$, there exists some $\delta>0$ such that*

$$\sum_{i=1}^{n}\left|u\left(t_i'\right)-u\left(t_i\right)\right|<\varepsilon,$$

whenever $\{[t_i,t_i'] : i=1,\ldots,n\}$ is a finite collection of mutually disjoint subintervals of $[a,b]$ with

$$\sum_{i=1}^{n}\left|t_i'-t_i\right|<\delta.$$

Proposition 1.2.1 *The following conditions on a real-valued function u on a compact interval $[a,b]$ are equivalent*

1. *u is absolutely continuous;*
2. *there exists a Lebesgue integrable function v on $[a,b]$ such that*

$$u(t)=u(a)+\int_a^t v(s)ds,$$

for all t in $[a,b]$.

Definition 1.2.3 *[2] For $n\in N$, we denote by $\mathcal{AC}^n[0,1]$ The space $u(t)$ which have continuous derivatives up to order $n-1$ on $[0,1]$ such that $D^{(n-1)}u(t)$ is absolutely continuous:*

$$\mathcal{AC}^n[0,1]=\left\{u|[0,1]\to\mathbb{R} \text{ and } \left(D^{n-1}u\right)(t) \text{ is absolutely continuous in } [0,1]\right\}.$$

Lemma 1.2.1 *Let $\alpha>0, n=[\alpha]+1$. Assume that $u\in L^1(0,1)$ with a fractional integration of order $n-\alpha$ that belongs to $\mathcal{AC}^n[0,1]$. Then the equality*

$$\left(I_{0+}^{\alpha}D_{0+}^{\alpha}u\right)(t)=u(t)-\sum_{k=1}^{n}\frac{\left(\left(I_{0+}^{n-\alpha}u\right)(t)\right)^{(n-k)}|t=0}{\Gamma(\alpha-k+1)}t^{\alpha-k}$$

holds almost everywhere on $[0,1]$.

Definition 1.2.4 *[2] Given $\beta>0$ and $N=[\beta]+1$ we can define a linear space*

$$C^{\beta}[0,1]=\{u(t)|u(t)=I_{0+}^{\beta}x(t)+c_1t^{\beta-1}+\cdots+c_{N-1}t^{\beta-(N-1)}, x\in C[0,1], t\in[0,1]\},$$

where $c_i\in\mathbb{R},\ i=1,\ldots,N-1$.

Lemma 1.2.2 *[2] $C^{\beta}[0,1]$ is a Banach space with the norm*

$$\|u\|_{C^{\beta}} = \left\|D_{0+}^{\beta} u\right\|_{\infty} + \cdots + \left\|D_{0+}^{\beta-(N-1)} u\right\|_{\infty} + \|u\|_{\infty}.$$

Lemma 1.2.3 *[2] $F \subset \mathcal{C}^{\beta}[0,1]$ is a sequentially compact set if and only if F is uniformly bounded and equicontinuous. Here uniformly bounded means there exists $M > 0$ such that for every $u \in F$*

$$\|u\|_{\mathcal{C}^{\beta}} = \left\|D_{0+}^{\beta} u\right\|_{\infty} + \cdots + \left\|D_{0+}^{\beta-(N-1)} u\right\|_{\infty} + \|u\|_{\infty} < M,$$

and equicontinuous means that

$\forall \varepsilon > 0, \exists \delta > 0,$ *for all* $t_1, t_2 \in [0,1], |t_1 - t_2| < \delta, u \in F, i \in \{0, \cdots, N-1\},$

there hold

$$|u(t_1) - u(t_2)| < \varepsilon, \quad \left|D_{0+}^{\beta-i} u(t_1) - D_{0+}^{\beta-i} u(t_2)\right| < \varepsilon.$$

Definition 1.2.5 *[5] We define the space $\mathcal{C}^1(\overline{\Omega})$ as follows: $\mathcal{C}^1(\overline{\Omega}) = \{u \in \mathcal{C}(\overline{\Omega}) : u$ can be extended to a function $\overline{u}$ on an open set $\Omega_1 \supset \overline{\Omega}$ in such a way that $\overline{u}$ has continuous first-order partial derivatives on $\Omega_1\}$.*

1.3 The Coincidence Degree Theory of Mawhin

1.3.1 Brouwer Degree

Let $\Omega \subset \mathbb{R}^n$ be an open and bounded, $u : \Omega \to \mathbb{R}^n$ a function such that $u \in \mathcal{C}^1(\overline{\Omega})$ and $p_0 \in \mathbb{R}^n$. We consider the following problem:

$$(\mathcal{P}) \quad \text{Find } t \in \Omega,\ u(t) = p_0.$$

Example 1.3.1 *(n=1) Let $\Omega =]0,1[$ and $u : \Omega \to \mathbb{R}$ be a function of class $\mathcal{C}^1([0,1])$, satisfies the following assumption*

$$\textit{For all solution } t \textit{ of } (\mathcal{P}),\ u'(t) \neq 0.$$

We introduce the next integer

$$d(p_0) = \begin{cases} \sum_{i \in I} \operatorname{sgn}(u'(t_i)), & \textit{if } \{t_i, i \in I \subset \mathbb{N}\} \textit{ is the set of solution of } (\mathcal{P}) \\ 0, & \textit{if the problem } (\mathcal{P}) \textit{ has not solution.} \end{cases} \tag{1.10}$$

Where, the integer d depends on the function u and the open set Ω.

We give illustrative examples.

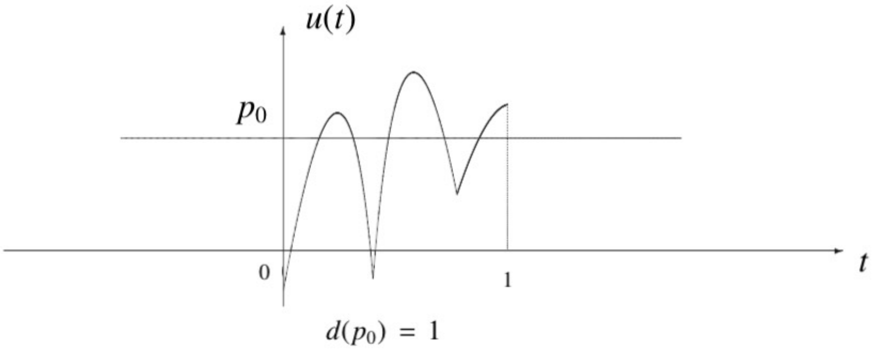

Figure 1.2

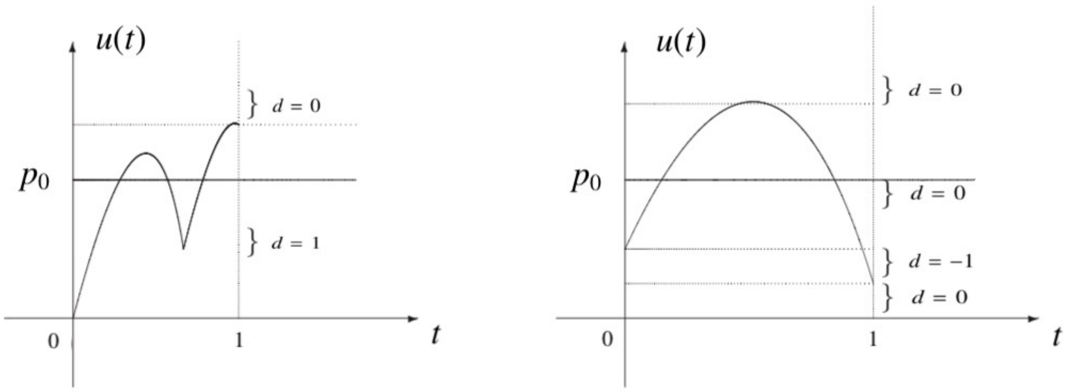

Figure 1.3

Remark 1.3.1

- *If $d(p_0) \neq 0$ the problem $(\mathcal{P})$ admits at least one solution.*
- *If $d(p_0) = 0$ the problem $(\mathcal{P})$ may or may not admit a solution.*

Continuous and Differentiable Functions

We begin with the following Bolzano's intermediate value theorem:

Theorem 1.3.1 *Let $u : [a, b] \to \mathbb{R}$ be a continuous function, then, for m between $u(a)$ and $u(b)$, there exists $t_0 \in [a, b]$ such that $u(t_0) = m$.*

Corollary 1.3.1 *Let $u : [a, b] \to \mathbb{R}$ be a continuous function such that $u(a)u(b) < 0$. Then there exists $t_0 \in (a, b)$ such that $u(t_0) = 0$.*

Corollary 1.3.2 *Let $u : [a, b] \to [a, b]$ be a continuous function. Then there exists $t_0 \in [a, b]$ such that $u(t_0) = t_0$.*

Definition 1.3.1 *Let $\Omega \subset \mathbb{R}^n$ be an open subset. then a function $u : \Omega \to \mathbb{R}^n$ is differentiable at $t_0 \in \Omega$ if there is a matrix $u'(t_0)$ such that*

$$u(t_0 + h) = u(t_0) + u'(t_0)h + \Theta(h), \tag{1.11}$$

where $t_0 + h \in \Omega$ and $\frac{|\Theta(h)|}{|h|}$ tends to zero as $|h| \to 0$.

- If $u \in \mathcal{C}^1(\overline{\Omega})$ is differentiable at t_0, we call

$$J_u(t_0) = \det u'(t_0), \tag{1.12}$$

  the Jacobian of u at t_0.

- If $J_u(t_0) = 0$, then t_0 is said to be a critical point of u. The set of all critical points of u in $\overline{\Omega}$ is denoted by $S_u(\overline{\Omega})$, where

$$S_u(\overline{\Omega}) = \{t \in \Omega : J_u(t) = 0\}. \tag{1.13}$$

Construction of Brouwer Degree

Let $\Omega \in \mathbb{R}^n$ be open and bounded and $u \in \mathcal{C}^1(\overline{\Omega})$. If $p \notin u(\partial\Omega)$, then the Brouwer degree $\deg(u, \Omega, p)$ is a tool that describes the number of solutions for equation $u(t) = p$.

Theorem 1.3.2 *[3] Let $u \in \mathcal{C}^1(\overline{\Omega})$, $p \in \mathbb{R}^n$ be given with $p \notin u(S_u)$. Then the set $u^{-1}(p)$ is either finite or empty.*

Definition 1.3.2 *[7] Let $u \in \mathcal{C}^1(\overline{\Omega})$, $p \in \mathbb{R}^n$ be given with $p \notin u(\partial\Omega)$, and $p \notin u(S_u)$. The Brouwer degree of u at p with respect to Ω, $\deg(u, \Omega, p)$, is defined by*

$$\deg(u, \Omega, p) = \sum_{t \in u^{-1}(p)} \operatorname{sgn} J_u(t)$$

where $\deg(u, \Omega, p) = 0$ if $u^{-1}(p) = \emptyset$.

Theorem 1.3.3 *[5] If $p \in \Omega$, then $\deg(I, \Omega, p) = 1$. If $p \notin \overline{\Omega}$, then $\deg(I, \Omega, p) = 0$.*

Example 1.3.2 *Let $\Omega = B(0, 2)$ and*

$$u(t) = \left(t_1^3 - 1, t_1 + 3t_2\right), \quad t \in \mathbb{R}^2$$

For this function we have $u^{-1}((0, 0)) = \{(1, -1/3)\}$ and

$$u'(x) = \begin{bmatrix} 3t_1^2 & 0 \\ 1 & 3 \end{bmatrix}$$

Thus $\operatorname{sgn} J_u((1, -1/3) = 1 = \deg(u, B(0, 2), 0)$.

Definition 1.3.3 *[7] Let $\Omega \subset \mathbb{R}^n$ be open and bounded and $u \in \mathcal{C}^1(\overline{\Omega})$. If $p \notin u(\partial\Omega)$, and $p \notin u(S_u)$. Then we define*

$$\deg(u, \Omega, p) = \deg(v, \Omega, p),$$

where $v \in \mathcal{C}^1(\overline{\Omega})$ and $|v - u| < d(p, u(\partial\Omega))$.

Theorem 1.3.4 *[7] Let $\Omega \subset \mathbb{R}^n$ be an open bounded subset and $u : \overline{\Omega} \to \mathbb{R}^n$ be a continuous mapping. If $p \notin u(\partial\Omega)$ then $\deg(u, \Omega, p)$ satisfy the following properties:*

1. ***(Solvability.)*** *If $\deg(u, \Omega, p) \neq 0$, then $u(t) = p$ has a solution in Ω.*

2. ***(Homotopy).*** *Let $H(\lambda, \cdot) : [0, 1] \times \overline{\Omega} \to \mathbb{R}^n$ be continuous and such that $H(\lambda, t) \neq p$, $\lambda \in [0, 1]$, $t \in \partial\Omega$. Then $\deg(H(\lambda, \cdot), \Omega, p)$ is a constant on $\lambda \in [0, 1]$.*

3. ***(Additivity).*** *Suppose that Ω_1, Ω_2 are two disjoint open subsets of Ω and $p \notin u\left(\overline{\Omega} - (\Omega_1 \cup \Omega_2)\right)$. Then*

$$\deg(u, \Omega, p) = \deg(u, \Omega_1, p) + \deg(u, \Omega_2, p).$$

Theorem 1.3.5 *Let $u : \overline{B(0, R)} \subset \mathbb{R}^n \to \overline{B(0, R)}$ be a continuous mapping. If $|u(t)| \leq R$ for all $t \in \partial B(0, R)$, then u has a fixed point in $\overline{B(0, R)}$.*

Proof. We assume that $t \neq u(t)$ for all $t \in \partial B(0, R)$. Put

$$H(\lambda, t) = t - \lambda u(t) \text{ for all } (\lambda, t) \in [0, 1] \times \overline{B(0, R)}.$$

Then

$$0 \neq H(\lambda, t) \text{ for all } [0, 1] \times \partial B(0, R).$$

Where $\deg(H(\lambda, \cdot), \Omega, 0)$ is a constant on $\lambda \in [0, 1]$. Therefore, we have

$$\deg(H(\lambda, \cdot), B(0, R), 0) = \deg(I - u, B(0, R), 0) = \deg(I, B(0, R), 0) = 1.$$

Where

$$|u(t)| < d(0, I(\partial B(0, R))) < d(0, \partial B(0, R)) \leqslant R.$$

Hence u has a fixed point in $\overline{B(0, R)}$. ■

Theorem 1.3.6 *[7] Let $C \subset \mathbb{R}^n$ be a nonempty bounded closed convex subset and $u : C \to C$ be a continuous mapping. Then u has a fixed point in C.*

Theorem 1.3.7 ***(Borsuk's Theorem)**. let $\Omega \subset \mathbb{R}^n$ be open bounded and symmetric with $0 \in \Omega$. If $u \in \mathcal{C}(\overline{\Omega})$ is odd and $0 \notin u(\partial\Omega)$, then $\deg(u, \Omega, 0)$ is odd.*

1.3.2 Leray-Schauder Degree

In 1934, Leray and Schauder generalized Brouwer degree theory to an infnite Banach space.

Lemma 1.3.1 *Let X be a real Banach space, $\Omega \subset X$ be an open bounded subset and $M : \overline{\Omega} \to X$ be a continuous compact mapping without a fixed point in $\partial\Omega$, so if $\varepsilon > 0$ such that $\|u - Mu\| > 4\varepsilon$ for all $u \in \partial\Omega$. Then, for any $\varepsilon > 0$, there exist a finite dimensional space E_ε and a continuous mapping $M_\varepsilon : \overline{\Omega} \to E_\varepsilon$ such that*

$$\|M_\varepsilon u - Mu\| \leqslant \varepsilon \quad \text{for all } u \in \overline{\Omega},$$

$$\|u - M_\varepsilon u\| \geqslant 3\varepsilon \quad \text{for all } u \in \partial\Omega.$$

We can define the Leray-Schauder degree of $I - M$ by the approximation M_ε

Definition 1.3.4 *Let X be a Banach space, $\Omega \subset X$ be an open bounded set and $M : \overline{\Omega} \to X$ be a continous compact mapping without a fixed point on $\partial\Omega$, let $\varepsilon > 0$, $E_\varepsilon \subset X$ and $M_\varepsilon : \overline{\Omega} \to E_\varepsilon$ given by lemma 1.3.1.*
We consider F a finite dimensional space containing E_ε, such that $\Omega_F = F \cap \Omega \neq \emptyset$. We define the Leray-Schauder degree by

$$\deg(I - M, \Omega, p) := \deg\left(I_F - M_\varepsilon, \Omega_F, p\right),$$

where $p \in E_\varepsilon$.

Theorem 1.3.8 *[5] Let $p \notin (I - M)(\partial\Omega)$. If $\deg(I - M, \Omega, p) \neq 0$, then there exists $u_0 \in \Omega$ such that $(I - M)u_0 = p$.*

Theorem 1.3.9 *[7] The Leray-Schauder degree has the following properties:*

1. ***(Normality).*** $\deg(I, \Omega, 0) = 1$ *if and only if* $0 \in \Omega$.
2. ***(Solvability).*** *If* $\deg(I - M, \Omega, 0) \neq 0$, *then* $Mu = u$ *has a solution in* Ω.
3. ***(Homotopy).*** *Let* $H(\lambda, u) : [0, 1] \times \overline{\Omega} \to X$ *be continuous compact and* $H(\lambda, u) \neq u$ *for all* $(\lambda, u) \in [0, 1] \times \partial\Omega$. *Then* $\deg\left(I - H(\lambda, \cdot), \Omega, 0\right)$ *doesn't depend on* $\lambda \in [0, 1]$.

Lemma 1.3.2 *[4] The Leray-Schauder degree of a linear isomorphism is equal to ± 1.*

1.3.3 On the Coincidence Degree Of Mawhin

Mawhin studied a class of mappings of the form $L + N$, where L is a Fredholm mapping of index zero and N is a nonlinear mapping, which it called a L-compact mapping. These two concepts will be discussed later.

Algebraic Preliminaries

Let X and Z be two vector spaces, the domain of operator L, $\operatorname{dom} L$ is a linear subspace of X, and $L : \operatorname{dom} L \subset X \to Z$ is a linear operator. Assume that the operators

$$P : X \to X, \quad Q : Z \to Z,$$

linear projection operators such that the chain

$$X \xrightarrow{P} \operatorname{dom} L \xrightarrow{L} Z \xrightarrow{Q} Z \tag{1.14}$$

is exact, that is, $\operatorname{Im} P = \operatorname{Ker} L$ and $\operatorname{Im} L = \operatorname{Ker} Q$.

Let us define the restriction of L to $\operatorname{dom} L \cap \operatorname{Ker} P$ as $L_P : \operatorname{dom} L \cap \operatorname{Ker} P \to \operatorname{Im} L$.

Lemma 1.3.3 *[2] L_P is an algebraic isomorphism.*

Now, let us define $K_P := L_P^{-1}$, where $K_P : \operatorname{Im} L \subset Z \to \operatorname{dom} L \cap \operatorname{Ker} P$ is one-to-one, onto, and $PK_P = 0$.

Lemma 1.3.4 *[4]*

1. *On* $\operatorname{Im} L$, *we have* $LK_P = L(I - P)K_P = L_P(I - P)K_P = I$.
2. *On* $\operatorname{dom} L$, *we have* $K_P L = K_P L(I - P) = K_P L_p(I - P) = (I - P)$.

Definition 1.3.5 *Let $z \in Z$ and $\operatorname{Im}(L)$ is a subspace of Z. Then $z + \operatorname{Im}(L)$ is the subset of Z defined by*

$$z + \operatorname{Im}(L) = \{z + \overline{z} : \ \overline{z} \in \operatorname{Im}(L)\}.$$

Example 1.3.3 *Let $\operatorname{Im}(L) = \{(x, 2x) \in \mathbb{R}^2 : \ x \in \mathbb{R}\}$ Then $\operatorname{Im}(L)$ is the line in $\mathbb{R}^2$ through the origin with slope 2. Thus*

$$(17, 20) + Im(L)$$

is the line in $\mathbb{R}^2$ that contains the point $(17, 20)$ and has slope 2.

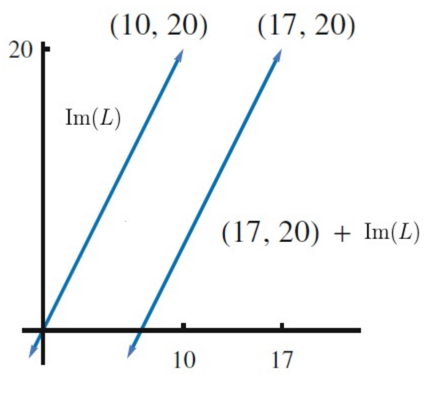

Figure 1.4

Definition 1.3.6

- *An affine subset of* Z *is a subset of* Z *of the form* $z + \mathrm{Im}(L)$ *for some* $z \in Z$ *and some subspace* $\mathrm{Im}(L)$ *of* Z.

- *For* $z \in Z$ *and* $\mathrm{Im}(L)$ *a subspace of* Z*, the affine subset* $z + \mathrm{Im}(L)$ *is said to be parallel to* $\mathrm{Im}(L)$.

Example 1.3.4 *In Example 1.3.3 above, all the lines in* $\mathbb{R}^2$ *with slope* 2 *are parallel to Im*(L).

Definition 1.3.7 *The cokernel of a linear mapping of vector spaces* $L : \mathrm{dom}\, L \to Z$ *is the quotient space* $Z/\,\mathrm{Im}(L)$ *of the codomain of* L *by the image of* L.

Definition 1.3.8 *Let* $\mathrm{Im}(L)$ *is a subspace of* Z*. Then the quotient space* $Z/\,\mathrm{Im}(L)$ *is the set of all affine subsets of* Z *parallel to* $\mathrm{Im}(L)$*. In other words,*

$$\mathrm{Coker}\, L = Z/\,\mathrm{Im}(L) = \{z + \mathrm{Im}(L) : z \in Z\}.$$

Example 1.3.5 *If Im*$(L) = \{(x, 2x) \in \mathbb{R}^2 : \; x \in \mathbb{R}\}$ *then* $\mathbb{R}^2/\,\mathrm{Im}\, L$ *is the set of all lines in* $\mathbb{R}^2$ *that have slope* 2.

Definition 1.3.9 *Let* $\mathrm{Im}(L)$ *is a subspace of* Z*. The canonic surjection operator* Π *is the linear* $\Pi : Z \to \mathrm{Coker}\, L$ *defined by*

$$\Pi(z) = z + \mathrm{Im}(L) \; for \;\; z \in Z.$$

Definition 1.3.10 *Let* Z *is finite-dimensional and* $\mathrm{Im}(L)$ *is a subspace of* Z*. Then*

$$\dim \mathrm{Coker}\, L = \dim Z/\,\mathrm{Im}(L)) = \dim Z - \dim \mathrm{Im}(L).$$

Lemma 1.3.5 *[4] The canonic surjection operator* Π *is linear and* $\mathrm{Ker}\,\Pi = \mathrm{Ker}\, Q$.

Proposition 1.3.1 *If there exists an one-to-one operator* $\Lambda : \mathrm{Coker}\, L \to \ker L$*, then*

$$Lu = z, \quad z \in Z \tag{1.15}$$

will be equivalent to

$$(I - P)u = (\Lambda\Pi + K_{P,Q})\, z. \tag{1.16}$$

Here, the operator $K_{P,Q} : Z \to X$ *is defined as*

$$K_{P,Q} = K_P(I - Q).$$

Proof. Since $\operatorname{Im} L = \ker Q = \ker \Pi$, then for $z \in \operatorname{Im} L$ we have $Qz = 0$ and $\Lambda \Pi z = 0$. From here, it is seen that

$$\begin{aligned} Lu = z &\iff Lu = z - Qz \\ &\iff K_P L u = K_P(z - Qz) \\ &\iff (I - P)u = K_P(I - Q)z \\ &\iff (I - P)u = (\Lambda \Pi + K_P(I - Q))\, z. \end{aligned}$$

■

Definition of Coincidence Degree for Some Linear Perturbations of Fredholm Mappings

Let X and Z be two real Banach spaces, $\Omega \subset X$ an open, bounded subset of X and $\overline{\Omega}$ an closure of Ω. Let us assume that the operators

$$L : \operatorname{dom} L \subset X \to Z, \quad N : \overline{\Omega} \subset X \to Z \tag{1.17}$$

satisfy the following conditions:

(i) L is linear and $\operatorname{Im} L$ is an closed subset of Z.

(ii) $\operatorname{Ker} L$ and $\operatorname{Coker} L = Z / \operatorname{Im} L$ are finite dimensional spaces and $\dim \ker L = \dim \operatorname{Coker} L$.

(iii) the operator $N : \overline{\Omega} \subset X \to Z$ is continuous and $\Pi N(\Omega)$ is bounded.

(iv) the operator $K_{P,Q} N : \overline{\Omega} \to Z$ is compact on $\overline{\Omega}$.

Definition 1.3.11 *The operator L which satisfies the conditions (i) and (ii) will be called as Fredholm operator of index zero.*

Definition 1.3.12 *The operator $N : \overline{\Omega} \to Z$ which satisfies the conditions (iii) and (iv) will be called L-compact operator.*

Theorem 1.3.10 *[8] Let Z be a Banach space. If the operator L is a Fredholm operator of index zero then there exist continuous projections $P : X \to X$ and $Q : Z \to Z$ such that the chain*

$$X \xrightarrow{P} \operatorname{dom} L \xrightarrow{L} Z \xrightarrow{Q} Z \tag{1.18}$$

will be exact.

Proposition 1.3.2 *[8] The element $u \in \operatorname{dom} L \cap \overline{\Omega}$ is a solution of the operator equation (1) if and only if it satisfies*

$$(I - P)u = (\Lambda\Pi + K_{P,Q})\, Nu. \tag{1.19}$$

In other words, the set of solutions of (1) is equal to the set of fixed points of the operator $M : \overline{\Omega} \to X$ defined by

$$M = P + (\Lambda\Pi + K_{P,Q})\, N. \tag{1.20}$$

Here, $\Lambda : \operatorname{Coker} L \to \ker L$ is any isomorphism.

Proposition 1.3.3 *[2] Assume that the conditions (i)–(iv) hold. Then, the operator M is compact on $\overline{\Omega}$.*

Lemma 1.3.6 *[4] If $0 \notin (L - N)(\operatorname{Dom} L \cap \partial\Omega)$ then the Leray-Schauder degree $(I - M, \Omega, 0)$ is well defined.*

Definition 1.3.13 *[4] If the operators L and N satisfy the conditions $(i) - (iv)$ and Lemma 1.3.6 then the coincidence degree of L and N in Ω defined by*

$$\deg((L, N), \Omega) = \deg(I - M, \Omega, 0). \tag{1.21}$$

Theorem 1.3.11 *[8] Assume that the conditions (i) to (iv) and Lemma 1.3.6 are satisfied. Then coincidence degree satisfies the following basic properties.*

1. ***(Existence theorem)**. If $\deg[(L, N), \Omega] \neq 0$, then $0 \in (L - N)(\operatorname{dom} L \cap \Omega)$.*

2. ***(Excision property)**. If $\Omega_0 \in \Omega$ is an open set such that $(L - N)^{-1}(0) \in \Omega_0$, then*

$$\deg[(L, N), \Omega] = \deg\left[(L, N), \Omega_0\right]. \tag{1.22}$$

3. ***(Additivity property)**. If $\Omega = \Omega_1 \cup \Omega_2$ with Ω_1 and Ω_2 are open, bounded, disjoint subsets of X, then*

$$\deg[(L, N), \Omega] = \deg\left[(L, N), \Omega_1\right] + \deg\left[(L, N), \Omega_2\right]. \tag{1.23}$$

4. ***(Invariance under homotopy property)**. If the operator*

$$\begin{aligned} \widetilde{N} &: \overline{\Omega} \times [0, 1] \to Z \\ (u, \lambda) &\longmapsto \widetilde{N}(u, \lambda) \end{aligned} \tag{1.24}$$

is $L-$compact in $\overline{\Omega} \times [0, 1]$ and such that for each $\lambda \in [0, 1]$, $0 \notin [L - \widetilde{N}(\cdot, \lambda)](\operatorname{dom} L \cap \partial\Omega)$, then coincidence degree $\deg[(L, N(\cdot, \lambda)), \Omega]$, is independent of λ in $[0, 1]$. In particular

$$\deg[(L, N(\cdot, 1)), \Omega] = \deg[(L, N(\cdot, 0)), \Omega]. \tag{1.25}$$

Theorem 1.3.12 *[4]* ***(Mawhin's theory of coincidence)****. Let L be a Fredholm operator of index zero and let N be $L-$compact on $\overline{\Omega}$. Assume that the following conditions are satisfied:*

(i) $Lu \neq \lambda Nu$ *for every* $(u, \lambda) \in [(\operatorname{dom} L \setminus \operatorname{Ker} L) \cap \partial\Omega)] \times (0, 1)$.

(ii) $Nu \notin \operatorname{Im} L$ *for every* $u \in \operatorname{Ker} L \cap \partial\Omega$.

(iii) $\deg(QN_{|\operatorname{Ker} L}, \Omega \cap \operatorname{Ker} L, 0) \neq 0$,

where $Q : X \to X$ *is a projection as above with* $\operatorname{Im} L = \operatorname{Ker} Q$*. Then the equation* $Lu = Nu$ *has at least one solution in* $\operatorname{dom} L \cap \overline{\Omega}$.

Chapter 2

Existence Results for a Boundary Value Problem at Resonance

Contents

2.1 Motivation

Let X and Z be two Banach spaces. Consider the operators

$$L : X \to Z, \quad N : X \to Z.$$

where L being a linear operator and N nonlinear operator. Let the operator equation

$$Lu = Nu. \tag{2.1}$$

We can write the fractional boundary value problems with the form (2.1). If L is invertible, or $\operatorname{Ker} L = \{0\}$, (2.1) is called non-resonant problem. Otherwise, if $\operatorname{Ker} L$ is not a trivial space, then it is called resonant problem.

Example 2.1.1 ***(Non-resonant case)***. *We define the boundary value problem at non-resonance as*

$$\begin{array}{c} D^{\alpha}_{0+}u(t) = f\left(t, u(t)\right), \quad 0 \leq t \leq 1 \\ D^{\alpha-1}u(0) = D^{\alpha-2}u(1) = 0, \quad u(1) = 0 \end{array}. \tag{2.2}$$

Where $2 < \alpha \leqslant 3$. *Let* $X = \mathcal{C}([0,1], \mathbb{R})$, *with the norm* $\|u\|_{\infty}$.

- *Define L to be the linear operator from X to X as*

$$Lu = D^{\alpha}_{0+}u, \quad u \in X.$$

Thus $\operatorname{Ker}(L) = \{0\}$.

Example 2.1.2 ***(Resonant case)***. *We define the boundary value problem at resonance as*

$$\begin{array}{c} D^{\alpha}_{0+}u(t) = f\left(t, u(t), Du(t)\right), \quad 0 < t < 1 \\ I^{2-\alpha}_{0+}u\left(t\right)|_{t=0} = 0, \quad \beta u(\mu) = u(1) \end{array}. \tag{2.3}$$

where $1 < \alpha \leqslant 2$, $0 < \mu < 1$, *and* $\beta \in \mathbb{R}$. *Let*

$$Z = L^1[0,1], \quad X = \mathcal{C}^{\alpha-1}[0,1]$$

with the norm

$$\|u\|_{\mathcal{C}^{\alpha-1}} = \left\|D^{\alpha-1}_{0+}u\right\|_{\infty} + \|u\|_{\infty}.$$

Then X is a Banach space.

- *Define L to be the linear operator from* $\operatorname{dom}(L) \cap X$ *to Z with*

$$\operatorname{dom}(L) = \big\{u \in \mathcal{C}^{\alpha-1}[0,1]$$
$$|D^{\alpha}_{0+}u \in L^1(0,1), D^{\alpha-2}_{0+}u(0) = 0, \ \beta u(\mu) = u(1)\},$$

and

$$Lu = D^{\alpha}_{0+}u, \quad u \in \operatorname{dom}(L).$$

Thus $\operatorname{Ker}(L) = \{ct^{\alpha-1} | c \in \mathbb{R}\}$.

2.2 Application with the Coincidence Degree of Mawhin

In this section, the following fractional order ordinary differential equation boundary value problem:

$$\begin{aligned} D_{0+}^{\alpha}u(t) &= f\left(t, u(t), D_{0+}^{\alpha-1}u(t)\right) + e(t), \quad 0 \le t \le 1 \\ I_{0+}^{2-\alpha}u\left(t\right)\big|_{t=0} &= 0, \quad D_{0+}^{\alpha-1}u(1) = \textstyle\sum_{i=1}^{m-2}\beta_i D_{0+}^{\alpha-1}u\left(\eta_i\right) . \end{aligned} \tag{2.4}$$

is considered, where $1 < \alpha \leqslant 2$, is a real number, $f : [0,1] \times \mathbb{R}^2 \to \mathbb{R}$ is continuous and $e \in L^1[0,1]$, and $\eta_i \in (0,1)$, are given constants such that $\sum_{i=1}^{m-2}\beta_i = 1$.

By using the coincidence degree theory, some existence results of solutions are established.

2.2.1 Existence Result

Here, we always suppose that $1 < \alpha \le 2$ is a real number and $\sum_{i=1}^{m-2}\beta_i = 1$.

Let

$$Z = L^1[0,1], \quad X = \mathcal{C}^{\alpha-1}[0,1]$$

with the norm

$$\|u\|_{\mathcal{C}^{\alpha-1}} = \left\|D_{0+}^{\alpha-1}u\right\|_{\infty} + \|u\|_{\infty}.$$

Then X is a Banach space.

Given a function u such that

$$D_{0+}^{\alpha}u = f(t) \in L^1(0,1) \text{ and } I_{0+}^{2-\alpha}u\left(t\right)\big|_{t=0} = 0.$$

There holds $u \in \mathcal{C}^{\alpha-1}[0,1]$. In fact, with lemma 1.2.1, one has

$$u(t) = I_{0+}^{\alpha}f(t) + c_1 t^{\alpha-1} + c_2 t^{\alpha-2}.$$

Where

$$c_1 = \frac{\left(\left(I_{0+}^{2-\alpha}u\right)(t)\right)'\big|_{t=0}}{\Gamma(\alpha)}, \quad c_2 = \frac{I_{0+}^{2-\alpha}u\left(t\right)\big|_{t=0}}{\Gamma(\alpha-1)}.$$

Combine with $I_{0+}^{2-\alpha}u\left(t\right)\big|_{t=0} = 0$ there is $c_2 = 0$. So,

$$\begin{aligned} u(t) &= I_{0+}^{\alpha}f(t) + c_1 t^{\alpha-1} \\ &= I_{0+}^{\alpha}f(t) + I_{0+}^{\alpha-1}c_1\Gamma(\alpha) \\ &= I_{0+}^{\alpha-1}\left[I_{0+}^{1}f(t) + c_1\Gamma(\alpha)\right]. \end{aligned}$$

Because

$$\begin{aligned} I_{0+}^{\alpha-1}c_1\Gamma(\alpha) &= \frac{c_1\Gamma(\alpha)}{\Gamma(\alpha-1)}\int_0^t (t-s)^{\alpha-2}ds \\ &= \frac{c_1\Gamma(\alpha)}{(\alpha-1)\Gamma(\alpha-1)}(t-s)^{\,(\alpha-1)}\Big|_0^t \\ &= c_1 t^{\alpha-1}. \end{aligned}$$

Thus $u \in \mathcal{C}^{\alpha-1}[0,1]$.

- Define L to be the linear operator from $\mathrm{dom}(L) \cap X$ to Z with

$$\mathrm{dom}(L) = \{u \in \mathcal{C}^{\alpha-1}[0,1] \\ |D^{\alpha}_{0+}u \in L^1(0,1), I^{2-\alpha}_{0+}u(0) = 0, D^{\alpha-1}_{0+}u(1) = \sum_{i=1}^{m-2} \beta_i D^{\alpha-1}_{0+}u(\eta_i)\},$$

and

$$Lu = D^{\alpha}_{0+}u, \quad u \in \mathrm{dom}(L). \tag{2.5}$$

- Define $N : X \to Z$ by

$$Nu(t) = f\left(t, u(t), D^{\alpha-1}_{0+}u(t)\right) + e(t), \quad t \in [0,1].$$

Then boundary value problem (2.4) can be written as

$$Lu = Nu.$$

Lemma 2.2.1 *Let L be defined as (2.5), then*

$$\mathrm{Ker}(L) = \left\{ct^{\alpha-1} | c \in \mathbb{R}\right\} \tag{2.6}$$

and

$$\mathrm{Im}(L) = \left\{y \in Z | \sum_{i=1}^{m-2} \beta_i \int_{\eta_i}^{1} y(s)ds = 0\right\}. \tag{2.7}$$

Proof.

- By Lemma 1.2.1, Lemma 1.1.1, $D^{\alpha}_{0+}u(t) = 0$ has solution,

$$D^{\alpha}_{0+}u(t) = 0.$$

$$\begin{aligned}
I^{\alpha}_{0+}D^{\alpha}_{0+}u(t) &= 0. \\
u(t) &= \frac{\left(I^{2-\alpha}_{0+}u(t)\right)'\Big|_{t=0}}{\Gamma(\alpha)} t^{\alpha-1} + \frac{I^{2-\alpha}_{0+}u(t)|_{t=0}}{\Gamma(\alpha-1)} t^{\alpha-2} \\
&= \frac{\left(\frac{d}{dx}\right)^2 I^{2-(\alpha-1)}_{0+}u(t)|_{t=0}}{\Gamma(\alpha)} t^{\alpha-1} + \frac{I^{2-\alpha}_{0+}u(t)|_{t=0}}{\Gamma(\alpha-1)} t^{\alpha-2} \\
&= \frac{D^{\alpha-1}_{0+}u(t)|_{t=0}}{\Gamma(\alpha)} t^{\alpha-1} + \frac{I^{2-\alpha}_{0+}u(t)|_{t=0}}{\Gamma(\alpha-1)} t^{\alpha-2}.
\end{aligned}$$

Combine with $I^{2-\alpha}_{0+}u(t)|_{t=0} = 0$. So

$$u(t) = ct^{\alpha-1},$$

where

$$c = \frac{D_{0+}^{\alpha-1}u\left(t\right)\big|_{t=0}}{\Gamma(\alpha)}.$$

Hence

$$\mathrm{Ker}(L) = \left\{ct^{\alpha-1} | c \in \mathbb{R}\right\}.$$

- Assume that $y \in \mathrm{Im}\,(L)$, where

$$\mathrm{Im}\,(L) = \left\{y \in Z |\ y = D_{0+}^{\alpha}u(t) \text{ for some } u \in \mathrm{dom}\,(L)\right\}.$$

Since $y \in \mathrm{Im}\,(L)$, there exists a function $u \in \mathrm{dom}\,(L)$ such that $y(t) = D_{0+}^{\alpha}u(t)$. Then we have

$$I_{0+}^{\alpha}y(t) = u(t) - c_1t^{\alpha-1} - c_2t^{\alpha-2}$$

where

$$c_1 = \frac{D_{0+}^{\alpha-1}u\left(t\right)\big|_{t=0}}{\Gamma(\alpha)}, \quad c_2 = \frac{I_{0+}^{2-\alpha}u\left(t\right)\big|_{t=0}}{\Gamma(\alpha-1)}.$$

By the boundary condition $I_{0+}^{2-\alpha}u\left(t\right)\big|_{t=0}$, one has $c_2 = 0$. So,

$$u(t) = I_{0+}^{\alpha}y(t) + c_1t^{\alpha-1}$$

and by lemma 1.1.1.

$$\begin{aligned} D_{0+}^{\alpha-1}u(t) &= D_{0+}^{\alpha-1}I_{0+}^{\alpha}y(t) + D_{0+}^{\alpha-1}(c_1t^{\alpha-1}) \\ &= \left(\frac{d}{dx}\right)^2 I_{0+}^{2-(\alpha-1)}I_{0+}^{\alpha}y(t) + c_1\Gamma(\alpha) \\ &= I_{0+}^{1-\alpha}I_{0+}^{\alpha}y(t) + c_1\Gamma(\alpha) \\ &= I_{0+}^{1}y(t) + c_1\Gamma(\alpha). \end{aligned}$$

Where, for $t = 1$.

$$c_1\Gamma(\alpha) = D_{0+}^{\alpha-1}u(1) - I_{0+}^{1}y(1).$$

In view of the condition $D_{0+}^{\alpha-1}u(1) = \sum_{i=1}^{m-2}\beta_iD_{0+}^{\alpha-1}u\left(\eta_i\right)$, we have

$$\begin{aligned} D_{0+}^{\alpha-1}u(1) &= \sum_{i=1}^{m-2}\beta_iD_{0+}^{\alpha-1}u\left(\eta_i\right) \\ &= \sum_{i=1}^{m-2}\beta_i\left\{I_{0+}^{1}y(\eta_i) + D_{0+}^{\alpha-1}u(1) - I_{0+}^{1}y(1)\right\} \\ &= \sum_{i=1}^{m-2}\beta_i\left(I_{0+}^{1}y(\eta_i) - I_{0+}^{1}y(1)\right) + \sum_{i=1}^{m-2}\beta_iD_{0+}^{\alpha-1}u(1) \\ &= \sum_{i=1}^{m-2}\beta_i\left(I_{0+}^{1}y(\eta_i) - I_{0+}^{1}y(1)\right) + D_{0+}^{\alpha-1}u(1)\sum_{i=1}^{m-2}\beta_i \\ &= \sum_{i=1}^{m-2}\beta_i\left(I_{0+}^{1}y(\eta_i) - I_{0+}^{1}y(1)\right) + D_{0+}^{\alpha-1}u(1) \end{aligned}$$

it is equal to

$$\begin{aligned}\sum_{i=1}^{m-2}\beta_i\left(I^1_{0+}y(\eta_i)-I^1_{0+}y(1)\right)=0 \ &\Rightarrow\ \sum_{i=1}^{m-2}\beta_i\left(\int_0^{\eta_i}y(s)ds-\int_0^1 y(s)ds\right)=0\\ &\Rightarrow\ \sum_{i=1}^{m-2}\beta_i-\left(\int_{\eta_i}^0 y(s)ds+\int_0^1 y(s)ds\right)=0\\ &\Rightarrow\ \sum_{i=1}^{m-2}\beta_i\int_{\eta_i}^1 y(s)ds=0.\end{aligned}$$

thus, we obtain (2.7).

On the other hand, suppose $y\in Z$ and satisfies:

$$\sum_{i=1}^{m-2}\beta_i\int_{\eta_i}^1 y(s)ds=0.$$

Let $u(t)=I^{\alpha}_{0+}y(t)$, then we have

$$\begin{aligned}D^{\alpha}_{0+}u(t) &= D^{\alpha}_{0+}I^{\alpha}_{0+}y(t)\\ &= y(t)\end{aligned}$$

thus $D^{\alpha}_{0+}u(t)\in L^1(0,1)$. And

$$\begin{aligned}I^{2-\alpha}_{0+}u(0) &= I^{2-\alpha}_{0+}I^{\alpha}_{0+}y(0)\\ &= 0.\end{aligned}$$

In fact, with $D^{\alpha-1}_{0+}u(t)=I^1_{0+}y(t)$, one has $D^{\alpha-1}_{0+}u(1)=\sum_{i=1}^{m-2}\beta_i D^{\alpha-1}_{0+}u\left(\eta_i\right)$, where

$$\begin{aligned}D^{\alpha-1}_{0+}u(1) &= \sum_{i=1}^{m-2}\beta_i D^{\alpha-1}_{0+}u\left(\eta_i\right)\\ &= \sum_{i=1}^{m-2}\beta_i I^1_{0+}y(\eta_i)\\ &= \sum_{i=1}^{m-2}\beta_i\int_0^{\eta_i}y(s)ds\\ &= \sum_{i=1}^{m-2}\beta_i\left(\int_0^1 y(s)ds+\int_1^{\eta_i}y(s)ds\right)\\ &= \sum_{i=1}^{m-2}\beta_i\left(\int_0^1 y(s)ds-\int_{\eta_i}^1 y(s)ds\right)\\ &= \sum_{i=1}^{m-2}\beta_i\int_0^1 y(s)ds-\sum_{i=1}^{m-2}\beta_i\int_{\eta_i}^1 y(s)ds\\ &= \int_0^1 y(s)ds\sum_{i=1}^{m-2}\beta_i\\ &= I^1_{0+}y(1).\end{aligned}$$

Therefore, $u\in\mathrm{dom}(L)$ and $D^{\alpha}_{0+}u(t)=y(t)$. So, $y\in\mathrm{Im}(L)$.

■

Lemma 2.2.2 *There exist* $k \in \{0, 1, \cdots, m-2\}$ *satisfies*

$$\sum_{i=1}^{m-2} \beta_i \eta_i^{k+1} \neq 1$$

Proof. Suppose it is not true, for any $k \in \mathbb{N}$ we have

$$\sum_{i=1}^{m-2} \beta_i \eta_i^{k+1} = 1.$$

that means

$$\begin{pmatrix} \eta_1 & \eta_2 & \cdots & \eta_{m-2} \\ \eta_1^2 & \eta_2^2 & \cdots & \eta_{m-2}^2 \\ \vdots & \vdots & \ddots & \vdots \\ \eta_1^{m-2} & \eta_2^{m-2} & \dots & \eta_{m-2}^{m-2} \end{pmatrix} \begin{pmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_{m-2} \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}$$

it is equal to

$$\begin{cases} \beta_1\eta_1 + \beta_2\eta_2 + \ldots + \beta_{m-2}\eta_{m-2} = 1 \\ \beta_1\eta_1^2 + \beta_2\eta_2^2 + \ldots + \beta_{m-2}\eta_{m-2}^2 = 1 \\ \vdots \\ \beta_1\eta_1^{m-2} + \beta_2\eta_2^{m-2} + \ldots + \beta_{m-2}\eta_{m-2}^{m-2} = 1 \\ \beta_1\eta_1^{m-1} + \beta_2\eta_2^{m-1} + \ldots + \beta_{m-2}\eta_{m-2}^{m-1} = 1 \end{cases} \Leftrightarrow \begin{cases} \beta_1\eta_1 + \beta_2\eta_2 + \ldots + \beta_{m-2}\eta_{m-2} - 1 = 0 \\ \beta_1\eta_1^2 + \beta_2\eta_2^2 + \ldots + \beta_{m-2}\eta_{m-2}^2 - 1 = 0 \\ \vdots \\ \beta_1\eta_1^{m-2} + \beta_2\eta_2^{m-2} + \ldots + \beta_{m-2}\eta_{m-2}^{m-2} - 1 = 0 \\ \beta_1\eta_1^{m-1} + \beta_2\eta_2^{m-1} + \ldots + \beta_{m-2}\eta_{m-2}^{m-1} - 1 = 0 \end{cases}$$

it is equal to

$$\begin{pmatrix} \eta_1 & \eta_2 & \cdots & \eta_{m-2} & 1 \\ \eta_1^2 & \eta_2^2 & \cdots & \eta_{m-2}^2 & 1 \\ \vdots & \vdots & \ddots & \vdots & \\ \eta_1^{m-2} & \eta_2^{m-2} & \dots & \eta_{m-2}^{m-2} & 1 \\ \eta_1^{m-1} & \eta_2^{m-1} & \cdots & \eta_{m-2}^{m-1} & 1 \end{pmatrix} \begin{pmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_{m-2} \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix}$$

it is equal to

$$\begin{pmatrix} 1 & \eta_1 & \cdots & \eta_{m-3} & \eta_{m-2} \\ 1 & \eta_1^2 & \cdots & \eta_{m-3}^2 & \eta_{m-2}^2 \\ \vdots & \vdots & \ddots & \vdots & \\ 1 & \eta_1^{m-2} & \dots & \eta_{m-3}^{m-2} & \eta_{m-2}^{m-2} \\ 1 & \eta_1^{m-1} & \cdots & \eta_{m-2}^{m-1} & \eta_{m-2}^{m-1} \end{pmatrix} \begin{pmatrix} -1 \\ \beta_1 \\ \vdots \\ \beta_{m-3} \\ \beta_{m-2} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix}.$$

In fact, with the Vandermonde Determinant is not equal to zero and it is invertible, one has

$$
\begin{aligned}
V_{m-1} &= \begin{vmatrix} 1 & \eta_1 & \cdots & \eta_{m-3} & \eta_{m-2} \\ 1 & \eta_1^2 & \cdots & \eta_{m-3}^2 & \eta_{m-2}^2 \\ \vdots & \vdots & \ddots & \vdots & \\ 1 & \eta_1^{m-2} & \ldots & \eta_{m-3}^{m-2} & \eta_{m-2}^{m-2} \\ 1 & \eta_1^{m-1} & \cdots & \eta_{m-2}^{m-1} & \eta_{m-2}^{m-1} \end{vmatrix} \\
&= \begin{vmatrix} 1 & \eta_1 & \eta_2 & \cdots & \eta_{m-2} \\ 1-\eta_1 & 0 & \eta_2(\eta_2-\eta_1) & \cdots & \eta_{m-2}(\eta_{m-2}-\eta_1) \\ \vdots & \vdots & \vdots & & \\ 1-\eta_1 & 0 & \eta_2^{m-3}(\eta_2-\eta_1) & \ldots & \eta_{m-2}^{m-3}(\eta_{m-2}-\eta_1) \\ 1-\eta_1 & 0 & \eta_2^{m-2}(\eta_2-\eta_1) & \cdots & \eta_{m-2}^{m-2}(\eta_{m-2}-\eta_1) \end{vmatrix} \\
&= \eta_1 \begin{vmatrix} 1 & \eta_2 & \cdots & \eta_{m-2} \\ 1-\eta_1 & \eta_2(\eta_2-\eta_1) & \cdots & \eta_{m-2}(\eta_{m-2}-\eta_1) \\ \vdots & \vdots & \vdots & \\ 1-\eta_1 & \eta_2^{m-3}(\eta_2-\eta_1) & \ldots & \eta_{m-2}^{m-3}(\eta_{m-2}-\eta_1) \\ 1-\eta_1 & \eta_2^{m-2}(\eta_2-\eta_1) & \cdots & \eta_{m-2}^{m-2}(\eta_{m-2}-\eta_1) \end{vmatrix} \\
&= \eta_1(1-\eta_1)(\eta_2-\eta_1)\cdots(\eta_{m-2}-\eta_1) \begin{vmatrix} 1 & \eta_2 & \cdots & \eta_{m-2} \\ 1 & \eta_2 & \cdots & \eta_{m-2} \\ \vdots & \vdots & \vdots & \\ 1 & \eta_2^{m-3} & \ldots & \eta_{m-2}^{m-3} \\ 1 & \eta_2^{m-2} & \cdots & \eta_{m-2}^{m-2} \end{vmatrix} \\
&= \eta_1(1-\eta_1)(\eta_2-\eta_1)\cdots(\eta_{m-2}-\eta_1)V_{m-2} \\
&= \eta_1(1-\eta_1)(\eta_2-\eta_1)\cdots(\eta_{m-2}-\eta_1)[\eta_2(1-\eta_2)(\eta_3-\eta_2)\cdots(\eta_{m-2}-\eta_2)]V_{m-3} \\
&= \prod_{1\le i,j\le m-2} \eta_j(1-\eta_j)(\eta_i-\eta_j).
\end{aligned}
$$

Hence

$$
\begin{pmatrix} -1 \\ \beta_1 \\ \vdots \\ \beta_{m-3} \\ \beta_{m-2} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix} \tag{2.8}
$$

there is a contradiction. ■

Lemma 2.2.3 *[2] $L : \mathrm{dom}(L) \cap X \to Z$ is a Fredholm operator of index zero. Furthermore, the linear continuous projector operators $Q : Z \to Z$ and $P : X \to X$ can be defined by*

$$Qu = C_u t^k, \quad \text{for every } u \in Z.$$

$$Pu(t) = D_{0+}^{\alpha-1} u\,(t)\big|_{t=0}\, t^{\alpha-1}, \quad \text{for every } u \in X,$$

where

$$C_u = \frac{\sum_{i=1}^{m-2} \beta_i \int_{\eta_i}^{1} u(s)ds}{(k+1)\left(1 - \sum_{i=1}^{m-2} \beta_i \eta_i^{k+1}\right)}.$$

Here $k \in \{0, 1, \cdots, m-2\}$ satisfies $\sum_{i=1}^{m-2} \beta_i \eta_i^{k+1} \neq 1$. And the linear operator

$$K_P : \mathrm{Im}(L) \to \mathrm{dom}(L) \cap \mathrm{Ker}(P)$$

can be written by

$$K_P(y) = I_{0+}^{\alpha} y(t).$$

Furthermore

$$\|K_P(y)\|_{C^{\alpha-1}} \leq \left(1 + \frac{1}{\Gamma(\alpha)}\right) \|y\|_1, \text{ for all } y \in \mathrm{Im}(L).$$

Lemma 2.2.4 *[2] For given $e \in L^1[0,1]$, $K_P(I-Q)N : X \to X$ is completely continuous.*

Theorem 2.2.1 *Let $f : [0,1] \times \mathbb{R}^2 \to \mathbb{R}$ be continuous. Assume that*

(A_1) *There exists functions $a, b, c, r \in L^1[0,1]$, and constant $\theta \in [0,1)$ such that for all $(x,y) \in \mathbb{R}^2, t \in [0,1]$ either*

$$|f(t,x,y)| \leq a(t)|x| + b(t)|y| + c(t)|y|^{\theta} + r(t). \tag{2.9}$$

Or else

$$|f(t,x,y)| \leq a(t)|x| + b(t)|y| + c(t)|x|^{\theta} + r(t). \tag{2.10}$$

(A_2) *There exists constant $M > 0$ such that for $u \in \mathrm{dom}(L)$, if $\left|D_{0+}^{\alpha-1}u(t)\right| > M$ for all $t \in [0,1]$, then*

$$\sum_{i=1}^{m-2} \beta_i \int_{\eta_i}^{1} \left[f\left(s, u(s), D_{0+}^{\alpha-1}u(s)\right) + e(s)\right] ds \neq 0.$$

(A_3) *There exists $M^* > 0$ such that for any $c \in \mathbb{R}$, if $|c| > M^*$ then either*

$$c\left(\sum_{i=1}^{m-2} \beta_i \int_{\eta_i}^{1} \left[f\left(s, cs^{\alpha-1}, c\Gamma(\alpha)\right) + e(s)\right] ds\right) < 0.$$

Or else

$$c\left(\sum_{i=1}^{m-2} \beta_i \int_{\eta_i}^{1} \left[f\left(s, cs^{\alpha-1}, c\Gamma(\alpha)\right) + e(s)\right] ds\right) > 0.$$

Then, for every $e \in L^1[0,1]$, the boundary value problem (2.4), has at least one solution in $\mathcal{C}^{\alpha-1}[0,1]$ provided that

$$\|a\|_1 + \|b\|_1 < \frac{1}{\overline{C}},$$

where $\overline{C} = \Gamma(\alpha) + 2 + \frac{1}{\Gamma(\alpha)}$.

Proof. Let

$$\Omega_1 = \{u \in \operatorname{dom}(L)\setminus \operatorname{Ker}(L) | Lu = \lambda Nu \text{ for some } \lambda \in (0,1)\}.$$

Then for $u \in \Omega_1$, $Lu = \lambda Nu$, and $Nu \in Im(L)$, hence

$$\sum_{i=1}^{m-2} \beta_i \int_{\eta_i}^{1} \left[f\left(s, u(s), D_{0+}^{\alpha-1}u(s)\right) + e(s)\right] ds = 0.$$

Thus, from $(A2)$, there exists $t_0 \in [0,1]$ such that $\left|D_{0+}^{\alpha-1}u(t)\right|_{t=t_0}| \leq M$. For $u \in \Omega_1$, there holds $D_{0+}^{\alpha-1}u \in C^{\alpha-1}[0,1], D_{0+}^{\alpha}u \in (L^1(0,1))$.

By Lemma 1.1.2.

$$DD_{0+}^{\alpha-1}u(t) = D_{0+}^{\alpha}u(t).$$

So,

$$D_{0+}^{\alpha-1}u\ (t)|_{t=0} = D_{0+}^{\alpha-1}u(t) - I_{0+}^{1}D_{0+}^{\alpha}u(t).$$

There exists $t_0 \in [0,1]$ such that

$$D_{0+}^{\alpha-1}u\ (t)|_{t=0} = D_{0+}^{\alpha-1}u\ (t)|_{t=t_0} - I_{0+}^{1}D_{0+}^{\alpha}u\ (t)|_{t=t_0}$$

Thus,

$$\begin{aligned} \left|D_{0+}^{\alpha-1}u(t)\right|_{t=0}| &\leq M + \left\|D_{0+}^{\alpha}u(t)\right\|_1 \\ &\leq M + \|Lu\|_1 \\ &\leq M + \|Nu\|_1. \end{aligned} \tag{2.11}$$

Again for $u \in \Omega_1$, $u \in \operatorname{dom}(L)\setminus \operatorname{Ker}(L)$, then $(I-P)u \in \operatorname{dom}(L) \cap \operatorname{Ker}(P)$ and $LPu = 0$. Thus from Lemma 2.2.3 , we have

$$\begin{aligned} \|(I-P)u\|_{\mathcal{C}^{\alpha-1}} &= \|K_P L(I-P)u\|_{C^{\alpha-1}} \\ &\leqslant \left(1 + \frac{1}{\Gamma(\alpha)}\right) \|L(I-P)u\|_1 \\ &= \left(1 + \frac{1}{\Gamma(\alpha)}\right) \|Lu\|_1 \\ &\leqslant \left(1 + \frac{1}{\Gamma(\alpha)}\right) \|Nu\|_1. \end{aligned} \tag{2.12}$$

From (2.11), (2.12), we have

$$\begin{aligned}
\|u\|_{\mathcal{C}^{\alpha-1}} &\leq \|Pu\|_{C^{\alpha-1}} + \|(I-P)u\|_{C^{\alpha-1}} \\
&= \|D_{0+}^{\alpha-1}Pu\|_\infty + \|Pu\|_\infty + \|(I-P)u\|_{C^{\alpha-1}} \\
&= (\Gamma(\alpha)+1)\left|D_{0+}^{\alpha-1}u(t)\big|_{t=0}\right| + \|(I-P)u\|_{C^{\alpha-1}} \\
&\leq (\Gamma(\alpha)+1)\left(M + \|Nu\|_1\right) + \left(1+\frac{1}{\Gamma(\alpha)}\right)\|Nu\|_1 \\
&= (\Gamma(\alpha)+1)M + \left(\Gamma(\alpha)+2+\frac{1}{\Gamma(\alpha)}\right)\|Nu\|_1 \\
&= (\Gamma(\alpha)+1)M + \overline{C}\|Nu\|_1. \qquad (2.13)
\end{aligned}$$

Where

$$\overline{C} = \Gamma(\alpha)+2+\frac{1}{\Gamma(\alpha)},$$

and

$$\begin{aligned}
|Nu| &= \left|f\left(t,u(t),D_{0+}^{\alpha-1}u(t)\right)+e(t)\right| \\
&\leq \left|f\left(t,u(t),D_{0+}^{\alpha-1}u(t)\right)\right| + |e(t)| \\
&\leq a(t)|u(t)| + b(t)|D_{0+}^{\alpha-1}u(t)| + c(t)|D_{0+}^{\alpha-1}u(t)|^\theta + r(t) + |e(t)| \\
&\leq a(t)\|u(t)\|_\infty + b(t)\|D_{0+}^{\alpha-1}u(t)\|_\infty + c(t)\|D_{0+}^{\alpha-1}u(t)\|_\infty^\theta + r(t) + |e(t)|.
\end{aligned}$$

If (2.9) holds, then from (2.13), we get

$$\|u\|_{\mathcal{C}^{\alpha-1}} \leq \overline{C}\left[\|a\|_1\|u\|_\infty + \|b\|_1\left\|D_{0+}^{\alpha-1}u\right\|_\infty + \|c\|_1\left\|D_{0+}^{\alpha-1}u\right\|_\infty^\theta + \|r\|_1 + \|e\|_1\right] + (\Gamma(\alpha)+1)M. \qquad (2.14)$$

Thus, from $\|u\|_\infty \leq \|u\|_{\mathcal{C}^{\alpha-1}}$ and (2.14), we obtain

$$\|u\|_\infty \leq \frac{\overline{C}}{1-\overline{C}\|a\|_1}\left[\|b\|_1\left\|D_{0+}^{\alpha-1}u\right\|_\infty + \|c\|_1\left\|D_{0+}^{\alpha-1}u\right\|_\infty^\theta + \|r\|_1 + \|e\|_1 + \frac{(\Gamma(\alpha)+1)M}{\overline{C}}\right]. \qquad (2.15)$$

Again, from (2.14), (2.15), one has

$$\begin{aligned}
\|D_{0+}^{\alpha-1}u\|_\infty &\leq \overline{C}\left[\|a\|_1\|u\|_\infty + \|b\|_1\left\|D_{0+}^{\alpha-1}u\right\|_\infty + \|c\|_1\left\|D_{0+}^{\alpha-1}u\right\|_\infty^\theta + \|r\|_1 + \|e\|_1\right] + (\Gamma(\alpha)+1)M. \\
&\leq +\frac{\overline{C}}{1-\overline{C}\|a\|_1}\|b\|_1\left\|D_{0+}^{\alpha-1}u\right\|_\infty + \frac{\overline{C}}{1-\overline{C}\|a\|_1}\|c\|_1\left\|D_{0+}^{\alpha-1}u\right\|_\infty^\theta \\
&\quad + \frac{\overline{C}}{1-\overline{C}\|a\|_1}\left[\|r\|_1 + \|e\|_1 + \frac{(\Gamma(\alpha)+1)M}{\overline{C}}\right].
\end{aligned}$$

Hence

$$\left\|D_{0+}^{\alpha-1}u\right\|_\infty \leq \frac{\overline{C}\|c\|_1}{1-\overline{C}\left(\|a\|_1+\|b\|_1\right)}\left\|D_{0+}^{\alpha-1}u\right\|_\infty^\theta + \frac{\overline{C}}{1-\overline{C}\left(\|a\|_1+\|b\|_1\right)}\left[\|r\|_1 + \|e\|_1 + \frac{(\Gamma(\alpha)+1)M}{\overline{C}}\right]. \qquad (2.16)$$

Since $\theta \in [0,1)$, from the above last inequality, there exists $M_1 > 0$ such that

$$\left\| D_{0+}^{\alpha-1} u \right\|_\infty \leq M_1, \tag{2.17}$$

thus from (2.16) and (2.15), there exists $M_2 > 0$ such that

$$\|u\|_\infty \leq M_2, \tag{2.18}$$

hence

$$\begin{aligned} \|u\|_{C^{\alpha-1}} &= \|u\|_\infty + \left\| D_{0+}^{\alpha-1} u \right\|_\infty \\ &\leq M_1 + M_2. \end{aligned}$$

Therefore $\Omega_1 \subset X$ is bounded.

If (2.10) holds, similar to the above argument, we can prove that Ω_1 is bounded too.

Let

$$\Omega_2 = \{u \in \mathrm{Ker}(L) | Nu \in \mathrm{Im}(L)\}.$$

For $u \in \Omega_2$, there is

$$u \in \mathrm{Ker}(L) = \left\{ u \in \mathrm{dom}(L) | u = ct^{\alpha-1}, c \in \mathbb{R}, t \in [0,1] \right\},$$

and

$$Nu \in \mathrm{Im}(L),$$

thus

$$\sum_{i=1}^{m-2} \beta_i \int_{\eta_i}^1 \left[f\left(s, cs^{\alpha-1}, c\Gamma(\alpha)\right) + e(s) \right] ds = 0.$$

From $(A2)$, there exists $t_0 \in [0,1]$ such that $\left| D_{0+}^{\alpha-1} u(t) \right|_{t=t_0} | \leq M$, where $u \in \mathrm{dom}(L)$, we get $|c| \leq \dfrac{M}{\Gamma(\alpha)}$, thus Ω_2 is bounded in X.

Next, according to the condition $(A3)$, for any $c \in \mathbb{R}$, if $|c| > M^*$, then either

$$c \left(\sum_{i=1}^{m-2} \beta_i \int_{\eta_i}^1 \left[f\left(s, cs^{\alpha-1}, c\Gamma(\alpha)\right) + e(s) \right] ds \right) < 0. \tag{2.19}$$

Or else

$$c \left(\sum_{i=1}^{m-2} \beta_i \int_{\eta_i}^1 \left[f\left(s, cs^{\alpha-1}, c\Gamma(\alpha)\right) + e(s) \right] ds \right) > 0. \tag{2.20}$$

If (2.19) holds, let

$$\Omega_3 = \{u \in Ker(L) | -\lambda Vu + (1-\lambda) QNu = 0, \lambda \in [0,1]\} \tag{2.21}$$

where $V : \text{Ker}(L) \to \text{Im}(Q)$ is the linear isomorphism given by $V\left(ct^{\alpha-1}\right) = ct^k, \forall c \in \mathbb{R}, t \in [0,1]$. For $u = c_0 t^{\alpha-1} \in \Omega_3$,

$$\lambda c_0 t^k = (1-\lambda)\left(\sum_{i=1}^{m-2} \beta_i \int_{\eta_i}^{1} \left[f\left(s, c_0 s^{\alpha-1}, c_0\Gamma(\alpha)\right) + e(s)\right] ds\right). \tag{2.22}$$

If $\lambda = 1$, then $c_0 = 0$. Otherwise, if $|c_0| > M^*$, in view of (2.19), one has

$$c_0(1-\lambda)\left(\sum_{i=1}^{m-2} \beta_i \int_{\eta_i}^{1} \left[f\left(s, c_0 s^{\alpha-1}, c_0\Gamma(\alpha)\right) + e(s)\right] ds\right) < 0, \tag{2.23}$$

which contradicts to $\lambda c_0^2 \geq 0$. Thus

$$\Omega_3 \subset \left\{u \in \text{Ker}(L) |\ u = ct^{\alpha-1},\ |c| \leq M^*\right\} \tag{2.24}$$

is bounded in X.

If (2.20) holds, then define the set

$$\Omega_3 = \left\{u \in \text{Ker}(L) | \lambda V u + (1-\lambda) Q N u = 0, \lambda \in [0,1]\right\}, \tag{2.25}$$

here V as in above. Similar to above argument, we can show that 3 is bounded too.

In the following, we shall prove that all conditions of Theorem 1.3.12 are satisfied. Set Ω be a bounded open set of X such that $\bigcup_{i=1}^{3} \overline{\Omega}_i \subset \Omega$.

By Lemma 2.2.4, $K_P(I-Q)N : \overline{\Omega} \to X$ is compact, thus N is L-compact on $\overline{\Omega}$. Then by above arguments, we have

(*i*) $Lu \neq \lambda N u$ for every $(u, \lambda) \in [(\text{dom } L \setminus \text{Ker } L) \cap \partial\Omega)] \times (0,1)$;

(*ii*) $Nu \notin \text{Im } L$ for every $u \in \text{Ker } L \cap \partial\Omega$;

Finally, we will prove that (*iii*) of Theorem 1.3.12 is satisfied. Let

$$H(u, \lambda) = \pm \lambda V u + (1-\lambda) Q N u. \tag{2.26}$$

According to the above argument, we know

$$H(u, \lambda) \neq 0, \text{ for all } u \in \text{Ker}(L) \cap \partial\Omega. \tag{2.27}$$

Thus, by the homotopy property of degree

$$\begin{aligned}
\deg\left(Q\ N|_{Ker(L)}, \Omega \cap \text{Ker}(L), 0\right) &= \deg(H(\cdot,0), \Omega \cap \text{Ker}(L), 0) \\
&= \deg(H(\cdot,1), \Omega \cap Ker(L), 0) \\
&= \deg(\pm V, \Omega \cap \text{Ker}(L), 0) \\
&\neq 0.
\end{aligned}$$

Then by Theorem 1.3.12, $Lu = Nu$ has at least one solution in $\text{dom}(L) \cap \overline{\Omega}$, so that the problem (2.4) has one solution in $C^{\alpha-1}[0,1]$. ■

2.2.2 Example

Example 2.2.1 *Consider the boundary value problem*

$$\begin{aligned} D_{0+}^{\frac{3}{2}}u(t) = \tfrac{1}{10}\sin(u(t)) + \tfrac{1}{10}D_{0+}^{\frac{1}{2}}u(t) + 3\sin\left(D_{0+}^{\frac{1}{2}}u(t)\right)^{\frac{1}{3}} + 1 + \cos^2 t \quad 0<t<1 \\ I_{0+}^{\frac{1}{2}}u(0) = 0, \quad D_{0+}^{\alpha-1}u(1) = 6D_{0+}^{\frac{1}{2}}u\left(\tfrac{1}{3}\right) - 5D_{0+}^{\frac{1}{2}}u\left(\tfrac{1}{2}\right) \end{aligned} \tag{2.28}$$

Let $\beta_1 = 6, \beta_2 = -5, \eta_1 = \frac{1}{3}, \eta_2 = \frac{1}{2}$ *and*

$$f(t,x,y) = \frac{\sin x}{10} + \frac{y}{10} + 3\sin\left(y^{\frac{1}{3}}\right), \quad e(t) = 1 + \cos^2 t \tag{2.29}$$

then

$$\beta_1 + \beta_2 = 1. \quad |f(t,x,y)| \le \frac{|x|}{10} + \frac{|y|}{10} + 3|y|^{\frac{1}{3}} \tag{2.30}$$

Again, taking $a(t) = b(t) \equiv \frac{1}{10}$, *then*

$$\|a\|_1 + \|b\|_1 = \frac{1}{5} < \frac{1}{\Gamma\left(\frac{3}{2}\right) + 2 + \frac{1}{\Gamma\left(\frac{3}{2}\right)}} \approx \frac{1}{4} \tag{2.31}$$

For any $u \in C^{\frac{1}{2}} \cap I_{0+}^{\frac{3}{2}}\left(L^1[0,1]\right)$, *For* $M = 52$, *assume* $\left|D_{0+}^{\frac{1}{2}}u(t)\right| > M$ *holds for any* $t \in [0,1]$. *Since the continuity of* $D_{0+}^{\frac{1}{2}}u$, *then either* $D_{0+}^{\frac{1}{2}}u(t) > M$ *or* $D_{0+}^{\frac{1}{2}}u(t) < -M$ *holds for any* $t \in [0,1]$. *If* $D_{0+}^{\frac{1}{2}}u(t) > M$ *holds for any* $t \in [0,1]$, *then*

$$f\left(t, u(t), D_{0+}^{\frac{1}{2}}u(t)\right) + e(t) \ge \frac{M-21}{10} > 0, \tag{2.32}$$

so

$$\begin{aligned} & 6\int_{\frac{1}{3}}^{1}\left[f\left(s,u(s),D_{0+}^{\frac{1}{2}}u(s)\right)+e(s)\right]ds - 5\int_{\frac{1}{2}}^{1}\left[f\left(s,u(s),D_{0+}^{\frac{1}{2}}u(s)\right)+e(s)\right]ds \\ > & \int_{\frac{1}{3}}^{1}\left[f\left(s,u(s),D_{0+}^{\frac{1}{2}}u(s)\right)+e(s)\right]ds \\ \ge & \frac{2(M-21)}{30} > 0 \end{aligned}$$

If $D_{0+}^{\frac{1}{2}}u(t) < -M$ *hold for any* $t \in [0,1]$, *then*

$$f\left(t, u(t), D_{0+}^{\frac{1}{2}}u(t)\right) + e(t) \le \frac{51-M}{10} < 0 \tag{2.33}$$

so

$$\begin{aligned} & 6\int_{\frac{1}{3}}^{1}\left[f\left(s,u(s),D_{0+}^{\frac{1}{2}}u(s)\right)+e(s)\right]ds - 5\int_{\frac{1}{2}}^{1}\left[f\left(s,u(s),D_{0+}^{\frac{1}{2}}u(s)\right)+e(s)\right]ds \\ < & \int_{\frac{1}{3}}^{1}\left[f\left(s,u(s),D_{0+}^{\frac{1}{2}}u(s)\right)+e(s)\right]ds \\ < & \int_{\frac{1}{3}}^{1}\left[f\left(s,u(s),D_{0+}^{\frac{1}{2}}u(s)\right)+e(s)\right]ds \\ \le & \frac{2(51-M)}{30} < 0 \end{aligned}$$

Thus, the condition $(A2)$ *holds. Again, taking* $M^* = \frac{52}{\Gamma(3/2)}$, *for any* $k \in \mathbb{R}$, *if* $|c| > M^*$, *we have*

$$c\left(6\int_{\frac{1}{3}}^{1}\left[f\left(s, cs^{\frac{1}{2}}, c\Gamma\left(\frac{3}{2}\right)\right) + e(s)\right] ds - 5\int_{\frac{1}{2}}^{1}\left[f\left(s, cs^{\frac{1}{2}}, c\Gamma\left(\frac{3}{2}\right)\right) + e(s)\right] ds\right) > 0 \quad (2.34)$$

So, the condition $(A3)$ *holds. Thus, with Theorem 2.2.1, the boundary value problem (2.28), has at least one solution in* $C^{\frac{1}{2}}[0,1]$.

Bibliography

[1] S. Axler, *Linear Algebra Done Right Third Edition,* Department of Mathematics San Francisco State University San Francisco, CA, USA, 2015.

[2] Z.B. Bai, *On Solutions of some Fractional m-point Boundary Value Problem at Resonance,* Elect. J. Qualit. Theo. Diff. Equa, No. 37, 2010, 1-15.

[3] S. Djebali, *Degré Topologique,* Département de Mathématiques, ENS-Kouba, Alger, Algérie, 2007.

[4] R.E. Gaines and J.L. Mawhin: *Coincidence Degree, and Nonlinear Differential Equations,* Springer-Verlag Berlin. Heidelberg, 1977.

[5] A. G. Kartsatos: *Advanced Ordinary Differential Equations,* Mancorp, 1993.

[6] C. Milici, Gh. Draganescu and J.T. Machado, *Introduction to Fractional Differential Equations,* Nonlinear Systems and Complexity, 2019.

[7] D. O'Regan, Y. Je Cho and Y. Che, *Topological Degree Theory and Applications,* Taylor and Francis, Chapman and Hall/CRC, Florida, Fla, USA, 2006.

[8] A. Sırma and S. Sevgin, *A Note on Coincidence Degree Theory,* Abstract and Applied Analysis, No. 370946, 2012, 1-18.